

Title	On free quotients of complete intersection Calabi-Yau manifolds
Creators	Braun, Volker
Date	2010
Citation	Braun, Volker (2010) On free quotients of complete intersection Calabi-Yau manifolds. Journal of High Energy Physics, 2011 (4). ISSN 1029-8479
URL	https://dair.dias.ie/id/eprint/254/
DOI	DIAS-STP-10-03

On Free Quotients of Complete Intersection Calabi-Yau Manifolds

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Abstract

In order to find novel examples of non-simply connected Calabi-Yau threefolds, free quotients of complete intersections in products of projective spaces are classified by means of a computer search. More precisely, all automorphisms of the product of projective spaces that descend to a free action on the Calabi-Yau manifold are identified.

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1 Introduction

Almost all Calabi-Yau manifolds that we know about are simply connected. For example, the largest known class of Calabi-Yau threefolds was classified in [1, 2] and consists of 3-d hypersurfaces in 4-d toric varieties. The ambient toric varieties correspond to (usually numerous) subdivisions of the normal fans of 473,800,776 reflexive 4-d polyhedra. Only 16 of those lead to Calabi-Yau hypersurfaces with non-trivial fundamental group [3], which moreover ends up being either $\pi_1(X) = \mathbb{Z}_2, \mathbb{Z}_3, \text{ or } \mathbb{Z}_5$.

Are non-simply connected Calabi-Yau manifolds genuinely rare or is this simply a case of “searching under the lamppost”? Note that, to each non-simply connected manifold X , there is associated a unique simply connected manifold, its universal cover \tilde{X} , with a free $\pi_1(X)$ action. Moreover, by modding out this free action we can recover the original manifold. This suggests that one should search for free actions on already known Calabi-Yau manifolds in order to find new ones with non-vanishing fundamental group. This approach has been successful for a long time [4, 5, 6, 7], and produced quite a number of manifolds of phenomenological interest for heterotic string compactifications.

A very convenient subset of (simply-connected) Calabi-Yau manifolds are the 7890 complete intersections in products of projective spaces (CICY). Not only are they small enough in number to be easily handled with a modern computer, but their ambient spaces also come with a rather evident automorphism group. They have been a source for free group actions for a long time [8, 9, 10, 11]. In a painstaking manual search [12] most of the free group actions were actually found. However, some remained hidden including a very curious three-generation manifold [13] with minimal Hodge numbers $h^{11}(X) = 1$, $h^{21}(X) = 4$. Another application of the free CICY quotients is that, in contrast to the simply-connected CICYs, they contain examples of ample rigid divisors that are useful for moduli stabilization [14]. In the remainder of this paper, we will perform an exhaustive search through the automorphisms of products of projective spaces and classify all that restrict to a free action on the complete intersection Calabi-Yau threefolds. A similar search can be performed for more general complete intersections in toric varieties, but we leave this for future work.

Before delving into the classification, we would like to apologize to the reader for the horrendous technicalities that lie ahead. It is strongly recommended to start with the results in Table 1 on page 24 and their discussion in Section 5. The list of all free group actions is included in the source code of this paper which can be obtained from the arXiv server, see Appendix C for more details.

2 The Classification

2.1 CICY Group Actions

The goal of this paper is to classify group actions on Calabi-Yau threefolds that are complete intersections in products of projective spaces (CICY). Moreover, we will only consider group actions that come from group actions on the ambient space $\prod_i \mathbb{P}^{d_i-1}$. That is, we only consider group actions that are combinations of

1. Projective-linear action on the individual factors \mathbb{P}^{d_i-1} , and

2. Permutations¹ of the factor \mathbb{P}^{d_i-1} .

In other words, we only allow group actions that are represented by linear transformations on the combined homogeneous coordinates. These are also the group actions of physical interest for the construction of (equivariant) monad bundles, see [15, 16, 17, 18]. In general, there are also non-linear group actions. However, in special cases we classify actually all possible group actions. For example, when the Calabi-Yau manifold in question is given by its Kodaira embedding² $X \subset \mathbb{P}^{d-1}$, then all actions are linear. In particular, any group action on the Quintic in \mathbb{P}^4 is of the type we are considering.

Recall the standard notation for the degrees of the transverse polynomials defining a CICY manifold. This is just a matrix (c_{ij}) such that the j -th polynomial is of homogeneous degree c_{ij} in the homogeneous coordinates of the i -th projective space. For the group action to descend to the complete intersection the individual polynomials need not be preserved, only their common zero set must be. In particular, if multiple polynomials of the same degree occur then they might be transformed into non-trivial linear combinations.

This is why we will use a slightly different notation where the degrees (and, hence, the diffeomorphism type) of the CICY is defined by a configuration matrix with pairwise different columns

$$\begin{array}{cccc}
 & \vec{p}_1 & \vec{p}_2 & \cdots & \vec{p}_m \\
 \mathbb{P}_1 \stackrel{\text{def}}{=} \mathbb{P}^{d_1-1} & c_{11} & c_{12} & \cdots & c_{1m} \\
 \mathbb{P}_2 \stackrel{\text{def}}{=} \mathbb{P}^{d_2-1} & c_{21} & c_{22} & \cdots & c_{2m} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 \mathbb{P}_n \stackrel{\text{def}}{=} \mathbb{P}^{d_n-1} & c_{n1} & c_{n2} & \cdots & c_{nm}
 \end{array} , \tag{1}$$

meaning that

- The ambient space is $\prod_{i=1}^n \mathbb{P}_i$
- The CICY is cut out by m vectors of equations \vec{p}_j each having $\delta_j \in \mathbb{Z}_{>}$ components.
- Each component of the equation vector \vec{p}_j is a homogeneous polynomial of degree $c_{ij} \in \mathbb{Z}_{\geq}$ in the d_i homogeneous coordinates of the i -th factor \mathbb{P}_i .

¹Called external in [9], but we will not use this notation in the following.

²That is, there is a (invariant but not necessarily equivariant) line bundle \mathcal{L} on X such that the $d = h^0(X, \mathcal{L})$ global sections s_α do not vanish simultaneously and separate points and tangent directions. That is, $x \mapsto [s_0(x) : \cdots : s_d(x)]$ defines an embedding into \mathbb{P}^{d-1} .

Obviously $n - 3 = \sum \delta_j$ for threefolds. Moreover, the vanishing of the first Chern class is equivalent to

$$d_i = \sum_{j=1}^m c_{ij} \delta_j \quad \forall i = 1, \dots, n. \quad (2)$$

However, the group and index theory we will use is independent of the dimension and Chern class and could be applied to more general complete intersections.

To formalize this notion of group action, let us define

Definition 1 (CICY groups). *A CICY group is a quadruple (C, G, π_r, π_c) where*

- $C = (d_i, c_{ij}, \delta_j)_{i=1..n, j=1..m}$ is the configuration matrix of a CICY,
- G is a group,
- $\pi_r : G \rightarrow P_{row}$ is a permutation action on the n rows, and
- $\pi_c : G \rightarrow P_{col}$ is a permutation action on the r columns

such that the configuration matrix is invariant under the permutations. That is,

$$c_{i,j} = c_{\pi_r(g)(i), \pi_c(g)(j)} \quad \forall g \in G, \quad (3)$$

the number d_i of homogeneous coordinates of $\mathbb{P}_i = \mathbb{P}^{d_i-1}$ is constant on orbits of P_{row} , and the number of components δ_j of p_j is constant on orbits of P_{col} .

Lemma 1. π_c is uniquely determined by (C, G, π_r) if it exists.

Now, a representation of a CICY group is the collection of matrices, one for each group element and each projective space, acting on the homogeneous coordinates. One must ensure that permutations interchange the different projective space and equation vectors. Note that this is the same structure for the rows and columns. Therefore, let us define

Definition 2 (π -representation). *A (linear) π -representation is a quadruple $(G, \pi, \vec{d}, \gamma)$ where*

- $\pi : G \rightarrow P$ is a permutation action of G on $\{1, \dots, n\}$.
- $\gamma_i : G \rightarrow GL(d_i, \mathbb{C})$ is a map satisfying

$$\gamma_{\pi(h)(i)}(g) \gamma_i(h) = \gamma_i(gh) \quad \forall g, h \in G, i \in \{1, \dots, n\}. \quad (4)$$

In other words, the $d_i \times d_i$ matrices $\gamma_i(g)$ can be assembled into an ordinary group representation by block matrices $\gamma(g)$ of the form

$$\gamma(g) = \mathbf{P}(\pi(g), \vec{d}) \operatorname{diag}(\gamma_1(g), \dots, \gamma_n(g)) \quad (5)$$

where $\mathbf{P}(\pi(g), \vec{d})$ is the permutation matrix corresponding to the permutation $\pi(g)$ acting on $\{1, \dots, n\}$ but with entries being rectangular matrices $\mathbf{0}_{d_i \times d_j}$ and (square) identity matrices $\mathbf{1}_{d_i \times d_j}$ instead of 0 and 1.

A projective π -representation (G, π, γ) is one where $\gamma_i : G \rightarrow PGL(d_i)$. This is the case of interest to us, since homogeneous coordinates as well as the zero sets of polynomials do not depend on overall \mathbb{C}^\times factors.

Let us formalize the data required to define a group action on a CICY manifold;

Definition 3 (CICY group action). *A CICY group action is a tuple $(C, G, \pi_r, \gamma, \pi_c, \rho)$ such that*

- $C = (d_i, c_{ij}, \delta_j)_{i=1..n, j=1..m}$ is the configuration matrix of a CICY,
- (C, G, π_r, π_c) is a CICY group, and
- $(G, \pi_r, \vec{d}, \gamma)$ and $(G, \pi_c, \vec{\delta}, \rho)$ are π -representations.

A CICY group action defines an action on the combined homogeneous coordinates

$$\vec{z} \stackrel{\text{def}}{=} [z_{1,1} : \dots : z_{1,d_1} \mid z_{2,1} : \dots : z_{2,d_2} \mid \dots \mid z_{n,1} : \dots : z_{n,d_n}] \quad (6)$$

of $\prod \mathbb{P}_i$. This action induces a π -representation on the combined polynomial equations

$$\vec{p} \stackrel{\text{def}}{=} (\vec{p}_1, \dots, \vec{p}_m) = (p_{1,1}, \dots, p_{1,\delta_1}; \dots; p_{r,1}, \dots, p_{r,\delta_m}). \quad (7)$$

We say that the polynomials defining the CICY are invariant under the group action if this induced action on the equations equals the representation (G, π_c, ρ) . In other words, the composition

$$\rho^{-1}(g) \vec{p}(\gamma(g)\vec{z}) = \vec{p}(\vec{z}) \quad \forall \gamma \in G \quad (8)$$

leaves the polynomials invariant. That is, the (G, π_c, ρ) action cancels out the non-trivial action on the polynomials.

