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BY

STEPHEN O'BRIEN

AND

JOHN L. SYNGE

INSTITIÚID ÁRD-LÉINN BHAILE ÁTHA CLIATH
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Summary: On the surface of a body such as the earth, some of the components of the stress-energy tensor T^m_n change abruptly, and there are discontinuities in some of the derivatives of the metric tensor g_{mn}. The purpose of this paper is to make a general investigation of the jump conditions for T^m_n and the derivatives of g_{mn} across any 3-space of discontinuity in the Riemannian 4-space of general relativity. The procedure is to take a metric tensor depending on a parameter ϵ and associate a boundary layer of thickness 2ϵ with a selected 3-space Σ_3; then, by letting ϵ tend to zero, we produce a discontinuity on Σ_3 and derive the jump conditions on Σ_3 by a limiting process, based on certain hypotheses of boundedness.

1. Introduction

A problem in mathematical physics is not completely stated by writing down partial differential equations. Boundary conditions must be specified, and also jump conditions across surfaces on which some of the unknown quantities or their derivatives may be discontinuous.

To illustrate what this means, consider the problem of determining the Newtonian gravitational potential V of a distribution of matter for which the density is a constant ρ_1 in a volume R_1, a different constant ρ_2 in a volume

R_2 enclosing R_1, and zero in infinite space R_3 outside R_2. It is true that there is an integral formula giving V directly, but if we look at the problem from the standpoint of field theory, we start by writing down the partial differential equations

$$\begin{aligned} \Delta V &= 4\pi \rho_1 \quad \text{in } R_1, \\ \Delta V &= 4\pi \rho_2 \quad \text{in } R_2, \\ \Delta V &= 0 \quad \text{in } R_3. \end{aligned} \tag{1.1}$$

These equations do not determine V. We must add to them the boundary condition that V tends to zero at infinity, and the jump conditions

$$V \text{ and } \partial V/\partial n \text{ continuous across } \Sigma \text{ and } \Sigma', \tag{1.2}$$

where Σ is the surface separating R_1 and R_2 and Σ' the surface separating R_2 and R_3.

The conditions (1.2) may be viewed in three different ways. First, in a field theory of gravitation which admits the existence of discontinuities from the beginning, they must be written down alongside the field equations (1.1) as distinct hypotheses, ranking equal with the field equations in importance. Secondly, they may be derived from the field equations with the aid of some subsidiary hypotheses by a limiting process in which the surfaces Σ and Σ' are replaced by thin boundary layers in which the density changes rapidly but continuously, Thirdly, both (1,1) and (1.2) may be derived from a hypothesis as to the continuity of V and an integral law of the form

$$\int \frac{\partial V}{\partial n}\, dS = 4\pi \int \rho \, d\tau, \tag{1.3}$$

connecting the integral of the normal derivative of V over a surface S with the integral of density throughout its interior, this law holding for all surfaces S, even those containing surfaces of discontinuity of the density.

A similar situation arises in the general theory of relativity. We write down the familiar field equations

$$G_{mn} = -\kappa T_{mn}, \qquad (1.4)$$

where G_{mn} are the components of the Einstein tensor (functions of g_{mn} and their first and second derivatives), κ is a gravitational constant, and T_{mn} the components of the stress-energy tensor. We may regard (1.4) as partial differential equations for the determination of the g_{mn}, the tensor T_{mn} being assigned, subject to the conservation conditions

$$T^{m}_{n|m} = 0, \qquad (1.5)$$

implied by (1.4). (The vertical stroke denotes covariant differentiation.)

If the given distribution of T_{mn} is continuous, the determination of g_{mn} from (1.4) would require at least the statement of boundary conditions at infinity. On account of the non-linearity of (1.4), the whole question of the data necessary to determine a solution is a very difficult one, and we have no intention of attempting to discuss it here. Our question is a much simpler one. We think of a 3-space Σ_3 in space-time across which some of the components of T_{mn} are discontinuous (e.g. the history of the surface of the earth). It is quite certain that the field equations (1.4) cannot determine g_{mn} in the absence of a statement as to what happens when we cross Σ_3, and we ask the question: What are the jump conditions on g_{mn} and their partial derivatives, and also the jump conditions on T_{mn}, if some of the components of T_{mn} change abruptly on crossing Σ_3?

As in the Newtonian problem discussed above, there are three ways of treating this matter. First, we might extract from our physical intuition jump conditions which seem natural and write them down as a hypothesis beside the field equations (1.4). Secondly, we might introduce a boundary layer across which all quantities change continuously, and proceed to a limit in which the thickness of the layer tends to zero. Thirdly, we might replace the field equations (1.4) by some comprehensive integral formula analogous to (1.3), and obtain from it (combined with the hypothesis that g_{mn} are continuous) both (1.4) and the remaining jump conditions.

We lack the physical intuition required in the first way, and we have no idea what integral law to take in order to proceed in the third way; accordingly we shall follow the second way, that of the boundary layer.

2. Notation and coordinate system

We shall use Latin suffixes for the range 1, 2, 3, 4 and Greek for the range 1, 2, 3, with summation for repeated suffixes in both cases.

