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Studies in the Non-Symmetric Generalization of the Theory of Gravitation I

BY

E. SCHRÖDINGER

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STUDIES IN THE NON-SYMMETRIC GENERALIZATION OF THE THEORY OF GRAVITATION I.

By E. SCHRÖDINGER.

Summary: The field equations are solved for weak fields. Given a weak Maxwellian field, the gravitational field can be found by quadratures. It is entirely different from what older theories would let one expect. Moreover to Maxwell's equations a condition for the four-current is added, viz. that it has to be the gradient of an invariant which satisfies D'Alembert's wave equation. Several possible analogues of the matter tensor are discussed and computed for weak Maxwellian fields including charges. The approximation reached here is insufficient and will have to be extended in order to reveal the reaction of the fields on their sources.

Introduction.

According to Einstein's famous theory of 1916 the gravitational field in empty space is mathematically described as follows. A symmetrical fundamental tensor g_{ik} shall have vanishing covariant derivative with respect to a symmetrical affinity Γ^i_{kl}, which by this demand is uniquely determined as the Christoffel-bracket affinity $\left\{ {i \atop k\ l} \right\}$, formed of the g_{ik} and their first derivatives. The contracted curvature tensor R_{ik} of this affinity shall vanish. (A later version of the theory replaces $R_{ik} = 0$ by $R_{ik} = \lambda g_{ik}$; but the constant λ of the dimension $[\text{coordinate}]^{-2}$ is so small on any "human" scale that for most purposes the "cosmical term" can be dropped.)

The non-symmetric generalization of this theory, first pointed out by Einstein[1] and Einstein and Straus[2], consists essentially in dropping the two restrictions "symmetrical", underlined above. At first an undesirable freedom turns up as to how to generalize the notion of "covariant derivative" with respect to a non-symmetric affinity; but this question is unequivocally decided by the demand that the Γ^i_{kl} should again be uniquely determined as functions of the g_{ik} and their first derivatives and go over into the Christoffel-brackets, when g_{ik} becomes symmetric. The two versions as regards λ $(= 0$ or $\neq 0)$ remain. But another dilemma is more momentous. The theory, as described in the preceding sentences of this paragraph reads

$$g_{\underset{+-}{ik};l} \equiv g_{ik,l} - g_{sk}\Gamma^s_{il} - g_{is}\Gamma^s_{lk} = 0 \qquad (1)$$

$$R_{ik} \equiv -\frac{\partial \Gamma^s_{ik}}{\partial x_s} + \frac{\partial \Gamma^s_{is}}{dx_k} + \Gamma^s_{it}\Gamma^t_{sk} - \Gamma^s_{ts}\Gamma^t_{ik} = \lambda g_{ik} . \qquad (2)$$

Now the equations (1), by determining the Γ^i_{kl}, determine a basic vector field with components $\Gamma_k = \Gamma^i_{\underset{\vee}{ki}}$ (the hook $\vee$ is short for "skew part of"). This vector field vanishes, of course, in the symmetric case. It would seem not at all unnatural that it should not do so in the general case, where the demand

$$\Gamma^i_{\underset{\vee}{ki}} = 0 \qquad (3)$$

is indeed a severe further restriction. We shall, however, here follow that more restrictive form of the theory (suggested by Einstein) which adjoins (3) on equal footing to the field equations (1) and (2). This entails that

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1) A. Einstein, Journal of Mathematics, 46, p.578, 1945.

2) A. Einstein and E.G. Straus, ibid., 47, p.731, 1946.

the indispensable (first) set of Maxwell's equations

$$\mathfrak{g}^{\overset{ik}{\vee}}{}_{,i} = 0 \qquad (4)$$

follows directly from (1) without any further complication.[3)]

The equations (1), (2), (3) with the immediate consequence (4) represent so much the simplest generalization of Einstein's theory of pure gravitation that it is imperative to study its possibilities as closely as one can. Such investigation will carry in different direction according to what accomplishment one expects from the theory. One may hope that exact solutions, involving strong fields, will reveal the nature of the ultimate particles. I do not believe this, mainly because I do not believe the ultimate particles to be identifiable individuals that could be described in this fashion. Moreover in the symmetric theory (i.e. in Einstein's theory of 1916) the exact solutions, involving strong fields, have disclosed the ingenuity of the mathematicians who discovered them, but nothing more. Not only would their application to the ultimate particles teach us nothing about the latter; but none of the great successes of the theory depended on those ingenious solutions. All the results, confirmed by observation, could be worked out, though with less elegance and much more trouble, by a mathematician who could handle only routine methods of approximation.

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3) $\mathfrak{g}^{ik}$ is defined as $g^{ik}\sqrt{-g}$, where g is the determinant of the g_{ik}, and the g^{ik} are defined by $g^{ik}g_{il} = g^{ki}g_{li} = \delta^k_l$. For a comprehensive survey see my paper Proc.R.Irish Acad. 51 (A), p.163, 1947. The complication alluded to above is the necessity of distinguishing between $^*\Gamma^i_{kl}$ and Γ^i_{kl}, the latter intervening in (2), the former in (1) and (3).

In the present theory an assiduous application of such methods to weak fields is bound to tell us something on the interlacing of three things, gravitational field, electromagnetic field, and electric charges, all three of which here spring from one basic conception (so much so that for <u>strong</u> fields the sharp distinction between them would probably disappear). One may hope that this will provide a better foundation to the quantum mechanical treatment of the fields, which at present is based on a number of classical or pseudoclassical field theories of independent origin, cemented together by "interaction terms". Macroscopic experience, embodied in classical theories such as Maxwell's, guides our choice as regards both the basic field equations and the interaction terms, but still leaves much arbitrariness. The most powerful general restriction is, of course, Lorentz-invariance. Is it too much to expect safer guidance from a unified theory based from the outset on the principle of general invariance?

This hope is not abated, but strengthened by the fact that the present theory, as we shall see, is not even in first approximation a simple replica of what one gets by introducing "matter" in the form of a Maxwellian field into the Riemannian manifold that represents pure gravitation in the symmetric theory. A momentous discrepancy is revealed by a brief consideration of equations (1) and (2). From (1) the Γ's are linear functions of the first derivatives $g_{ik,l}$, just as they are in the symmetric case. It is therefore easy to see that in the first member of (2), i.e. in the Einstein tensor, all terms contain two derivations, being either linear in the second derivatives of the g_{ik}, or quadratic in the first. If the skew field, $g_{\underset{\vee}{ik}}$, vanishes you get just the Einstein tensor of the $\left\{ {i \atop k\;l} \right\}$. If the symmetric field, $g_{\underline{ik}}$, is Galilean, only terms containing two derivations on the $g_{\underset{\vee}{ik}}$ (being either linear in their second or quadratic in their first derivatives) can survive. They become an additive supplement to the Einstein-tensor in the case that both the $g_{\underset{\vee}{ik}}$ and the deviations of the $g_{\underline{ik}}$ from (-1,-1,-1,1) are <u>small</u>. Hence the said $g_{\underset{\vee}{ik}}$-terms, according to the 10 symmetric components of equations (2), constitute the "matter tensor" by which the electromagnetic field "generates" a gravitational field. One would expect in these places the components of Maxwell's stress-energy-momentum tensor, the familiar quadratic forms of the <u>non-differentiated</u> $g_{\underset{\vee}{ik}}$. But this is obviously <u>ultra vires</u> of a theory like ours, which from its fundamental structure must yield here an entity that might loosely be termed an "Einstein tensor, formed of the $g_{\underset{\vee}{ik}}$" with regard to its dependence on the second and first derivatives of the latter.

