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Counting BPS Operators in $\mathcal{N}=4$ SYM

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The free field partition function for a generic $U(N)$ gauge theory, where the fundamental fields transform in the adjoint representation, is analysed in terms of symmetric polynomial techniques. It is shown by these means how this is related to the cycle polynomial for the symmetric group and how the large N result may be easily recovered. Higher order corrections for finite N are also discussed in terms of symmetric group characters. For finite N, the partition function involving a single bosonic fundamental field is recovered and explicit counting of multi-trace quarter BPS operators in free $\mathcal{N}=4$ super Yang Mills discussed, including a general result for large N. The partition function for quarter BPS operators in the chiral ring of $\mathcal{N}=4$ super Yang Mills is analysed in terms of plane partitions. Asymptotic counting of BPS primary operators with differing R-symmetry charges is discussed in both free $\mathcal{N}=4$ super Yang Mills and in the chiral ring. Also, general and explicit expressions are derived for $SU(2)$ gauge theory partition functions, when the fundamental fields transform in the adjoint, for free field theory.

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1. Introduction

For the past while there has been intense interest in finite N partition functions for Yang Mills theories, especially in super-symmetric ones, particularly with regard to their construction for BPS states and the counting thereof [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15]. Much attention has been devoted to this issue for $\mathcal{N} = 4$ super Yang Mills theory, it being by now the archetypal example of a conformal field theory for which we have a dual description in terms of string theory, by means of the AdS/CFT conjecture [16]. An example is in the assiduous efforts that have been made to explain the entropy of certain BPS black holes in $\mathrm{AdS}_5 \times S^5$ [17,18,19,20] in terms of microscopic counting of dual operators in $\mathcal{N} = 4$ super Yang Mills, with gauge group $SU(N)$, by means of partition functions [1]. While this problem remains unsolved to date, essentially due to the difficulty of defining what is meant by these dual operators for finite N, there are other equally interesting sectors of (super) Yang Mills theories where more progress with counting has been made, examples being in free $\mathcal{N} = 4$ super Yang Mills and in chiral ring sectors involving BPS operators.

For $\mathcal{N} = 4$ super Yang Mills, the half BPS sector consists of multi-trace operators involving a single bosonic operator, Z. Similarly, the quarter BPS sector consists of multi-trace operators involving two bosonic operators Z, Y while the eighth consists of ones involving three bosonic operators Z, Y, X and two fermionic operators $\lambda, \bar{\lambda}$. (All these operators are here assumed to belong to the Lie algebra of $U(N)$.)

For the chiral ring, the commuting/anti-commuting of these operators is at the heart of why we can write very concise and elegant generating functions for the finite N multi-trace partition functions [1]. Analysis of these partition functions, in terms of the counting of operators, has become a sophisticated industry where such approaches as the so-called 'Plethystic Program' have provided substantial results [3,9].

This paper is devoted largely to the issue of partition functions for free field theory and particularly to the counting of gauge invariant multi-trace operators for the case of two bosonic fundamental fields. This is of relevance to the quarter BPS sector of $\mathcal{N} = 4$ super Yang Mills in the free field limit when the operators Z, Y do not commute, in contrast to the chiral ring when they do.

The partition function for a free, massless quark-gluon gas was computed long ago [21]. This involved taking particle statistics into account using coherent state techniques and then imposing the gauge singlet condition by integrating over the relevant gauge group.

With some modifications to the expression thus derived we may write the multi-trace partition function for some generic bosonic/fermionic fundamental fields in terms of an integral over the gauge group, involving the single particle partition function [22,23]. This is the starting point here.

For $U(N)$ gauge theories we may easily write down the integral, though, even for this case, its evaluation is far from simple. One approach (which we adapt here for the $SU(2)$ case) is to rewrite the expression in terms of an N-fold contour integral, whereby it may in principle be evaluated by summing the contributions from poles inside origin centred unit discs, in each of the N complex planes - similar techniques have been used in [10]. Due to the number of poles this becomes unfeasible for higher values of N. Another approach is to use the fact that the complex integral that interests us provides for an inner product for symmetric polynomials - see Macdonald [24], pp. 363 - 372, for a related discussion. Taking this point more seriously reveals an alternative route to evaluating the free field theory partition function which exposes not only the large N case in an almost trivial way, but also how and where this differs from the finite N case.

This treatment also reveals an alternative interpretation of the free field partition function at finite N - it is related to a gauge group average of the cycle polynomial for the symmetric permutation group (after a certain identification of 'letters' with gauge group valued variables). This point is not dwelt on further here though makes the connection between the partition function and Polya enumeration explicit. For single trace operators at large N this connection has already been made [22,25] for $\mathcal{N} = 4$ super Yang Mills, whereby the partition function for single trace operators is related to the cycle polynomial for the cyclic permutation group.[1]

Another issue is how to use the expression for the free field partition function to give explicit counting of gauge invariant multi-trace operators in a Yang-Mills theory, with gauge group $U(N)$. The case for one bosonic fundamental field has been widely discussed and, for finite N, the operators are counted by partition numbers of non-negative integers into at most N parts (which is here denoted by $p_N(n)$ - no closed formula for these numbers for arbitrary N, n exist, though they have a 'nice' generating function). Here, this result is re-derived, from a symmetric function perspective, by employing the well known Cauchy-Littlewood formula.

To proceed further with counting, for the quarter BPS sector of $\mathcal{N} = 4$ super Yang Mills, for instance, character methods prove to be both natural and indispensable. Char-

[1] For more recent applications of symmetric polynomials and Polya enumeration to supersymmetric quantum mechanical models, analysed in terms of Fock space methods, see [26,27,28].

acters in relation to conformal field theories prove to be very convenient for encoding the allowable representations [15,29,30,31,32,33] and for studying related partition functions, [34,35,36,33]. For $\mathcal{N}=4$ super Yang Mills, it was shown in [33] that, if we are to distinguish among primary operators with differing conformal dimensions, spins and R-symmetry charges, such counting is most easily achieved using reductions of the full $\mathcal{N}=4$ superconformal characters, in certain limits that isolate corresponding sectors of short/semi-short operators. (One such limit corresponds to the index constructed in [1].) This point may be easily illustrated for quarter BPS primary operators in $\mathcal{N}=4$ super Yang Mills, the case of two bosonic fundamental fields here. (See [37] for an explicit construction and counting of quarter BPS operators.) For $\mathcal{N}=4$ super Yang Mills, the counting of quarter BPS primary operators is complicated by the fact that, if we are to keep track of differing R-symmetry representations, any partition function restricted to this sector must be expanded in terms of $U(2)$ characters (or two-variable Schur polynomials). Denoting some partition function restricted to this sector by $\mathcal{Z}(t,u)$, where t,u are letters corresponding to the fields Z,Y, then by expanding

$$\mathcal{Z}(t,u)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathcal{N}_{(n,m)}\,s_{(n,m)}(t,u)\,,\qquad s_{(n,m)}(t,u)=\frac{t^{n+1}u^m-t^m u^{n+1}}{t-u}\,,\tag{1.1}$$

in terms of two-variable Schur polynomials $s_{(n,m)}(t,u)$, we obtain the numbers $\mathcal{N}_{(n,m)}$ of gauge invariant quarter BPS operators belonging to the $[m,n{-}m,m]$ $SU(4)_R$ R-symmetry representation and having conformal dimension $n+m$ (so that they are superconformal primary highest weight states in the corresponding quarter BPS supermultiplets). (The case $m=0$ counts gauge invariant half BPS primary operators.)

Here, the free field partition function is thus expanded in terms of Schur polynomials, depending on the same variables as the one particle partition function, the two boson case being a specialisation. This is quite naturally achieved using the Cauchy-Littlewood formula (and, if we include fermions, another formula due to Littlewood). Generally, we may obtain a result that relates the counting numbers to a sum over Kronecker coefficients. These arise naturally in the theory of the symmetric permutation group, though remain somewhat mysterious from a combinatorial perspective.

Specialising to the two boson case, a recursive procedure is employed here for the counting of multi-trace quarter BPS operators in free field theory at finite N. This issue was given considerable discussion for the large N case in [33] - here the results of [33] are generalised in terms of a generating function that may be employed to count quarter BPS operators for any R-symmetry charges, at large N. Asymptotic counting is also addressed in the latter case for the numbers $\mathcal{N}_{(n,m)}$ in (1.1) for large n and fixed m.

To complete the discussion, counting of quarter BPS operators in the chiral ring of

$\mathcal{N} = 4$ super Yang Mills is investigated in terms of expanding over $U(2)$ characters as in (1.1). An explicit formula is given for the corresponding finite N partition function, with a short combinatorial interpretation given in terms of plane partitions, and specialised to large N. For the latter case, the exponential behaviour of the numbers $\mathcal{N}_{(n,m)}$ in (1.1) is found for n and m both comparably large. This behaviour is consistent with a special case addressed in [3,9]. By way of completion, a similar discussion for an arbitrary number of bosonic fundamental fields in the chiral ring is included.

Two appendices are included; the first establishes some notation used for partitions and gives some standard results for the symmetric group and symmetric polynomials, the second gives some tables of numbers of quarter BPS operators in free $\mathcal{N} = 4$ super Yang Mills, with gauge group $U(N)$, for which explicit formulae are given in the main text. Footnotes contain further details and points of clarification.

2. Free Field Partition Functions

We start from the single particle partition function which is here denoted by $f(\mathrm{t})$ for some variables[2] $\mathrm{t} = (t_1, t_2 \ldots)$. The general form of $f(\mathrm{t})$ is

$$f(\mathrm{t}) = \sum_i a_i t_i \,, \tag{2.1}$$

where each t_i is a letter corresponding to a fundamental field and a_i are signs, being $+1$ for a bosonic field, or -1 for a fermionic field.

For compact gauge Lie group G, the multi-trace partition function is then given by [21], (see [22,23] for refinements,)

$$\mathcal{Z}_G(\mathrm{t}) = \int_G \mathrm{d}\mu_G(g) \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} f(\mathrm{t}^n) \chi_R(g^n) \Big) \,, \tag{2.2}$$

involving the Haar (or G-invariant, or Hurwitz) measure $\mathrm{d}\mu_G(g)$ for $g \in G$ (so that $\int_G \mathrm{d}\mu_G(g) F(g) = \int_G \mathrm{d}\mu_G(g) F(gh) = \int_G \mathrm{d}\mu_G(g) F(hg)$ for all $h \in G$ and $\int_G \mathrm{d}\mu_G(g) = 1$) and where $\chi_R(g)$ is the character for the R representation of G, assuming that the fundamental fields transform in identical gauge group representations R.

[2] In what follows roman letters are used to denote a collection of variables and, for $\mathrm{x} = (x_1, \ldots, x_i)$, $\mathrm{y} = (y_1, \ldots, y_j)$ for example, the shorthand x^α is used to mean $({x_1}^\alpha, \ldots, {x_i}^\alpha)$ and $\mathrm{z} = \mathrm{xy}$ to mean $\mathrm{z} = (z_{11}, \ldots, z_{ij})$ where $z_{rs} = x_r y_s$. The latter convenient notation has been used by Macdonald [24].

For $G = U(N)$, so that for any matrix $U \in U(N)$ we may write $U = V\Theta V^\dagger$, where V is a unitary matrix and $\Theta = \mathrm{diag.}(e^{i\theta_1}, \ldots, e^{i\theta_N})$, $0 \leq \theta_i < 2\pi$, and for some $F(U) = F(\Theta)$, independent of V, then we may write

$$\int_{U(N)} \mathrm{d}\mu_{U(N)}(U) F(U) = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \prod_{j=1}^{N} \mathrm{d}\theta_j \prod_{1\leq k<l\leq N} |e^{i\theta_k} - e^{i\theta_l}|^2 \, F(\Theta)\,, \tag{2.3}$$

which is, of course, related to the Weyl parametrisation of $U(N)$. Thus, for such $F(U)$, the left-hand side of (2.3) simplifies to an integral over the N torus. Of course, as $\chi_R(U)$ generally depends on linear combinations of $\mathrm{tr}(U^j)^k \mathrm{tr}(U^{\dagger l})^m$ for various non-negative integers j, k, l, m then any function of $\chi_R(U)$ is an example of such an $F(U)$.

We are interested in the case where $R = \mathrm{Adj.}$ is the adjoint representation so that for $U(N)$ we have that $\chi_{\mathrm{Adj.}}(U) = \mathrm{tr}\, U\, \mathrm{tr}\, U^\dagger$ (while for $SU(N)$ then $\chi_{\mathrm{Adj.}}(U) = \mathrm{tr}\, U\, \mathrm{tr}\, U^\dagger - 1$). For $U(N)$ we then find that, using (2.2) with (2.3),

$$\mathcal{Z}_{U(N)}(\mathrm{t}) = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \prod_{j=1}^{N} \mathrm{d}\theta_j \prod_{1\leq k<l\leq N} |e^{i\theta_k} - e^{i\theta_l}|^2 \, \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} f(\mathrm{t}^n) \sum_{j,k=1}^{N} e^{in(\theta_j - \theta_k)} \Big)\,. \tag{2.4}$$

We may write (2.4) as an N-fold contour integral by first making the variable change $z_j = e^{i\theta_j}$ so that the integrals in (2.4) are around unit circles in each z_j complex plane and then we obtain

$$\mathcal{Z}_{U(N)}(\mathrm{t}) = \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i} \Delta(\mathrm{z})\Delta(\mathrm{z}^{-1}) \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} f(\mathrm{t}^n) p_n(\mathrm{z}) p_n(\mathrm{z}^{-1}) \Big)\,, \tag{2.5}$$

where $\Delta(\mathrm{z}) = \prod_{1\leq i<j\leq N}(z_i - z_j)$ is the Vandermonde determinant and $p_n(\mathrm{z}) = \sum_{i=1}^{N} {z_i}^n$ is a power symmetric polynomial - see appendix A for a brief discussion of symmetric polynomials. This integral may then in principle be evaluated by deforming the contours so as to extract the residues at poles within the discs $|z_j| < 1$, $1 \leq j \leq N$.

A crucial observation is that, for some N variable symmetric polynomials $g(\mathrm{z}), h(\mathrm{z})$, then

$$\big\langle g, h \big\rangle_N = \big\langle h, g \big\rangle_N = \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i} \Delta(\mathrm{z})\Delta(\mathrm{z}^{-1}) g(\mathrm{z}) h(\mathrm{z}^{-1})\,, \tag{2.6}$$

acts as an inner product - this is easy to see in terms of Schur polynomials which provide an orthonormal basis for symmetric polynomials. The reader may now wish to peruse appendix A where notation regarding partitions and a short discussion of symmetric polynomials is included.