Let y^r be a coordinate system in a domain R_4 of space-time. A set of four equations

$$y^r = f^r (x^1, x^2, x^3, x^4) \tag{2.1}$$

may be regarded as defining a singly infinite set of 3-spaces in R_4, the parametric equations of any one of these 3-spaces being given by putting x^4 = constant, with x^ρ as parameters. We shall denote by Σ_3 the 3-space with equation $x^4 = 0$, or

$$y^r = f^r (x^1, x^2, x^3, 0). \tag{2.2}$$

This is the 3-space which we shall take as the 3-space of discontinuity. It divides (we shall suppose) R_4 into two parts, R^+_4 with $x^4 > 0$ and R^-_4 with

$x^4 < 0$ (Fig. 1).

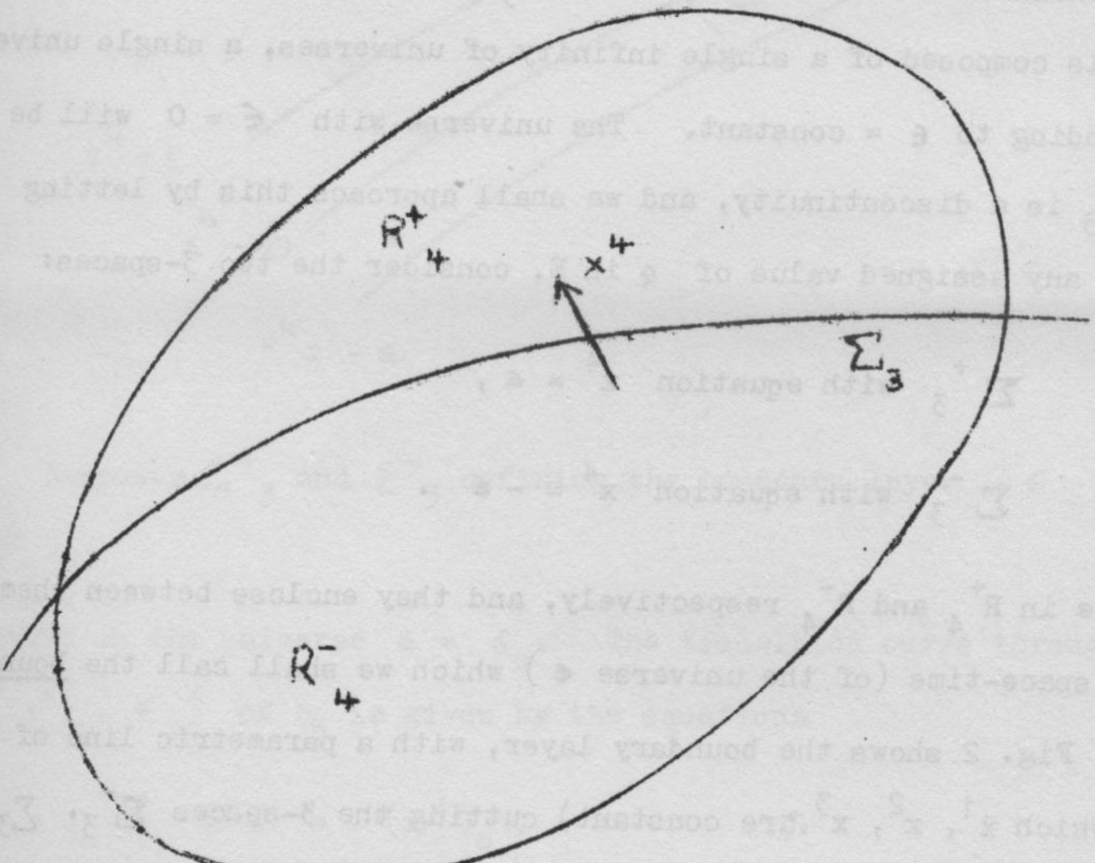

Fig. 1: R_4 divided into R^+_4 and R^-_4 by Σ_3 ($x^4 = 0$)

We shall henceforth use x^r as coordinates in R_4, the coordinates y^r serving merely to introduce them. Note that there is no implication that x^4 is a time-like coordinate; indeed no mention has been made of the metric tensor so far in relation to the coordinate system.

3. The boundary layer

We now introduce a parameter ϵ which takes values in the range $\epsilon_0 \geqslant \epsilon \geqslant 0$, where ϵ_0 is a positive constant, fixed once for all. We shall denote by E the range $\epsilon_0 \geqslant \epsilon > 0$, the value $\epsilon = 0$ being thus excluded from the range E.

We think of a 5-dimensional space S_5 in which x^r and ϵ are coordinates. Thus S_5 is composed of a single infinity of universes, a single universe corresponding to ϵ = constant. The universe with $\epsilon = 0$ will be that in which Σ_3 is a discontinuity, and we shall approach this by letting $\epsilon \longrightarrow 0$.