This has an interesting consequence for the universal constants that are involved, when the equations are expressed in C.G.S-units. In the symmetric theory, where the matter-tensor T_{ik} (say, in energy-units) is introduced explicitly, it is multiplied by

$$\kappa = \frac{8\pi k}{c^4} = 2.073 \times 10^{-48}\ g^{-1}cm^{-1}sec^2 , \qquad (5)$$

which gives it the required dimensions cm^{-2} of the Einstein tensor. In the present theory one must clearly regard the "geometrical" $g_{\underset{\vee}{ik}}$ as the electromagnetic field, measured in some (probably very big) universal unit, say <u>b</u> in C.G.S. Thus instead of κ the constant b^{-2} turns up in the quadratic terms (which are the leading ones). Since b^2 is an energy density ($g\ cm^{-1}sec^{-2}$) the factor b^{-2} has the dimension

$$[b^{-2}] = g^{-1}\ cm\ sec^2 , \qquad (6)$$

which differs from (5) by the square of a length. If one puts

$$b^{-2} = \varkappa l^2 = 2.073 \times 10^{-48} \, l^2 , \qquad (7)$$

one is inclined to think that l must be a universal length of some importance. What may it mean? Well, the fact that the gravitational effect of an electromagnetic field depends on derivatives of the field-strength, means, broadly speaking, that the effect is enhanced for short wavelength or high frequency and reduced for long waves. Our l indicates - again very broadly speaking - the order of magnitude of the wave length for which the gravitational effect is of the same order as judged heretofore.

These considerations may suffice to indicate the strangeness and novelty of the present theory. They raise a host of questions which one cannot hope to decide without going into many more details about its concrete consequences.

1. Radiation Field without Charges.

The 64 equations (1), ordinary linear equations with respect to the 64 Γ's, determine the latter uniquely as rational functions of the g's and their first derivatives. But the routine solution, which expresses each Γ as the quotient of two determinants of rank 64 is practically beyond control, it is just impossible to handle. For many purposes the following procedure is useful. One splits (1) into the symmetric and skew-symmetric parts, writing them thus

$$g_{\underline{ik},l} - g_{\underset{\vee}{sk}} \Gamma^s_{\underset{\vee}{il}} - g_{\underset{\vee}{is}} \Gamma^s_{\underset{\vee}{lk}} = g_{\underline{sk}} \Gamma^s_{\underline{il}} + g_{\underline{is}} \Gamma^s_{\underline{lk}} \qquad (1,1)$$

$$g_{\underset{\vee}{ik},l} - g_{\underset{\vee}{sk}} \Gamma^s_{\underline{il}} - g_{\underset{\vee}{is}} \Gamma^s_{\underline{lk}} = g_{\underline{sk}} \Gamma^s_{\underset{\vee}{il}} + g_{\underline{is}} \Gamma^s_{\underset{\vee}{lk}} . \qquad (1,2)$$

Envisage the first equation. Following a well-known routine, multiply it by $\frac{1}{2}$ and add to it, member by member, the two equations obtained by cyclic permutations (ikl), after multiplying them by $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Then on the right only one term survives. Now introduce the symmetric tensor h^{ik} by *)

$$h^{ik} g_{\underline{il}} = \delta^k_l \tag{1,3}$$

(which implies that we assume the determinant of the $g_{\underline{ik}}$ to be $\neq 0$). This enables us to obtain the expressions (1,4) for the $\Gamma^i_{\underline{kl}}$. Exactly the same procedure, applied to (1,2), gives (1,5):

$$\Gamma^i_{\underline{kl}} = \left\{ {i \atop k\ l} \right\} - h^{im}\left(g_{\underset{\vee}{ls}} \Gamma^s_{\underset{\vee}{km}} + g_{\underset{\vee}{ks}} \Gamma^s_{\underset{\vee}{lm}}\right) \tag{1,4}$$

$$\Gamma^i_{\underset{\vee}{kl}} = \left\langle {i \atop k\ l} \right\rangle + h^{im}\left(g_{\underset{\vee}{ls}} \Gamma^s_{\underline{km}} - g_{\underset{\vee}{ks}} \Gamma^s_{\underline{lm}}\right) . \tag{1,5}$$

While the curly bracket is precisely the Christoffel-symbol of the $g_{\underline{ik}}$,

$$\left\{ {i \atop k\ l} \right\} = \frac{1}{2} h^{im}\left(g_{\underline{ml},k} + g_{\underline{km},l} - g_{\underline{kl},m}\right) , \tag{1,6}$$

the pointed bracket stands for a somewhat analogous expression formed from the $g_{\underset{\vee}{ik}}$

$$\left\langle {i \atop k\ l} \right\rangle = \frac{1}{2} h^{im}\left(g_{\underset{\vee}{km},l} + g_{\underset{\vee}{ml},k} + g_{\underset{\vee}{kl},m}\right) . \tag{1,7}$$

Notice that the third term just <u>fails</u> to continue the cyclic permutation. A sometimes useful remark is that the second member of (1,5) could also be written

$$\frac{1}{2} h^{im}\left(g_{\underset{\vee}{km};l} + g_{\underset{\vee}{ml};k} + g_{\underset{\vee}{kl};m}\right) , \tag{1,8}$$

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*) We must not call it $g^{\underline{ik}}$, because this is something else.

the semicolon meaning the invariant derivative with respect to the symmetric affinity $\Gamma^{i}_{\underline{kl}}$.

The expressions (1,4) and (1,5) are exact, but at first sight not much seems to be gained by them, since the first only expresses the *affinity* $\Gamma^{i}_{\underline{kl}}$ by the *tensor* $\Gamma^{i}_{\underset{\vee}{kl}}$, and the second the other way round. They are useful *inter alia* for investigating solutions *in the neighbourhood* of a symmetric $g_{\underline{ik}}$-field. For from (1,5), if the $g_{\underset{\vee}{ik}}$ are *small*, the components of the *tensor* are small of the same order. Hence, from (1,4) the symmetric affinity differs from the Christoffel-brackets only by quantities of the second order. Thus, using in (1,5) the Christoffel-brackets for $\Gamma^{s}_{\underline{km}}$ etc., one gets the tensor, with an error of the third order; and if this is used in (1,4), one gets $\Gamma^{i}_{\underline{kl}}$ with an error of the fourth order. Thus, by alternating substitutions, both parts, and hence the whole Γ^{i}_{kl}, is developed in a series of ascending powers of the $g_{\underset{\vee}{ik}}$; the rule that produces the next term from the preceding one could easily be established. The explicit approximations for the Γ^{i}_{kl} have then to be subjected to the equations (2).