The General and Large N Cases for $U(N)$

For application of inner products to (2.5) we have that, in terms of power symmetric polynomials $p_\lambda(\mathrm{z})$ for partitions λ,

$$\begin{aligned}\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}f(\mathrm{t}^n)p_n(\mathrm{z})p_n(\mathrm{z}^{-1})\Big) &= \prod_{n=1}^{\infty}\sum_{a_n=0}^{\infty}\frac{1}{n^{a_n}a_n!}f(\mathrm{t}^n)^{a_n}p_n(\mathrm{z})^{a_n}p_n(\mathrm{z}^{-1})^{a_n}\,,\\ &= \sum_{\lambda}\frac{1}{z_\lambda}f_\lambda(\mathrm{t})p_\lambda(\mathrm{z})p_\lambda(\mathrm{z}^{-1})\,,\end{aligned} \tag{2.7}$$

with the definitions of

$$z_\lambda = \prod_{n=1}^{\infty} n^{a_n}a_n!\,,\qquad f_\lambda(\mathrm{t}) = \prod_{n=1}^{\infty} f(\mathrm{t}^n)^{a_n}\,, \tag{2.8}$$

being in terms of the frequency representation of λ, $(1^{a_1}, 2^{a_2}, \ldots)$ with $\sum_{n\geq 1} n\, a_n = |\lambda|$, the weight of the partition λ (note that the frequency representation of λ is simply a convenient re-ordering of the parts of λ).

In (2.7) the numbers z_λ have a standard combinatorial interpretation - for a given permutation $\sigma \in \mathcal{S}_m$ with a_1 1-cycles, a_2 2-cycles etc., so that $\sum_{n\geq 1} n\, a_n = m = |\lambda|$, then $z_\lambda = \prod_{n\geq 1} n^{a_n}a_n!$ is the size of the centraliser $Z_\sigma = \{\tau; \tau \in \mathcal{S}_m,\ \tau\sigma\tau^{-1} = \sigma\}$ of $\sigma \in \mathcal{S}_m$. (This may be easily seen as under conjugation of σ by τ then τ can permute the cycles of length n among themselves in $a_n!$ ways and/or render a cyclic rotation on each of the individual cycles in n^{a_n} ways.) More details of the symmetric group are to be found in appendix A.

We may immediately observe that (2.7) represents a sum over cycle polynomials of the symmetric group $\mathcal{S}_m$. This is given by, for letters $u_1, u_2, \ldots u_m$,[3]

$$C_m(\mathrm{u}) = \sum_{a_1,\ldots,a_m\geq 0}\delta_{a_1+2a_2+\ldots+ma_m,m}\prod_{n=1}^{m}\frac{1}{n^{a_n}a_n!}\,{u_n}^{a_n} = \sum_{\lambda\vdash m}\frac{1}{z_\lambda}u_\lambda\,, \tag{2.9}$$

[3] This formula is easy to see from the definition of the cycle polynomial for a subgroup G of $\mathcal{S}_m$. This is given by

$$\frac{1}{|G|}\sum_{g\in G}{u_1}^{j_1(g)}\cdots {u_m}^{j_m(g)} = \frac{1}{|G|}\sum_{K_g}|K_g|{u_1}^{j_1(g)}\cdots {u_m}^{j_m(g)}\,,$$

where $j_i(g)$ denotes the number of i cycles in the unique decomposition of g into disjoint cycles and K_g denotes the conjugacy classes of G with class representatives g. The size of the conjugacy class K_g is given by $|K_g| = |G|/|Z_g|$ where Z_g is the centraliser of $g \in G$. For the present case then, $G = \mathcal{S}_m$, $|\mathcal{S}_m| = m!$, and $|Z_\sigma| = z_\lambda$, where λ gives the cycle structure of $\sigma \in \mathcal{S}_m$, and thus, for the corresponding conjugacy class K_λ, $|K_\lambda| = m!/z_\lambda$.

where '$\lambda \vdash m$' means that λ is any partition of m - see appendix A for notation - and

$$u_\lambda = \prod_{n=1}^{m} {u_n}^{a_n} \,, \tag{2.10}$$

in terms of the frequency representation of λ above. Identifying $u_n = f(\mathrm{t}^n) p_n(\mathrm{z}) p_n(\mathrm{z}^{-1})$ then we may rewrite

$$\mathcal{Z}_{U(N)}(\mathrm{t}) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i} \Delta(\mathrm{z}) \Delta(\mathrm{z}^{-1}) C_m(\mathrm{u}) \,, \tag{2.11}$$

the sum of the $U(N)$ group averages of each of the cycle polynomials $C_m(\mathrm{u})$. (Physically, the interpretation is that the cycle index for the symmetric permutation group accounts for particle statistics while integration over the gauge group imposes the gauge singlet condition. For purposes of clarity, the $U(N)$ case has been focused upon here, though from the form of (2.2) it is easy to see how this generalises for other gauge groups whereby the letters $u_n = f(\mathrm{t}^n) \chi_R(g^n)$ for the fundamental fields transforming in identical gauge group representations, R.)

Directly from (2.7), in terms of the inner product (2.6), then

$$\mathcal{Z}_{U(N)}(\mathrm{t}) = \sum_{\lambda} \frac{1}{z_\lambda} f_\lambda(\mathrm{t}) \big\langle p_\lambda, p_\lambda \big\rangle_N = \sum_{\lambda} \frac{1}{z_\lambda} f_\lambda(\mathrm{t}) \sum_{\substack{\mu \vdash |\lambda| \\ \ell(\mu) \leq N}} \big(\chi^\mu_\lambda\big)^2 \,, \tag{2.12}$$

where on the right-hand side of (2.12) we have used an expression for the inner product of two power symmetric polynomials expressed in terms of the characters of the symmetric group, given in appendix A. (Here '$\ell(\mu)$' means the number of non-zero parts of the partition μ.)

Using a result of appendix A (essentially orthogonality relations for symmetric group characters), (2.12) may be rewritten as

$$\mathcal{Z}_{U(N)}(\mathrm{t}) = \sum_{\substack{\lambda \\ |\lambda| \leq N}} f_\lambda(\mathrm{t}) + \sum_{\substack{\lambda \\ |\lambda| > N}} \frac{1}{z_\lambda} f_\lambda(\mathrm{t}) \sum_{\substack{\mu \vdash |\lambda| \\ \ell(\mu) \leq N}} \big(\chi^\mu_\lambda\big)^2 \,. \tag{2.13}$$

In the large N limit, $\mathcal{Z}_{U(N)}(\mathrm{t})$ simplifies considerably as only the first term in (2.13) need be considered. Using the frequency representation of λ then

$$\mathcal{Z}_{U(\infty)}(\mathrm{t}) = \sum_{\lambda} f_\lambda(\mathrm{t}) = \prod_{n=1}^{\infty} \sum_{a_n=0}^{\infty} f(\mathrm{t}^n)^{a_n} = \prod_{n=1}^{\infty} \frac{1}{1 - f(\mathrm{t}^n)} \,, \tag{2.14}$$

a result which has been obtained using Polya counting methods for single trace operators and saddle point approximations [22,23].

Higher order corrections in $|\lambda|$, the weight of the partition λ, to (2.13) may be obtained by successive evaluation of $\sum_{\substack{\mu\vdash|\lambda|\\ \ell(\mu)\le N}}\big(\chi^\mu_\lambda\big)^2$. One method is to employ the Murnaghan-Nakayama Rule, used to compute χ^μ_λ using skew hooks and Young diagrams. (A readable account of the Murnaghan-Nakayama Rule may be found in [38], though of course it is explained in many standard textbooks that discuss the symmetric group.)

For the case of $|\lambda| = N{+}1$ then we may observe that,

$$\sum_{\substack{\mu\vdash N+1\\ \ell(\mu)\le N}}\big(\chi^\mu_\lambda\big)^2 = \sum_{\substack{\mu\vdash N+1\\ \ell(\mu)\le N+1}}\big(\chi^\mu_\lambda\big)^2 - (\chi^\nu_\lambda)^2 = z_\lambda - (\chi^\nu_\lambda)^2\,, \qquad \nu = (1^{N+1})\,, \tag{2.15}$$

since the partition $\nu = (1^{N+1})$ is the only one excluded among those partitions μ of $N{+}1$ with $\ell(\mu)\le N$. By applying the Murnaghan-Nakayama Rule we may determine, for $\nu = (1^L)$,

$$(\chi^\nu_\lambda)^2 = \begin{cases} 1 & \text{for } |\lambda| = L \\ 0 & \text{otherwise}\end{cases}\,, \tag{2.16}$$

since χ^ν_λ in this case is just a sign. (This may be easily seen as there is only one possible way to remove successive skew hooks, which in this case are just column Young diagrams of length λ_i, from the (1^L) column Young diagram to leave one of normal shape, in this case, another column Young diagram.) Thus, using (2.15) with (2.16) in (2.13), we obtain

$$\mathcal{Z}_{U(N)}(\mathrm{t}) = \sum_{\substack{\lambda\\ |\lambda|\le N+1}} f_\lambda(\mathrm{t}) - \sum_{\substack{\lambda\\ |\lambda|=N+1}}\frac{1}{z_\lambda}f_\lambda(\mathrm{t}) + \sum_{\substack{\lambda\\ |\lambda|>N+1}}\frac{1}{z_\lambda}f_\lambda(\mathrm{t})\sum_{\substack{\mu\vdash|\lambda|\\ \ell(\mu)\le N}}\big(\chi^\mu_\lambda\big)^2\,. \tag{2.17}$$

By a similar line of argument we may do the same for the case of $|\lambda| = N{+}2$. We have that, for $\nu_1 = (1^{N+2})$ and $\nu_2 = (2, 1^{N+1})$,

$$\sum_{\substack{\mu\vdash N+2\\ \ell(\mu)\le N}}\big(\chi^\mu_\lambda\big)^2 = \sum_{\substack{\mu\vdash N+2\\ \ell(\mu)\le N+2}}\big(\chi^\mu_\lambda\big)^2 - (\chi^{\nu_1}_\lambda)^2 - (\chi^{\nu_2}_\lambda)^2 = z_\lambda - (\chi^{\nu_1}_\lambda)^2 - (\chi^{\nu_2}_\lambda)^2\,. \tag{2.18}$$

We may determine, for $\nu = (2, 1^L)$,

$$(\chi^\nu_\lambda)^2 = \begin{cases} (a_1-1)^2 & \text{for } |\lambda| = \sum_{n\ge1} n\,a_n = L{+}2 \\ 0 & \text{otherwise}\end{cases}\,. \tag{2.19}$$

Using (2.16), (2.19) with (2.18) in (2.17) then we obtain

$$\begin{aligned}\mathcal{Z}_{U(N)}(\mathrm{t}) = {}& \sum_{\substack{\lambda\\ |\lambda|\le N+2}} f_\lambda(\mathrm{t}) - \sum_{\substack{\lambda\\ N+1\le|\lambda|\le N+2}}\frac{1}{z_\lambda}f_\lambda(\mathrm{t}) - \sum_{\substack{\lambda\\ |\lambda|=N+2}}\frac{1}{z'_\lambda}f_\lambda(\mathrm{t}) \\ & + \sum_{\substack{\lambda\\ |\lambda|>N+2}}\frac{1}{z_\lambda}f_\lambda(\mathrm{t})\sum_{\substack{\mu\vdash|\lambda|\\ \ell(\mu)\le N}}\big(\chi^\mu_\lambda\big)^2\,,\end{aligned} \tag{2.20}$$

where, in the frequency representation of λ,

$$\frac{1}{z'_\lambda} = \frac{(a_1-1)^2}{z_\lambda} = \Big(\frac{1}{a_1!} - \frac{1}{(a_1-1)!} + \frac{1}{(a_1-2)!}\Big)\Big/ \prod_{n=2}^{\infty} n^{a_n} a_n!\,. \tag{2.21}$$

We may proceed in this manner to compute explicit higher order corrections though this becomes cumbersome save for the first few cases as shown. (For $|\lambda| > N+2$ the corrections will always involve contributions from (2.16) and (2.19) as well as extra ones coming from $\sum_{\substack{\mu\vdash|\lambda|\\ \ell(\mu)\le|\lambda|}} \big(\chi^\mu_\lambda\big)^2 - \sum_{\substack{\mu\vdash|\lambda|\\ \ell(\mu)\le N}} \big(\chi^\mu_\lambda\big)^2$.)

The One Boson Case for $U(N)$

For the case of one bosonic fundamental field (applicable to half BPS operators for $\mathcal{N}=4$ super Yang Mills), we have $f(t)=t$ in (2.5), so that we may write

$$\mathcal{Z}_{U(N)}(t) = \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i} \Delta(\mathrm{z})\Delta(\mathrm{z}^{-1}) \prod_{j,k=1}^{N} \frac{1}{1-tz_j {z_k}^{-1}}\,. \tag{2.22}$$

To evaluate this integral we may use the Cauchy-Littlewood formula,

$$\prod_{i=1}^{L}\prod_{j=1}^{M} \frac{1}{1-x_iy_j} = \sum_{\substack{\lambda\\ \ell(\lambda)\le \min.\{L,M\}}} s_\lambda(x_1,\ldots,x_L)s_\lambda(y_1,\ldots,y_M)\,, \tag{2.23}$$

where the sum on the right-hand side is over all partitions λ such that the corresponding Young diagrams have no more than min.$\{L,M\}$ rows, $\ell(\lambda)\le$ min.$\{L,M\}$. With $x_i = tz_i$, $y_i = {z_i}^{-1}$, $i=1,\ldots,N$, in (2.23), so that $s_\lambda(tz_1,\ldots,tz_N) = t^{|\lambda|}s_\lambda(\mathrm{z})$, and employing also (2.6) and the orthonormality of Schur polynomials, we may easily obtain,

$$\mathcal{Z}_{U(N)}(t) = \sum_{\substack{\lambda\\ \ell(\lambda)\le N}} t^{|\lambda|}\big\langle s_\lambda, s_\lambda\big\rangle_N = \sum_{\substack{\lambda\\ \ell(\lambda)\le N}} t^{|\lambda|}\,. \tag{2.24}$$

By changing summation variables so that $\lambda_i - \lambda_{i+1} = a_i, i=1,\ldots,N{-}1, \lambda_N = a_N$ then we may write

$$\mathcal{Z}_{U(N)}(t) = \sum_{a_1,\ldots,a_N=0}^{\infty} t^{a_1+2a_2+\ldots+Na_N} = P_N(t)\,, \tag{2.25}$$

where[4]

$$P_N(t) = \prod_{i=1}^{N} \frac{1}{1-t^i}\,. \tag{2.26}$$

[4] $1/P_\infty(t) = \prod_{n=1}^{\infty}(1-t^n)$ is commonly called the Euler function, denoted by $\Phi(t)$. $P_\infty(t) = \sum_{n=0}^{\infty} p(n)t^n$ acts as a generating function for the number of unordered partitions of n, $p(n)$. Note that $p_N(n) = p(n)$ for $n \le N$, i.e. the number of partitions of n into no more than N parts is the same as the total number of partitions of n so long as $n\le N$.