Theorem 1. *Fix a CICY group (C, G, π_r, π_c) , and a projective π -representation $(G, \pi_r, \vec{d}, \gamma)$ acting on the homogeneous coordinates. Then the zero set $\{\vec{p} = 0\} \subset \prod \mathbb{P}_i$ is invariant if and only if there is a CICY action $(C, G, \pi_r, \gamma, \pi_c, \rho)$ leaving the polynomials invariant.*

Finally, note that the invariant polynomials can be easily computed by the usual Reynolds operator, that is, summing over orbits of the group.

Example 1 (A CICY group action). Consider the CICY #20,

$$\left[\begin{array}{c|cccc} \mathbb{P}^1 & 0 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 0 & 0 & 2 \\ \mathbb{P}^4 & 2 & 2 & 1 & 0 \end{array} \right] = \begin{array}{c} \mathbb{P}_1 \stackrel{\text{def}}{=} \mathbb{P}^1 \\ \mathbb{P}_2 \stackrel{\text{def}}{=} \mathbb{P}^1 \\ \mathbb{P}_3 \stackrel{\text{def}}{=} \mathbb{P}^1 \\ \mathbb{P}_4 \stackrel{\text{def}}{=} \mathbb{P}^4 \end{array} \begin{array}{c} (p_1) \\ (p_2) \\ (p_3) \\ (p_4) \end{array} \begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline 0 & 0 & 2 \\ \hline 0 & 0 & 2 \\ \hline 2 & 1 & 0 \\ \hline \end{array} = C. \quad (9)$$

In particular, the numbers of homogeneous coordinates corresponding to each row are $d = (2, 2, 2, 5)$, and the numbers of equations corresponding to each column are $\delta = (2, 1, 1)$.

Now, let us consider the group $\mathbb{Z}_4 = \{1, g, g^2, g^3\}$ generated by g . One possible CICY group for the configuration matrix C is (C, G, π_r, π_c) with the permutation actions³

$$\pi_r(g) \stackrel{\text{def}}{=} (2, 3), \quad \pi_c(g) \stackrel{\text{def}}{=} (). \quad (10)$$

An example of a CICY group action is $(C, \mathbb{Z}_4, \pi_r, \gamma, \pi_c, \rho)$ with the representations generated by

$$\gamma(g) \stackrel{\text{def}}{=} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & -i \end{pmatrix}, \quad (11)$$

$$\rho(g) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus (1) \oplus (i).$$

A basis for the invariant polynomial vectors is

$$\begin{array}{l} (0, 0, 0, z_2 z_3 z_4 z_5 z_6), \quad (0, 0, 0, z_2 z_3^2 z_6^2 - z_2 z_4^2 z_5^2), \\ (0, 0, 0, z_1 z_4^2 z_6^2), \quad (0, 0, 0, z_1 z_3^2 z_6^2 + z_1 z_4^2 z_5^2), \\ (0, 0, 0, z_1 z_3^2 z_5^2), \quad (0, 0, z_2 z_{10}, 0), \\ (0, 0, z_2 z_9, 0), \quad (0, 0, z_1 z_{11}, 0), \\ (0, z_{10}^2, 0, 0), \quad (0, z_9 z_{10}, 0, 0), \\ (0, z_9^2, 0, 0), \quad (0, z_7 z_8, 0, 0), \\ (z_{11}^2, 0, 0, 0), \quad (z_{10} z_{11}, 0, 0, 0), \\ (z_9 z_{11}, 0, 0, 0), \quad (z_8^2, 0, 0, 0), \\ (z_7^2, 0, 0, 0). \end{array} \quad (12)$$

One can show that a sufficiently generic linear combination cuts out a smooth fixed-point free CICY threefold.

³I am using cycle notation for the permutations.

2.2 Classification Algorithm

Using index theory one can show [8, 9] that any free group action on one of the 7890 CICYs has $|G| \leq 64$. Hence, there are only a finite number of possible CICY groups. Moreover, there is only a finite number of distinct group representations for fixed dimension. Therefore, there is only a finite number of free CICY group actions, and we can, in principle, enumerate all of them:

```

1: FreeActions = {}
2: for all CICY configuration matrices  $C$  do
3:   for all CICY groups  $(C, G, \pi_r, \pi_c)$  such that  $|G| \leq 64$  do
4:     for all  $\pi$ -representations  $(G, \pi_r, \vec{d}, \gamma)$  and  $(G, \pi_c, \vec{\delta}, \rho)$  do
5:        $\vec{p}$  = random linear combination of  $(C, G, \pi_r, \gamma, \pi_c, \rho)$ -invariant polynomials
6:        $X = \{\vec{p} = 0\} \subset \prod \mathbb{P}_i$ 
7:       if  $X$  is fixed-point free and  $X$  is smooth then
8:         Add  $(C, G, \pi_r, \gamma, \pi_c, \rho)$  to FreeActions
9:       end if
10:    end for
11:  end for
12: end for
13: return FreeActions

```

Although finite, working through this algorithm is far out of reach of present-day capabilities. Enumerating all $(G, \pi_r, \vec{d}, \gamma)$ and all $(G, \pi_c, \vec{\delta}, \rho)$ representations is feasible, but their Cartesian product often exceeds 10^{10} pairs. Moreover, checking for fixed points and, in particular, smoothness requires Gröbner basis computations that can take from seconds to multiple days on a modern desktop computer⁴ even using the algorithmic improvements outlined below.

The key to classifying the free actions is to compute the character-valued indices of a sample of equivariant line bundles. These must be of a certain “free” type, otherwise the group action cannot be free on the CICY manifold. Moreover, these character-valued indices can be computed without explicitly constructing the representations or polynomials. In Section 3 we will introduce a generalization of Schur covers that is necessary to compute characters of projective π -representations, and in Section 4 we will show how to compute the indices using character theory alone.

One still needs a few optimizations to classify all free CICY quotients. These include

- Knowing the group G lets us identify line bundles that must be equivariant. The ordinary (not character-valued) index must be divisible by $|G|$, yielding stronger restrictions than indices that only depend on the configuration matrix.

⁴All computations in this paper were done on a 2.66GHz Intel Core i7 processor and 12 GiB of RAM.

- The π -representation $(G, \pi_r, \vec{d}, \gamma)$ and $(G, \pi_c, \vec{\delta}, \rho)$ can be decomposed into blocks corresponding to the $\text{img}(\pi)$ -orbits. The list of all “big” representations is just the Cartesian product of all the representations corresponding to the individual $\text{img}(\pi)$ -orbits.
- In many CICYs there are a few line bundles whose character-valued index does not depend on all of the blocks of the $(G, \pi_r, \vec{d}, \gamma)$ and $(G, \pi_c, \vec{\delta}, \rho)$ -representations. By testing these line bundles first, we can eliminate some choices for the contributing blocks without going through the whole Cartesian product.
- Smoothness and absence of fixed points can be checked much faster over finite fields. Choosing the wrong finite field or the wrong invariant polynomial may yield false negatives, but a positive answer is definite. By repeating the test with different finite fields and a different linear combination of invariant polynomials, we can make false negatives highly unlikely.
- As we will show in detail in Section 3, one can enumerate the $(G, \pi_r, \vec{d}, \gamma)$ and $(G, \pi_c, \vec{\delta}, \rho)$ -representations using characters. The explicit representation matrices are only required to check for fixed points and smoothness, but not to compute the character-valued indices.

Using these ideas, we present the improved Algorithm 1. I implemented this classification algorithm using the GAP and SINGULAR computer algebra systems [19, 20, 21]. The whole program completed within a few months of run time.

3 Group Actions

3.1 Projective Representations

Recall that a (linear) representation of a group G is a map

$$r : G \rightarrow GL(n, \mathbb{C}), \quad r(g)r(h) = r(gh) \quad \forall g, h \in G. \quad (13)$$

The matrices $r(g)$ clearly depend on the chosen basis, but representations that merely differ by a coordinate transformation should be regarded as the same. An obvious invariant of the representation r is its character

$$\chi_r : G \rightarrow \mathbb{C}^\times, \quad g \mapsto \text{Tr} \left(r(g) \right). \quad (14)$$

Recall some well-known properties of the characters:

- $\chi_r(g) = \chi_r(h^{-1}gh)$ depends only on the conjugacy class of $g \in G$.
- There is a one-to-one correspondence between irreducible representations and their characters.

Algorithm 1 Classifying the smooth free CICY quotients

```
1: FreeActions = {}
2: for all CICY configuration matrices  $C$  do
3:   for all CICY groups  $(C, G, \pi_r, \pi_c)$  such that
        $|G|$  divides all indices of  $C$  and every subgroup of  $G$  acts freely do
4:     if topological index of some  $G$ -equivariant bundle is not divisible by  $|G|$  then
5:       continue with next CICY group
6:     end if
7:     Find generalized Schur cover  $\tilde{G} \rightarrow G$ 
8:      $\Gamma =$  all (linear)  $\pi$ -representations  $(\tilde{G}, \pi_r, \vec{d}, \gamma)$ 
9:      $R =$  all (linear)  $\pi$ -representations  $(\tilde{G}, \pi_c, \vec{\delta}, \rho)$ 
10:    for all  $\mathcal{L}$  in a sample of invariant line bundles do
11:      for all  $(\tilde{G}, \pi_r, \vec{d}, \gamma) \in \Gamma$  and  $(\tilde{G}, \pi_c, \vec{\delta}, \rho) \in R$ 
          that are not already ruled out by a previous line bundle  $\mathcal{L}$  do
12:        Compute the character-valued index  $\chi(\mathcal{L})$ 
13:        if  $\chi(\mathcal{L})$  is not of the free type then
14:           $\{(C, \tilde{G}, \pi_r, \gamma, \pi_c, \rho)$  cannot act freely}
15:          continue with next representation
16:        end if
17:        Compute the twist  $\tau$ , a character of  $\ker(\tilde{G} \rightarrow G)$ 
18:        if  $\tau = 1$  and  $|G| \nmid \dim \chi(\mathcal{L})$  then  $\{\tau = 1$  means  $\mathcal{L}$  is  $G$ -equivariant}
19:          continue with next representation
20:        end if
21:        Construct the explicit representation matrices for  $\gamma, \rho$ .
22:        for many finite fields  $\mathbb{F}$  do
23:           $\vec{p} =$  random  $\mathbb{F}$ -linear combination of  $(C, \tilde{G}, \pi_r, \gamma, \pi_c, \rho)$ -invariants
24:           $X = \{\vec{p} = 0\} \subset \prod \mathbb{F}\mathbb{P}_i$ 
25:          if  $X$  is fixed-point free and  $X$  is smooth then
26:            Add  $(C, G, \pi_r, \gamma, \pi_c, \rho)$  to FreeActions
27:          end if
28:        end for
29:      end for
30:    end for
31:  end for
32: end for
33: return FreeActions
```

Clearly, it is desirable to work with the characters instead of (isomorphism) classes of representations. However, this requires that all representations are linear, and not just projective.