We shall use this first to derive a solution which corresponds to a weak, but otherwise arbitrary field of radiation without charge and current.

On account of (3) the set of equations whose approximate treatment we have just explained has the consequence (4), which is obviously Maxwell's *first* set, in other words it states the vanishing of the *magnetic* four-current. It will thus be seen that in the present theory, at variance with common usage, the magnetic field is represented by the components that have an index 4, the electric field by those that have none. The vanishing of the *electric* four-current is *not* a field-equation, but a condition we now impose to specify our solutions. We choose a "small" six-vector φ_{ik} for which *exactly*

$$\varphi_{ik,l} + \varphi_{kl,i} + \varphi_{li,k} = 0 \qquad (1,9)$$

and put

$$\underset{\vee}{g_{ik}} = \varphi_{ik} \,. \qquad (1,10)$$

We anticipate that the $\underline{g_{ik}}$ deviate from their Galilean values only by quantities of the second order, or rather that solutions can be found for which this is the case; but we shall have to prove it. For the moment we <u>assume</u>

$$g_{ik} = \varepsilon_i \, \delta_{ik} + \varphi_{ik} + \gamma_{ik} \,, \qquad (1,11)$$

where

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1 \,, \qquad \varepsilon_4 = 1 \,, \qquad \gamma_{ik} = \gamma_{ki} \qquad (1,12)$$

and the γ's are of the second order.*) It is then easily seen that

$$\mathfrak{g}^{\underset{\vee}{ik}} = \varepsilon_i \, \varepsilon_k \, \varphi_{ik} + O_3 \,, \qquad (1,13)$$

so that the field equations (4) demand

$$\varepsilon_i \, \varphi_{ik,i} = 0 \,, \qquad (1,14)$$

third order quantities being neglected. If (1,9) is multiplied by ε_i and differentiated with respect to x_i, one gets, with regard to (1,14),

$$\varepsilon_i \, \varphi_{kl,i,i} = 0 \,, \qquad (1,15)$$

which is the D'Alembert equation for every component.

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*) The symbol ε_i stands outside the summation convention. Its index <u>always</u> agrees with another one in the same term, as in (1,11) and (1,13), but this in itself is <u>not</u> to indicate a summation. Only when the other index itself is a dummy, as in (1,14) and (1,15), summation applies as usual.

So far the φ-field is just an arbitrary charge-free radiation field, governed by Maxwell's equations in empty space, (1,9) and (1,14). We shall see that no further restrictions need be imposed on it. It only remains to determine the γ's accordingly.

In (1,4) the curly brackets are of the order of the γ's, thus by assumption of second order, and so are the other terms. Hence from (1,5)

$$\Gamma^{i}_{\underset{\vee}{kl}} = \left\langle {i \atop k\,l} \right\rangle ,$$

neglecting third order terms. But then from (1,7) and (1,9)

$$\Gamma^{i}_{\underset{\vee}{kl}} = \varepsilon_i \, \varphi_{kl,i} , \tag{1,16}$$

so that (3) is satisfied, in virtue of (1,14). From (1,4) we get

$$\Gamma^{i}_{\underline{kl}} = \left\{ {i \atop k\,l} \right\} - \varepsilon_i \, \varepsilon_s \, (\varphi_{ls} \, \varphi_{ki,s} + \varphi_{ks} \, \varphi_{li,s}) \tag{1,17}$$

with

$$\left\{ {i \atop k\,l} \right\} = \frac{1}{2} \, \varepsilon_i \, (\gamma_{il,k} + \gamma_{ki,l} - \gamma_{kl,i}) . \tag{1,18}$$

With these expressions we have to set up equations (2), where we drop, of course, the cosmical term, so just $R_{kl} = 0$. If here we split the Γ's into their symmetric and skew parts, equations (3) entail a considerable simplification and we get

$$R_{kl}(\Gamma^{i}_{\underline{kl}}) - \Gamma^{i}_{\underset{\vee}{kl};i} - \Gamma^{\alpha}_{\underset{\vee}{k\beta}} \, \Gamma^{\beta}_{\underset{\vee}{l\alpha}} = 0 , \tag{1,19}$$

where the first term means the Einstein tensor formed of the $\Gamma^{i}_{\underline{kl}}$, and the semicolon rigorously refers to them, but may in our approximation be replaced by a comma, i.e. by plain differentiation. Here the third term is symmetric in k and l, the second term, which is skew, vanishes by (1,16) and (1,15), and the skew part of the first term

$$\Gamma^{i}_{\underline{ki},l} - \Gamma^{i}_{\underline{li},k} = 0 ,$$

as a quite general and exact consequence of (1). Indeed if (1) is multiplied by g^{ik} and contracted, $\Gamma^{i}_{\underline{li}}$ turns out to be the derivative with respect to x_l of the square root of the determinant of the g_{ik} (not of the $g_{\underline{ik}}$; that would be $\left\{ {i \atop l\,i} \right\}$). So the skew part of (1,19) is satisfied and we are left with the 10 symmetric equations

$$R_{kl}(\Gamma^{i}_{\underline{kl}}) - \Gamma^{\alpha}_{k\check{\beta}}\Gamma^{\beta}_{l\check{\alpha}} = 0 \qquad (1,20)$$

as the only conditions for the 10 γ_{ik}, given the Maxwellian radiation field φ_{ik}. We write them explicitly, dropping a fourth-order contribution in the first term of (1,20),

$$-\Gamma^{i}_{\underline{kl},i} + \Gamma^{i}_{\underline{ki},l} - \Gamma^{\alpha}_{k\check{\beta}}\Gamma^{\beta}_{l\check{\alpha}} = 0 . \qquad (1,21)$$

In all that follows the relations (1,9), (1,14), (1,15) must always be remembered. We shall not quote them every time we use them. The evaluation of (1,21) according to (1,16) - (1,18) is straightforward, except for the following not quite obvious passage. By contracting (1,17) and (1,18) with respect to i and l one finds

$$\Gamma^{i}_{\underline{ki}} = \frac{1}{2}\varepsilon_i\gamma_{ii,k} - \varepsilon_i\varepsilon_s\varphi_{is}\varphi_{ki,s} . \qquad (1,22)$$

Now, in order that the second term in (1,21) be symmetric in k and l, as it must, the last term in (1,22) must also be a derivative with respect to x_k, which it does not *prima facie* appear to be. But from (1,9)

$$\varepsilon_i \varepsilon_s \varphi_{is} \varphi_{ki,s} = - \varepsilon_i \varepsilon_s \varphi_{is} \varphi_{is,k} - \varepsilon_i \varepsilon_s \varphi_{is} \varphi_{sk,i} .$$

If in the last term you first exchange the <u>notation</u> of the dummies i and s, and then commute the two pairs of "skew subscripts" simultaneously, it proves equal to the term on the left. Hence

$$\varepsilon_i \varepsilon_s \varphi_{is} \varphi_{ki,s} = -\frac{1}{4} \varepsilon_i \varepsilon_s (\varphi_{is} \varphi_{is})_{,k} . \qquad (1,23)$$

We shall have to use this relation frequently.