For any assigned value of ϵ in E, consider the two 3-spaces:

$$\begin{aligned} &\Sigma^+_3 \quad \text{with equation} \quad x^4 = \epsilon, \\ &\Sigma^-_3 \quad \text{with equation} \quad x^4 = -\epsilon . \end{aligned} \tag{3.1}$$

These lie in R^+_4 and R^-_4 respectively, and they enclose between them a portion of space-time (of the universe ϵ) which we shall call the boundary layer. Fig. 2 shows the boundary layer, with a parametric line of x^4 (along which x^1, x^2, x^3 are constant) cutting the 3-spaces Σ^-_3, Σ_3, Σ^+_3 at P^-, P, P^+ respectively. As $\epsilon \longrightarrow 0$ the boundary layer collapses on Σ_3, but of course this limiting process demands consideration, not of a single universe (ϵ fixed), but of an infinity of universes ($\epsilon \longrightarrow 0$).

We need a representation of the boundary layer in S_5. Along P^-PP^+, as in Fig. 2, x^ρ are fixed, and there are only two variable quantities, x^4 and ϵ. Thus we may draw a diagram as in Fig. 3, with x^4 as abscissa and ϵ as ordinate. The boundary layer appears as the shaded triangle, bounded by the three lines, $x^4 = \epsilon$, $x^4 = -\epsilon$, $\epsilon = \epsilon_0$.

The process by which we make the boundary layer collapse on Σ_3 is a rather delicate one. We need transition curves in S_5, leading one universe into another as ϵ changes, and finally leading to the universe $\epsilon = 0$ in which there is a discontinuity. We define the curves as follows. Let $(x^r)_0$

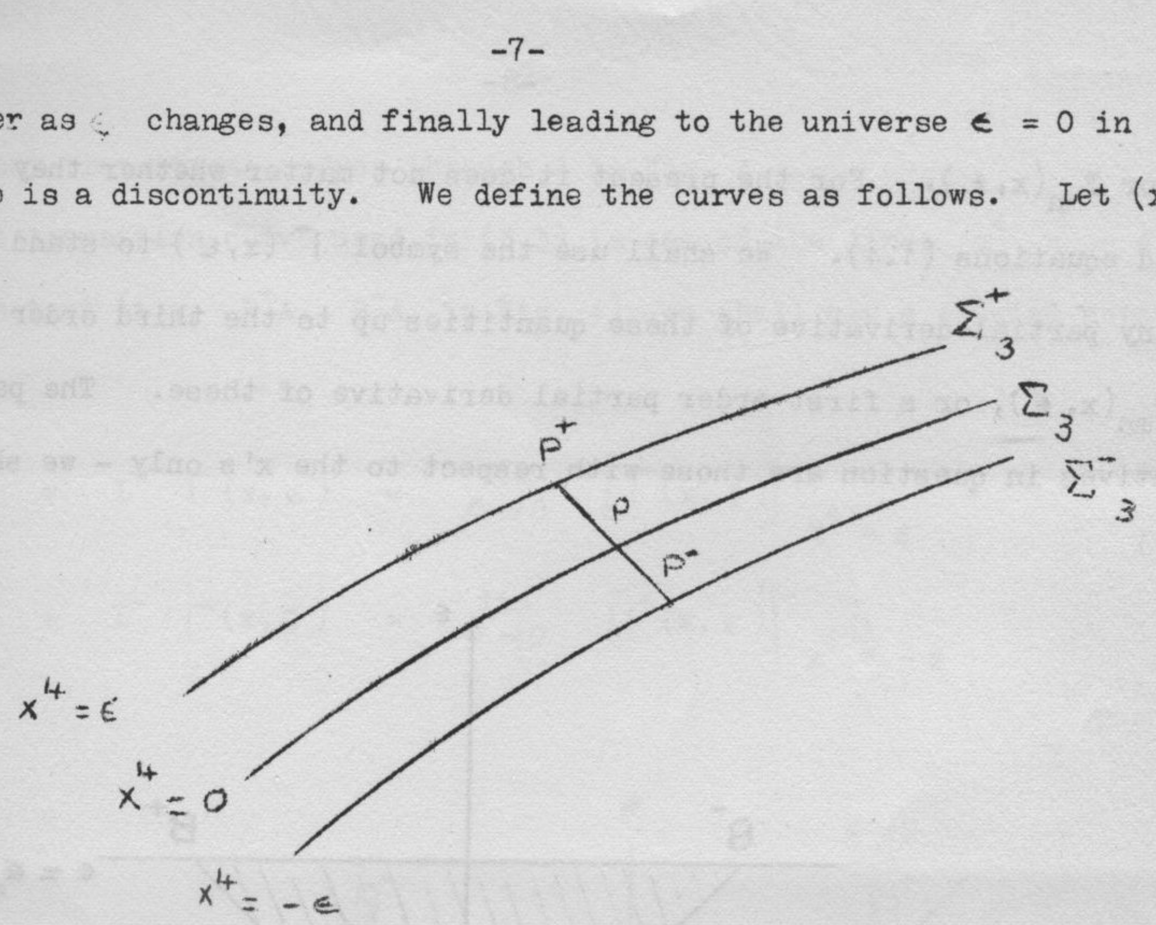

Fig. 2: 3-spaces Σ^+_3 and Σ^-_3 defining the boundary layer $-\epsilon \leq x^r \leq \epsilon$

be any event in the universe $\epsilon = \epsilon_0$. The transition curve through the point $[(x^r)_0, \epsilon_0]$ of S_5 is given by the equations

$$\begin{aligned} x^\rho &= (x^\rho)_0 , \\ x^4 - (x^4)_0 &= \epsilon - \epsilon_0 \quad \text{if } (x^4)_0 \geq \epsilon_0 , \\ x^4 - (x^4)_0 &= -(\epsilon - \epsilon_0) \quad \text{if } (x^4)_0 \leq -\epsilon_0 , \\ x^4 / (x^4)_0 &= \epsilon / \epsilon_0 \quad \text{if } |(x^4)_0| \leq \epsilon_0 . \end{aligned} \tag{3.2}$$