Consider the following example of a projective representation,

Example 2 (A projective representation). *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{ (\begin{smallmatrix} + \\ + \end{smallmatrix}), (\begin{smallmatrix} + \\ - \end{smallmatrix}), (\begin{smallmatrix} - \\ + \end{smallmatrix}), (\begin{smallmatrix} - \\ - \end{smallmatrix}) \}$ and*

$$r \left(\begin{smallmatrix} + \\ - \end{smallmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad r \left(\begin{smallmatrix} - \\ + \end{smallmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (15)$$

Thought of as $PGL(2)$ -matrices, r is a projective representation of G . However, the matrices $r \left(\begin{smallmatrix} + \\ - \end{smallmatrix} \right), r \left(\begin{smallmatrix} - \\ + \end{smallmatrix} \right) \in GL(2)$ generate the group D_8 , so r is not a (linear) representation of G . Moreover, one cannot turn r into a representation by multiplying $r \left(\begin{smallmatrix} + \\ - \end{smallmatrix} \right), r \left(\begin{smallmatrix} - \\ + \end{smallmatrix} \right)$ by fixed overall phases.

As is obvious from the example, if one wants to work with linear instead of projective representations one can lift them to linear representations, but at the cost of having to enlarge the group. Clearly, there is an epimorphism from the enlarged group \tilde{G} to the original group G by making everything projective again. This means that

$$1 \longrightarrow K \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1 \quad (16)$$

is a central extension, that is, the kernel K is in the center of \tilde{G} . In other words, $K \subset \tilde{G}$ are the commutators that are non-trivial in \tilde{G} but become trivial when mapped into G .

Thanks to Schur [22, 23] we know that, for any finite group G , there is a finite covering group \tilde{G} such that there is a one-to-many⁵ correspondence between

- projective representations $r : G \rightarrow PGL(n)$ and
- twisted representations, that is linear representations $\tilde{r} : \tilde{G} \rightarrow GL(n)$ such that $\tilde{r}(k) \sim \mathbf{1}_{n \times n}$ for all $k \in K$.

Any such group is called a “hinreichend ergänzte Gruppe” (sufficient⁶ extension) or of surjective type. If \tilde{G} is of minimal size, then it is called a “Darstellungsgruppe” (representation group) or Schur cover. In general, a Schur cover is not uniquely determined.

A twisted representation $\tilde{r} : \tilde{G} \rightarrow GL(n)$ determines a one-dimensional representation $\tau : K \rightarrow \mathbb{C}^\times$ via $\tilde{r}(k) = \tau(k)\mathbf{1}_{n \times n}$. Multiplying \tilde{r} with a one-dimensional representation of \tilde{G} also multiplies τ , so we should identify the orbits under this action. This leads to

⁵Many because if $\psi : \tilde{G} \rightarrow \mathbb{C}^\times$ is a one-dimensional representation then \tilde{r} and $\psi\tilde{r}$ correspond to the same projective representation.

⁶Note that if \tilde{G} is a sufficient extension, then $\tilde{G} \times H$ is sufficient as well. So there are infinitely many sufficient extensions.

Definition 4 (Twist of a twisted representation). *Consider a central extension eq. (16) and let \tilde{r} be a twisted representation. Then we say that⁷*

$$\tau = \frac{1}{\dim \tilde{r}} \tilde{r}|_K = \frac{1}{\dim \tilde{r}} \text{Res}_K^{\tilde{G}}(\tilde{r}) \in \text{Hom}(K, \mathbb{C}^\times) / \text{Res}_K^{\tilde{G}} \text{Hom}(\tilde{G}, \mathbb{C}^\times) \quad (17)$$

is the “twist” of \tilde{r} . It is a one-dimensional representation of K modulo the multiplicative action of the restrictions of one-dimensional representations of \tilde{G} .

In Appendix A, we will remark on the connection between the twisted representations and the more standard approach towards projective representations using group cohomology. However, this is not necessary to understand the remainder of this paper.

Evidently, sums of representations with the same twist are again twisted representations and correspond to a projective representation; The sum of representations with different twists is not a twisted representation. Finally, if $\tau = 1$ is (equivalent to) the trivial representation, then the corresponding projective representation is actually linear.

Example 3 (Continuation from Example 2). *A Schur cover of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is D_8 , leading to the central extension*

$$1 \longrightarrow \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \longrightarrow \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1 \quad (18)$$

The group D_8 has four 1-dimensional irreps (of twist $\tau = 1$) and one 2-dimensional irrep of twist $\tau(-\mathbf{1}_{2 \times 2}) = -1$.

3.2 Induction and Restriction

Using Schur covers and characters solves the problem of enumerating all projective representations in an efficient manner. However, we need to generalize it to representations in products of projective spaces where some group elements act by permutations.

For the reminder of this subsection, let us only consider linear π -representations $(G, \pi, \vec{d}, \gamma)$, see Definition 2. Moreover, for simplicity let us assume that the permutation action of $\text{img}(\pi)$ is transitive, that is, forms only a single orbit $\{1, \dots, n\}$. Note that this implies that the dimension vector $\vec{d} = (d, \dots, d)$ is constant. By decomposing an arbitrary π -representation into a direct sum we can always reduce to the single-orbit case.

Now, G acts on the index set $\{1, \dots, n\}$ via $\pi : G \rightarrow P$. Some of the group elements of G will leave $1 \in \{1, \dots, n\}$ invariant. Let us denote this stabilizer by

$$G_1 \stackrel{\text{def}}{=} \left\{ g \in G \mid \pi(g)(1) = 1 \right\}. \quad (19)$$

⁷Hom will always denote group homomorphisms in this paper.

The restriction of the first block γ_1 of γ to G_1 is an actual representation of G_1 , as this subgroup does not permute it.

One can recover the whole representation matrix γ from $\gamma_1|_{G_1}$ as follows. First, fix a choice of group elements $g_1 \stackrel{\text{def}}{=} 1$, $g_i \in G$, $i = 2, \dots, n$, such that $\pi(g_i)(1) = i$. By the assumption of $P = \pi(G)$ having only a single orbit, we can always find such $\{g_1, g_2, \dots, g_n\}$. This allows us to factorize any group element into

$$\forall g \in G, \forall 1 \leq i \leq n \quad \exists h \in G_1 : \quad g = g_{\pi(g)(i)} \circ h \circ g_i^{-1} \quad (20)$$

Due to the choice $g_1 \stackrel{\text{def}}{=} 1$ the representation matrix $\gamma_1(g_1) = \mathbf{1}_{d \times d}$. Since g_i , $i = 2, \dots, n$ maps the first block to the i -th block, we can choose coordinates on the i -th block such that

$$\gamma_1(g_i) = \mathbf{1}_{d \times d} \quad \forall i = 1, \dots, n. \quad (21)$$

Using eq. (5), we can expand any group representation matrix as

$$\begin{aligned} \gamma(g) &= \gamma(g_{\pi(g)(i)}) \circ \gamma(h) \circ \gamma(g_i^{-1}) \\ &= \mathbf{P}(\pi(g_{\pi(g)(i)}), \vec{d}) \text{diag}(\gamma_1(g_{\pi(g)(i)}), \dots, \gamma_n(g_{\pi(g)(i)})) \\ &\quad \mathbf{P}(\pi(h), \vec{d}) \text{diag}(\gamma_1(h), \dots, \gamma_n(h)) \\ &\quad \text{diag}(\gamma_1(g_i)^{-1}, \dots, \gamma_n(g_i)^{-1}) \mathbf{P}(\pi(g_i)^{-1}, \vec{d}) \end{aligned} \quad (22)$$

Evaluating the permutation matrices, we see that the i -th block of $\gamma(g)$ is

$$\gamma_i(g) = \gamma_1(g_{\pi(g)(i)}) \gamma_1(h) \gamma_1(g_i)^{-1} = \gamma_1(h) \quad \forall i = 1, \dots, n. \quad (23)$$

Hence, $\gamma'_1 = \gamma_1|_{G_1}$ determines the whole π -representation γ .

Example 4 (Induction). *Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm ij\}$ and $\pi(i) = (1, 2)$, $\pi(j) = ()$. Then the stabilizer $(Q_8)_1 = \{j^\ell | \ell = 0, \dots, 3\} \simeq \mathbb{Z}_4$. Pick the representation*

$$\gamma'_1 : (Q_8)_1 \rightarrow \mathbb{C}^\times, \quad \gamma'_1(j^\ell) = \exp\left(\frac{2\pi i \ell}{4}\right) \quad (24)$$

Now, let us choose $g_1 = 1$, $g_2 = i$. The π -representation $(G, \pi, (1, 1), \gamma)$ thus generated is given by

$$\gamma(i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma(j) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (25)$$

This construction that is called induction. It takes a representation $\gamma'_1 : G_1 \rightarrow GL(d, \mathbb{C})$ of a subgroup $G_1 \subset G$ and constructs a larger representation

$$\gamma = \text{Ind}_{G_1}^G(\gamma'_1) : G \rightarrow GL\left(\frac{d|G|}{|G_1|}, \mathbb{C}\right). \quad (26)$$

To summarize, we have shown

Theorem 2 (Defining data of a π -representation). *A linear π -representation $(G, \pi, \vec{d}, \gamma)$ such that $\text{img}(\pi)$ has a single orbit is, up to linear coordinate changes, uniquely determined by*

- *The permutation P acting on $\{1, \dots, n\}$,*
- *a group homomorphism $\pi : G \rightarrow P$,*
- *the dimension $d \in \mathbb{Z}$ of a single block, and*
- *a linear representation $\gamma'_1 : G_1 \rightarrow GL(d)$.*

The corresponding π -representation is then

$$\left(G, \pi, \underbrace{(d, \dots, d)}_n, \text{Ind}_{G_1}^G(\gamma'_1) \right). \quad (27)$$

Finally, note that there is an inner product on the group characters,

$$(\chi, \psi) = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)} \in \mathbb{Z}. \quad (28)$$

With respect to this inner product, induction and restriction⁸ are adjoint functors. That is, given a subgroup $H \subset G$ and characters χ of H and ψ of G ,

$$\left\langle \text{Ind}_H^G(\chi), \psi \right\rangle = \left\langle \chi, \text{Res}_H^G(\psi) \right\rangle. \quad (29)$$

Therefore, the character of an induced representation can be computed without explicitly constructing the induced representation.