The result of the whole evaluation of (1,21) is

$$\frac{1}{2} \varepsilon_i (\gamma_{kl,i,i} + \gamma_{ii,k,l} - \gamma_{li,k,i} - \gamma_{ki,l,i}) +$$

$$+ \varepsilon_i \varepsilon_s \varphi_{ks,i} \varphi_{li,s} + \frac{1}{4} \varepsilon_i \varepsilon_s (\varphi_{si} \varphi_{si})_{,k,l} = 0 . \qquad (1,24)$$

The γ-terms are, of course, simply the Einstein tensor of a nearly-Galilean metric, well known from the theory of weak gravitational fields e.g. of gravitational waves of infinitesimal amplitude. The φ-terms occupy the place where in the old theory the matter tensor would be stuck in. It is <u>their</u> structure that interests us most, because it is entirely novel. (The solution for γ is performed by well known routine methods, we shall give it in due course.)

One property is obligatory in the φ-terms, because the γ-terms have it: when readjusted by "subtracting the half-spur on the diagonal" they must have vanishing divergence. This is a welcome check on possible mistakes in sign or numerical coefficients. Let us put for abbreviation

$$\varepsilon_i \varepsilon_s \varphi_{ks,i} \varphi_{li,s} + \frac{1}{4} \varepsilon_i \varepsilon_s (\varphi_{si} \varphi_{si})_{,k,l} = \Phi_{kl} . \qquad (1,25)$$

Then <u>we must find</u>

$$\varepsilon_k \left(\Phi_{kl} - \tfrac{1}{2} \delta_{kl} \varepsilon_l \varepsilon_m \Phi_{mm} \right)_{,k} = 0 ,$$

or

$$\varepsilon_k \Phi_{kl,k} - \tfrac{1}{2} \varepsilon_m \Phi_{mm,l} = 0 . \qquad (1,26)$$

Now we have

$$\begin{aligned} \varepsilon_k \Phi_{kl,k} &= \varepsilon_k \varepsilon_i \varepsilon_s \varphi_{ks,i} \varphi_{li,s,k} + \tfrac{1}{4} \varepsilon_k \varepsilon_i \varepsilon_s (\varphi_{si} \varphi_{si})_{,k,k,l} \\ &= \tfrac{1}{4} \varepsilon_k \varepsilon_i \varepsilon_s (\varphi_{si} \varphi_{si})_{,k,k,l} , \qquad (1,27) \end{aligned}$$

the first term on the right vanishing by symmetry. On the other hand

$$\varepsilon_m \Phi_{mm} = \varepsilon_m \varepsilon_i \varepsilon_s \varphi_{ms,i} \varphi_{mi,s} + \tfrac{1}{4} \varepsilon_m \varepsilon_i \varepsilon_s (\varphi_{si} \varphi_{si})_{,m,m} .$$

In the first term on the right

$$\varepsilon_i \varepsilon_s \varphi_{ms,i} \varphi_{mi,s} = \varepsilon_i \varepsilon_s (\varphi_{ms,i} \varphi_{mi})_{,s} = \tfrac{1}{4} \varepsilon_i \varepsilon_s (\varphi_{mi} \varphi_{mi})_{,s,s} ,$$

where we have used (1,23). So

$$\varepsilon_m \Phi_{mm} = \tfrac{1}{2} \varepsilon_m \varepsilon_i \varepsilon_s (\varphi_{si} \varphi_{si})_{,m,m} . \qquad (1,28)$$

This and (1,27) proves (1,26).

The last relation is in itself of interest. For according to it the spur of

$$\Phi_{kl} - \tfrac{1}{2} \delta_{kl} \varepsilon_l \varepsilon_m \Phi_{mm} ,$$

which here plays the rôle of the stress-energy-momentum tensor, turns out to be

$$\varepsilon_k (\Phi_{kk} - \tfrac{1}{2} \delta_{kk} \varepsilon_k \varepsilon_m \Phi_{mm}) = - \varepsilon_m \Phi_{mm} = - \tfrac{1}{2} \varepsilon_m \varepsilon_i \varepsilon_s (\varphi_{si} \varphi_{si})_{,m,m} . \qquad (1,29)$$

This is the D'Alembertian of the invariant (the $E^2 - H^2$ of elementary theory) and does not vanish in general, as the spur of Maxwell's energy tensor does. This leads to the astonishing conclusion that in the present theory a pure, charge-free Maxwellian field of radiation is capable of producing a gravitational field which according to the old theory could only be produced by matter other than an electromagnetic field. This raises the hope that in this theory we may be able to picture ordinary matter without sticking it in explicitly.

Now let us attend to the solution of (1,24), which using the notation (1,25) we write

$$\frac{1}{2}\,\varepsilon_i(\gamma_{kl,i,i} + \gamma_{ii,k,l} - \gamma_{li,k,i} - \gamma_{ki,l,i}) + \Phi_{kl} = 0 \qquad (1,30)$$

For later use we note its contraction, which from (1,28) is

$$\varepsilon_i\,\varepsilon_k(\gamma_{kk,i,i} - \gamma_{ki,k,i}) + \frac{1}{2}\,\varepsilon_k\,\varepsilon_i\,\varepsilon_s\,(\varphi_{is}\,\varphi_{is})_{,k,k} = 0\,. \qquad (1,31)$$

We contemplate an infinitesimal change of frame

$$x_l = x'_l + a_l(x'_m)\;; \qquad (1,32)$$

the functions a_l are to be of the same order as the γ's. This does not change the φ_{kl}, nor Φ_{kl} perceptibly, but it does change the γ's, which in any frame must mean the deviations of the $g_{\underline{ik}}$ from $\varepsilon_i\,\delta_{ik}$. One easily finds

$$\gamma'_{ik} = \gamma_{ik} + \varepsilon_k\,a_{k,i} + \varepsilon_i\,a_{i,k}\,, \qquad (1,33)$$

and from it the following two relations

$$\varepsilon_i(\gamma'_{ik,i} - \gamma'_{ii,k}) = \varepsilon_i(\gamma_{ik,i} - \gamma_{ii,k}) + \varepsilon_i \varepsilon_k a_{k,i,i} - a_{i,i,k}$$

(1,34a)

and

$$\varepsilon_i(\gamma'_{ik,i} - \tfrac{1}{2}\gamma'_{ii,k}) = \varepsilon_i(\gamma_{ik,i} - \tfrac{1}{2}\gamma_{ii,k}) + \varepsilon_i \varepsilon_k a_{k,i,i} .$$

(1,34b)