Thus the transition curves form a congruence if S_5, one curve passing through each point. They appear as straight lines in Fig. 4, where AB^+ and AB^- are the edges of the boundary layer, as in Fig. 3.

We now introduce in S_5 a metric tensor $g_{mn}(x, \epsilon)$ and a stress-energy tensor $T_{mn}(x, \epsilon)$. For the present it does not matter whether they satisfy the field equations (1.4). We shall use the symbol $\Gamma(x, \epsilon)$ to stand for $g_{mn}(x, \epsilon)$, or any partial derivative of these quantities up to the third order inclusive, or $T_{mn}(x, \epsilon)$, or a first order partial derivative of these. The partial derivatives in question are those with respect to the x's only - we shall not use

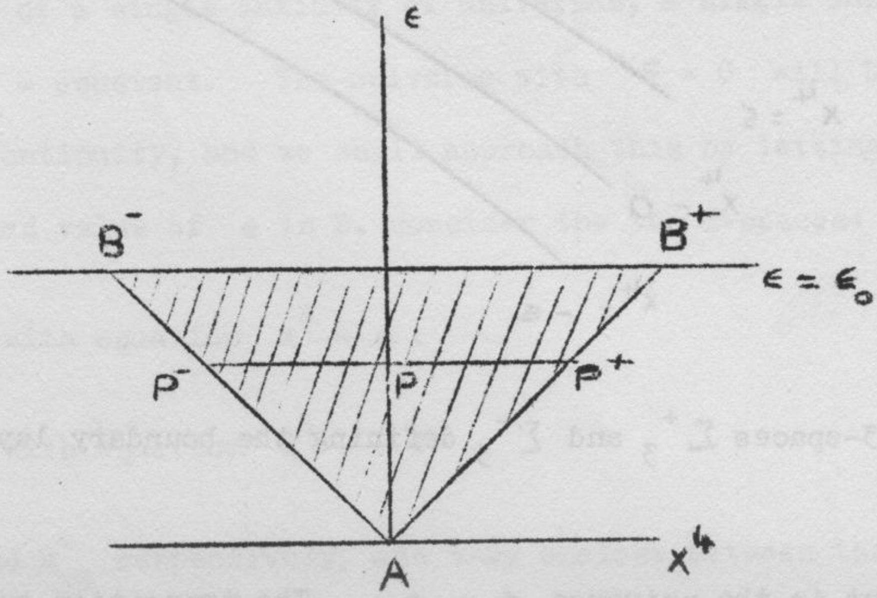

Fig. 3: Representation of the boundary layer in S_5

partial derivatives with respect to ϵ.

We define a limiting process L as follows:

$$L\,\Gamma(x, \epsilon) = \text{limit of } \Gamma(x, \epsilon) \text{ as we proceed along a transition curve to } \epsilon = 0. \quad (3.3)$$

Thus $L\,\Gamma(x, \epsilon)$ is determined when we are given starting values of x^r and ϵ; it is equally a function of the values of x^r at the point where the transition curve meets $\epsilon = 0$, unless at this final point we have $x^4 = 0$; in this latter case we expect $L\,\Gamma(x, \epsilon)$ to be undetermined, since there are infinitely many transitions curves through A in Fig. 4.

If the transition curve used in (3.3) is one of the lines $x^4 = \pm \epsilon$, x^ρ = constant (i.e. B^+A, B^-A in Fig. 4), we shall use a special notation:

$$\Gamma^+ = L^+ \Gamma(x, \epsilon) = \lim_{\epsilon \to 0} \left[\Gamma(x, \epsilon)\right]_{x^4 = \epsilon} \tag{3.4}$$

$$\Gamma^- = L^- \Gamma(x, \epsilon) = \lim_{\epsilon \to 0} \left[\Gamma(x, \epsilon)\right]_{x^4 = -\epsilon} .$$