Of course this is nothing other than the generating function for the number $p_N(n)$ of partitions of n into no more than N parts since, by definition,

$$\sum_{\substack{\lambda \\ \ell(\lambda)\leq N}} \delta_{|\lambda|,n} = p_N(n)\,, \tag{2.27}$$

so that by the above

$$\sum_{n=0}^{\infty} p_N(n) t^n = P_N(t)\,. \tag{2.28}$$

This makes explicit the connection between $\mathcal{Z}_{U(N)}(t)$ and the partition numbers $p_N(n)$.

The $SU(2)$ Gauge Group Case

Here we first consider $f(\mathrm{t}) = \sum_{j=1}^{k} t_j$ in (2.1) so that the variables $0 \leq t_i < 1$ represent k bosons in the single particle partition function. For such fields transforming in the adjoint representation of $SU(2)$ then (2.2) simplifies significantly. For any $U \in SU(2)$ we may write $U = V\Theta V^\dagger$, where V is unitary and $\Theta = \mathrm{diag.}(e^{i\theta}, e^{-i\theta})$, for $0 \leq \theta < 2\pi$, so that for $F(U) = F(\theta)$, then in usual Weyl parametrisation,

$$\begin{aligned}\int_{SU(2)} \mathrm{d}\mu_{SU(2)}(U) F(U) &= \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\theta\, \sin^2\theta\; F(\theta) \\ &= \frac{1}{4\pi}\int_0^{4\pi} \mathrm{d}\theta\, \big(1-\cos\theta\big)\, F(\tfrac{\theta}{2}) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \big(1-\cos\theta\big)\, F(\tfrac{\theta}{2})\,,\end{aligned} \tag{2.29}$$

where $F(\theta) = F(\theta+\pi)$ is assumed in writing the last line. In the present case, $F(U) = F(\theta) = \sum_{n\geq 1} f(\mathrm{t}^n)\chi_{\mathrm{Adj.}}(U^n)/n$, where $\chi_{\mathrm{Adj.}}(U) = \mathrm{tr}(U)\mathrm{tr}(U^\dagger) - 1 = e^{2i\theta} + e^{-2i\theta} + 1 = 2\cos 2\theta + 1$, so that

$$\mathcal{Z}_{SU(2)}(\mathrm{t}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, (1-\cos\theta) \prod_{j=1}^{k} \frac{1}{(1-t_j)(1-t_j e^{i\theta})(1-t_j e^{-i\theta})}\,. \tag{2.30}$$

Making the variable change $z = e^{i\theta}$, and using that $F(\theta) = F(-\theta)$ is even, then

$$\mathcal{Z}_{SU(2)}(\mathrm{t}) = \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z}(1-z) \prod_{j=1}^{k} \frac{1}{(1-t_j)(1-t_j z)(1-t_j z^{-1})}\,, \tag{2.31}$$

where the integral is around the unit circle $|z| = 1$. The residues in (2.31) may be easily computed since all the relevant (simple) poles in the disc $|z| < 1$ occur at the points $z = t_j$. Thus

$$\mathcal{Z}_{SU(2)}(\mathrm{t}) = \sum_{i=1}^{k} \frac{{t_i}^{k-1}}{1-{t_i}^2} \prod_{\substack{j=1\\ j\neq i}}^{k} \frac{1}{(t_i - t_j)(1-t_i t_j)(1-t_j)}\,. \tag{2.32}$$

This partition function has an interesting interpretation from a group theory perspective. We may write,

$$\frac{1+t}{(1-tz)(1-tz^{-1})} = \sum_{n=0}^{\infty} \chi_n(z) t^n \,, \tag{2.33}$$

where

$$\chi_j(z) = \frac{z^{j+\frac{1}{2}} - z^{-j-\frac{1}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \,, \qquad j \in \tfrac{1}{2}\mathbb{Z} \,, \tag{2.34}$$

is an $SU(2)$ character, corresponding to the spin j irreducible representation, R_j. Now the integral in (2.31) acts as an $SU(2)$ inner product,

$$\langle \chi_j, \chi_k \rangle = \frac{1}{2\pi i} \oint \frac{\mathrm{d}z}{z} (1-z) \chi_j(z) \chi_k(z^{-1}) = \delta_{jk} \,, \tag{2.35}$$

for $j, k \in \mathbb{N}$. Thus, from (2.31) with (2.33) and (2.35),

$$\prod_{j=1}^{k} (1 - {t_j}^2) \mathcal{Z}_{SU(2)}(\mathrm{t}) = \sum_{n_1, \ldots, n_k = 0}^{\infty} \langle \chi_{n_1} \cdots \chi_{n_k}, 1 \rangle {t_1}^{n_1} \cdots {t_k}^{n_k} \,, \tag{2.36}$$

acts as a generating function for the number of singlets in the decomposition of the $SU(2)$ representation $R_{n_1} \otimes \cdots \otimes R_{n_k}$.[5] By using that $\chi_n(z) = \sum_{j=-n}^{n} z^j$ we may use the Cauchy residue theorem to compute explicitly that

$$\langle \chi_{n_1} \cdots \chi_{n_k}, 1 \rangle = \sum_{j_1=0}^{2n_1} \cdots \sum_{j_k=0}^{2n_k} (\delta_{j_1+\cdots+j_k, n_1+\cdots+n_k} - \delta_{j_1+\cdots+j_k, n_1+\cdots+n_k+1}) \,. \tag{2.37}$$

If we modify the one particle partition function to include k bosons and $\bar{k}$ fermions and hence consider (2.1) in the form $f(t, \bar{t}) = \sum_{j=1}^{k} t_j - \sum_{\bar{j}=1}^{\bar{k}} \bar{t}_{\bar{j}}$ then we may similarly as above evaluate

$$\mathcal{Z}_{SU(2)}(\mathrm{t}, \bar{\mathrm{t}}) = \frac{1}{2\pi i} \oint \frac{\mathrm{d}z}{z} (1-z) \prod_{\substack{1 \le j \le k \\ 1 \le \bar{j} \le \bar{k}}} \frac{(1-\bar{t}_{\bar{j}})(1-\bar{t}_{\bar{j}} z)(1-\bar{t}_{\bar{j}} z^{-1})}{(1-t_j)(1-t_j z)(1-t_j z^{-1})} \,, \tag{2.38}$$

[5] Generating functions for products of Lie algebra representations have been considered elsewhere, in [39] for instance. A generating function for the number of singlets in n products of the fundamental times n products of the anti-fundamental representations for $SU(N)$ was found by Gessel [40] in terms of Toeplitz determinants involving Bessel functions. See also [41] for a nice physics oriented discussion of similar issues. The special case of $R_{\frac{1}{2}} \otimes \cdots \otimes R_{\frac{1}{2}}$ ($2n$ products of the fundamental) for $SU(2)$, contains a Catalan number, $\frac{1}{n+1}\binom{2n}{n}$, of singlet representations.

where the contour is around the unit disc $|z|=1$. So long as $k>\bar{k}$ then (2.38) receives contributions from only those simple poles at $z=t_j$ so that for this case we obtain

$$\mathcal{Z}_{SU(2)}(\mathrm{t},\bar{\mathrm{t}})\Big|_{k>\bar{k}} = \sum_{i=1}^{k} \frac{{t_i}^{k-\bar{k}-1}}{1-{t_i}^2} \prod_{\substack{1\le\bar{j}\le\bar{k}\\ 1\le j\le k, j\neq i}} \frac{(t_i-\bar{t}_{\bar{j}})(1-t_i\bar{t}_{\bar{j}})(1-\bar{t}_{\bar{j}})}{(t_i-t_j)(1-t_it_j)(1-t_j)}\,. \tag{2.39}$$

For $k\le\bar{k}$ then (2.38) also receives contributions from poles at $z=0$. For instance

$$\mathcal{Z}_{SU(2)}(\mathrm{t},\bar{\mathrm{t}})\Big|_{k=\bar{k}} = \sum_{i=1}^{k} \frac{1}{t_i(1-{t_i}^2)} \prod_{\substack{1\le j,\bar{j}\le k\\ j\neq i}} \frac{(t_i-\bar{t}_{\bar{j}})(1-t_i\bar{t}_{\bar{j}})(1-\bar{t}_{\bar{j}})}{(t_i-t_j)(1-t_it_j)(1-t_j)} + \prod_{1\le j,\bar{j}\le k} \frac{\bar{t}_{\bar{j}}(1-\bar{t}_{\bar{j}})}{t_j(1-t_j)}\,, \tag{2.40}$$

where the last term on the right-hand side of (2.40) comes from the simple pole at $z=0$.

These formulae should be useful for computing the multi-trace partition functions, for fundamental fields transforming in an $SU(2)$ gauge group, in other sectors of $\mathcal{N}=4$ super Yang Mills. For instance, after a suitable identification of the variables $t_j,\bar{t}_{\bar{j}}$ with variables in single particle partition functions for semi-short sectors of $\mathcal{N}=4$ super Yang Mills, described in detail in [33], then (2.40) should allow for an explicit expression for corresponding multi-trace partition functions. They may also be useful for computing the $\mathcal{N}=4$ superconformal index of [1] for $SU(2)$ gauge group, or at least for restrictions of it such as described in [33] or [42].

3. Counting Operators in Free $\mathcal{N}=4$ Super Yang Mills

In this section, the counting of half and quarter BPS operators for free $\mathcal{N}=4$ super Yang Mills, when the fundamental fields transform in the adjoint representation of $U(N)$, is discussed in some detail.

Counting Operators Directly

We may, of course, proceed to count multi-trace half and quarter BPS primary operators directly, in terms of the fundamental fields, Z, for half BPS operators and Z,Y, for quarter BPS operators.

(Z,Y) forms a $U(2)$ doublet, where $U(2)$ has generators given by a subset of the $SU(4)_R$ generators, $H_i,E_{i\pm}$, $1\le i\le 3$, where H_i are the Cartan sub-algebra generators and $E_{i\pm}$ are ladder operators satisfying (in the Chevalley-Serre basis) $[H_i,E_{j\pm}]=\pm K_{ij}E_{j\pm}$, with $[K_{ij}]$ being the usual $SU(4)$ Cartan matrix. The $U(2)$ generators consist of the $SU(2)$ generators $H_2, E_{2\pm}$, where explicitly $[(H_1, H_2, H_3), E_{2\pm}] = \mp(1,-2,1)E_{2\pm}$, along with the generator $H_1{+}H_2{+}H_3$, whose eigenvalues give the conformal dimensions in this case, $[H_1{+}H_2{+}H_3, (Z,Y)] = (Z,Y)$ [33]. Explicitly, we have that $[E_{2+}, Z] = 0$, $[E_{2-}, Z] = Y$, $[(H_1, H_2, H_3), Z] = (0,1,0)Z$, $[(H_1, H_2, H_3), Y] = (1,-1,1)Y$ so that an operator involving n Z's and m Y's transforms in the $[m, n{-}m, m]$ $SU(4)_R$ R-symmetry representation.

For k-trace half BPS primary operators transforming in the $[0,n,0]$ $SU(4)_R$ R-symmetry representation, with conformal dimension n, then in terms of the fundamental field Z a basis is provided by,

$$\mathrm{tr}(Z^{n_1}) \cdots \mathrm{tr}(Z^{n_k})\,, \quad \sum_{i=1}^{k} n_i = n\,. \tag{3.1}$$

We have that, due to trace identities for finite N, $\mathrm{tr}(Z^n)$ for $n > N$ is expressible in terms of a sum over multi-trace operators of the form (3.1), for $k > 1$, and thus, a minimal basis for multi-trace half BPS primary operators consists of (3.1) for all $1 \leq k \leq n$ and with every $n_i \leq N$, ordered so that $n_1 \geq n_2 \geq \ldots \geq n_k \geq 0$, i.e. so that $(n_1, \ldots, n_k)$ is a partition of n where each part $n_i \leq N$. With this restriction, the number of multi-trace half BPS primary operators for a given n is

$$\mathcal{N}_{(n)} = p_N(n)\,, \tag{3.2}$$

since the number $p_N(n)$, in (2.27), of partitions of n into $\leq N$ parts is the same as the number of partitions of n in which each part is $\leq N$ - see [43] for a simple proof employing generating functions.

For quarter BPS operators belonging to the $[m, n{-}m, m]$ $SU(4)_R$ R-symmetry representation, a basis for k-trace operators is

$$\mathrm{tr}\Big(\prod_j Z^{n_{1j}} Y^{m_{1j}}\Big) \cdots \mathrm{tr}\Big(\prod_j Z^{n_{kj}} Y^{m_{kj}}\Big)\,, \quad \sum_{i,j} n_{ij} = n\,, \quad \sum_{i,j} m_{ij} = m\,, \tag{3.3}$$

where there is a choice of ordering in each trace. (Note that the $m = 0$ case corresponds to the half BPS case already considered.) Using the basis provided by (3.3) for all allowable k, then to avoid over-counting of multi-trace quarter BPS operators, the cyclicity of each trace and also trace identities for finite N must be accounted for. Assuming that this is done, let $\mathcal{M}_{(n,m)}$ denote the number of elements in this minimal basis for multi-trace operators of the form (3.3). Then, to obtain the number $\mathcal{N}_{(n,m)}$ of multi-trace quarter BPS *primary* operators in the $SU(4)_R$ representation $[m, n{-}m, m]$, the number of $U(2)$ descendants, in the $SU(4)_R$ representation $[m, n{-}m, m]$, of multi-trace quarter BPS primary operators, in

$SU(4)_R$ representations $[j, n{+}m{-}2j, j]$, $0 \le j \le m{-}1$, must be subtracted from $\mathcal{M}_{(n,m)}$. (These descendants arise due to the relation $[E_{2-}, Z] = Y$. Acting with $(E_{2-})^{m-j}$ on the highest weight state in the $SU(4)_R$ representation $[j, n{+}m{-}2j, j]$ we obtain a descendant in the $SU(4)_R$ representation $[m, n{-}m, m]$.) The number of such $U(2)$ descendants coincides with $\mathcal{N}_{(n+m-j,j)}$, the number of corresponding primary operators. In this way, we obtain that

$$\mathcal{M}_{(n,m)} = \mathcal{N}_{(n,m)} + \mathcal{N}_{(n+1,m-1)} + \ldots + \mathcal{N}_{(n+m-1,1)} + \mathcal{N}_{(n+m)} \,, \tag{3.4}$$

so that $\mathcal{N}_{(n,m)} = \mathcal{M}_{(n,m)} - \mathcal{M}_{(n+1,m-1)}$ may be obtained recursively for each m.