3.3 Generalized Schur Covers

Similar to the usual case of projective representations, we can turn projective representations into linear representations by enlarging the group. The basic recipe is the same as in Subsection 3.1: Given a projective representation $\gamma : G \rightarrow \prod_i PGL(d_i)$, we can pick generators g_1, \dots, g_k of G and matrices $\gamma(g_i) \in \prod_i GL(d_i)$ that generate γ projectively. As a matrix group, the $\gamma(g_i)$ generate a potentially larger group

$$\tilde{G} \stackrel{\text{def}}{=} \langle \gamma(g_1), \dots, \gamma(g_k) \rangle \quad (30)$$

which maps onto G in the tautological way $\tilde{G} \rightarrow G, \gamma(g_i) \mapsto g_i$.

⁸Restriction is just the ordinary pullback $\text{Res}_H^G(\chi) \stackrel{\text{def}}{=} \chi|_H : H \rightarrow \mathbb{C}^\times$ of a character $\chi : G \rightarrow \mathbb{C}^\times$ to a subgroup $H \subset G$.

However, there are some differences. Most notably, the short exact sequence

$$1 \longrightarrow K \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1 \quad (31)$$

is no longer a central extension; In fact, the kernel $K \subset \tilde{G}$ not only consists of matrices proportional to the identity matrix, but also of the form $\bigoplus \zeta_i \mathbf{1}_{d_i \times d_i}$ with not all $\zeta_i \in \mathbb{C}^\times$ being equal. Nevertheless, the induction construction reviewed in Subsection 3.2 still works: A projective representation of the stabilizer G_1 determines a twisted representation of its ordinary Schur cover \tilde{G}_1 , which induces a multi-twisted⁹ representation of \tilde{G} corresponding to a multi-projective representation of G . That way, we can find a finite cover \tilde{G} for each finite group. However, \tilde{G} can be strictly larger than the ordinary Schur cover:

Example 5 (A generalized Schur cover). *Consider the group $G = \mathbb{Z}_4 \times \mathbb{Z}_4 = \{(a, b) | 0 \leq a, b \leq 3\}$ acting on CICY #21 via*

$$\begin{array}{c}
\begin{array}{cc}
(p_1) & (p_2) \\
\mathbb{P}_1 = \mathbb{P}^2 & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \\
\mathbb{P}_2 = \mathbb{P}^2 & \begin{array}{|c|c|} \hline 0 & 2 \\ \hline \end{array} \\
\mathbb{P}_3 = \mathbb{P}^2 & \begin{array}{|c|c|} \hline 2 & 0 \\ \hline \end{array} \\
\mathbb{P}_4 = \mathbb{P}^2 & \begin{array}{|c|c|} \hline 0 & 2 \\ \hline \end{array} \\
\mathbb{P}_5 = \mathbb{P}^2 & \begin{array}{|c|c|} \hline 2 & 0 \\ \hline \end{array}
\end{array}
\stackrel{\text{def}}{=} C,
\end{array}
\quad
\begin{array}{l}
\pi_r(1, 0) = (2, 3)(4, 5) \\
\pi_r(0, 1) = (2, 4)(3, 5) \\
\pi_c(1, 0) = (1, 2) \\
\pi_c(0, 1) = ().
\end{array}
\quad (32)$$

This defines the CICY group (C, G, π_r, π_c) .

A freely acting projective CICY group action is $(C, G, \pi_r, \gamma, \pi_c, \rho)$ with the representation matrices

$$\begin{aligned}
\gamma(1, 0) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, & \gamma(0, 1) &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 \end{pmatrix}, \\
\rho(1, 0) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \rho(0, 1) &= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.
\end{aligned}
\quad (33)$$

A basis for the 3-dimensional space of invariant homogeneous polynomials is

$$\begin{aligned}
\bar{p}^{(1)} &\stackrel{\text{def}}{=} \left(z_2 z_5 z_6 z_9 z_{10}, \quad -z_1 z_3 z_4 z_7 z_8 \right) \\
\bar{p}^{(2)} &\stackrel{\text{def}}{=} \left(z_1 z_5^2 z_{10}^2 + z_1 z_6^2 z_9^2, \quad -z_2 z_3^2 z_8^2 - z_2 z_4^2 z_7^2 \right), \\
\bar{p}^{(3)} &\stackrel{\text{def}}{=} \left(z_1 z_5^2 z_9^2 + z_1 z_6^2 z_{10}^2, \quad -z_2 z_3^2 z_7^2 - z_2 z_4^2 z_8^2 \right)
\end{aligned}
\quad (34)$$

⁹We call a representation $\gamma : G \rightarrow \prod PGL(d_i)$ multi-projective. Lifting it to a linear representation yields a (non-unique) multi-twisted representation $\tilde{\gamma} : G \rightarrow \prod GL(d_i)$.

and one can show that a generic linear combination defines a fixed-point free smooth Calabi-Yau threefold.

Clearly, $|G| = 16$. A Schur cover, that is, a smallest group that linearizes any projective G -representation, is the Heisenberg group $\mathbb{Z}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ and has 64 elements. This group is also sufficient to linearize the column π -representations. However, it is insufficient to linearize the row π -representation $(G, \pi_r, (2, 2, 2, 2, 2), \gamma)$. The matrices $\gamma(1, 0)$, $\gamma(0, 1)$ generate a matrix group of order 256. Linearizing every row and column π -representation simultaneously requires a covering group of order 512.

4 Character-Valued Indices

4.1 Invariant and Equivariant Line Bundles

Consider a line bundle \mathcal{L} on a complex manifold X with a group G acting on X . Although we are primarily interested in free actions, we will also consider group actions with fixed points for the purposes of this subsection.

The line bundle \mathcal{L} is invariant if $g^*\mathcal{L} \simeq \mathcal{L}$ for all $g \in G$. If $\text{Pic}^0(X) = 1$, as is the case for proper Calabi-Yau threefolds, the line bundles are classified by their first Chern class. In that case \mathcal{L} is invariant if and only if

$$c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})^G. \quad (35)$$

Each isomorphism $g^*\mathcal{L} \simeq \mathcal{L}$ defines a linear map

$$\gamma(g) : H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}). \quad (36)$$

However, the linear maps $\gamma(g)$ need not be a group homomorphism, that is, $\gamma(g)\gamma(h) \neq \gamma(gh)$. Therefore, the representation matrices $\gamma(g)$ generate a covering group \tilde{G} with kernel K ,

$$1 \longrightarrow K \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1. \quad (37)$$

In the case where $X = \prod \mathbb{P}^{d_i}$ is the ambient space of a product of projective spaces, the short exact sequence is of course identical to eq. (31).

A line bundle is equivariant if it is invariant and the representation matrices do form a representation of the group G acting on the base space. Note that

- Not every G -invariant line bundle is G -equivariant.
- Every G -invariant line bundle is \tilde{G} -equivariant for some sufficient extension $\tilde{G} \rightarrow G$. The kernel K acts trivially on the base space X .
- Every G -invariant line bundle is \mathbb{Z}_k -equivariant for every cyclic subgroup $\mathbb{Z}_k \subset G$.

4.2 Implications of Freeness

Recall the generalization of the Lefschetz fixed point theorem to holomorphic vector bundles [24]: Given a bundle \mathcal{V} over X and a holomorphic map $f : X \rightarrow X$ with isolated¹⁰ fixed points together with an isomorphism $F : f^*\mathcal{V} \rightarrow \mathcal{V}$. Then this implies an action on the bundle-valued cohomology groups via the double pull-back

$$\begin{array}{ccc}
 & H^i(X, f^*\mathcal{V}) & \\
 f^* \nearrow & & \searrow F^* \\
 H(f, F) : H^i(X, \mathcal{V}) & \xrightarrow{\quad} & H^i(X, \mathcal{V}).
 \end{array} \tag{38}$$

Like the vector spaces $H^i(X, \mathcal{V})$, this map can depend on moduli. However, the Euler characteristic

$$\chi(f, F) \stackrel{\text{def}}{=} \sum_i (-1)^i \text{Tr} H^i(f, F) = \sum_{P \in X^f} \frac{\text{Tr} F_P}{\det(1 - df_P)}. \tag{39}$$

is invariant under deformations and can be computed from data localized at the fixed point set X^f alone.

We always defined group actions on CICY manifolds X via linear π -representation $(\tilde{G}, \pi_r, \vec{d}, \gamma)$. Clearly, this defines maps $\gamma(g) : X \rightarrow X$. Moreover, by not only defining the projective action but also the linearized action on the homogeneous coordinates, we implicitly define isomorphisms $\gamma(k)^*(\mathcal{L}) \rightarrow \mathcal{L}$ on any G -invariant holomorphic line bundle \mathcal{L} . Therefore, we have a well-defined action of $(\tilde{G}, \pi_r, \vec{d}, \gamma)$ on the bundle cohomology groups $H^i(X, \mathcal{L})$. By setting

$$\chi(\mathcal{L})(g) \stackrel{\text{def}}{=} \sum_i (-1)^i \text{Tr}_{H^i(X, \mathcal{L})} (\gamma(g)^*) \quad \forall g \in \tilde{G} \tag{40}$$

we can extend the holomorphic Euler characteristic to a one-dimensional representation of \tilde{G} . Clearly, evaluating at $1 \in \tilde{G}$ simplifies to the usual holomorphic Euler characteristic. Using the fixed point theorem, we conclude that if \mathcal{L} is G -invariant ($\Rightarrow \tilde{G}$ -equivariant) and $g \in \tilde{G}$ acts freely on X , then $\chi(\mathcal{L})(g) = 0$.

If \mathcal{L} is already G -equivariant and G acts freely, then we furthermore learn that X/G is a smooth manifold with holomorphic line bundle \mathcal{L}/γ . In this case, $\chi(\mathcal{L})(1) = |G| \chi(X/G, \mathcal{L}/\gamma)$ must be divisible by the order $|G|$ of the group.

Definition 5 (Free type of a character). *Consider a G -action on a CICY X defined by an extension*

$$1 \longrightarrow K \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1 \tag{41}$$

and a linear CICY group action $(C, \tilde{G}, \pi_r, \gamma, \pi_c, \rho)$. We say that the character-valued index $\chi(\mathcal{L}) : \tilde{G} \rightarrow \mathbb{C}^\times$ of a G -invariant holomorphic line bundle is of free type if

¹⁰The case of non-isolated fixed points is essentially similar [25]. We only restrict to isolated fixed points for ease of presentation.