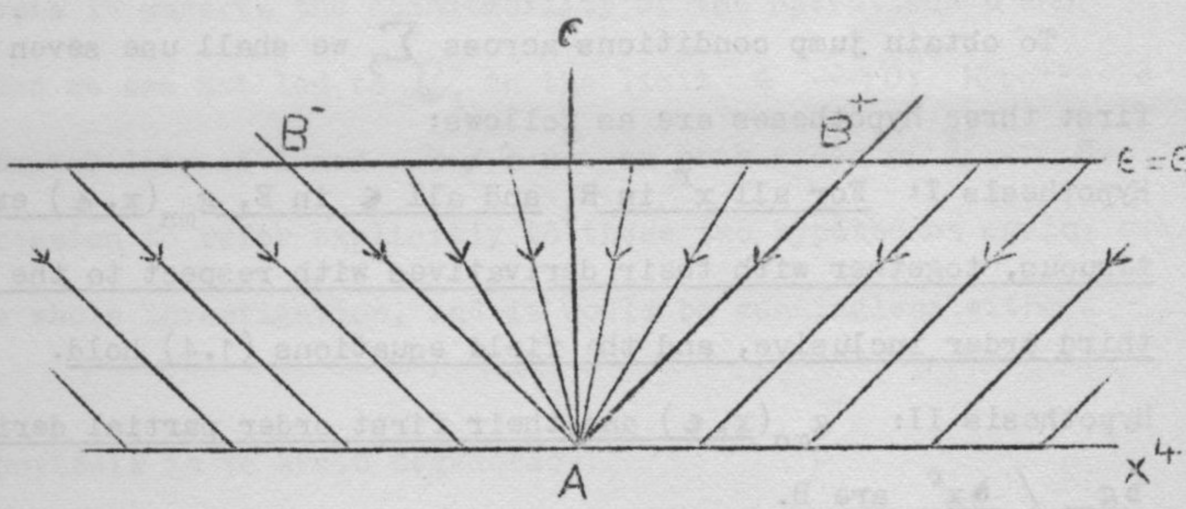

Fig. 4: Transition curves in S_5

Thus Γ^+ and Γ^- are functions of x^1, x^2, x^3 on Σ_3.

As a convenient shorthand, we shall use the letter B to indicate that a quantity $\Gamma(x, \epsilon)$ is bounded above and below for all x^r in R_4 and all ϵ in E. We shall also use the letter B to stand for any such quantity; no confusion is likely to arise from this loose notation. The property B is of importance in relation to the limiting process L.

The following is true:

Theorem I: <u>If</u> $\partial \Gamma(x, \epsilon) / \partial x^4$ <u>is B, then, for common values of</u> x^ρ ,

$$\Gamma^{+} = \Gamma^{-}. \tag{3.5}$$

To see this, we note that

$$\left[\Gamma(x,\epsilon)\right]_{x^4=\epsilon} - \left[\Gamma(x,\epsilon)\right]_{x^4=-\epsilon} = \int_{-\epsilon}^{\epsilon} \frac{\partial}{\partial x^4}\Gamma(x,\epsilon)\,dx^4, \tag{3.6}$$

and from this (3.5) follows immediately on letting ϵ tend to zero.

4. Hypotheses

To obtain jump conditions across Σ_3 we shall use seven hypotheses. The first three hypotheses are as follows:

Hypothesis I: For all x^r in R_4 and all ϵ in E, $g_{mn}(x,\epsilon)$ exist and are continuous, together with their derivatives with respect to the x's up to the third order inclusive, and the field equations (1.4) hold.

Hypothesis II: $g_{mn}(x,\epsilon)$ and their first order partial derivatives $\partial g_{mn}/\partial x^r$ are B.

Hypothesis III: The components T_{mn} of the stress-energy tensor are B.

Note that in Hypothesis I the value $\epsilon = 0$ is not included; naturally, because once the discontinuity has been established, some derivatives will no longer exist at points on it, and at these points the field equations become meaningless. Hypothesis II is suggested by the bounded character of the gradient of the Newtonian potential, and Hypothesis III by the boundedness of density.

The next two hypotheses are

Hypothesis IV: For any quantity $\Gamma(x,\epsilon)$, as defined in Section 3, the limit $L\,\Gamma(x,\epsilon)$ of (3.3) exists and

$$L \frac{\partial}{\partial x^r} \Gamma(x, \epsilon) = \frac{\partial}{\partial x^r} L \Gamma(x, \epsilon) \tag{4.1}$$

provided that the transition curve along which L is calculated does not lead to $x^4 = 0$ for $\epsilon = 0$.

Hypothesis V: Γ^+ and Γ^-, as defined in (3.4), exist as functions of x^1, x^2, x^3, and

$$\left(\frac{\partial \Gamma}{\partial x^\rho}\right)^+ = \frac{\partial \Gamma^+}{\partial x^\rho}, \quad \left(\frac{\partial \Gamma}{\partial x^\rho}\right)^- = \frac{\partial \Gamma^-}{\partial x^\rho} \tag{4.2}$$

Thus Hypothesis IV asserts the commutability of the operations L and $\partial / \partial x^r$, provided we are not led to Σ_3 in the limit $\epsilon \longrightarrow 0$; Hypothesis V asserts the commutability of L and $\partial / \partial x^\rho$ on both sides of Σ_3. We shall not have occasion to refer explicitly to these two hypotheses again, but they underlie the whole investigation, and it would be meaningless without them.

The next hypothesis is to avoid degeneracy:

Hypothesis VI: Det $g_{mn}(x, \epsilon)$ is bounded away from zero for all x^r in R_4 and all ϵ in E.

The last hypothesis is to exclude rapid oscillations in Σ_3:

Hypothesis VII: If any quantity $\Gamma(x, \epsilon)$ is B, then $\partial \Gamma(x, \epsilon) / \partial x^\rho$ is also B.