We use them for specializing our transformation a_1 in two different ways. The first will be used much later - we insert it here only as a digression to save our repeating the same kind of argument then. We can not choose the a_1 so that the first member of (1,34a) vanishes, because it has a non-vanishing divergence, from (1,31), which holds with the same φ_{is} in all our frames. But we can try to demand

$$\varepsilon_i(\gamma'_{ik,i} - \gamma'_{ii,k}) = \tfrac{1}{2}\varepsilon_i \varepsilon_s (\varphi_{is}\varphi_{is})_{,k} \qquad (1,35)$$

which leads to the condition for a_k

$$\tfrac{1}{2}\varepsilon_i \varepsilon_s(\varphi_{is}\varphi_{is})_{,k} + \varepsilon_i(\gamma_{ii,k} - \gamma_{ik,i}) = \varepsilon_i \varepsilon_k a_{k,i,i} - a_{i,i,k} .$$

(1,36)

This turns into an inhomogeneous D'Alembert equation, if we make the accessory demand

$$a_{i,i} = 0 , \qquad (1,37)$$

analogous to Maxwell's auxiliary condition $(\operatorname{div} A + \dot{A} = 0)$ in the theory of the electromagnetic potential. From this familiar theory it is known that (1,37) will be satisfied, if we choose for a_k the retarded potentials of the first member of (1,36); it will be satisfied because this first member has according to (1,31) vanishing divergence. The conclusion is that - if we have a solution of (1,30) - a frame can always be found, in which (1,35) holds. This we shall use much later. I apologise for the digression, which is hereby ended.

To establish a solution of (1,30) we need a <u>different</u> specialization of a_k, suggested by (1,34b). This is much simpler. We can obviously obtain

$$\varepsilon_i \left(\gamma'_{ik,i} - \frac{1}{2}\gamma'_{ii,k}\right) = 0 \qquad (1,38)$$

by choosing for a_k <u>any</u> solution of

$$0 = \varepsilon_i \left(\gamma_{ik,i} - \frac{1}{2}\gamma_{ii,k}\right) + \varepsilon_i \varepsilon_k a_{k,i,i} \,. \qquad (1,39)$$

In the primed frame (1,30) is greatly simplified, it reads

$$\frac{1}{2}\varepsilon_i \gamma'_{kl,i,i} + \Phi_{kl} = 0 \,, \qquad (1,40)$$

where Φ_{kl} needs no dash, because it has not changed. Any solution of (1,30) can be transformed into a solution of (1,40) that satisfies (1,38). So we lose nothing in generality by adopting the simplified form (1,40) right away and restricting attention to those solutions that satisfy (1,38) - others are of no interest whatever.

We have to shew that such solutions in general exist. Take for γ'_{kl} in (1,40) the retarded potential, and readjust the equation by subtracting its half-spur on the diagonal:

$$\frac{1}{2}\varepsilon_i\left(\gamma'_{kl} - \frac{1}{2}\delta_{kl}\varepsilon_l\varepsilon_m\gamma'_{mm}\right)_{,i,i} + \Phi_{kl} - \frac{1}{2}\delta_{kl}\varepsilon_l\varepsilon_m\Phi_{mm} = 0 \,.$$

We have shewn above, (1,26), that the Φ-terms have vanishing divergence. Though it is now a question of a tensor-divergence, the same reasoning applies that we used just before in connection with (1,37), borrowing from the elementary theory of retarded potentials; we must only, as it were, apply the argument four times over. It follows that the expression in the round bracket, being the retarded potential of something that has vanishing divergence, has itself vanishing divergence. The four equations which state this fact are precisely (1,38). This really finishes our problem: together with an arbitrary Maxwellian field, the retarded potential solutions of (1,40) are an (approximate) solution of the field-equations.

The solution is, of course, not unique. But two different solutions obviously differ only by a system of gravitational waves of small amplitude, familiar from the symmetric theory. They too are most conveniently investigated in a frame such as we have used; they then take the form of 10 arbitrary wave-functions, interrelated by the four relations (1,38).

It would be interesting to know whether, by superposing on to our solution a suitable system of gravitational waves, one could obtain in general one that gives the γ_{kl} as functions of the local φ_{kl} and their derivatives. I have neither been able to find it, nor to prove definitely that it is impossible. But I suspect the latter.

An alternative method of solving (1,40) is to give oneself φ_{kl} as a Fourier integral, that is to resolve the electromagnetic field into plane waves. Then one obtains the Fourier representation of Φ_{kl} from (1,25) and that of the γ's from (1,40). The condition (1,38) must be checked. An essential feature of this method, which I intend to follow up in a later section, is that for a single plane wave $\Phi_{kl} = 0$. The field-producing "matter" is constituted, as far as our present analysis goes, by the contributions of pairs of plane waves.

2. Charges.

If the four-current does not vanish, (1,9) has to be superseded by the definition of the four-current

$$\varphi_{ik,l} + \varphi_{kl,i} + \varphi_{li,k} = s_{ikl} . \qquad (2,1)$$

This is the only primary modification, but it entails many others. While (1,14) stays

$$\varepsilon_i \varphi_{ik,i} = 0 , \qquad (2,2)$$

(1,15) becomes at first

$$\varepsilon_i \varphi_{kl,i,i} = \varepsilon_i s_{ikl,i} \qquad (2,3)$$

(but it will be restored). Instead of (1,16) we get now

$$\Gamma^{i}_{\underset{\vee}{kl}} = \left\langle {i \atop k\ l} \right\rangle = \varepsilon_i \varphi_{kl,i} - \frac{1}{2} \varepsilon_i s_{kli} , \qquad (2,4)$$

and instead of (1,17)

$$\Gamma^{i}_{\underline{kl}} = \left\{ {i \atop k\ l} \right\} - \varepsilon_i \varepsilon_s(\varphi_{ls} \varphi_{ki,s} + \varphi_{ks} \varphi_{li,s}) + \frac{1}{2} \varepsilon_i \varepsilon_s(\varphi_{ls} s_{kis} + \varphi_{ks} s_{lis}), \qquad (2,5)$$

while (1,18), of course, stays.

From (2,4) and (2,2) the condition (3) is satisfied. In (1,19) nothing is changed; as before, the second term is the only skew constituent, however it no longer vanishes automatically, but imposes the condition

$$\varepsilon_i s_{kli,i} = 0 , \qquad (2,6)$$

from (2,4) and (2,3). (This, by the way, restores (2,3) to its original form (1,15).) The convenient rule (1,23) becomes now

$$\varepsilon_i \varepsilon_s \varphi_{is} \varphi_{ki,s} = -\frac{1}{4}\varepsilon_i \varepsilon_s(\varphi_{is} \varphi_{is})_{,k} + \frac{1}{2}\varepsilon_i \varepsilon_s \varphi_{is} s_{kis} . \tag{2,7}$$

This causes no additional terms in $\Gamma^{i}_{\underline{ki}}$, which from (2,5) and (2,7) is the same as from (1,22) and (1,23), viz.

$$\Gamma^{i}_{\underline{ki}} = \frac{1}{2}\varepsilon_i \gamma_{ii,k} + \frac{1}{4}\varepsilon_i \varepsilon_s(\varphi_{is} \varphi_{is})_{,k} . \tag{2,8}$$

The evaluation of (1,21) is now easy. Many terms cancel. The net result, superseding (1,24) is:

$$\frac{1}{2}\varepsilon_i(\gamma_{kl,i,i} + \gamma_{ii,k,l} - \gamma_{il,k,i} - \gamma_{ki,l,i}) + \varepsilon_i \varepsilon_s \varphi_{ks,i} \varphi_{li,s} +$$
$$+ \frac{1}{4}\varepsilon_i \varepsilon_s(\varphi_{si} \varphi_{si})_{,k,l} + \frac{1}{4}\varepsilon_i \varepsilon_s s_{kis} s_{lis} = 0 . \tag{2,9}$$

The only modification is the addition of the last term.