- $\chi(\mathcal{L})(g) = 0 \quad \forall g \in \tilde{G} - K$, and
- if \mathcal{L} is G -equivariant, then $\frac{1}{|G|}\chi(\mathcal{L})(1) \in \mathbb{Z}$.

Clearly, if the G -action is free then the index is always of free type.

4.3 (Anti-)Symmetrizations and Induction

As we discussed in Subsection 3.2, the induction extends the group action on the homogeneous coordinates of a single projective space to the permutation orbit. Although this unambiguously defines the group action on the combined homogeneous coordinates, it is not quite what we need to compute the cohomology of line bundles on the product of projective spaces.

Example 6 (Induction vs. Cohomology). *Consider the permutation action as in Example 4. Now, let us start with the representation*

$$\gamma'_1 : (Q_8)_1 \rightarrow GL(3, \mathbb{C}), \quad \gamma'_1(j^\ell) = \text{diag}(1, i^\ell, (-1)^\ell). \quad (42)$$

The induced Q_8 -representation $\gamma = \text{Ind}_{(Q_8)_1}^{Q_8}(\gamma'_1)$ is

$$\gamma(i) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma(j) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix}. \quad (43)$$

Now, consider $(Q_8, \pi, (3, 3), \gamma)$ as a π -representation acting on $\mathbb{P}_{[x_0:x_1:x_2]}^2 \times \mathbb{P}_{[y_0:y_1:y_2]}^2 = \mathbb{P}_1 \times \mathbb{P}_2$. Using the standard identification between sections of $\mathcal{O}(1)$ and homogeneous coordinates, we identify the representations

$$\begin{aligned} \gamma'_1 &= H^0(\mathbb{P}_1, \mathcal{O}(1)), \\ \gamma &= \text{Ind}_{(Q_8)_1}^{Q_8}(\gamma'_1) = H^0(\mathbb{P}_1, \mathcal{O}(1)) \oplus H^0(\mathbb{P}_2, \mathcal{O}(1)) \\ &= H^0(\mathbb{P}_1 \times \mathbb{P}_2, \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)) = \text{span}_{\mathbb{C}} \{x_0, x_1, x_2, y_0, y_1, y_2\}. \end{aligned} \quad (44)$$

But we would like to know the cohomology of an invariant line bundle, for example

$$H^0(\mathbb{P}_1 \times \mathbb{P}_2, \mathcal{O}(1, 1)) = H^0(\mathbb{P}_1, \mathcal{O}(1)) \otimes H^0(\mathbb{P}_2, \mathcal{O}(1)) = \text{span}_{\mathbb{C}} \{x_i y_j \mid 0 \leq i, j \leq 2\}. \quad (45)$$

The problem is that the induction procedure Ind_H^G adds (as direct sum \oplus) the H -representations in order to get the G -representations, but for the purposes of computing the cohomology groups of projective spaces we should multiply them (form the symmetrized tensor product \odot). Hence, we are led to define a new operation

Definition 6 (SymInd and AltInd). Let $H \subset G$ and $\gamma'_1 : H \rightarrow GL(n)$ a representation of H . We know that the induced representation is of the form eq. (5)

$$\text{Ind}_H^G(\gamma'_1)(g) = \mathbf{P}(\pi(g), \vec{d}) (\gamma_1(g) \oplus \cdots \oplus \gamma_n(g)). \quad (46)$$

Let us define the associated operations

$$\begin{aligned} \text{SymInd}_H^G(\gamma'_1)(g) &= \gamma_1(g) \odot \cdots \odot \gamma_n(g), \\ \text{AltInd}_H^G(\gamma'_1)(g) &= (-1)^{|\pi(g)|} \gamma_1(g) \odot \cdots \odot \gamma_n(g), \end{aligned} \quad (47)$$

where $|\pi(g)|$ is the signature of the permutation $\pi(g)$. If the representation is \mathbb{Z}_2 -graded, then we furthermore define

$$\text{GrInd}_H^G(\gamma_1) = \begin{cases} \text{SymInd}_H^G(\gamma'_1) & \text{if } \gamma'_1 \text{ is even,} \\ \text{AltInd}_H^G(\gamma'_1) & \text{if } \gamma'_1 \text{ is odd.} \end{cases} \quad (48)$$

Clearly, this definition of SymInd/AltInd does not refer to specific coordinates and therefore extends to operations on group characters. In Appendix B, we will present explicit formulas that are necessary to efficiently compute the character-valued indices that appear in the CICY group classification algorithm.

Let us further note that the definition of SymInd is exactly what is needed to compute the cohomology groups of line bundles on products of projective spaces:

Example 7 (Continuation of Example 6).

Conj.classes(Q_8)	1	i	j	-1	ij
Ind(γ'_1)	6	0	0	2	0
SymInd(γ'_1)	9	1	1	1	1
AltInd(γ'_1)	9	-1	1	1	-1

(49)

The cohomology of the line bundle $\mathcal{O}(1, 1)$ is¹¹

$$h^0(\mathbb{P}_1 \times \mathbb{P}_2, \mathcal{O}(1, 1)) = \text{SymInd}_{(Q_8)_1}^{Q_8}(\gamma'_1) \quad (50)$$

as a Q_8 -character. Note that γ'_1 and the permutation action are precisely the defining data for the π -representation, see Theorem 2.

¹¹In the context of G -manifolds and G -equivariant vector bundles, we write $H^\bullet(\cdots)$ for the G -representation on the cohomology and $h^\bullet(\cdots)$ for the corresponding G -character.

If we have a general π -representation $(G, \pi_r, \vec{d}, \gamma)$ acting on $\prod_{k=1}^n \mathbb{P}_k = \prod \mathbb{P}^{d_k}$, then we have to split the product into π_r -orbits and apply the SymInd construction to each orbit. Let us define the index set and its π_r -orbits to be

$$S_n \stackrel{\text{def}}{=} \{1, \dots, n\} = \{1, \dots\} \cup \dots \cup \{\dots, n\} = \bigcup_{G\{i\} \in S_n/G} G\{i\}. \quad (51)$$

By abuse of notation, we denote by i also the embedding of the i -th factor \mathbb{P}_i in the product,

$$i : \mathbb{P}_i \longrightarrow \prod_{k=1}^n \mathbb{P}_k. \quad (52)$$

Finally, note that exchanging two odd-degree cohomology groups incurs an extra minus sign. Therefore, the character-valued cohomology of a G -equivariant line bundle \mathcal{L} is

$$h^\bullet\left(\prod_{k=1}^n \mathbb{P}_k, \mathcal{L}\right) = \prod_{G\{i\} \in S_n/G} \text{GrInd}_{G_i}^G \left(h^\bullet(\mathbb{P}_i, i^* \mathcal{L}) \right), \quad (53)$$

where GrInd is symmetric or anti-symmetric depending on the mod-2 cohomological degree of $h^\bullet(\mathbb{P}_i, i^* \mathcal{L})$. The corresponding character-valued Euler characteristic is

$$\chi\left(\prod_{k=1}^n \mathbb{P}_k, \mathcal{L}\right) = \sum_{\vec{q}} (-1)^{\sum_{G\{i\} \in S_n/G} [G:G_i] q_i} \prod_{G\{i\} \in S_n/G} \text{GrInd}_{G_i}^G \left(h^{q_i}(\mathbb{P}_i, i^* \mathcal{L}) \right), \quad (54)$$

where the summation over all possible degree vectors $\vec{q} \in \mathbb{Z}^{|S_n/G|}$ has, of course, only finitely many non-zero summands.

4.4 The Koszul Spectral Sequence

Consider a complete intersection cut out by m transverse polynomials. Each polynomial equation $p_i = 0$ defines a divisor

$$D_j \stackrel{\text{def}}{=} \{p_j = 0\} \subset \prod_{i=1}^n \mathbb{P}_i. \quad (55)$$

An immediate consequence of a complete intersection $X \subset \prod \mathbb{P}_i$ is that we have a Koszul resolution¹²

$$0 \longrightarrow \mathcal{O}(-\sum D_j) \longrightarrow \dots \longrightarrow \bigoplus_{j < k} \mathcal{O}(-D_j - D_k) \longrightarrow \bigoplus \mathcal{O}(-D_j) \longrightarrow \underline{\mathcal{O}} \longrightarrow 0. \quad (56)$$

That is, the above sequence is exact everywhere except at the underlined entry. At that position, the cohomology is \mathcal{O}_X . In other words, the Koszul complex is equivalent

¹²By \mathcal{O} we will always denote the trivial line bundle on the ambient space $\prod \mathbb{P}_i$.

to \mathcal{O}_X in the derived category, and we can interchange them for the purposes of computing bundle cohomology. After tensoring with a line bundle \mathcal{L} , the associated hypercohomology spectral sequence reads

$$E_1^{-p,q} = H^q\left(\prod \mathbb{P}_i, \bigoplus_{1 \leq j_1 < \dots < j_p \leq m} \mathcal{O}(-D_{j_1} - \dots - D_{j_p}) \otimes \mathcal{L}\right) \Rightarrow H^{-p+q}(X, \mathcal{L}|_X) \quad (57)$$

Note that all non-vanishing entries are in the second quadrant. To evaluate all the higher differentials in the spectral sequence is, of course, a lot of work. However, any non-trivial differential removes the same subspace from the even and from the odd cohomology groups, leaving the Euler characteristic invariant. Therefore, we can compute the character-valued index already from the E_1 -tableau by pretending that all higher differentials vanish. One obtains

$$\chi(X, \mathcal{L}|_X) = \sum_{1 \leq j_1 < \dots < j_p \leq m} (-1)^p \chi\left(\mathcal{O}(-D_{j_1} - \dots - D_{j_p}) \otimes \mathcal{L}\right) = \sum_{p,q} (-1)^{p+q} E_1^{-p,q} \quad (58)$$

A good way of dealing with the indices $1 \leq j_1 < \dots < j_p \leq n$ in the resolution is to consider them as basis elements of the (formal) exterior algebra generated by the polynomials $p_{j_1} \wedge \dots \wedge p_{j_p}$.