Note that this does not apply to differentiation with respect to x^4.

5. Jump conditions for g_{mn} and their first derivatives

By Hypothesis II, $\partial g_{mn}(x, \epsilon) / \partial x^4$ is B, and so by Theorem I we have

Theorem II: The metric tensor is continuous across Σ_3, or

$$g^+_{mn} = g^-_{mn} . \tag{5.1}$$

Differentiation with respect to the parameters in Σ_3 gives

Theorem III: <u>All partial derivatives of g_{mn} with respect to x^1, x^2, x^3 are continuous across Σ_3; in particular</u>

$$\frac{\partial g^+_{mn}}{\partial x^\rho} = \frac{\partial g^-_{mn}}{\partial x^\rho}, \quad \frac{\partial^2 g^+_{mn}}{\partial x^\rho \, \partial x^\sigma} = \frac{\partial^2 g^-_{mn}}{\partial x^\rho \, \partial x^\sigma} . \tag{5.2}$$

What we have so far obtained in the way of jump conditions in Theorems II and III is not very serious; they do not really go beyond the simple assumption of the continuity of g_{mn}, and that we might be prepared to make ad hoc. What we are now coming to is less obvious.

We turn to Hypothesis I and integrate the field equations (1.4) along the path P^-PP^+ of Fig. 2. This gives

$$\int_{-\epsilon}^{\epsilon} (G_{mn} + \kappa T_{mn}) \, dx^4 = 0. \tag{5.3}$$

Now T_{mn} is B, by Hypothesis III, and so this gives

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} G_{mn} \, dx^4 = - \kappa \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} T_{mn} \, dx^4 = 0. \tag{5.4}$$

From the field equations we also have

$$\int_{-\epsilon}^{\epsilon} (g^{mn} G_{mn} + \kappa g^{mn} T_{mn}) \, dx^4 = 0, \tag{5.5}$$

and hence

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} g^{mn} G_{mn} \, dx^4 = 0. \tag{5.6}$$

The Einstein tensor G_{mn} is connected with the Ricci tensor R_{mn} and the invariant R by the formulae (cf. Syngc and Schild, 1949, p. 89)

$$G_{mn} = R_{mn} - \frac{1}{2} g_{mn} R, \quad R = g^{mn} R_{mn}, \quad g^{mn} G_{mn} = - R, \tag{5.7}$$

and so we get from (5.4) and (5.6) the equations

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} (R_{mn} - \frac{1}{2} g_{mn} R) \, dx^4 = 0, \tag{5.8}$$

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} R \, dx^4 = 0. \tag{5.9}$$

Now any parts of these integrands which are B will disappear on taking the limits. To explore this, we write out the formula for the Ricci tensor in terms of the Riemann tensor:

$$R_{mn} = g^{pq} R_{pmnq},$$

$$R_{pmnq} = \frac{1}{2} \left(\frac{\partial^2 g_{pq}}{\partial x^m \, \partial x^n} + \frac{\partial^2 g_{mn}}{\partial x^p \, \partial x^q} - \frac{\partial^2 g_{pn}}{\partial x^m \, \partial x^q} - \frac{\partial^2 g_{mq}}{\partial x^p \, \partial x^n} \right) +$$

$$+ g^{rs} \left([pq,r] \, [mn,s] - [pn,r] \, [mq,s] \right). \tag{5.10}$$

Now we know from Hypotheses II, VI and VII that the following are B:

$$g_{mn}, \quad g^{mn}, \quad \frac{\partial g_{mn}}{\partial x^r}, \quad [mn,r], \quad \frac{\partial^2 g_{mn}}{\partial x^r \, \partial x^\sigma}, \tag{5.11}$$

and so the only parts of the integrands in (5.8) and (5.9) not necessarily B are those which involve <u>two</u> differentiations with respect to x^4. All components of R_{pmnq} are B except those with two 4's:

$$R_{4\mu\nu4} = R_{\mu44\nu} = -R_{4\mu4\nu} = -R_{\mu4\nu4} = \frac{1}{2}\frac{\partial^2 g_{\mu\nu}}{(\partial x^4)^2} + B. \tag{5.12}$$

Hence

$$R_{\mu\nu} = g^{rs} R_{r\mu\nu s} = g^{44} R_{4\mu\nu4} + B = \frac{1}{2} g^{44} \frac{\partial^2 g_{\mu\nu}}{(\partial x^4)^2} + B,$$

$$R_{\mu4} = g^{rs} R_{r\mu4s} = g^{4\sigma} R_{4\mu4\sigma} + B = -\frac{1}{2} g^{4\nu} \frac{\partial^2 g_{\mu\nu}}{(\partial x^4)^2} + B,$$

$$R_{44} = g^{rs} R_{r44s} = g^{\rho\sigma} R_{\rho44\sigma} + B = \frac{1}{2} g^{\mu\nu} \frac{\partial^2 g_{\mu\nu}}{(\partial x^4)^2} + B,$$

$$R = (g^{\mu\nu} g^{44} - g^{4\mu} g^{4\nu}) \frac{\partial^2 g_{\mu\nu}}{(\partial x^4)^2} + B. \tag{5.13}$$