We may illustrate by counting all multi-trace quarter BPS primary operators in the $[1, n{-}1, 1]$ R-symmetry representation. In this case a basis for $k{+}1$-trace operators is provided by

$$\mathrm{tr}(Z^{n_1}) \cdots \mathrm{tr}(Z^{n_k}) \mathrm{tr}(Z^j Y) \,, \qquad \sum_{i=1}^{k} n_i = n - j \,. \tag{3.5}$$

Cyclicity of traces implies that we may arrange Y as shown, to avoid over-counting. $U(N)$ trace identities imply, similarly as for the half BPS case, that a minimal basis for multi-trace operators requires $j < N$ and each $n_i \le N$ in (3.5) for every $1 \le k \le n{-}j$, so that $(n_1, \ldots, n_k)$ forms a partition of $n{-}j$, with every part $\le N$. Thus, by a similar argument as for the half BPS case, $\mathcal{M}_{(n,1)} = \sum_{j=0}^{N-1} p_N(n-j)$. Finally, to ensure that only primary operators are counted then we must subtract off contributions from descendants of half BPS primary operators in the $[0, n{+}1, 0]$ $SU(4)_R$ representation, of which there are $p_N(n{+}1)$. Using (3.4) with (3.2) we then conclude that

$$\mathcal{N}_{(n,1)} = \sum_{j=0}^{N-1} p_N(n-j) - p_N(n{+}1) \,, \tag{3.6}$$

gives the number of multi-trace quarter BPS primary operators in the $[1, n{-}1, 1]$ R-symmetry representation.

Counting in this fashion becomes more difficult for greater m and now a procedure is described employing symmetric polynomials to find a generating function for the numbers of multi-trace quarter BPS primary operators in the $[m, n{-}m, m]$ $SU(4)_R$ representation, for $m = 0, 1, 2$ at finite N and for any n, m at large N. This generating function is subsequently used to provide asymptotic counting for fixed m, large n in the large N limit.

Counting Operators via Expansion of Partition Functions in Schur Polynomials

For k bosonic fundamental fields, we may take $f(\mathrm{t}) = \sum_{j=1}^{k} t_j$ in (2.1) so that (2.5) may be written as

$$\mathcal{Z}_{U(N)}(\mathrm{t}) = \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i} \Delta(\mathrm{z}) \Delta(\mathrm{z}^{-1}) \prod_{j=1}^{k} \prod_{r,s=1}^{N} \frac{1}{1 - t_j z_r {z_s}^{-1}} \,. \tag{3.7}$$

Often it is the case that such partition functions should be expanded in terms of $s_\lambda(\mathrm{t})$, the k variable Schur polynomial labelled by partitions λ. An example is provided by (1.1) for counting multi-trace quarter BPS operators. We may use the Cauchy-Littlewood formula (2.23) to expand in this way, to obtain

$$\mathcal{Z}_{U(N)}(\mathrm{t}) = \sum_{\substack{\lambda \\ \ell(\lambda)\leq \min.\{k,N^2\}}} \mathcal{N}_\lambda \, s_\lambda(\mathrm{t}) \,, \tag{3.8}$$

where

$$\mathcal{N}_\lambda = \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i} \Delta(\mathrm{z})\Delta(\mathrm{z}^{-1}) \, s_\lambda(\mathrm{z}\mathrm{z}^{-1}) \,, \tag{3.9}$$

where $\mathrm{z}\mathrm{z}^{-1}$ has components $z_i {z_j}^{-1}$, $1 \leq i, j \leq N$.

From Macdonald [24] we have that

$$s_\lambda(\mathrm{x}\mathrm{y}) = \sum_{\mu,\nu \vdash |\lambda|} \gamma^\lambda_{\mu\nu} s_\mu(\mathrm{x}) s_\nu(\mathrm{y}) \,, \tag{3.10}$$

in terms of Kronecker coefficients,

$$\begin{aligned} \gamma^\lambda_{\mu\nu} =& \frac{1}{|\lambda|!} \sum_{\sigma \in \mathcal{S}_{|\lambda|}} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma) \\ =& \sum_{\rho \vdash |\lambda|} \frac{1}{z_\rho} \chi^\lambda_\rho \, \chi^\mu_\rho \, \chi^\nu_\rho \,, \end{aligned} \tag{3.11}$$

being a sum over irreducible $\mathcal{S}_{|\lambda|}$ characters evaluated at $\sigma \in \mathcal{S}_{|\lambda|}$, related to a sum over irreducible $\mathcal{S}_{|\lambda|}$ characters evaluated on the conjugacy classes labelled by the partitions ρ in the second line. Using (2.6) along with the orthonormality property of Schur polynomials we find that

$$\mathcal{N}_\lambda = \sum_{\substack{\mu \vdash |\lambda| \\ \ell(\mu) \leq N}} \gamma^\lambda_{\mu\mu} \,. \tag{3.12}$$

The situation becomes much more involved if we include also $\bar{k}$ fermionic fields, so that (2.1) may be written in the form $f(t,\bar{t}) = \sum_{j=1}^{k} t_j - \sum_{\bar{\jmath}=1}^{\bar{k}} \bar{t}_{\bar{\jmath}}$, and attempt to expand $\mathcal{Z}_{U(N)}(\mathrm{t},\bar{\mathrm{t}})$ in terms of products of Schur polynomials $s_\lambda(\mathrm{t}) s_\mu(\bar{\mathrm{t}})$. Such expansion is required for counting, for instance, for the free field partition function in the eighth BPS sector of $\mathcal{N} = 4$ super Yang Mills. In this case the partition function is expanded, analogous to (1.1), in terms of $SU(2|3)$ characters, which may be expressed in terms of a linear combination of products of two-variable and three-variable Schur polynomials. (See

[33] for a discussion of counting for the eighth BPS sector along these lines.) Including fermions, (3.7) becomes modified by

$$\mathcal{Z}_{U(N)}(\mathrm{t},\bar{\mathrm{t}}) = \frac{1}{(2\pi i)^N N!}\oint \prod_{i=1}^{N}\frac{\mathrm{d}z_i}{z_i}\Delta(\mathrm{z})\Delta(\mathrm{z}^{-1}) \prod_{\substack{1\le j\le k\\ 1\le \bar{\jmath}\le \bar{k}}}\prod_{r,s=1}^{N}\frac{1-\bar{t}_{\bar{\jmath}}z_r {z_s}^{-1}}{1-t_j z_r {z_s}^{-1}}\,. \tag{3.13}$$

To achieve the expansion, we may use the Cauchy-Littlewood formula (2.23) along with another formula of Littlewood,

$$\prod_{i=1}^{L}\prod_{j=1}^{M}(1+x_i y_j) = \sum_{\substack{\lambda\\ \ell(\lambda)\le L,\ell(\tilde{\lambda})\le M}} s_\lambda(x_1,\ldots,x_L)\, s_{\tilde{\lambda}}(y_1,\ldots,y_M)\,, \tag{3.14}$$

where $\tilde{\lambda}$ is the partition conjugate to λ (where the rows and columns of the Young diagram corresponding to λ are interchanged) and where the sum is restricted to those λ whereby the corresponding Young diagrams have at most L rows, $\ell(\lambda)\le L$, and M columns, $\ell(\tilde{\lambda})\le M$. We may thus write

$$\mathcal{Z}_{U(N)}(\mathrm{t},\bar{\mathrm{t}}) = \sum_{\substack{\lambda\\ \ell(\lambda)\le \min.\{k,N^2\}}}\ \sum_{\substack{\mu\\ \ell(\mu)\le\bar{k},\ell(\tilde{\mu})\le N^2}} \mathcal{N}_{\lambda,\mu}\, s_\lambda(\mathrm{t})\, s_\mu(\bar{\mathrm{t}})\,, \tag{3.15}$$

where

$$\mathcal{N}_{\lambda,\mu} = \frac{(-1)^{|\mu|}}{(2\pi i)^N N!}\oint\prod_{i=1}^{N}\frac{\mathrm{d}z_i}{z_i}\Delta(\mathrm{z})\Delta(\mathrm{z}^{-1})\, s_\lambda(\mathrm{z}\mathrm{z}^{-1}) s_{\tilde{\mu}}(\mathrm{z}\mathrm{z}^{-1})\,. \tag{3.16}$$

Obviously these numbers are considerably more involved than those in (3.9). We may of course use (3.10) again to interpret (3.16) in terms of Kronecker coefficients.

Counting Quarter BPS Operators by Symmetric Polynomial Methods

The two bosonic fundamental field case is now focused upon.[6] In particular, the numbers $\mathcal{N}_{(n,m)}$ in (1.1) are evaluated using results of the last sub-section.

[6] The two boson case leads to an interesting generalisation of an identity in [44] involving Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$, the coefficients that appear in the decomposition $s_\lambda(\mathrm{x})s_\mu(\mathrm{x}) = \sum_\nu c^\nu_{\lambda\mu}s_\nu(\mathrm{x})$. With $f(\mathrm{t})$ in (2.1) given by $f(\mathrm{t}) = t_1+t_2$, and expanding appropriately the corresponding integrand in (3.7) using (2.23); then using (2.6), the orthonormality of Schur polynomials and the result (2.14), we obtain (note that $c^\nu_{\lambda\mu}=0$ if $|\nu|\ne|\lambda|+|\mu|$)

$$\mathcal{Z}_{U(\infty)}(t_1,t_2) = \sum_{\lambda,\mu} {t_1}^{|\lambda|}{t_2}^{|\mu|}\langle s_\lambda s_\mu, s_\lambda s_\mu\rangle_\infty = \sum_{\substack{\lambda,\mu,\nu\\ \nu\vdash|\lambda|+|\mu|}} {t_1}^{|\lambda|}{t_2}^{|\mu|}(c^\nu_{\lambda\mu})^2 = \prod_{n\ge1}\frac{1}{1-{t_1}^n-{t_2}^n}$$

which reduces to Theorem 4.1 of [44] if we take $t_1=t_2=t$.

We may proceed to evaluate $\mathcal{N}_\lambda$ recursively. The simplest case is for $\mathcal{N}_\lambda = \mathcal{N}_{(n)}$, whereby introducing a formal variable t then it is clear, by (2.23) with (2.22), (2.25) and (2.26), that

$$\sum_{n=0}^{\infty} \mathcal{N}_{(n)} t^n = \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i} \Delta(\mathrm{z})\Delta(\mathrm{z}^{-1}) \prod_{j,k=1}^{N} \frac{1}{1 - t z_j {z_k}^{-1}} = P_N(t)\,, \tag{3.17}$$

so that, by (2.28), $\mathcal{N}_{(n)}$ is given by (3.2).

More generally to evaluate $\mathcal{N}_{(n,m)}$ from (3.9) we may use, for $\mathrm{y} = \mathrm{z}\mathrm{z}^{-1}$,

$$\begin{aligned} s_{(m)}(\mathrm{y}) s_{(n)}(\mathrm{y}) &= s_{(n,m)}(\mathrm{y}) + s_{(n+1,m-1)}(\mathrm{y}) + \ldots + s_{(n+m-1,1)}(\mathrm{y}) + s_{(n+m)}(\mathrm{y})\,, \\ s_{(m)}(\mathrm{z}\mathrm{z}^{-1}) &= \sum_{\substack{\mu \vdash m \\ \ell(\mu) \leq N}} s_\mu(\mathrm{z}) s_\mu(\mathrm{z}^{-1})\,, \end{aligned} \tag{3.18}$$

where the expression in the first line of (3.18) may be easily seen using Young tableaux multiplication rules while (2.23) determines the expression in the second line. ¿From (3.9) with (3.18), we may find a useful generating function, in terms of a formal variable t, for the numbers in (3.4) as follows,

$$\begin{aligned} \mathcal{F}_N^{(m)}(t) &= \sum_{n=0}^{\infty} \mathcal{M}_{(n,m)} t^n \\ &= \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i} \Delta(\mathrm{z})\Delta(\mathrm{z}^{-1})\, s_{(m)}(\mathrm{z}\mathrm{z}^{-1}) \sum_{n=0}^{\infty} s_{(n)}(\mathrm{z}\mathrm{z}^{-1}) t^n \\ &= \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i} \Delta(\mathrm{z})\Delta(\mathrm{z}^{-1}) \sum_{\substack{\mu \vdash m \\ \ell(\mu) \leq N}} s_\mu(\mathrm{z}) s_\mu(\mathrm{z}^{-1}) \prod_{j,k=1}^{N} \frac{1}{1 - t z_j {z_k}^{-1}} \\ &= \sum_{\substack{\mu \vdash m \\ \ell(\mu) \leq N}} \sum_{\lambda} t^{|\lambda|} \big\langle s_\lambda s_\mu, s_\lambda s_\mu \big\rangle_N\,, \end{aligned} \tag{3.19}$$

so that we may write

$$\mathcal{N}_{(n,m)} = \frac{1}{2\pi i} \oint \frac{\mathrm{d}t}{t} \Big(\frac{1}{t^n} \mathcal{F}_N^{(m)}(t) - \frac{1}{t^{n+1}} \mathcal{F}_N^{(m-1)}(t) \Big)\,, \tag{3.20}$$

which allows for recursive determination of $\mathcal{N}_{(n,m)}$.