Example 8 (Koszul resolution). *By abbreviating $\mathcal{O}(-D_{j_1} - \dots - D_{j_p}) = \mathcal{O}_{j_1 \wedge \dots \wedge j_p}$ we can write the Koszul complex for $r = 3$ transverse polynomials as*

$$0 \longrightarrow \mathcal{O}_{1 \wedge 2 \wedge 3} \xrightarrow{(p_3, p_2, p_1)} \mathcal{O}_{1 \wedge 2} \oplus \mathcal{O}_{1 \wedge 3} \oplus \mathcal{O}_{2 \wedge 3} \xrightarrow{\begin{pmatrix} p_2 & -p_1 & 0 \\ -p_3 & 0 & p_1 \\ 0 & p_3 & -p_2 \end{pmatrix}} \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_3 \xrightarrow{\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}} \underline{\mathcal{O}} \longrightarrow 0. \quad (59)$$

4.5 Equivariant Koszul

4.5.1 No Permutations

First, let us assume that there are no permutations, but only a linear G -action on each projective space and each polynomial. Then we can easily compute the cohomology of each line bundle $H^q(\prod \mathbb{P}_i, \mathcal{O}_{j_1 \wedge \dots \wedge j_p})$ as a G -representation, using the notation of Example 8. However, if the polynomial equations are not G -invariant, then the index must depend on their transformation as well! Following the maps through the Koszul resolution until we end up at the homological degree-0 piece, we see that $\mathcal{O}_{j_1 \wedge \dots \wedge j_p}$ ends up being multiplied by $p_{j_1}, p_{j_2}, \dots, p_{j_p}$. Therefore, its contribution to the character-valued index must be $p_{j_1} \cdots p_{j_p} \chi(\mathcal{O}_{j_1 \wedge \dots \wedge j_p})$, where we consider the polynomials as G -characters.

4.5.2 With Permutations

This gets more complicated when we consider the case where the G -action permutes the polynomials by a permutation action $\pi : G \rightarrow P$. Since the polynomials appear with different signs in the maps of the Koszul resolution, permuting them yields an extra minus sign corresponding to the signature of the permutation. Therefore, the contribution to the character-valued index is

$$\begin{aligned}\chi(\mathcal{L}|_X) &= \sum (-1)^p p_{j_1} \wedge \cdots \wedge p_{j_p} \chi\left(\mathcal{O}_{j_1 \wedge \cdots \wedge j_p} \otimes \mathcal{L}\right) \\ &= \sum_{\wedge \vec{j} \in \Lambda_m} (-1)^p p_{\wedge \vec{j}} \chi\left(\mathcal{O}_{\wedge \vec{j}} \otimes \mathcal{L}\right),\end{aligned}\tag{60}$$

where we used the notation

$$\Lambda_m \stackrel{\text{def}}{=} \left\{ j_1 \wedge \cdots \wedge j_p \mid 0 \leq p \leq m, 1 \leq j_1 < \cdots < j_p \leq m \right\}\tag{61}$$

for the standard basis of anti-symmetrized indices and

$$p_{\wedge \vec{j}} = p_{\wedge (j_1, \dots, j_p)} \stackrel{\text{def}}{=} p_{j_1} \wedge \cdots \wedge p_{j_p}\tag{62}$$

for the exterior powers of the polynomials thought of as group characters. However, the above equation for $\chi(\mathcal{L}|_X)$ is only useful if the multi-index $\wedge \vec{j} = j_1 \wedge \cdots \wedge j_p$ is invariant under the permutation action; Otherwise, the group action will exchange different summands and we still do not have a closed expression for the index.

To write a general equation, we have to decompose the multi-indices into orbits of the permutation action and choose representatives

$$\Lambda_r/G \stackrel{\text{def}}{=} \left\{ [\wedge \vec{j}(1)], [\wedge \vec{j}(2)], \dots \right\} = \left\{ \pm j_1 \wedge \cdots \wedge j_p \mid 0 \leq p \leq n \right\} / \langle \pm, G \rangle.\tag{63}$$

Each bundle $\mathcal{O}_{\wedge \vec{j}}$ is then fixed under

$$G_{\wedge \vec{j}} = \text{Stab}_{\wedge \vec{j}}(G) = \left\{ g \in G \mid \pi(g)(\wedge \vec{j}) = \pm \wedge \vec{j} \right\},\tag{64}$$

and, therefore,

$$\chi(\mathcal{O}_{\wedge \vec{j}} \otimes \mathcal{L}) : G_{\wedge \vec{j}} \longrightarrow \mathbb{C}^\times\tag{65}$$

is a character of the stabilizer.

The other $G_{\wedge \vec{j}}$ -character that enters the index formula is $p_{\wedge \vec{j}}$. However, each individual polynomial p_j is a character of its stabilizer G_j which, in general, neither contains nor is contained in $G_{\wedge \vec{j}}$. To proceed further, we have to decompose the G -invariant index sets into G -orbits of a single index,

$$\wedge \vec{j} = j_1 \wedge \cdots \wedge j_p = (j_1 \wedge \cdots) \wedge (\cdots) \wedge \cdots \wedge (\cdots \wedge j_p) = \bigwedge_{j \in \vec{j}/G} \left(\wedge G(j) \right)\tag{66}$$

Now, consider the orbit $G(j)$ generated by j . To compute the $p_{\wedge G(j)}$ as a character of $G_{\wedge \vec{j}}$ we only need knowledge of one of the polynomials (say, p_j) and the permutation action of the group. One obtains that

$$p_{\wedge \vec{j}} = \prod_{j \in \vec{j}/G} \text{AltInd}_{G_j \cap G_{\wedge \vec{j}}}^{G_{\wedge \vec{j}}} \left(\text{Res}_{G_j \cap G_{\wedge \vec{j}}}^{G_j}(p_j) \right) \quad (67)$$

as a character of $G_{\wedge \vec{j}}$.

Finally, summing over the Λ_m/G -orbits and keeping track of how the permutation acts on the summands is nothing but the induction from the stabilizer $G_{\wedge \vec{j}}$ to the full group G . Therefore, we can write a closed expression for the character-valued index as

$$\chi(\mathcal{L}|_X) = \sum_{\wedge \vec{j} \in \Lambda_m/G} (-1)^{|\wedge \vec{j}|} \text{Ind}_{G_{\wedge \vec{j}}}^G \left(p_{\wedge \vec{j}} \chi(\mathcal{O}_{\wedge \vec{j}} \otimes \mathcal{L}) \right). \quad (68)$$

4.5.3 General Case

In the most general case, the group G acts on the polynomials not only via permutations, but also by forming non-trivial linear combinations if the degrees allow for it. As in the CICY case, we group polynomials of the same degree into vectors \vec{p}_j . Moreover, we assign multiplicities $1 \leq |j| \leq \dim(\vec{p}_j)$ to each index, constant on permutation orbits, in order to keep track of $|j|$ -fold exterior powers $\wedge_{|j|} \vec{p}_j \stackrel{\text{def}}{=} \vec{p}_j \wedge \cdots \wedge \vec{p}_j$ contributing to the character-valued index. Here, the exterior powers are graded by $|j| \bmod 2$.

Hence, the index set of interest is

$$\Lambda_m \stackrel{\text{def}}{=} \left\{ i_1 \wedge \cdots \wedge i_k \mid 0 \leq \sum |j_\ell| \leq m, 1 \leq i_1 < \cdots < i_k \leq m \right\} \quad (69)$$

The permutation action on the multi-indices-with-multiplicities can then again be grouped into orbits

$$\Lambda_m/G = \bigcup_{\wedge \vec{j} \in \Lambda_m/G} \left\{ \wedge \vec{j} \right\} = \bigcup_{\wedge \vec{j} \in \Lambda_m/G} \left\{ \bigwedge_{j \in \vec{j}/G} (\wedge G(j)) \right\} \quad (70)$$

Putting everything together, the closed form expression for the character-valued index is

$$\begin{aligned} \chi(\mathcal{L}|_X) &= \sum (-1)^{\sum |j_\ell|} \left(\bigwedge_{|j_1|} p_{j_1} \right) \wedge \cdots \wedge \left(\bigwedge_{|j_k|} p_{j_k} \right) \chi(\mathcal{O}_{j_1 \wedge \cdots \wedge j_k} \otimes \mathcal{L}) \\ &= \sum_{\wedge \vec{j} \in \Lambda_m/G} (-1)^{|\wedge \vec{j}|} \text{Ind}_{G_{\wedge \vec{j}}}^G \left[\chi(\mathcal{O}_{\wedge \vec{j}} \otimes \mathcal{L}) \prod_{j \in \vec{j}/G} \text{GrInd}_{G_j \cap G_{\wedge \vec{j}}}^{G_{\wedge \vec{j}}} \left(\bigwedge_{|j|} \text{Res}_{G_j \cap G_{\wedge \vec{j}}}^{G_j}(\vec{p}_j) \right) \right], \quad (71) \end{aligned}$$

where the grading in GrInd is $|j| \bmod 2$.

4.6 Character-Valued Index

Let us now apply the Koszul resolution to the CICYs. Using (71), the index of a line bundle on the Calabi-Yau threefold is determined by the character-valued cohomology groups on the ambient space and group theoretic information about the column CICY group action. For each term in the resolution, we then apply eq. (54) in order to compute the cohomology groups on the ambient space from the row CICY group action. We use the following notation:

- $\tilde{G}_i = \text{Stab}_{\{i\}}(\pi_r) = \{g \in \tilde{G} \mid \pi_r(g)(j) = j\}$ is the stabilizer of the i -th row under the action of the row permutations.
- $\tilde{G}_j = \text{Stab}_{\{j\}}(\pi_c) = \{g \in \tilde{G} \mid \pi_c(g)(j) = j\}$ is the stabilizer of the j -th column under the action of the column permutations.
- $\tilde{G}_{\wedge \vec{j}} = \text{Stab}_{\wedge \vec{j}}(\pi_c) = \{g \in \tilde{G} \mid \pi_c(g)(\wedge \vec{j}) = \pm \wedge \vec{j}\}$ is the stabilizer of $\mathcal{O}_{\wedge \vec{j}}$ in the Koszul resolution.
- The homogeneous coordinates of the i -th projective space \mathbb{P}_i form a (linear) representation of \tilde{G}_i . Let us denote the restriction to the subgroup $\tilde{G}_i \cap \tilde{G}_{\wedge \vec{j}}$ by $\text{Res}_{\tilde{G}_i \cap \tilde{G}_{\wedge \vec{j}}}^{\tilde{G}_i}(\mathbb{P}_i)$.