Thus we get from (5.9)

$$\lim_{\epsilon\to0} \int_{-\epsilon}^{\epsilon} (g^{\mu\nu} g^{44} - g^{4\mu} g^{4\nu}) \frac{\partial^2 g_{\mu\nu}}{(\partial x^4)^2} dx^4 = 0, \tag{5.14}$$

or

$$\lim_{\epsilon\to0} \left[(g^{\mu\nu} g^{44} - g^{4\mu} g^{4\nu}) \frac{\partial g_{\mu\nu}}{\partial x^4} \right]_{-\epsilon}^{\epsilon} - \lim_{\epsilon\to0} \int_{-\epsilon}^{\epsilon} \frac{\partial}{\partial x^4} (g^{\mu\nu} g^{44} - g^{4\mu} g^{4\nu}) \frac{\partial g_{\mu\nu}}{\partial x^4} dx^4 = 0. \tag{5.15}$$

This integrand is B and so the limit of the integral is zero; this gives us the result that

$$(g^{\mu\nu}\ g^{44}\ -\ g^{4\mu}\ g^{4\nu})\ \frac{\partial g_{\mu\nu}}{\partial x^4}\quad \text{is continuous across } \Sigma_3. \qquad (5.16)$$

We now treat (5.8) in the same way. By (5.13) it gives the ten equations

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\left[g^{44}\ \frac{\partial^2 g_{\mu\nu}}{(\partial x^4)^2}\ -\ g_{\mu\nu}\ (g^{\rho\sigma}\ g^{44}\ -\ g^{4\rho}\ g^{4\sigma})\ \frac{\partial^2 g_{\rho\sigma}}{(\partial x^4)^2}\right]\, dx^4 = 0,$$

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\left[g^{4\nu}\ \frac{\partial^2 g_{\mu\nu}}{(\partial x^4)^2}\ +\ g_{4\mu}\ (g^{\rho\sigma}\ g^{44}\ -\ g^{4\rho}\ g^{4\sigma})\ \frac{\partial^2 g_{\rho\sigma}}{(\partial x^4)^2}\right]\, dx^4 = 0,$$

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\left[g^{\mu\nu}\ \frac{\partial^2 g_{\mu\nu}}{(\partial x^4)^2}\ -\ g_{44}\ (g^{\rho\sigma}\ g^{44}\ -\ g^{4\rho}\ g^{4\sigma})\ \frac{\partial^2 g_{\rho\sigma}}{(\partial x^4)^2}\right]\, dx^4 = 0.$$

(5.17)

On integrating by parts as in (5.15), we are left with B integrands, and so we find that the following quantities are continuous across Σ_3:

$$g^{44}\ \frac{\partial g_{\mu\nu}}{\partial x^4}\ -\ g_{\mu\nu}\ (g^{\rho\sigma}\ g^{44}\ -\ g^{4\rho}\ g^{4\sigma})\ \frac{\partial g_{\rho\sigma}}{\partial x^4},$$

$$g^{4\nu}\ \frac{\partial g_{\mu\nu}}{\partial x^4}\ +\ g_{4\mu}\ (g^{\rho\sigma}\ g^{44}\ -\ g^{4\rho}\ g^{4\sigma})\ \frac{\partial g_{\rho\sigma}}{\partial x^4}, \qquad (5.18)$$

$$g^{\mu\nu}\ \frac{\partial g_{\mu\nu}}{\partial x^4}\ -\ g_{44}\ (g^{\rho\sigma}\ g^{44}\ -\ g^{4\rho}\ g^{4\sigma})\ \frac{\partial g_{\rho\sigma}}{\partial x^4}.$$

Now since g_{mn} are continuous across Σ_3, so also are g^{mn}, and so, when we combine (5.18) with (5.16) we see that

$$g^{44}\frac{\partial g_{\mu\nu}}{\partial x^4}, \quad g^{4\nu}\frac{\partial g_{\mu\nu}}{\partial x^4}, \quad g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^4} \quad \text{are continuous across } \Sigma_3. \tag{5.19}$$

This result includes (5.16).

We have now to distinguish a particular case from the general case. The particular case is that in which $g^{44} = 0$ on Σ_3, the general case that in which $g^{44} \neq 0$ on Σ_3. What does this particular case mean? In a Riemannian 4-space with a metric of signature $+ + + -$ or $- - - +$ (space-time is of one or other type, according to the convention we choose) there exists at each point an elementary null cone with equation $g_{mn}\,dx^m\,dx^n = 0$, and there are 3-spaces with the property that they are tangent to the elementary null cono at each of their points. Such 3-spaces we call <u>null surfaces</u> (Synge and McConnell 1928). If the equation of a 3-space is written $\varphi(x^1, x^2, x^3, x^4) = 0$, the condition that it shall be a null surface is

$$g^{mn}\frac{\partial\varphi}{\partial x^m}\frac{\partial\varphi}{\partial x^n} = 0 \quad \text{for } \varphi = 0. \tag{5.20}$$