The interesting novel feature is (2,6) which restricts the distribution of the four-current in an unexpected way. The four-current, owing to the unwonted allotment of indices to the field components, presents itself in the present theory primarily as an antisymmetric tensor of third rank s_{ikl}. To grasp the meaning of (2,6), let us just for the moment return to the customary notation with the help of the antisymmetric density ε^{iklm} (which must not be confused with the Galilean metric ε_i). We get from (2,6)

$$\varepsilon_i \varepsilon^{rskl} s_{kli,i} = 0 . \tag{2,10}$$

Here k and l ($\neq$ k) are "neither r nor s", so that there are - disregarding the trivial repetition that results from exchanging k and 1 - only <u>two</u> terms, viz. $i = r$ and $i = s$. If we <u>suspend the summation rule for a moment</u>, the equation reads

$$(\varepsilon_r \varepsilon^{rskl} s_{klr})_{,r} + (\varepsilon_s \varepsilon^{rskl} s_{kls})_{,s} = 0 .$$

This we multiply by $-\varepsilon_r \varepsilon_s$ and permute the superscripts in a certain way:

$$(\varepsilon_s \varepsilon^{sklr} s_{klr})_{,r} - (\varepsilon_r \varepsilon^{rkls} s_{kls})_{,s} = 0 . \qquad (2,11)$$

Here the summation rule is still suspended and $sklr$ are a definite permutation of 1234. But now we may restore the summation rule <u>inside</u> both the brackets, for this only amounts to multiplying our equation by $3! = 6$. Moreover we put quite generally (of course <u>with</u> summation rule)

$$p_m = \frac{1}{6} \varepsilon_m \varepsilon^{mnpq} s_{npq} . \qquad (2,12)$$

Then (2,11) becomes

$$p_{s,r} - p_{r,s} = 0 . \qquad (2,13)$$

Moreover from the definition of s_{ikl} by (2,1) it is easy to shew that

$$\varepsilon_m p_{m,m} = \frac{1}{6} \varepsilon^{mnpq} s_{npq,m} = 0 , \qquad (2,14)$$

which is the equation of continuity of the charge and current. According to (2,13) the four-dimensional curl of p_s vanishes. We know that this means it is a gradient, say

$$p_s = \Phi_{,s} . \qquad (2,15)$$

From (2,14) Φ must satisfy

$$\varepsilon_m \Phi_{,m,m} = 0 . \qquad (2,16)$$

So the restriction imposed on the flow of electricity boils down to this: <u>the four-current is the gradient of an invariant wave-function</u> (i.e. of a solution of D'Alembert's equation).

The solution of (2,9) follows exactly the pattern of (1,24), or (1,30), the latter form comprising (2,9) provided we now supplement the definition of Φ_{kl}, formerly given by (1,25), thus:

$$\Phi_{kl} = \varepsilon_i \varepsilon_s \varphi_{ks,i} \varphi_{li,s} + \frac{1}{4} \varepsilon_i \varepsilon_s (\varphi_{si} \varphi_{si})_{,k,l} + \frac{1}{4} \varepsilon_i \varepsilon_s s_{kis} s_{lis} . \tag{2,17}$$

I emphasized before the remarkable fact that its spur does not vanish even in the charge-free case. I wish to supply the general value of this spur, and also the proof that (1,26) continues to hold, as of course it must. We have now

$$\varepsilon_m \Phi_{mm} = \varepsilon_m \varepsilon_i \varepsilon_s \varphi_{ms,i} \varphi_{mi,s} + \frac{1}{4} \varepsilon_m \varepsilon_i \varepsilon_s (\varphi_{si} \varphi_{si})_{,m,m} + \frac{1}{4} \varepsilon_m \varepsilon_i \varepsilon_s s_{mis} s_{mis} . \tag{2,18}$$

The first term on the right we transform, paying attention to (2,2), (2,7), (2,6) and (2,1):

$$\varepsilon_m \varepsilon_i \varepsilon_s (\varphi_{ms,i} \varphi_{mi})_{,s} = \varepsilon_m \varepsilon_i \varepsilon_s \left[\frac{1}{4}(\varphi_{im} \varphi_{im})_{,s} - \frac{1}{2} \varphi_{im} s_{sim}\right]_{,s} =$$

$$= \varepsilon_m \varepsilon_i \varepsilon_s \left[\frac{1}{4}(\varphi_{im} \varphi_{im})_{,s,s} - \frac{1}{2} \varphi_{im,s} s_{sim}\right]$$

$$= \frac{1}{4} \varepsilon_m \varepsilon_i \varepsilon_s (\varphi_{im} \varphi_{im})_{,s,s} - \frac{1}{6} \varepsilon_m \varepsilon_i \varepsilon_s s_{sim} s_{sim} .$$

This gives

$$\varepsilon_m \Phi_{mm} = \frac{1}{2} \varepsilon_m \varepsilon_i \varepsilon_s (\varphi_{si} \varphi_{si})_{,m,m} + \frac{1}{12} \varepsilon_m \varepsilon_i \varepsilon_s s_{mis} s_{mis} , \tag{2,19}$$

which supplements (1,28).