The character-valued index of $\mathcal{L}|_X$ on the Calabi-Yau threefold X is then

$$\begin{aligned} \chi(\mathcal{L}|_X) = & \sum_{\wedge \vec{j} \in \Lambda_m/G} \sum_{\vec{q} \in \mathbb{Z}^{|S_n/\tilde{G}_{\wedge \vec{j}}|}} (-1)^{|\wedge \vec{j}| + \sum_{\tilde{G}_{\wedge \vec{j}}\{i\} \in S_n/\tilde{G}_{\wedge \vec{j}}} [\tilde{G}:\tilde{G}_i] q_i} \\ & \text{Ind}_{\tilde{G}_{\wedge \vec{j}}}^{\tilde{G}} \left\{ \left[\prod_{\tilde{G}_{\wedge \vec{j}}\{i\} \in S_n/\tilde{G}_{\wedge \vec{j}}} \text{GrInd}_{\tilde{G}_i \cap \tilde{G}_{\wedge \vec{j}}}^{\tilde{G}_{\wedge \vec{j}}} h^{q_i} \left(\text{Res}_{\tilde{G}_i \cap \tilde{G}_{\wedge \vec{j}}}^{\tilde{G}_i}(\mathbb{P}_i), i^*(\mathcal{O}_{\wedge \vec{j}} \otimes \mathcal{L}) \right) \right] \right. \\ & \left. \times \left[\prod_{j \in \vec{j}/\tilde{G}} \text{GrInd}_{\tilde{G}_j \cap \tilde{G}_{\wedge \vec{j}}}^{\tilde{G}_{\wedge \vec{j}}} \left(\bigwedge_{|j|} \text{Res}_{\tilde{G}_j \cap \tilde{G}_{\wedge \vec{j}}}^{\tilde{G}_j}(\vec{p}_j) \right) \right] \right\}. \quad (72) \end{aligned}$$

The importance of the above formula is that it expresses the index using precisely the defining data of a CICY group action and only group characters (instead of explicit representations).

5 Calabi-Yau Groups

I ran the classification algorithm and found group actions allowed by indices on 195 CICY configurations. Usually, there is more than one action of the same group for any given CICY configuration. It is difficult to distinguish truly distinct actions

Group	$ G $	ID	N_f	N_{f+s}	CICY #
\mathbb{Z}_2	2	1	166	166	...
\mathbb{Z}_3	3	1	31	31	6, 14, 18, 26, 242, 536, 1215, 1306, 2104, 3388, 3406, 3413, 3620, 4415, 5967, 5982, 6021, 6024, 6502, 7206, 7240, 7246, 7247, 7300, 7664, 7669, 7800, 7808, 7810, 7878, 7884
\mathbb{Z}_4	4	1	23	23	19, 20, 21, 30, 95, 480, 2564, 2568, 2572, 2639, 5301, 5452, 6826, 6836, 6927, 6947, 7246, 7300, 7484, 7735, 7745, 7861, 7862
$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	2	40	40	15, 19, 20, 21, 22, 480, 2357, 2534, 2564, 2566, 2568, 2640, 5256, 5301, 5302, 5421, 5452, 6715, 6784, 6788, 6826, 6828, 6829, 6836, 6927, 6947, 7435, 7447, 7462, 7484, 7487, 7491, 7522, 7714, 7735, 7745, 7819, 7823, 7861, 7862
\mathbb{Z}_5	5	1	5	5	4335, 6655, 7447, 7761, 7890
\mathbb{Z}_6	6	2	4	4	6, 7206, 7246, 7300
\mathbb{Z}_8	8	1	7	7	19, 21, 2564, 6836, 6947, 7861, 7862
$\mathbb{Z}_4 \times \mathbb{Z}_2$	8	2	11	11	19, 21, 2564, 2568, 6836, 6927, 6947, 7735, 7745, 7861, 7862
Q_8	8	4	7	7	19, 21, 2564, 6836, 6947, 7861, 7862
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	8	5	1	1	7861
$\mathbb{Z}_3 \times \mathbb{Z}_3$	9	2	6	6	14, 7240, 7669, 7808, 7878, 7884
\mathbb{Z}_{10}	10	2	3	3	4335, 7447, 7761
$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	12	1	2	2	7246, 7300
\mathbb{Z}_{12}	12	2	2	2	7246, 7300
$\mathbb{Z}_4 \times \mathbb{Z}_4$	16	2	5	3	21, 6836, 6947 , 7861, 7862
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	16	4	5	5	21, 6836, 6947, 7861, 7862
$\mathbb{Z}_8 \times \mathbb{Z}_2$	16	5	5	5	21, 6836, 6947, 7861, 7862
$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$	16	6	2	2	21, 7862
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	16	10	1	1	7861
$\mathbb{Z}_2 \times Q_8$	16	12	5	3	21, 6836, 6947 , 7861, 7862
$\mathbb{Z}_{10} \times \mathbb{Z}_2$	20	5	1	1	7447
$\mathbb{Z}_5 \times \mathbb{Z}_5$	25	2	2	1	7890, 7761
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$	32	2	1	1	7861
$\mathbb{Z}_8 \times \mathbb{Z}_4$	32	3	1	1	7861
$\mathbb{Z}_8 \rtimes \mathbb{Z}_4$	32	4	1	1	7861
$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	32	5	1	1	7861
$\mathbb{Z}_8 \rtimes \mathbb{Z}_4$	32	13	1	1	7861
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	32	21	1	1	7861
$\mathbb{Z}_2 \times (\mathbb{Z}_4 \rtimes \mathbb{Z}_4)$	32	23	1	1	7861
$\mathbb{Z}_4 \rtimes Q_8$	32	35	1	1	7861
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times Q_8$	32	47	1	1	7861
$\mathbb{Z}_{10} \times \mathbb{Z}_5$	50	5	1	0	7761
$\mathbb{Z}_8 \times \mathbb{Z}_8$	64	2	1	0	7861
$\mathbb{Z}_8 \rtimes \mathbb{Z}_8$	64	3	1	0	7861
$(\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	64	68	1	0	7861
$(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_4$	64	72	1	0	7861
$\mathbb{Z}_8 \rtimes Q_8$	64	179	1	0	7861

Table 1: The free group actions on CICYs. The ~~stricken-out~~ numbers are non-smooth CICY. The “ID” field is the GAP `IdSmallGroup` of the group. N_f is the number of CICY configurations admitting a free action, and N_{f+s} is the number of smooth CICY admitting a free action.

from those that are related by an automorphism of the manifold. For example, the two free $\mathbb{Z}_3 \times \mathbb{Z}_3$ actions on the CICY #19 investigated in [26] and [27, 28, 29, 30] yield quotients with different complex structures, but are neither distinguished by topological invariants like Betti numbers nor by Gromov-Witten invariants, at least not by those that have been computed so far. With this caveat in mind, the CICY configurations admitting free group actions are listed in Table 1. Note that, in a few cases indicated by a stricken-out CICY number in the table, all linear combinations of invariant polynomials fail to be transverse. These define free group actions on singular CICY threefolds. Moreover, note that most 2-groups are realized on the CICY #7861, the complete intersection of 4 quadrics in \mathbb{P}^7 . These were classified previously¹³ in [32, 33, 31].

An obvious question is whether we can guess any restrictions on allowed groups by looking at the list of examples. General properties of these groups are reviewed in Table 2. Recall that, for finite groups,

$$\begin{array}{ccccccc}
 & & \text{polycyclic} & & & & \\
 & \Leftarrow & & \Leftarrow & & & \\
 \text{solvable} & & & & \text{supersolvable} & \Leftarrow & \text{nilpotent} \Leftarrow \text{Abelian} \Leftarrow \text{cyclic} \\
 & \Leftarrow & & \Leftarrow & & & \\
 & & \text{monomial} & & & &
 \end{array} \quad (73)$$

Note that the dicyclic group quotient investigated in [13] is the only known non-nilpotent Calabi-Yau group. In Table 3, we describe the groups acting freely on smooth CICYs by giving a list of subgroups that must not occur. As there is a limit of $|G| \leq 64$ just because of topological indices, the forbidden subgroups of large order are presumably only an artifact of the finite sample of Calabi-Yau threefolds under consideration.

However, it is a curious observation that the dihedral group D_6 with 6 elements (a.k.a. the symmetric group on three letters S_3) and the dihedral group D_8 are not allowed¹⁴. Note that an ample divisor D (that is, a divisor in the dual of the Kähler cone) in a Calabi-Yau threefold X is a surface of general type. By the Lefschetz hyperplane theorem $\pi_1(D) = \pi_1(X)$. Focusing on the complete intersection of four quadrics in \mathbb{P}^7 (CICY #7861), the minimal ample divisor is a section of $\mathcal{O}(1)$, that is, a complete intersection of four quadrics in \mathbb{P}^6 . Beauville [32] constructed a free Q_8 action on this Calabi-Yau threefold and noted that the $\mathcal{O}(1)$ divisor on the quotient is a so-called Campedelli surface¹⁵ with $\pi_1(D) = Q_8$. It is known [34, 35] that Campedelli surfaces cannot have fundamental groups D_{2n} for $n \geq 3$.

Of course Campedelli surfaces are the very exception amongst ample divisors on CICYs. Moreover, any finite group can appear as the fundamental groups of a surface

¹³Note that the order-32 group $\mathbb{Z}_2 \times (\mathbb{Z}_4 \rtimes \mathbb{Z}_4) = \text{SmallGroup}(32, 23)$ is omitted in [31]

¹⁴In fact, only the exceptional dihedral groups $\mathbb{Z}_2 = D_2$ and $\mathbb{Z}_2^2 = D_4$ are allowed.

¹⁵A Campedelli surface S is a surface of general type with $K^2 = 2$ and $h^0(S, K) = 0$. Such a surface has $h^{1,1}(S) = 8$ but its fundamental group is not uniquely determined. The size of the fundamental group is limited to $\pi_1(S) \leq 9$.