The equation of Σ_3 is $x^4 = 0$, and so the condition that it shall be a null surface is

$$g^{44} = 0 \quad \text{for} \quad x^4 = 0. \tag{5.21}$$

If Σ_3 is not a null surface (the general case), then g^{44} does not vanish, and the continuity of the first of (5.19) gives us the result that

$$\frac{\partial g_{\mu\nu}}{\partial x^4} \quad \text{are continuous across } \Sigma_3. \tag{5.22}$$

But if Σ_3 is a null surface (particular case), then the continuity of the first of (5.19) is merely the trivial continuity 0 = 0, and we get, instead of the six conditions (5.22), only the four conditions

$$g^{4\nu} \frac{\partial g_{\mu\nu}}{\partial x^4} \quad \text{and} \quad g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^4} \quad \text{are continuous across } \Sigma_3. \tag{5.23}$$

Let us collect these results:

Theorem IV: <u>In general, i.e. when Σ_3 is not a null surface, the six derivatives $\partial g_{\mu\nu} / \partial x^4$ are continuous across the 3-space of discontinuity Σ_3 with equation $x^4 = 0$; or equivalently</u>

$$\left(\frac{\partial g_{\mu\nu}}{\partial x^4}\right)^+ = \left(\frac{\partial g_{\mu\nu}}{\partial x^4}\right)^- . \tag{5.24}$$

<u>If Σ_3 is a null surface, we have only the four jump conditions</u>

$$\left(g^{4\nu} \frac{\partial g_{\mu\nu}}{\partial x^4}\right)^+ = \left(g^{4\nu} \frac{\partial g_{\mu\nu}}{\partial x^4}\right)^- ,$$

$$\left(g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^4}\right)^+ = \left(g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^4}\right)^- . \tag{5.25}$$

The physical meaning of the expression " Σ_3 is a null surface" is that the discontinuity is being propagated in space-time like a light wave, for a null surface is the space-time representation of the wave propagation of light according to Huygens' principle. This would correspond for example to the advance of an electromagnetic shock wave into empty space, the tensor T_{mn} being the electromagnetic stress-energy tensor.

6. Jump conditions for T^m_n

Integration of the conservation equation (1.5) across the boundary layer gives

$$\int_{-\epsilon}^{\epsilon} T^m_{n|m} \, dx^4 = 0. \tag{6.1}$$

Now

$$T^m_{n|m} = \frac{\partial}{\partial x^m} T^m_n + \begin{Bmatrix} m \\ p\ m \end{Bmatrix} T^p_n - \begin{Bmatrix} p \\ n\ m \end{Bmatrix} T^m_p \,. \tag{6.2}$$

By Hypotheses III and VII, T^m_n and $\partial T^m_n / \partial x^\rho$ are B, and as in (5.11), the Christoffel symbols are B. Hence, going to the limit $\epsilon \longrightarrow 0$ in (6.1), we get

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \frac{\partial}{\partial x^4} T^4_n \, dx^4 = 0, \tag{6.3}$$

and so

$$(T^4_n)^+ = (T^4_n)^- \,. \tag{6.4}$$

Thus we have the result:

Theorem V: <u>Across a 3-space of discontinuity Σ_3, with equation $x^4 = 0$, the components T^4_n of the stress-energy tensor are continuous, as shown in (6.4).</u>

7. Conclusion

We have investigated the jump conditions across a 3-space Σ_3 with equation $x^4 = 0$ and found in Theorems II, IV and V that if Σ_3 is not a null surface (it may be either space-like or time-like), then the following 20 quantities are continuous across Σ_3:

$$g_{mn}, \quad \frac{\partial g_{\mu\nu}}{\partial x^4}, \quad T^4_n \qquad \text{(Latin 1, 2, 3, 4; Greek 1, 2, 3).} \tag{7.1}$$

This appears to leave open the possibility of jumps in the 16 quantities

$$\frac{\partial g_{4\mu}}{\partial x^4}, \quad \frac{\partial g_{44}}{\partial x^4}, \quad T^\rho_n \ . \tag{7.2}$$

But actually T^ρ_n are restricted by the symmetry of T_{mn}; we have

$$T_{mn} = g_{mr} T^r_n = g_{nr} T^r_m , \tag{7.3}$$

and so

$$g_{m\rho} T^\rho_n - g_{n\rho} T^\rho_m = - g_{m4} T^4_n + g_{n4} T^4_m \ . \tag{7.4}$$

But by (7.1) the right hand side is continuous across Σ_3; therefore

$$g_{m\rho} T^\rho_n - g_{n\rho} T^\rho_m \quad \text{are continuous across } \Sigma_3. \tag{7.5}$$

There are six conditions here.

The jump conditions on T^m_n may also be expressed in tensor form for a general coordinate system as

$$(T^m_n N^n)^+ = (T^m_n N^n)^- , \tag{7.6}$$

where N^r is a vector normal to Σ_3. We recognize a generalization to space-time of the familiar jump conditions for stress in the theory of elasticity.

We recall that if Σ_3 is a null surface, the six conditions (5.24) are replaced by the four conditions (5.25).

If we regard the field equations, not as partial differential equations for the determination of g_{mn}, but (as is sometimes done) as equations defining T_{mn}, then the jump conditions on T^m_n are to be regarded as conditions on the partial derivatives of g_{mn} of the first and second order.

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