Applying this to the case of $\mathcal{N}_\lambda = \mathcal{N}_{(n,1)}$ we have, from (3.18)

$$s_{(1)}(\mathrm{z}\mathrm{z}^{-1}) = s_{(1)}(\mathrm{z}) s_{(1)}(\mathrm{z}^{-1})\,, \tag{3.21}$$

so that, from (3.19),

$$\mathcal{F}_N^{(1)}(t) = \sum_{n=0}^{\infty} \big(\mathcal{N}_{(n,1)} + \mathcal{N}_{(n+1)}\big) t^n = \sum_{\substack{\lambda \\ \ell(\lambda)\le N}} t^{|\lambda|} \big\langle s_{(1)} s_\lambda, s_{(1)} s_\lambda \big\rangle_N \,. \tag{3.22}$$

Using (again, this may be easily seen from Young tableaux multiplication rules)

$$s_{(1)}(\mathrm{z}) s_\lambda(\mathrm{z}) = \sum_{r=1}^{N} s_{\lambda+\mathrm{e}_r}(\mathrm{z}) \,, \tag{3.23}$$

for $\{\mathrm{e}_r; 1 \le r \le N, \mathrm{e}_r \cdot \mathrm{e}_s = \delta_{rs}\}$ being usual orthonormal vectors, we find that

$$\mathcal{F}_N^{(1)}(t) = \sum_{n=0}^{\infty} \big(\mathcal{N}_{(n,1)} + \mathcal{N}_{(n+1)}\big) t^n = \sum_{\substack{\lambda \\ \ell(\lambda)\le N}} t^{|\lambda|} \sum_{r,s=1}^{N} \big\langle s_{\lambda+\mathrm{e}_r}, s_{\lambda+\mathrm{e}_s} \big\rangle_N \,. \tag{3.24}$$

Now for any partition λ, $\big\langle s_{\lambda+\mathrm{e}_r}, s_{\lambda+\mathrm{e}_s} \big\rangle_N$ vanishes unless $\mathrm{e}_r = \mathrm{e}_s$ for any r, s and $\lambda_{r-1} - \lambda_r > 0$ for $r = 2, \dots, N$, due to

$$s_{(\lambda_1,\dots,\lambda_{r-1},\lambda_r+1,\dots,\lambda_N)}(\mathrm{z}) = 0 \quad \text{for} \quad \lambda_{r-1} = \lambda_r \,, \quad r > 1 \,. \tag{3.25}$$

Changing summation variables to those in (2.25) then we have, with the definition (2.26),

$$\begin{aligned}\mathcal{F}_N^{(1)}(t) &= \sum_{n=0}^{\infty} \big(\mathcal{N}_{(n,1)} + \mathcal{N}_{(n+1)}\big) t^n = \sum_{a_1,\dots,a_N \ge 0} t^{a_1+\ldots+Na_N} + \sum_{r=1}^{N-1} \sum_{\substack{a_1,\dots,a_N \ge 0 \\ a_r \ge 1}} t^{a_1+\ldots+Na_N} \\ &= \sum_{i=0}^{N-1} t^i \sum_{a_1,\dots,a_N \ge 0} t^{a_1+\ldots+Na_N} = \frac{1}{1-t} P_{N-1}(t) \,.\end{aligned} \tag{3.26}$$

Thus, using (2.28), (3.2) with (3.26),[7]

$$\mathcal{N}_{(n,1)} = \sum_{j=0}^{n} p_{N-1}(j) - \mathcal{N}_{(n+1)} = \sum_{j=0}^{n} p_{N-1}(j) - p_N(n+1) \,. \tag{3.27}$$

[7] This formula agrees with (3.6) due to $\sum_{j=0}^{N-1} p_N(n-j) = \sum_{j=0}^{n} p_{N-1}(j)$ which follows because the corresponding generating functions match,

$$\sum_{n=0}^{\infty} \sum_{j=0}^{N-1} p_N(n-j) t^n = (1 + t + \ldots + t^{N-1}) P_N(t) = \frac{1}{1-t} P_{N-1}(t) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} p_{N-1}(j) t^n \,.$$

For the case of $\mathcal{N}_\lambda = \mathcal{N}_{(n,2)}$ we have that, from (3.18),

$$s_{(2)}(\mathrm{z}\mathrm{z}^{-1}) = s_{(2)}(\mathrm{z})s_{(2)}(\mathrm{z}^{-1}) + s_{(1,1)}(\mathrm{z})s_{(1,1)}(\mathrm{z}^{-1})\,, \tag{3.28}$$

so that, from (3.19), we have

$$\mathcal{F}^{(2)}_N(t) = \sum_{\substack{\lambda\\ \ell(\lambda)\le N}} t^{|\lambda|}\left(\left\langle s_{(2)}s_\lambda, s_{(2)}s_\lambda\right\rangle_N + \left\langle s_{(1,1)}s_\lambda, s_{(1,1)}s_\lambda\right\rangle_N\right)\,. \tag{3.29}$$

Using

$$\begin{aligned} s_{(2)}(\mathrm{z})s_\lambda(\mathrm{z}) &= \sum_{r=1}^{N} s_{\lambda+2\mathrm{e}_r}(\mathrm{z}) + \sum_{1\le r<s\le N} s_{\lambda+\mathrm{e}_r+\mathrm{e}_s}(\mathrm{z})\,,\\ s_{(1,1)}(\mathrm{z})s_\lambda(\mathrm{z}) &= \sum_{1\le r<s\le N} s_{\lambda+\mathrm{e}_r+\mathrm{e}_s}(\mathrm{z})\,, \end{aligned} \tag{3.30}$$

along with (3.25) and

$$s_{(\lambda_1,\dots,\lambda_{r-1},\lambda_r+2,\dots,\lambda_N)}(\mathrm{z}) = -s_{(\lambda_1,\dots,\lambda_r+1,\lambda_{r-1}+1,\dots,\lambda_N)}(\mathrm{z})\,, \tag{3.31}$$

for the cases where $\lambda_{r-1} = \lambda_r$, we may obtain, with the definition (2.26),

$$\mathcal{F}^{(2)}_N(t) = \frac{1-t^{N+1}}{(1-t)(1-t^2)}P_{N-1}(t) + \frac{1}{(1-t)(1-t^2)}P_{N-2}(t)\,, \tag{3.32}$$

where the first contribution comes from $\sum_\lambda t^{|\lambda|}\left\langle s_{(2)}s_\lambda, s_{(2)}s_\lambda\right\rangle_N$ while the second comes from $\sum_\lambda t^{|\lambda|}\left\langle s_{(1,1)}s_\lambda, s_{(1,1)}s_\lambda\right\rangle_N$. Since the partition number $p_k(-n) = 0$ for $n = 1,2,\dots$ we may write, using (2.28),[8]

$$\frac{1}{(1-t)(1-t^2)}P_k(t) = \sum_{n,i,j=0}^{\infty} p_k(n-i-2j)t^n\,. \tag{3.33}$$

Thus, from (3.32) with (3.27),

$$\begin{aligned} \mathcal{N}_{(n,2)} &= -\sum_{j=0}^{n+1} p_{N-1}(j)\\ &+\begin{cases} \sum_{i,j=0}^{\infty}\left(p_{N-2}(n-i-2j) + p_{N-1}(n-i-2j)\right) & \text{if } n\le N,\\ \sum_{i,j=0}^{\infty}\left(p_{N-2}(n-i-2j) + p_{N-1}(n-i-2j) - p_{N-1}(N+1-n-i-2j)\right) & \text{if } n\ge N+1. \end{cases} \end{aligned} \tag{3.34}$$

[8] This is a special case of the following: for any $f(n)$, $n\in\mathbb{Z}$, that satisfies $f(-n) = 0$, $n = 1,2,\dots$, then we may (at least formally) write

$$P_k(t)\sum_{n=0}^{\infty} f(n)t^n = \sum_{n,i_1,\dots,i_k=0}^{\infty} f(n-i_1-2i_2-\ldots-ki_k)t^n\,.$$

Tables of the numbers (3.2), (3.27) and (3.34) are given in appendix B for some few cases of n, N. Notice from these tables that the numbers $\mathcal{N}_{(n,m)}$ below the diagonal line $N \geq n+m$ for a given n are the same for all N. This is a general feature that derives from values of $\mathcal{N}_{(n,m)}$ for $N \geq n+m$, which numbers may be obtained from a corresponding generating function that is now constructed.

Using these techniques, we may provide a consistency check of (3.17), (3.26), (3.32) along with a general result for $\mathcal{N}_{(n,m)}$ for high enough values of N, $N \geq m+n$. This employs the orthogonality property of power symmetric polynomials $p_\lambda(\mathrm{z})$ (in the large N limit) along with

$$\begin{aligned} s_{(n)}(\mathrm{z}) &= \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda(\mathrm{z}) \\ &= \sum_{i_1,\ldots,i_n=0}^{\infty} \frac{1}{i_1! i_2! \cdots i_n!} \delta_{i_1+2i_2+\ldots+ni_n,n}\, p_1(\mathrm{z})^{i_1} \left(\tfrac{1}{2}p_2(\mathrm{z})\right)^{i_2} \cdots \left(\tfrac{1}{n}p_n(\mathrm{z})\right)^{i_n} . \end{aligned} \tag{3.35}$$

Using the trivial identity $p_\lambda(\mathrm{xy}) = p_\lambda(\mathrm{x})p_\lambda(\mathrm{y})$ then from (2.6), (3.19) with (3.35) we have

$$\begin{aligned} \mathcal{F}_\infty^{(m)}(t) &= \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \sum_{\mu \vdash m} \frac{1}{z_\lambda z_\mu} \langle p_\lambda p_\mu, p_\lambda p_\mu \rangle_\infty \, t^n \\ &= \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \sum_{\mu \vdash m} \frac{1}{z_\lambda z_\mu} \langle p_\nu, p_\nu \rangle_\infty \, t^n \\ &= \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \sum_{\mu \vdash m} \frac{z_\nu}{z_\lambda z_\mu} t^n , \end{aligned} \tag{3.36}$$

where for $(1^{a_1}, 2^{a_2}, \ldots)$ being the frequency representation of λ and $(1^{b_1}, 2^{b_2}, \ldots)$ being that of μ then ν has frequency representation $(1^{a_1+a_2}, 2^{a_2+b_2}, \ldots)$ so that $|\nu| = n+m$. This agrees with $\mathcal{F}_N^{(m)}(t)$ in a series expansion up to $O(t^{N-m})$ (since the last equation in (3.36) is also valid for finite N so long as $|\nu| = n+m \leq N$, by a result of appendix A). Now, since

$$\frac{z_\nu}{z_\lambda z_\mu} = \prod_{j=1}^{\infty} \frac{(a_j+b_j)!}{a_j! b_j!} , \tag{3.37}$$

we obtain from (3.36) that,

$$\begin{aligned} \mathcal{F}_\infty^{(m)}(t) &= \sum_{n=0}^{\infty} \sum_{a_1,\ldots,a_n=0}^{\infty} \sum_{b_1,\ldots,b_m=0}^{\infty} \delta_{a_1+\cdots+na_n,n} \delta_{b_1+\cdots+mb_m,m} \prod_{j=1}^{m} \frac{(a_j+b_j)!}{a_j! b_j!} t^n \\ &= \sum_{b_1,\ldots,b_m=0}^{\infty} \delta_{b_1+\cdots+mb_m,m} \prod_{j=1}^{m} \sum_{a_j=0}^{\infty} \frac{(a_j+b_j)!}{a_j! b_j!} t^{ja_j} \prod_{j>m} \frac{1}{1-t^j} \\ &= \sum_{b_1,\ldots,b_m=0}^{\infty} \delta_{b_1+\cdots+mb_m,m} \prod_{j=1}^{m} \frac{1}{(1-t^j)^{b_j+1}} \prod_{j>m} \frac{1}{1-t^j} . \end{aligned} \tag{3.38}$$

For the first few cases we have that, with $P_N(t)$ as defined in (2.26),

$$\mathcal{F}_\infty^{(m)}(t) = \begin{cases} P_\infty(t) & \text{for } m=0 \\ \frac{1}{1-t}P_\infty(t) & \text{for } m=1 \\ \frac{2}{(1-t)(1-t^2)}P_\infty(t) & \text{for } m=2 \end{cases}, \tag{3.39}$$

whose series expansion agrees with (3.17), (3.26), (3.32) up to $O(t^{N-m})$ for, respectively, $m=0,1,2$. We may use (3.20) with (3.36) to determine $\mathcal{N}_{(n,m)}$ exactly for $N \geq n+m$.

Asymptotic Counting of Quarter BPS Operators at Large N

Asymptotic counting for the one boson case in the large N limit, for which, with $P_N(t)$ as defined in (2.26), with $p(n)$ being the total number of (unordered) partitions of n,

$$\mathcal{Z}_{U(\infty)}(t) = P_\infty(t) = \sum_{n=1}^{\infty} p(n)t^n\,, \tag{3.40}$$

is the multi-trace partition function, entails finding an asymptotic value for the partition number $p(n)$ for 'large' n. This may be achieved by performing a saddle point approximation of $p(n) = \frac{1}{2\pi i}\oint \mathrm{d}t P_\infty(t)t^{-n-1}$. The function $P_\infty(t)$ has a 'large' singularity at $t=1$, but in addition has singularities at all other roots of unity - see [45] on the validity of ignoring these contributions asymptotically. This method was used by Hardy and Ramanujan to find their celebrated formula, here given in a less detailed form as,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\tfrac{2}{3}n}\right), \tag{3.41}$$

which was improved by Rademacher to give $p(n)$ exactly. Their method relied crucially on the modular properties of $P_\infty(t)$.