To prove (1,26) in the present case we form from (2,17)

$$\varepsilon_k \Phi_{kl,k} = \varepsilon_k \varepsilon_i \varepsilon_s (\varphi_{ks,i}\ \varphi_{li,s})_{,k} + \frac{1}{4}\varepsilon_k \varepsilon_i \varepsilon_s (\varphi_{si}\ \varphi_{si})_{,k,k,l} + $$
$$+ \frac{1}{4}\varepsilon_k \varepsilon_i \varepsilon_s\ s_{kis}\ s_{lis,k}\ , \qquad (2,20)$$

and from (2,19)

$$\frac{1}{2}\varepsilon_m \Phi_{mm,l} = \frac{1}{4}\varepsilon_m \varepsilon_i \varepsilon_s (\varphi_{si}\ \varphi_{si})_{,m,m,l} + \frac{1}{12}\varepsilon_m \varepsilon_i \varepsilon_s\ s_{mis}\ s_{mis,l}\ . \qquad (2,21)$$

We have to shew that the second members of the last two equations are equal. The first term on the right of (2,20) vanishes (one part by (2,2), the other by symmetry). The second term is the same as the first in (2,21). <u>It remains to be shewn</u> that the last terms, respectively, are equal, that is

$$\varepsilon_m \varepsilon_i \varepsilon_s\ s_{mis}\ s_{lis,m} = \frac{1}{3}\varepsilon_m \varepsilon_i \varepsilon_s\ s_{mis}\ s_{mis,l}\ . \qquad (2,22)$$

The first member can be transformed by using (2,1) and the equation of continuity that follows from (2,1): we get for this first member

$$\varepsilon_m \varepsilon_i \varepsilon_s (\varphi_{mi,s} + \varphi_{is,m} + \varphi_{sm,i})\ s_{lis,m} =$$
$$= \varepsilon_m \varepsilon_i \varepsilon_s\ \varphi_{mi,s}(s_{lis,m} + s_{lmi,s} + s_{lsm,i}) = \varepsilon_m \varepsilon_i \varepsilon_s\ \varphi_{mi,s}\ s_{mis,l}\ .$$

In the second member of (2,22) one may obviously replace s_{mis} by $3\ \varphi_{mi,s}$, on account of the antisymmetry of the other factor and the prescribed summations. This proves (2,22), and thereby the vanishing of the divergence of our present Φ_{kl} (when readjusted by subtracting its half-spur on the diagonal).

Let us survey the procedure by which a complete meaningful solution could be built up, as far as our present analysis goes. One first gives oneself an arbitrary solution Φ of D'Alembert's equation. This determines the four-current s_{ikl} by (2,15) and (2,12). Then one has to determine from Maxwell's equations (2,2) and (2,1) an electromagnetic field that may reasonably be regarded as "produced" by that s_{ikl}, and finally, from (2,9), a gravitational field $\check{\gamma}_{kl}$ that may reasonably be regarded as "produced" by the Maxwellian field.

This procedure can be accomplished by quadratures, but it leaves at every step wide liberty. One can see exactly to what stage of classical theories it corresponds: to determine the electromagnetic field, given the motion of the charges, and the gravitational field, given the distribution of matter. What is missing is the "back-coupling", the influence of both fields on the motion of the charges and that of the gravitational field on the electromagnetic field. Obviously our quadratic approximation is only a first step. To extend it to the next order will be a very laborious task. But it will have to be grappled with, if one wants to know what this theory really says about the interlacing of the fields.

3. The Energy Tensor.

The Φ_{kl} (readjusted by etc.; I shall sometimes suppress this phrase), being for weak fields the sources of the gravitational field, have a certain claim to be called the matter tensor for weak fields; but they have two competitors to this dignity, viz. arrays that present themselves in the general theory as natural analogues of the pseudotensor of the old theory, usually denoted there by $\mathfrak{t}^{l}_{m}$. These arrays are most conveniently described in terms of the following abbreviations

$$\Lambda_{ik} = \Gamma^{\alpha}_{i\beta}\Gamma^{\beta}_{\alpha k} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{ik} \tag{3,1a}$$

$$\Lambda = \mathfrak{g}^{ik}\Lambda_{ik} \tag{3,1b}$$

$$\Lambda^{\alpha}_{ik} = \Gamma^{\alpha}_{ik} - \delta^{\alpha}_{k}\Gamma^{\beta}_{i\beta} . \tag{3,1c}$$

We make a note that

$$R_{ik} = -\Lambda^{\alpha}_{ik,\alpha} + \Lambda_{ik} . \tag{3,2}$$

In terms of these Λ's the two arrays read, if the cosmical term is dropped and the field equations (1), (2), (3) are fulfilled,

$$\mathfrak{t}^{\alpha}_{l} = \tfrac{1}{2}(\Lambda^{\alpha}_{ik}\,\mathfrak{g}^{ik}{}_{,l} + \delta^{\alpha}_{l}\Lambda) , \tag{3,3}$$

and

$$\underset{A}{\mathfrak{t}}{}^{\alpha}_{l} = -\tfrac{1}{2}\mathfrak{g}^{ik}\Lambda^{\alpha}_{ik,l} . \tag{3,4}$$

The first reduces in the symmetric case to the familiar pseudotensor of Einstein, of which it is a generalization that I derived two years ago.[4] The second is a different generalization, a little suspect because it differs from the first even in the symmetric case, but somewhat suggested by the purely affine aspect[5] (that is why we have distinguished it here by an "A" under the $\mathfrak{t}$). In a later section I shall give a compre-

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4) Proc. Roy. Irish Acad., 52 (A), p.1, 1948; eqn. (4,6).

5) ibid. under press.

hensive survey of the derivations of these two pseudotensors, both of which have, of course, vanishing divergence. If this is known for the first, it is easily proved for the second. Indeed their difference

$$t^{\alpha}_{1} - \underset{\Lambda}{t}{}^{\alpha}_{1} = \frac{1}{2}(\mathfrak{g}^{ik}\Lambda^{\alpha}_{ik})_{,1} + \frac{1}{2}\delta^{\alpha}_{1}\Lambda$$

has the divergence

$$\frac{1}{2}(\mathfrak{g}^{ik}\Lambda^{\alpha}_{ik})_{\alpha,1} + \frac{1}{2}\Lambda_{,1},$$

which is zero; for

$$(\mathfrak{g}^{ik}\Lambda^{\alpha}_{ik})_{,\alpha} = \mathfrak{g}^{ik}\Lambda^{\alpha}_{ik,\alpha} + \mathfrak{g}^{ik}_{,\alpha}\Lambda^{\alpha}_{ik} = \Lambda - 2\Lambda = -\Lambda,$$

since

$$\mathfrak{g}^{ik}\Lambda^{\alpha}_{ik,\alpha} = \Lambda$$

and

$$\mathfrak{g}^{ik}_{,\alpha}\Lambda^{\alpha}_{ik} = -2\Lambda. \qquad (\Lambda)$$

The first follows from the field equation $R_{ik} = 0$, by (3,2) and (3,1b), the second from the facts that Λ is homogeneous of 2nd degree (indeed a quadratic form) in the $\mathfrak{g}^{ik}_{,\alpha}$ and that

$$\delta\Lambda = -\Lambda_{ik}\,\delta\mathfrak{g}^{ik} - \Lambda^{\alpha}_{ik}\,\delta(\mathfrak{g}^{ik}_{,\alpha}),$$

if Λ is regarded as a function of the 80 arguments $\mathfrak{g}^{ik}$ and $\mathfrak{g}^{ik}_{,\alpha}$, to be varied independently (except for the four linear relations (4) between the latter, which must be preserved in the variation; to derive the last relation, the Γ's must be expressed by the field equation (1)). This completes the proof, which I have inserted here merely as a digression.