Focusing now on the two bosonic fundamental field case for which, in the large N limit, (3.20) with (3.38) gives exact counting, at issue is first finding asymptotic values for the numbers $Q(n,m,\mathrm{b}) = Q(n,m,b_1,\ldots,b_m)$, with constraint equation $\sum_{j=1}^{m} jb_j = m$, defined by

$$\prod_{j=1}^{m} \frac{1}{(1-t^j)^{b_j+1}} \prod_{j>m} \frac{1}{1-t^j} = 1 + \sum_{n=1}^{\infty} Q(n,m,\mathrm{b})t^n\,. \tag{3.42}$$

Having found these we may then attempt to find the dominant contribution to (3.20) with (3.38) for large N. In order to give asymptotic values for $Q(n,m,\mathrm{b})$ we may follow [46] and apply a formula due to Meinardus which gives a general result for the generating function

$$\prod_{n=1}^{\infty}(1-t^n)^{-a_n} = 1 + \sum_{n=1}^{\infty} r(n)t^n\,. \tag{3.43}$$

A detailed version of Meinardus' theorem may be found in [46] but for purposes of brevity we may note that it implies that, as $n \to \infty$,

$$r(n) \sim C\, n^{\kappa}\, \exp\Big(\big(A\Gamma(\alpha+1)\zeta(\alpha+1)n^{\alpha}\big)^{1/(\alpha+1)}(\alpha+1)/\alpha \Big) \,, \tag{3.44}$$

where $\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$ is the Riemann zeta function and the constants C, κ, α, A are determined by the auxiliary Dirichlet series,

$$D(s) = \sum_{j=1}^{\infty} \frac{a_j}{j^s} \,, \tag{3.45}$$

which must converge for $\mathrm{Re}(s) > \alpha$, a positive real number, and possess an analytic continuation in the region $\mathrm{Re}(s) \geq c$, $-1 < c < 0$, such that, in this region, $D(s)$ is analytic except at a simple pole at $s = \alpha$ where it has residue A. In terms of α, A then

$$\begin{aligned} C &= \frac{1}{\sqrt{2\pi(1{+}\alpha)}} \big(A\Gamma(\alpha+1)\zeta(\alpha+1)\big)^{(1-2D(0))/2(\alpha+1)} \exp D'(0) \,, \\ \kappa &= (D(0) - 1 - \tfrac{1}{2}\alpha)/(\alpha+1) \,. \end{aligned} \tag{3.46}$$

Applying Meinardus' theorem to the case of (3.42), clearly we have

$$D(s) = \sum_{j=1}^{m} \frac{b_j}{j^s} + \zeta(s) \,, \tag{3.47}$$

so that, assuming $m = \sum_{j=1}^{m} j b_j$ is fixed, $D(s)$ has a simple pole at $s = \alpha = 1$ where it has residue $A = 1$. Using

$$D(0) = -\tfrac{1}{2} + \sum_{j=1}^{m} b_j \,, \qquad \exp D'(0) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{m} \frac{1}{j^{b_j}} \,, \tag{3.48}$$

then, from (3.44) with (3.46), we may easily determine that, as $n \to \infty$,

$$Q(n, m, \mathrm{b}) \sim \frac{1}{4n\sqrt{3}} \left(\frac{\sqrt{6n}}{\pi} \right)^{\sum_{j=1}^{m} b_j} \prod_{j=1}^{m} \frac{1}{j^{b_j}} \exp\left(\pi\sqrt{\tfrac{2}{3}n} \right) . \tag{3.49}$$

This reduces to (3.41) when $b_j = 0$, $1 \leq j \leq m$, whereby $Q(n, 0, \ldots, 0) = p(n)$. Using (3.20) for (3.38) with (3.42) and (3.49) then, as $n \to \infty$,

$$\begin{aligned} \mathcal{N}_{(n,m)} &= \sum_{\substack{b_1,\ldots,b_m \geq 0 \\ \sum_{j=1}^{m} j b_j = m}} Q(n, m, \mathrm{b}) - \sum_{\substack{b_1,\ldots,b_{m-1} \geq 0 \\ \sum_{j=1}^{m-1} j b_j = m-1}} Q(n{+}1, m{-}1, \mathrm{b}) \\ &\sim \frac{1}{4n\sqrt{3}} \left(\frac{\sqrt{6n}}{\pi} \right)^{m} \exp\left(\pi\sqrt{\tfrac{2}{3}n} \right) , \end{aligned} \tag{3.50}$$

since $Q(n,m,m,0,\ldots,0)$, for $b_1 = m$, $b_j = 0$, $j > 1$, dominates over all other terms in (3.50). This gives asymptotic values for the numbers in (1.1), for counting quarter BPS operators, transforming in $[m, n{-}m, m]$ $SU(4)_R$ representations, in the large N limit of free $\mathcal{N} = 4$ super Yang Mills, as previously described.

4. Counting Operators in the Chiral Ring of $\mathcal{N} = 4$ Super Yang Mills

For the purposes of counting operators in the chiral ring of $\mathcal{N} = 4$ super Yang Mills, we denote corresponding multi-trace partition functions by $\mathcal{C}_{U(N)}$(t).

The generating function for $\mathcal{C}_{U(N)}(t)$ for the case of one bosonic fundamental field has been written in the form [1,3,9]

$$\mathcal{C}(\nu, t) = \prod_{n=0}^{\infty} \frac{1}{1 - \nu t^n} = \sum_{N=0}^{\infty} \nu^N \mathcal{C}_{U(N)}(t)\,, \tag{4.1}$$

so that ν acts as a chemical potential for the rank of the gauge group $U(N)$. The equivalence $\mathcal{C}_{U(N)}(t) = \mathcal{Z}_{U(N)}(t) = P_N(t)$, with $\mathcal{Z}_{U(N)}(t)$ as in (2.25), is actually a special case of the q-Binomial theorem. Writing - see [43] for notation -

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})\,, \tag{4.2}$$

then the q-Binomial theorem is, for $|x|, |q| < 1$,

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}\,. \tag{4.3}$$

(Identifying $\nu = x$ and $q = t$ and setting $a = 0$ in (4.3), so that $1/(\nu;t)_\infty = \mathcal{C}(\nu,t)$ above and $1/(t;t)_N = P_N(t)$ in (2.26), then $\mathcal{C}_{U(N)}(t) = \mathcal{Z}_{U(N)}(t) = P_N(t)$ straightforwardly. This special case of the q-Binomial theorem is due to Euler.)

For the two boson case, so that the single particle partition function is given by $f(t,u) = t + u$ for some t, u, then the generating function for the finite N chiral ring partition function $\mathcal{C}_{U(N)}(t,u)$ is given by [1,3,9]

$$\mathcal{C}(\nu, t, u) = \prod_{n,m=0}^{\infty} \frac{1}{1 - \nu t^n u^m} = \sum_{N=0}^{\infty} \nu^N \mathcal{C}_{U(N)}(t,u)\,. \tag{4.4}$$

This function is more difficult to analyse in terms of counting though has been investigated by Stanley [47] in relation to partitions - there it has been dubbed the ‘double Eulerian’ generating function. Through use of the Cauchy-Littlewood formula, then we may expand $\mathcal{C}_{U(N)}(t,u)$ in terms of partitions of N as,

$$\mathcal{C}_{U(N)}(t,u) = \sum_{\lambda\vdash N} h_\lambda(t) h_\lambda(u)\,, \tag{4.5}$$

where

$$h_\lambda(t) = s_\lambda(1,t,t^2,\ldots)\,, \tag{4.6}$$

so that using an identity for Schur polynomials to be found in [24,47] then

$$\mathcal{C}_{U(N)}(t,u) = \sum_{\lambda\vdash N} \frac{\prod_{1\le i<j\le N}(1-t^{\lambda_i-\lambda_j+j-i})(1-u^{\lambda_i-\lambda_j+j-i})}{\prod_{i=1}^{N}(t;t)_{\lambda_i+N-i}(u;u)_{\lambda_i+N-i}}\,(tu)^{\sum_{i=1}^{N}(i-1)\lambda_i}\,. \tag{4.7}$$

(4.5) with (4.6) has a natural interpretation in terms of plane partitions in that, for π being all column-strict plane partitions of shape λ, $|\pi| = \sum_{i,j}\pi_{ij}$,[9]

$$h_\lambda(t) = s_\lambda(1,t,t^2,\ldots) = \sum_{\pi} t^{|\pi|}\,. \tag{4.8}$$

Obviously, (4.5) with (4.8) generalise for other chiral ring sectors. (For a different connection between the 'double Eulerian' generating function and major indices of permutations see [47], p. 385.) As an illustration of (4.5) with (4.8), we may consider the case $N = 2$ whereby $\lambda = (2,0),(1,1)$ gives the two possible partitions of 2. For $\lambda = (2,0)$ (corresponding to a Young diagram with a single row of two boxes) $\pi_{11} \ge \pi_{12} \ge 0$ gives all column-strict plane partitions of shape $(2,0)$, while for $\lambda = (1,1)$ (corresponding to a Young diagram with a single column of two boxes) then $\pi_{11} > \pi_{21} \ge 0$ gives all column-strict plane partitions of shape $(1,1)$. Thus,

$$\begin{aligned} h_{(2,0)}(t) &= \sum_{\substack{\pi_{11},\pi_{12}\ge 0\\ \pi_{11}\ge\pi_{12}}} t^{\pi_{11}+\pi_{12}} = \frac{1}{(1-t)(1-t^2)}\,, \\ h_{(1,1)}(t) &= \sum_{\substack{\pi_{11},\pi_{21}\ge 0\\ \pi_{11}>\pi_{21}}} t^{\pi_{11}+\pi_{21}} = \frac{t}{(1-t)(1-t^2)}\,, \end{aligned} \tag{4.9}$$

[9] See [47] for a detailed description of plane partitions. Briefly, a column-strict plane partition of shape λ is an array $\pi = (\pi_{ij})$ of non-negative integers with finitely many non-zero entries, that is arranged in a Young tableaux with shape λ - see appendix A - such that the numbers π_{ij} are weakly decreasing along each row, $\pi_{ij} \ge \pi_{i\,j+1} \ge 0$, and strictly decreasing down each column, $\pi_{ij} > \pi_{i+1\,j} \ge 0$. The sum of the parts of π is given by $|\pi| = \sum_{i,j}\pi_{ij}$. (Note that in contrast to the definition in [47], here we are allowing $\pi_{ij} = 0$, for some i,j, to be a part of the plane partition π with shape λ.)

so that, from (4.5) for $N = 2$,

$$\mathcal{C}_{U(2)}(t,u) = h_{(2,0)}(t)h_{(2,0)}(u) + h_{(1,1)}(t)h_{(1,1)}(u) = \frac{1+ut}{(1-t)(1-t^2)(1-u)(1-u^2)} \,, \quad (4.10)$$

which is the correct result as may be verified by extracting the ν^2 coefficient in an expansion of (4.4) up to $O(\nu^2)$.

In the large N limit,

$$\mathcal{C}_{U(\infty)}(t,u) = \prod_{\substack{n_1,n_2\geq 0\\ n_1+n_2>0}} \frac{1}{1-t^{n_1}u^{n_2}} \,, \quad (4.11)$$

upon which attention is shortly focused.

For the numbers $\mathcal{N}_{(n,m)} \to \widehat{\mathcal{N}}_{(n,m)}$ counting quarter BPS primary operators for the chiral ring of $\mathcal{N}=4$ super Yang Mills, belonging to $[m,n-m,m]$ $SU(4)_R$ R-symmetry representations, as in (1.1), we have [10]

$$\widehat{\mathcal{N}}_{(n,m)} = \frac{1}{8\pi^2}\oint\oint \mathrm{d}t\,\mathrm{d}u\,\mathcal{C}_{U(N)}(t,u)\, s_{(n,m)}(t^{-1},u^{-1})\,(t^{-1}-u^{-1})^2 \,. \quad (4.12)$$

These may be more conveniently evaluated in terms of the numbers in (3.4) $\mathcal{M}_{(n,m)} \to \widehat{\mathcal{M}}_{(n,m)}$, counting *all* chiral ring quarter BPS operators in the $[m,n{-}m,m]$ $SU(4)_R$ representation, given by

$$\widehat{\mathcal{M}}_{(n,m)} = \frac{1}{(2\pi i)^2}\oint\oint \mathrm{d}t\,\mathrm{d}u\;\mathcal{C}_{U(N)}(t,u)\;t^{-n-1}u^{-m-1} \,, \quad (4.13)$$

so that $\widehat{\mathcal{N}}_{(n,m)} = \widehat{\mathcal{M}}_{(n,m)} - \widehat{\mathcal{M}}_{(n+1,m-1)}$. Defining $\mathcal{P}_\lambda(n)$ to be the number of column-strict plane partitions π of shape λ so that $|\pi| = \sum_{i,j}\pi_{ij} = n$, then, from (4.5) with (4.8) and (4.13), $\widehat{\mathcal{M}}_{(n,m)} = \sum_{\lambda\vdash N}\mathcal{P}_\lambda(n)\mathcal{P}_\lambda(m)$. Thus,

$$\widehat{\mathcal{N}}_{(n,m)} = \sum_{\lambda\vdash N}\left(\mathcal{P}_\lambda(n)\mathcal{P}_\lambda(m) - \mathcal{P}_\lambda(n{+}1)\mathcal{P}_\lambda(m{-}1)\right) \,, \quad (4.14)$$

counts chiral ring quarter BPS primary operators in $SU(4)_R$ representations $[m,n-m,m]$ for any n,m at finite N.

Asymptotic Counting for Chiral Ring BPS Operators at Large N

For asymptotic counting of operators in the chiral ring of $\mathcal{N}=4$ super Yang Mills at large N, a relatively crude method is employed here which nevertheless captures the

[10] This formula employs the orthonormality relation of Schur polynomials described here and has appeared in a similar context in [33], appendix B.

exponential behaviour of counting numbers of interest. This method is based on saddle point approximations of functions near a dominant singularity - see [45] for a useful summary. (Often for physical applications in thermodynamics, e.g. for entropy formulae, we are interested only in the exponential behaviour of such numbers anyhow.)

To illustrate, we consider the one boson case in the large N limit again. We first find a convenient 'approximating function' as follows,

$$\begin{aligned} P_\infty(t) &= \prod_{n=1}^{\infty} \frac{1}{1-t^n} = \exp\Big(-\sum_{n=1}^{\infty} \ln(1-t^n)\Big) \\ &\sim \exp\Big(-\int_0^\infty \mathrm{d}s\, \ln(1-t^s)\Big) = \exp\Big(-\frac{\pi^2}{6\ln t}\Big)\,, \end{aligned} \tag{4.15}$$

which has an 'easier' singularity structure. (The approximation in the second step may be justified by the Euler-Maclaurin formula for approximating sums by integrals.) Using (4.15) then for large enough n,

$$p(n) = \frac{1}{2\pi i}\oint \mathrm{d}t\, P_\infty(t) t^{-n-1} \sim \frac{1}{2\pi i}\oint \mathrm{d}t\, e^{g(t)}\,, \qquad g(t) = -\frac{\pi^2}{6\ln t} - n\ln t\,. \tag{4.16}$$

We may approximate the latter integral for large n by noting that the dominant contribution is at the saddle point $t' = e^{-\pi/\sqrt{6n}} \sim 1$ for which

$$g(t') = \pi\sqrt{\tfrac{2}{3}n}\,, \quad g'(t') = 0\,, \quad g''(t') = \frac{2}{\pi}\sqrt{6n^3}\, e^{\pi\sqrt{2/3n}} = \alpha\,, \tag{4.17}$$

so that, for $t'' = t - t'$,

$$p(n) \sim e^{\pi\sqrt{2n/3}}\frac{1}{2\pi i}\oint \mathrm{d}t''\, e^{\frac{1}{2}\alpha t''^2} \sim e^{\pi\sqrt{2n/3}}\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}s\, e^{-\frac{1}{2}\alpha s^2} = \frac{1}{\sqrt{2\pi\alpha}}\, e^{\pi\sqrt{2n/3}}\,. \tag{4.18}$$

Thus,

$$\ln p(n) \sim \pi\sqrt{\tfrac{2}{3}n}\,, \tag{4.19}$$

which captures the correct behaviour of $\ln p(n)$ for large n, according to (3.41).

We may proceed analogously for the quarter BPS chiral ring multi-trace partition function at large N, (4.11), which we approximate by

$$\begin{aligned} \mathcal{C}_{U(\infty)}(t,u) &= \prod_{\substack{n_1,n_2\geq 0\\ n_1+n_2>0}} \frac{1}{1-t^{n_1}u^{n_2}} = \exp\Big(-\sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2>0}} \ln(1-t^{n_1}u^{n_2})\Big) \\ &\sim \exp\Big(-\int_0^\infty\int_0^\infty \mathrm{d}v\,\mathrm{d}w\ \ln(1-t^v u^w)\Big) = \exp\Big(\frac{\zeta(3)}{\ln t\,\ln u}\Big)\,. \end{aligned} \tag{4.20}$$

In this case we have, from (4.13),

$$\widehat{\mathcal{M}}_{(n,m)} \sim \frac{1}{(2\pi i)^2} \oint \oint \mathrm{d}t\, \mathrm{d}u \; e^{g(t,u)} \,, \tag{4.21}$$

where

$$g(t,u) = \frac{\zeta(3)}{\ln t \, \ln u} - n \ln t - m \ln u \,, \tag{4.22}$$

for n, m large. The dominant contribution to the integral, for n, m both large *and of the same order*, occurs about the point $(t', u') \sim (1,1)$ where

$$t' = e^{-(\zeta(3) m n^{-2})^{1/3}} \,, \qquad u' = e^{-(\zeta(3) n m^{-2})^{1/3}} \,, \tag{4.23}$$

for which

$$\begin{aligned} g(t',u') = 3\sqrt[3]{\zeta(3) nm} \,, \quad \frac{\partial}{\partial t} g(t,u)\Big|_{(t',u')} &= \frac{\partial}{\partial u} g(t,u)\Big|_{(t',u')} = 0 \,, \\ \frac{\partial^2}{\partial t^2} g(t,u)\Big|_{(t',u')} &= 2(\zeta(3)^{-1} n^5 m^{-1})^{\frac{1}{3}} \, e^{2(\zeta(3) m n^{-2})^{1/3}} = \alpha \,, \\ \frac{\partial^2}{\partial u^2} g(t,u)\Big|_{(t',u')} &= 2(\zeta(3)^{-1} m^5 n^{-1})^{\frac{1}{3}} \, e^{2(\zeta(3) n m^{-2})^{1/3}} = \beta \,, \\ \frac{\partial^2}{\partial t \partial u} g(t,u)\Big|_{(t',u')} &= (\zeta(3)^{-1} n^2 m^2)^{\frac{1}{3}} \, e^{(\zeta(3) m n^{-2})^{1/3} + (\zeta(3) n m^{-2})^{1/3}} = \gamma \,. \end{aligned} \tag{4.24}$$

So long as m, n are both large and of the same order, the saddle point approximation is justified and we obtain

$$\widehat{\mathcal{M}}_{(n,m)} \sim e^{3\sqrt[3]{\zeta(3) nm}} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{d}w \; e^{-\frac{1}{2}(\alpha v^2 + \beta w^2 + 2\gamma v w)} = h(\alpha,\beta,\gamma) e^{3\sqrt[3]{\zeta(3) nm}} \,, \tag{4.25}$$

where

$$h(\alpha,\beta,\gamma) = \frac{1}{2\pi\sqrt{\alpha\beta - \gamma^2}} = \frac{1}{2\pi\sqrt{3}} (\zeta(3) m^{-2} n^{-2})^{\frac{1}{3}} \, e^{-(\zeta(3) m n^{-2})^{1/3} - (\zeta(3) n m^{-2})^{1/3}} \,. \tag{4.26}$$

We thus have that for n, m both comparably large,

$$\ln \widehat{\mathcal{M}}_{(n,m)} \sim 3\sqrt[3]{\zeta(3) nm} \,, \tag{4.27}$$

so that

$$\ln \widehat{\mathcal{N}}_{(n,m)} = \ln \left(\widehat{\mathcal{M}}_{(n,m)} - \widehat{\mathcal{M}}_{(n+1,m-1)} \right) \sim \ln \widehat{\mathcal{M}}_{(n,m)} \sim 3\sqrt[3]{\zeta(3) nm} \,. \tag{4.28}$$

It is difficult to check the consistency of this result given the dearth of literature on these types of multi-variable generating functions and their asymptotic behaviour, however, we may consider the simpler function, also considered in [9],

$$\mathcal{C}_{U(\infty)}(t,t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^{n+1}} = \sum_{r=0}^{\infty} \mathcal{E}(r)\, t^r\,, \tag{4.29}$$

where, in terms of the counting numbers $\widehat{\mathcal{N}}_{(n,m)}$, from (1.1),

$$\mathcal{C}_{U(\infty)}(t,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} (n-m+1)\widehat{\mathcal{N}}_{(n,m)}\, t^{n+m} \quad \Rightarrow \quad \mathcal{E}(r) = \sum_{m=0}^{[\frac{1}{2}r]} (r-2m+1)\widehat{\mathcal{N}}_{(r-m,m)}\,. \tag{4.30}$$

Hence, $\mathcal{E}(r)$ counts quarter BPS primary operators in the chiral ring of $\mathcal{N}=4$ super Yang Mills, transforming in $[m,n-m,m]$ $SU(4)_R$ representations, with the same conformal dimensions $r=n+m$. Extracting the dominant contribution to $\ln\mathcal{E}(r)$ from (4.30), which occurs at the maximum value of m, $m_{\mathrm{M}} = [\frac{1}{2}r]$, and using (4.28), we obtain

$$\ln\mathcal{E}(r) \sim \ln\widehat{\mathcal{N}}_{(r-m_{\mathrm{M}},m_{\mathrm{M}})} \sim \tfrac{3}{2}\sqrt[3]{2\zeta(3)r^2}\,. \tag{4.31}$$

By considering (4.29) directly, we may employ Meinardus' theorem (described in the third section) to find the behaviour of $\ln\mathcal{E}(r)$ as $r\to\infty$. Note, however that Meinardus' theorem may not be applied directly to (4.29) since the corresponding auxiliary Dirichlet series (3.45), with $a_j = j+1$, has two simple poles. To overcome this difficulty we split (4.29) into a product of two functions, both separately amenable to application of Meinardus' theorem. One is the reciprocal of the Euler function, $P_\infty(t)$ in (4.15). The other, the MacMahon function, is given by

$$M(t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^n} = \sum_{r=0}^{\infty} q(r)\, t^r\,, \tag{4.32}$$

and has been considered in a similar context as here in [3].[11] Writing

$$\mathcal{C}_{U(\infty)}(t,t) = P_\infty(t)M(t)\,, \tag{4.33}$$

[11] The relation of $M(t)$ in (4.32) to plane partitions is given a description in [46]. Briefly, $q(r)$ gives the number of ordinary plane partitions π, so that $\pi_{ij} \geq \pi_{i+1\,j} > 0$, $\pi_{ij} \geq \pi_{i\,j+1} > 0$, with $|\pi| = \sum_{i,j}\pi_{ij} = r$. $p(r) < q(r)$ as ordinary partitions λ are a special case of plane partitions. In fact, the formula for $\ln q(r)$ found here is a special case of a more exact asymptotic formula first found by Wright [48] for the number of plane partitions $q(r)$ of the number r.

with $P_\infty(t)$ as in (4.15), so that, using (4.29),

$$\mathcal{E}(r) = \sum_{\substack{r_1, r_2 \geq 0 \\ r_1 + r_2 = r}} p(r_1) q(r_2) \,, \tag{4.34}$$

we may find the asymptotic behaviour of $\ln \mathcal{E}(r)$, as $r \to \infty$, by extracting the dominant contribution from (4.34) using the asymptotic behaviour of $p(r)$, $q(r)$. The auxiliary Dirichlet series for $M(t)$ in (4.32) is, from (3.45) with $a_j = j$,

$$D(s) = \zeta(s-1) \,, \tag{4.35}$$

which has a simple pole at $s = \alpha = 2$, at which the residue is $A = 1$. Thus, from (3.44),

$$\ln q(r) \sim \tfrac{3}{2} \sqrt[3]{2\zeta(3) r^2} \,. \tag{4.36}$$

This is consistent with (4.31) as the dominant contribution to $\ln \mathcal{E}(r)$ comes from the $r_1 = 0$ term in (4.34) (since $p(r) \ll q(r)$ as $r \to \infty$) so that $\ln \mathcal{E}(r) \sim \ln q(r)$.

It has not escaped attention that the method used here, to capture the exponential behaviour of asymptotic values for the numbers $\widehat{\mathcal{M}}_{(n,m)}$, may be easily extended to chiral ring sectors other than the quarter BPS one. Suppose, for simplicity, that Z_j, $1 \leq j \leq k-1$, are commuting bosonic fundamental fields, in the $U(N)$ Lie algebra, so that the single particle partition function is given by $f(\mathrm{t}) = \sum_{j=1}^{k-1} t_j$, in terms of the corresponding letters t_j. Let $\widehat{\mathcal{M}}_{(m_1,\ldots,m_{k-1})}$ denote the number of independent operators involving products of m_1 Z_1's, m_2 Z_2's etc. in corresponding multi-trace operators. The multi-trace partition function, in the large N limit, is given by,

$$\mathcal{C}_{U(\infty)}(\mathrm{t}) = \prod_{\substack{n_1,\ldots,n_{k-1} \geq 0 \\ n_1 + \ldots + n_{k-1} > 0}} \frac{1}{1 - {t_1}^{n_1} \cdots {t_{k-1}}^{n_{k-1}}} \,, \tag{4.37}$$

which may be crudely approximated by, similarly as before,[12]

$$\mathcal{C}_{U(\infty)}(\mathrm{t}) \sim \exp\Big(- \int_0^\infty \prod_{j=1}^{k-1} \mathrm{d}v_j \ln(1 - {t_1}^{v_1} \cdots {t_{k-1}}^{v_{k-1}}) \Big) = \exp\left(\frac{(-1)^{k+1} \zeta(k)}{\ln t_1 \cdots \ln t_{k-1}} \right) . \tag{4.38}$$

[12] For $\mathrm{Li}_n(x) = \sum_{j \geq 1} x^j / j^n$ being the usual Polylogarithm, with $\mathrm{Li}_n(1) = \zeta(n)$, $n > 1$, $\mathrm{Li}_n(0) = 0$, then with the convention $\mathrm{Li}_1(x) = -\ln(1-x)$, the following integral

$$\int_0^1 \frac{\mathrm{d}x}{x} \, \mathrm{Li}_n(zx) = \mathrm{Li}_{n+1}(z) \,,$$

may be useful for showing this, after a suitable change of variables.

Thus, without going into as much detail, for the analogue of (4.27) we have, (assuming m_j are all comparably large,)

$$\ln \widehat{\mathcal{M}}_{(m_1,\ldots,m_{k-1})} \sim g(t'_1,\ldots,t'_{k-1}) = k \sqrt[k]{\zeta(k)\, m_1 \cdots m_{k-1}}\,, \tag{4.39}$$

where $g(t'_1,\ldots,t'_{k-1})$ is the value of

$$g(t_1,\ldots,t_{k-1}) = \frac{(-1)^{k+1}\zeta(k)}{\ln t_1 \cdots \ln t_{k-1}} - m_1 \ln t_1 - \ldots - m_{k-1} \ln t_{k-1}\,, \tag{4.40}$$

at the saddle point $(t'_1,\ldots,t'_{k-1}) \sim (1,\ldots,1)$, where

$$(\ln t'_1,\ldots,\ln t'_{k-1}) = -\sqrt[k]{\zeta(k)\, m_1 \cdots m_{k-1}}\ (1/m_1,\ldots,1/m_{k-1})\,, \tag{4.41}$$

so that

$$\frac{\partial}{\partial t_j} g(t_1,\ldots,t_{k-1})\Big|_{(t'_1,\ldots,t'_{k-1})} = 0\,, \qquad j = 1,\ldots,k{-}1\,. \tag{4.42}$$

(4.39) is consistent with a result implied by Meinardus' theorem. The function,[13]

$$\mathcal{C}_{U(\infty)}(t,\ldots,t) \sim \prod_{n=1}^{\infty} (1-t^n)^{-n^{k-2}/(k-2)!} = \sum_{r=0}^{\infty} c(k,r) t^r\,, \tag{4.43}$$

has auxiliary Dirichlet series, from (3.45) with $a_j = j^{k-2}/(k-2)!$,

$$D(s) = \frac{1}{(k-2)!}\,\zeta(s+2-k)\,, \tag{4.44}$$

which has a simple pole at $s = \alpha = k{-}1$ at which the residue is $A = 1/(k-2)!$, so that, from (3.44),

$$\ln c(k,r) \sim \frac{k}{k{-}1} \sqrt[k]{(k{-}1)\,\zeta(k)\, r^{k-1}}\,. \tag{4.45}$$

(4.45) is precisely the result that may be obtained from (4.39) if we maximise the product $m_1 \cdots m_{k-1}$, subject to the constraint $\sum_{j=1}^{k-1} m_j = r$, for which the solution is $m_j = m_{\mathrm{M}} = r/(k-1)$ (relaxing the constraint that m_j be non-negative integers, which is irrelevant asymptotically), so that $\ln \widehat{\mathcal{M}}_{(m_{\mathrm{M}},\ldots,m_{\mathrm{M}})} \sim \ln c(k,r)$.

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[13] This may be easily seen from (4.37), as the number of solutions to $\sum_{j=1}^{k-1} m_j = n$, where m_j are non-negative integers, is the binomial number $\binom{n+k-2}{k-2}$ which, to leading order in large n, behaves like $n^{k-2}/(k{-}2)!$. More properly, we should split the product $\mathcal{C}_{U(\infty)}(t,\ldots,t)$ into pieces separately amenable to Meinardus' theorem, as for the prior case for $k = 3$, however, just as for that case, the numbers $c(k,r)$ dominate, and so other contributions are ignored here.