To compute our two pseudotensors for weak fields, including charges, we first evaluate the Λ's, (3,1), up to and including the second order. The last term in (3,1a) is, in virtue of (3), of third order, so we have

$$\Lambda_{ik} = \underset{\vee}{\Gamma^{\alpha}_{i\beta}} \underset{\vee}{\Gamma^{\beta}_{\alpha k}}$$

$$= -\varepsilon_{\alpha}\varepsilon_{\beta}(\varphi_{i\beta,\alpha} - \tfrac{1}{2} s_{i\beta\alpha})(\varphi_{k\alpha,\beta} - \tfrac{1}{2} s_{k\alpha\beta}) , \qquad (3,5)$$

and

$$\Lambda = -\varepsilon_i \varepsilon_{\alpha} \varepsilon_{\beta} (\varphi_{i\beta,\alpha} - \tfrac{1}{2} s_{i\beta\alpha})(\varphi_{i\alpha,\beta} - \tfrac{1}{2} s_{i\alpha\beta})$$

$$= -\varepsilon_i \varepsilon_{\alpha} \varepsilon_{\beta} \left[\tfrac{1}{4} (\varphi_{i\alpha}\varphi_{i\alpha})_{,\beta,\beta} - \tfrac{1}{12} s_{i\alpha\beta} s_{i\alpha\beta}\right] , \qquad (3,6)$$

by (2,7) and (2,1). The evaluation of (3,1c) gives by (2,4), (2,5) and (2,8):

$$\Lambda^{\alpha}_{ik} = \varepsilon_{\alpha}(\varphi_{ik,\alpha} - \tfrac{1}{2} s_{ik\alpha}) + \left\{ {\alpha \atop i\ k} \right\} - \varepsilon_{\alpha}\varepsilon_s(\varphi_{ks}\varphi_{i\alpha,s} + \varphi_{is}\varphi_{k\alpha,s}) +$$

$$+ \tfrac{1}{2}\varepsilon_{\alpha}\varepsilon_s(\varphi_{ks} s_{i\alpha s} + \varphi_{is} s_{k\alpha s}) - \delta^{\alpha}_{k}\left[\tfrac{1}{2}\varepsilon_{\beta}\gamma_{\beta\beta,i} + \tfrac{1}{4}\varepsilon_{\beta}\varepsilon_s(\varphi_{\beta s}\varphi_{\beta s})_{,i}\right] . \qquad (3,7)$$

Moreover, including quantities of the <u>first</u> order only, we have from (1,11) and (1,13)

$$\gamma^{ik} = \varepsilon_i \delta_{ik} + \varepsilon_i \varepsilon_k \varphi_{ik} . \qquad (3,8)$$

We turn to (3,3). In the first term on the right both factors are small and need therefore be considered in the first order only. It is convenient to pull the contravariant index down. From the last three equations we get

$$\underset{}{t}_{\alpha 1} = \frac{1}{2}(\varphi_{ik,\alpha} - \frac{1}{2} s_{ik\alpha}) \varepsilon_i \varepsilon_k \varphi_{ik,1} -$$

$$- \frac{1}{8} \varepsilon_\alpha \delta_{\alpha 1} \varepsilon_i \varepsilon_s \varepsilon_\beta \left[(\varphi_{is} \varphi_{is})_{,\beta\beta} - \frac{1}{3} s_{is\beta} s_{is\beta} \right] . \qquad (3,9)$$

No obvious simplification is possible, but we may notice that the second term on the right represents the half-spur of the first "subtracted on the diagonal". This must be so according to the relation marked (Λ) above.

The evaluation of (3,4) is a little more laborious, because the first factor is finite, so that the whole expression (3,7) must be taken into account. We obtain first, using also (1,18),

$$\underset{A}{t}_{\alpha 1} = - \frac{1}{2} \varepsilon_i \varepsilon_k \varphi_{ik}(\varphi_{ik,\alpha} - \frac{1}{2} s_{ik\alpha})_{,1} - \frac{1}{2} \varepsilon_i (\gamma_{\alpha i,i} - \gamma_{ii,\alpha})_{,1} +$$

$$+ \varepsilon_i \varepsilon_s (\varphi_{is} \varphi_{i\alpha,s})_{,1} - \frac{1}{2} \varepsilon_i \varepsilon_s (\varphi_{is} s_{i\alpha s})_{,1} + \frac{1}{8} \varepsilon_\beta \varepsilon_s (\varphi_{\beta s} \varphi_{\beta s})_{,\alpha,1} . \qquad (3,10)$$

We write the first term as follows

$$- \frac{1}{2} \varepsilon_i \varepsilon_k (\varphi_{ik} \varphi_{ik,\alpha} - \frac{1}{2} \varphi_{ik} s_{ik\alpha})_{,1} + \frac{1}{2} \varepsilon_i \varepsilon_k (\varphi_{ik,1} \varphi_{ik,\alpha} - \frac{1}{2} \varphi_{ik,1} s_{ik\alpha}) .$$

Moreover, according to (2,7)

$$\varepsilon_i \varepsilon_s \varphi_{is} \varphi_{i\alpha,s} - \frac{1}{2} \varepsilon_i \varepsilon_s \varphi_{is} s_{i\alpha s} = \frac{1}{4} \varepsilon_i \varepsilon_s (\varphi_{is} \varphi_{is})_{,\alpha} .$$

We shall explain presently that we can introduce a frame in which

$$\varepsilon_i (\gamma_{ii,\alpha} - \gamma_{\alpha i,i}) + \frac{1}{2} \varepsilon_i \varepsilon_s (\varphi_{is} \varphi_{is})_{,\alpha} + \frac{1}{4} \varepsilon_i \varepsilon_k \varphi_{ik} s_{ik\alpha} = 0 . \qquad (3,11)$$

In this frame we get

$$\underset{A}{\mathfrak{J}}_{\alpha 1} = \frac{1}{2}\varepsilon_i \varepsilon_k \Big[\varphi_{ik,1}\, \varphi_{ik,\alpha} - \frac{1}{4}(\varphi_{ik}\, \varphi_{ik})_{,\alpha,1} - \frac{1}{2}\varphi_{ik,1}\, s_{ik\alpha} + + \frac{1}{4}(\varphi_{ik}\, s_{ik\alpha})_{,1} \Big] . \qquad (3,12)$$

The first and the third terms coincide with the first part of (3,9), but the other terms in (3,12) are not diagonal as there. The structure of (2,17) is entirely different.

The demand (3,11) is the generalization, for non-vanishing current, of the demand (1,35). The proof that it can be fulfilled runs exactly analogously to the one we anticipated there in the simpler case, in order to refer to it now. The proof is based on the fact that the contracted field equations (2,9), after an easy reduction, enounce precisely the vanishing of the divergence of the first member of (3,11). This makes it possible to introduce a frame in which these 4 quantities themselves vanish. This frame is in general not the same as would simplify the γ-term in (2,9) (viz. reduce it to its first term) for the purpose of integration.

It is noteworthy that for a single plane wave all three analogues of the matter tensor, viz. (1,25) "readjusted", (3,9) and (3,12), vanish.

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