This is applicable to counting multi-trace operators in the eighth BPS chiral ring sector for $\mathcal{N} = 4$ super Yang Mills with fundamental fields Z, Y, X involving m_1 Z's, m_2 Y's, m_3 X's. Expanding the corresponding partition function (4.37), with $k = 4$, in terms of Schur polynomials $s_{(m_1,m_2,m_3)}(\mathrm{t})$, $m_1 \geq m_2 \geq m_3 \geq 0$, similar to (1.1), the expansion coefficients $\widehat{\mathcal{N}}_{(m_1,m_2,m_3)}$ count spinless multi-trace primary operators transforming in the $[m_2+m_3, m_1-m_2, m_2-m_3]$ $SU(4)_R$ R-symmetry representation, with conformal dimensions $m_1+m_2+m_3$ [33]. Just as in (4.28), asymptotically $\ln \mathcal{N}_{(m_1,m_2,m_3)} \sim \ln \mathcal{M}_{(m_1,m_2,m_3)}$. This counting, however, ignores contributions of the fermionic fields $\lambda, \bar{\lambda}$, which it may be important to include in order to give correct counting of eighth BPS chiral ring operators.

5. Conclusions

There are some obvious questions not answered by this work. The first is whether or not the approach in the second section using symmetric polynomials can give insight into thermodynamics at finite N, such as for the Hagedorn transition, for example. While it gives the large N expression (2.14) in an elementary way, its wider applicability or usefulness to such questions is unclear. The approach is undoubtedly useful for finding exact expressions for counting numbers (as in (3.2), (3.27) and (3.34) for quarter BPS operators) and (3.9), (3.16) may be useful for analysing counting for more complicated sectors of $\mathcal{N} = 4$ super Yang Mills, with gauge group $U(N)$.

The second question is how the arguments employing symmetric polynomial techniques here may be extended to other gauge groups, the most pertinent being perhaps $SU(N)$. Arguments here employing (2.6) and the orthonormality property of Schur polynomials should remain largely unaffected for $SU(N)$. Exact values for counting numbers obtained here should require some modification for $SU(N)$, though asymptotic values may be unchanged.

The third question concerns asymptotic values for counting numbers and how these may be improved. The asymptotic counting formulae given in such papers as [3,9] for chiral ring sectors are special cases of formulae such as those of Hardy and Ramanujan, Meinardus, etc., all of which derive from single variable generating functions. It is hoped that the expressions (3.50), (4.28), (4.39), given here for asymptotic counting of BPS operators, that distinguishes between differing R-symmetry charges, represents a serious attempt at going beyond consideration of single variable generating functions.[14] Improving upon these formulae will require more sophisticated techniques, perhaps along the lines

[14] After submission of the first version of this paper to the electronic archive, I received

used to find those of Hardy and Ramanujan or Meinardus and employing any modular properties of the multi-variable functions involved. This issue may also be important for microscopic counting for Black Holes, as the BPS solutions found thus far, for $\mathcal{N} = 4$ superconformal symmetry, depend on special values of R-symmetry charges [17,18,19,20] - see [50] for a related detailed discussion.

Thus far, the elegant results for finite N partition functions for chiral ring sectors have been interpreted from a largely geometric perspective - it may be interesting to investigate more how such results are related to the theory of random matrices and/or symmetric polynomials.

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an e-mail from Hai Lin pointing out an interesting comparison between (4.28) here and (2.14) of [49], obtained in quite a different context. The two formulae are essentially the same given the numerical value $3\sqrt[3]{\zeta(3)} = 3.189\ldots$, correct to three decimal places.

Appendix A. Partitions, symmetric group characters, symmetric polynomials and inner products

A generic partition λ is any finite or infinite sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers in decreasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ containing only finitely many non-zero terms. Often it is convenient to omit zero entries. The non-zero entries are called the parts of λ the number of which we denote by $\ell(\lambda)$. The sum of the parts of λ is called the weight of λ which we denote by $|\lambda| = \sum_i \lambda_i$. If $|\lambda| = L$ then λ is a partition of L and we write $\lambda \vdash L$. For convenience we sometimes write λ in its frequency representation which is a reordering of the entries in λ, indicating the number of times each successive non-negative integer occurs, $(1^{a_1}, 2^{a_2}, \ldots)$ so that exactly a_n of the parts of λ equal n and $|\lambda| = \sum_{n\geq 1} n\, a_n$.

In terms of standard Young diagrams, λ corresponds to a Young diagram of shape λ, with λ_1 boxes in the first row, λ_2 boxes in the second row etc.; the number of parts $\ell(\lambda)$ is simply the number of rows and the weight $|\lambda|$ is the total number of boxes.

For the symmetric group, $\mathcal{S}_N$, the irreducible representations are labelled by partitions $\lambda \vdash N$ - see [38] for a useful summary - so that, for $X^\lambda(\sigma)$, $\sigma \in \mathcal{S}_N$, being a corresponding matrix representation, then the character of $\sigma \in \mathcal{S}_N$ in the representation X^λ is $\chi^\lambda(\sigma) = \mathrm{tr}(X^\lambda(\sigma))$. The characters are class functions so that they take a constant value on conjugacy classes and, recalling that for $\mathcal{S}_N$ the conjugacy classes K_μ are labelled by partitions $\mu \vdash N$, corresponding to the cycle structure of a class representative, then $\chi^\lambda(\sigma) = \chi^\lambda_\mu$ for all $\sigma \in K_\mu$. With z_λ as defined in (2.8), a crucial property of $\mathcal{S}_N$ characters is the orthogonality of the matrix $[{z_\mu}^{-1/2}\chi^\lambda_\mu]_{\lambda\mu}$. This gives rise to the orthogonality relations, for $\lambda, \mu \vdash N$, (see also Ch. IV of [51] for a related discussion,)

$$\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N} \chi^\lambda(\sigma)\chi^\mu(\sigma) = \sum_{\nu\vdash N} \frac{1}{z_\nu}\chi^\lambda_\nu\, \chi^\mu_\nu = \delta_{\lambda\mu}\,, \tag{A.1}$$

and

$$\sum_{\nu\vdash N} \chi^\nu_\lambda\, \chi^\nu_\mu = z_\lambda \delta_{\lambda\mu}\,. \tag{A.2}$$

A convenient basis for N variable symmetric polynomials are Schur polynomials $s_\lambda(\mathrm{z}) = s_\lambda(z_1, \ldots, z_N)$ labelled by $\lambda = (\lambda_1, \ldots, \lambda_N)$. They may be expressed in a number of ways [24,47]. For convenience we write them as

$$s_\lambda(\mathrm{z}) = a_{\lambda+\rho}(\mathrm{z})/a_\rho(\mathrm{z})\,, \tag{A.3}$$

where ρ, the Weyl vector, is given by $\rho = (N-1, N-2, \dots, 1, 0)$ and

$$a_{\lambda+\rho}(\mathrm{z}) = \sum_{\sigma\in\mathcal{S}_N} \mathrm{sgn}(\sigma)\, {z_{\sigma(1)}}^{\lambda_1+N-1} \cdots {z_{\sigma(j)}}^{\lambda_j+N-j} \cdots {z_{\sigma(N)}}^{\lambda_N} = \det[{z_i}^{\lambda_j+N-j}] \,, \tag{A.4}$$

with

$$a_\rho(\mathrm{z}) = \det[{z_i}^{N-j}] = \prod_{1\le i<j\le N} (z_i - z_j) = \Delta(\mathrm{x}) \,, \tag{A.5}$$

being the Vandermonde determinant. Schur polynomials $s_\lambda(\mathrm{z})$ have a standard interpretation as corresponding to the characters of irreducible $U(N)$ (or, for $\prod_i z_i = 1$, $SU(N)$) Lie algebra representations. Here, λ gives the shape of the Young tableaux for the corresponding $U(N)$ Lie algebra representation.

For $\lambda = (\lambda_1, \dots, \lambda_N)$ and $\mu = (\mu_1, \dots, \mu_N)$ where $\lambda_i, \mu_i \in \mathbb{Z}$ then, from the definition of (2.6) along with (A.3),

$$\big\langle s_\lambda, s_\mu \big\rangle_N = \sum_{\sigma\in\mathcal{S}_N} \mathrm{sgn}(\sigma)\delta_{\lambda^\sigma\mu} = \sum_{\sigma\in\mathcal{S}_N} \mathrm{sgn}(\sigma)\delta_{\lambda\mu^\sigma} \,, \tag{A.6}$$

where, for any $\lambda' = (\lambda'_1, \dots, \lambda'_N)$, $\lambda'^\sigma = \sigma(\lambda'+\rho)-\rho$ is the shifted Weyl reflection of λ' by σ, with the action of $\mathcal{S}_N$ on λ' being given by $\sigma(\lambda'_1, \dots, \lambda'_N) = (\lambda'_{\sigma(1)}, \dots, \lambda'_{\sigma(N)})$. (Equation (A.6) is a reflection of $s_\lambda(\mathrm{x}) = \mathrm{sgn}(\sigma)s_{\lambda^\sigma}(\mathrm{x})$ for any partition λ and $\sigma \in \mathcal{S}_N$ - note that this property is useful for showing (3.25), (3.31). λ^σ has a standard interpretation in terms of $U(N)$ Lie algebra representations - for the Verma module with dominant integral highest weight having orthonormal basis labels λ, $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N \ge 0$, then λ^σ, for $\sigma \ne \mathrm{id}_{\mathcal{S}_N}$, are the orthonormal basis labels for the highest weights of all invariant sub-modules. This fact may be exploited to derive the Weyl character formula (A.3) for the irreducible $U(N)$ Lie algebra representation with dominant integral highest weight having orthonormal basis labels λ, or, alternatively, Young tableaux of shape λ.)

When λ, μ are partitions so that $\lambda_1 \ge \dots \ge \lambda_N \ge 0$ and $\mu_1 \ge \dots \ge \mu_N \ge 0$ then (A.6) reduces to a well defined inner product,

$$\big\langle s_\lambda, s_\mu \big\rangle_N = \delta_{\lambda\mu} \,, \tag{A.7}$$

so that in this case the Schur polynomials are orthonormal. Note that in order that $s_\lambda(\mathrm{x})$ be non-zero for some arbitrary partition λ then $\ell(\lambda) \le N$, so that (A.7) is zero for $\ell(\lambda) > N$ or $\ell(\mu) > N$.

Another basis for symmetric polynomials are the power symmetric polynomials, $p_\lambda(\mathrm{z})$, for $\lambda = (\lambda_1, \dots, \lambda_L) \vdash L$, which are defined by

$$p_\lambda(\mathrm{z}) = p_{\lambda_1}(\mathrm{z})p_{\lambda_2}(\mathrm{z}) \cdots p_{\lambda_L}(\mathrm{z}) \,, \qquad p_n(\mathrm{z}) = \sum_{i=1}^N {z_i}^n \,. \tag{A.8}$$

Note that there is no longer the restriction that $\ell(\lambda) \leq N$ as for Schur polynomials.

Symmetric group characters may be used to relate the two bases for symmetric polynomials [24,47] so that, with the definition of z_λ in (2.8),

$$s_\lambda(\mathrm{z}) = \sum_{\mu \vdash N} \frac{1}{z_\mu} \chi^\lambda_\mu \, p_\mu(\mathrm{z}) \,, \tag{A.9}$$

(a theorem of Frobenius) and, for $\lambda \vdash L$,

$$p_\lambda(\mathrm{z}) = \sum_{\substack{\mu \vdash L \\ \ell(\mu) \leq N}} \chi^\mu_\lambda \, s_\mu(\mathrm{z}) \,. \tag{A.10}$$

((A.9) with $\chi^{(N)}_\lambda = 1$ for all $\lambda \vdash N$ is useful for obtaining (3.35).)

Regarding the inner product (2.6), then using (A.10) along with (A.7), we then have that, for $\lambda \vdash L$, $\mu \vdash M$,

$$\big\langle p_\lambda, p_\mu \big\rangle_N = \delta_{LM} \sum_{\substack{\nu \vdash L \\ \ell(\nu) \leq N}} \chi^\nu_\lambda \chi^\nu_\mu \,. \tag{A.11}$$

Orthogonality of symmetric group characters implies, from (A.2), that for $|\lambda|, |\mu| \leq N$ then (A.11) simplifies to, with the definition of z_λ in (2.8),

$$\big\langle p_\lambda, p_\mu \big\rangle_N = z_\lambda \delta_{\lambda\mu} \,. \tag{A.12}$$

Appendix B. Tables

$\mathcal{N}_{(n)}$ $N \backslash n$	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1
2	2	2	3	3	4	4	5	5	6	6
3	2	3	4	5	7	8	10	12	14	16
4	2	3	5	6	9	11	15	18	23	27
5	2	3	5	7	10	13	18	23	30	37
6	2	3	5	7	11	14	20	26	35	44

Numbers of multi-trace half BPS primary operators, with conformal dimension n and belonging to $[0, n, 0]$ R-symmetry representations, for free $\mathcal{N} = 4$ SYM with $U(N)$ gauge group. (For every N there is one $[0, 0, 0]$ and $[0, 1, 0]$ representation - these are omitted above.)

$\mathcal{N}_{(n,1)}$ $N \backslash n$	2	3	4	5	6	7	8	9	10	11
2	1	1	2	2	3	3	4	4	5	5
3	1	2	4	5	8	10	13	16	20	23
4	1	2	5	7	12	16	23	30	40	49
5	1	2	5	8	14	20	30	41	57	74
6	1	2	5	8	15	22	34	48	69	92
7	1	2	5	8	15	23	36	52	76	104

Numbers of multi-trace quarter BPS primary operators, with conformal dimension $n+1$ and belonging to $[1, n-1, 1]$ R-symmetry representations, for free $\mathcal{N} = 4$ SYM with $U(N)$ gauge group. ($n = 0, 1$ cases are all zero.)

$\mathcal{N}_{(n,2)}$ $N \backslash n$	2	3	4	5	6	7	8	9	10	11
3	3	5	10	14	21	27	36	44	55	65
4	3	6	14	21	36	50	73	96	130	163
5	3	6	15	25	44	66	101	142	200	267
6	3	6	15	26	48	74	118	171	251	346
7	3	6	15	26	49	78	126	188	281	398
8	3	6	15	26	49	79	130	196	298	428

Numbers of multi-trace quarter BPS primary operators, with conformal dimension $n+2$ and belonging to $[2, n-2, 2]$ R-symmetry representations, for free $\mathcal{N} = 4$ SYM with $U(N)$ gauge group. ($n = 0, 1$ cases are all zero.)

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