$ G $	ID	N_f	N_{f+s}	Abelian invariants	Exponent	IsAbelian	IsCyclic	IsElementaryAbelian	IsNilpotentGroup	IsPerfectGroup	IsPolycyclicGroup	IsSupersolvableGroup	IsMonomialGroup	IsSimpleGroup	IsPGroup
2	1	166	166	[2]	2	Y	Y	Y	Y	N	Y	Y	Y	Y	Y
3	1	31	31	[3]	3	Y	Y	Y	Y	N	Y	Y	Y	Y	Y
4	1	23	23	[4]	4	Y	Y	N	Y	N	Y	Y	Y	N	Y
4	2	40	40	[2, 2]	2	Y	N	Y	Y	N	Y	Y	Y	N	Y
5	1	5	5	[5]	5	Y	Y	Y	Y	N	Y	Y	Y	Y	Y
6	2	4	4	[2, 3]	6	Y	Y	N	Y	N	Y	Y	Y	N	N
8	1	7	7	[8]	8	Y	Y	N	Y	N	Y	Y	Y	N	Y
8	2	11	11	[2, 4]	4	Y	N	N	Y	N	Y	Y	Y	N	Y
8	4	7	7	[2, 2]	4	N	N	N	Y	N	Y	Y	Y	N	Y
8	5	1	1	[2, 2, 2]	2	Y	N	Y	Y	N	Y	Y	Y	N	Y
9	2	6	6	[3, 3]	3	Y	N	Y	Y	N	Y	Y	Y	N	Y
10	2	3	3	[2, 5]	10	Y	Y	N	Y	N	Y	Y	Y	N	N
12	1	2	2	[4]	12	N	N	N	N	N	Y	Y	Y	N	N
12	2	2	2	[3, 4]	12	Y	Y	N	Y	N	Y	Y	Y	N	N
16	2	5	3	[4, 4]	4	Y	N	N	Y	N	Y	Y	Y	N	Y
16	4	5	5	[2, 4]	4	N	N	N	Y	N	Y	Y	Y	N	Y
16	5	5	5	[2, 8]	8	Y	N	N	Y	N	Y	Y	Y	N	Y
16	6	2	2	[2, 4]	8	N	N	N	Y	N	Y	Y	Y	N	Y
16	10	1	1	[2, 2, 4]	4	Y	N	N	Y	N	Y	Y	Y	N	Y
16	12	5	3	[2, 2, 2]	4	N	N	N	Y	N	Y	Y	Y	N	Y
20	5	1	1	[2, 2, 5]	10	Y	N	N	Y	N	Y	Y	Y	N	N
25	2	2	1	[5, 5]	5	Y	N	Y	Y	N	Y	Y	Y	N	Y
32	2	1	1	[4, 4]	4	N	N	N	Y	N	Y	Y	Y	N	Y
32	3	1	1	[4, 8]	8	Y	N	N	Y	N	Y	Y	Y	N	Y
32	4	1	1	[4, 4]	8	N	N	N	Y	N	Y	Y	Y	N	Y
32	5	1	1	[2, 8]	8	N	N	N	Y	N	Y	Y	Y	N	Y
32	13	1	1	[2, 4]	8	N	N	N	Y	N	Y	Y	Y	N	Y
32	21	1	1	[2, 4, 4]	4	Y	N	N	Y	N	Y	Y	Y	N	Y
32	23	1	1	[2, 2, 4]	4	N	N	N	Y	N	Y	Y	Y	N	Y
32	35	1	1	[2, 2, 2]	4	N	N	N	Y	N	Y	Y	Y	N	Y
32	47	1	1	[2, 2, 2, 2]	4	N	N	N	Y	N	Y	Y	Y	N	Y
50	5	1	0	[2, 5, 5]	10	Y	N	N	Y	N	Y	Y	Y	N	N
64	2	1	0	[8, 8]	8	Y	N	N	Y	N	Y	Y	Y	N	Y
64	3	1	0	[4, 8]	8	N	N	N	Y	N	Y	Y	Y	N	Y
64	68	1	0	[2, 2, 4]	4	N	N	N	Y	N	Y	Y	Y	N	Y
64	72	1	0	[2, 2, 4]	4	N	N	N	Y	N	Y	Y	Y	N	Y
64	179	1	0	[2, 2, 2]	8	N	N	N	Y	N	Y	Y	Y	N	Y

Table 2: Properties of the CICY groups.

Group	$ G $	ID	Abelian invariants	Exponent	IsAbelian	IsCyclic	IsElementaryAbelian	IsNilpotentGroup	IsPerfectGroup	IsPolycyclicGroup	IsSupersolvableGroup	IsMonomialGroup	IsSimpleGroup	IsPGroup
D_6	6	1	[2]	6	N	N	N	N	N	Y	Y	Y	N	N
D_8	8	3	[2, 2]	4	N	N	N	Y	N	Y	Y	Y	N	Y
\mathbb{Z}_9	9	1	[9]	9	Y	Y	N	Y	N	Y	Y	Y	N	Y
D_{10}	10	1	[2]	10	N	N	N	N	N	Y	Y	Y	N	N
A_4	12	3	[3]	6	N	N	N	N	N	Y	N	Y	N	N
$\mathbb{Z}_6 \rtimes \mathbb{Z}_2$	12	5	[2, 2, 3]	6	Y	N	N	Y	N	Y	Y	Y	N	N
\mathbb{Z}_{16}	16	1	[16]	16	Y	Y	N	Y	N	Y	Y	Y	N	Y
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	16	3	[2, 4]	4	N	N	N	Y	N	Y	Y	Y	N	Y
Q_{16}	16	9	[2, 2]	8	N	N	N	Y	N	Y	Y	Y	N	Y
\mathbb{Z}_2^4	16	14	[2, 2, 2, 2]	2	Y	N	Y	Y	N	Y	Y	Y	N	Y
$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	20	1	[4]	20	N	N	N	N	N	Y	Y	Y	N	N
\mathbb{Z}_{20}	20	2	[4, 5]	20	Y	Y	N	Y	N	Y	Y	Y	N	N
\mathbb{Z}_{25}	25	1	[25]	25	Y	Y	N	Y	N	Y	Y	Y	N	Y
$(\mathbb{Z}_2^2) \cdot (\mathbb{Z}_4 \times \mathbb{Z}_2)$	32	8	[2, 4]	8	N	N	N	Y	N	Y	Y	Y	N	Y
$Q_8 \rtimes \mathbb{Z}_4$	32	10	[2, 4]	8	N	N	N	Y	N	Y	Y	Y	N	Y
$\mathbb{Z}_4 \rtimes \mathbb{Z}_8$	32	12	[2, 8]	8	N	N	N	Y	N	Y	Y	Y	N	Y
$\mathbb{Z}_8 \rtimes \mathbb{Z}_4$	32	14	[2, 4]	8	N	N	N	Y	N	Y	Y	Y	N	Y
$\mathbb{Z}_4 \cdot D_8$	32	15	[2, 4]	8	N	N	N	Y	N	Y	Y	Y	N	Y
$\mathbb{Z}_4 \times Q_8$	32	26	[2, 2, 4]	4	N	N	N	Y	N	Y	Y	Y	N	Y
$(\mathbb{Z}_2^2) \cdot (\mathbb{Z}_2^3)$	32	32	[2, 2, 2]	4	N	N	N	Y	N	Y	Y	Y	N	Y
$\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	32	36	[2, 2, 8]	8	Y	N	N	Y	N	Y	Y	Y	N	Y
$\mathbb{Z}_2 \times (\mathbb{Z}_8 \rtimes \mathbb{Z}_2)$	32	37	[2, 2, 4]	8	N	N	N	Y	N	Y	Y	Y	N	Y
\mathbb{Z}_{50}	50	2	[2, 25]	50	Y	Y	N	Y	N	Y	Y	Y	N	N

Table 3: Groups that are not in the list of freely acting CICY groups. That is, the freely acting groups on smooth CICYs can be characterized as the groups of order 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 20, 25, 32, 50 that do not contain one of the groups above as a subgroup.

of general type S if one does not pose any restriction on the Chern numbers $\int_S c_1^2$ and $\int_S c_2$. Nevertheless, according to the classification result there are no free D_{2n} -actions, $n \geq 3$, on any CICY.

Acknowledgments

I would like to thank Philip Candelas, Rhys Davies, and Tony Pantev for useful discussions. I also would like to thank Frank Lübeck, Laurent Bartholdi, Willem de Graaf, and especially Alexander Hulpke for their GAP support, and Hans Schönemann for Singular support.

A Group Cohomology

The standard approach to a projective representation $r : G \rightarrow PGL(n)$ is by choosing a lift $\tilde{r}(g) \in GL(n)$ for each $g \in G$ and then noting that there is a function

$$c : G \times G \rightarrow \mathbb{C}^\times, \quad \tilde{r}(g)\tilde{r}(h) = c(g, h) \tilde{r}(gh) \quad \forall g, h \in G, \quad (74)$$

called the factor set. Associativity implies that c is a \mathbb{C}^\times -valued cocycle, and multiplying the matrices $\tilde{r}(g)$ by non-zero complex constants amounts to changing c by a coboundary. Therefore, the projective representation uniquely determines a group cohomology¹⁶ class $[c] \in H^2(G, \mathbb{C}^\times)$.

A short exact sequence eq. (16) defines a long exact sequence in cohomology,

$$\dots \longrightarrow H^1(\tilde{G}, \mathbb{C}^\times) \xrightarrow{R} H^1(K, \mathbb{C}^\times) \xrightarrow{\Delta} H^2(G, \mathbb{C}^\times) \xrightarrow{S} H^2(\tilde{G}, \mathbb{C}^\times) \longrightarrow \dots \quad (75)$$

The maps R, S are simply restriction (pull-back) via the maps in the short exact sequence. Furthermore, note that $H^1(-, \mathbb{C}^\times) = \text{Hom}(-, \mathbb{C}^\times)$ are precisely the one-dimensional representations.

Now, a sufficient extension is one where $S = 0$, that is, every factor set of a projective G -representation pulls back to the trivial factor set on \tilde{G} . This is equivalent to \tilde{G} linearizing every projective G -representation. In this case,

$$H^2(G, \mathbb{C}^\times) = \ker S = \text{img } \Delta = \text{coker } R. \quad (76)$$

But the cokernel of R is precisely the set of twist classes in Definition 4.

To summarize, the coboundary map Δ identifies twist classes with the factor sets of projective representations as long as we have chosen a sufficient extension $\tilde{G} \rightarrow G$. If one chooses \tilde{G} too small then the projective representations with $S \neq 0$ cannot be written as twisted representations.

¹⁶Group cohomology with coefficients M is the usual (topological) cohomology of its classifying space, $H^2(G, M) \stackrel{\text{def}}{=} H^2(BG, M)$.

B Character Formulas for SymInd/AltInd

If one were to naively follow Definition 6 in evaluating the character-valued (anti-) symmetric induction

$$\text{SymInd}_H^G, \text{AltInd}_H^G : \text{Hom}(H, \mathbb{C}^\times) \longrightarrow \text{Hom}(G, \mathbb{C}^\times) \quad (77)$$

then one would have to first construct a representation for the given H -character, compute the induced representation blocks, (anti-) symmetrize, and then compute the trace to obtain the resulting G -character. Obviously this is very inefficient, and we need an equation that works on the level of group characters only.

The key to deriving such an equation is that, given a H -representation γ'_1 , the (anti-) symmetrized induction $\text{SymInd}_H^G(\gamma'_1)$ is a sub-representation of $\text{Sym}^{[G:H]} \text{Ind}_H^G(\gamma'_1)$. Therefore, by subtracting the superfluous representations, there must be a formula of the form

$$\text{SymInd}_H^G(\chi) = \text{Sym}^{[G:H]} \left(\text{Ind}_H^G(\chi) \right) - \left(\dots \right) \quad (78)$$

only depending on the index $[G : H]$ of the subgroup H . Using the abbreviation $\text{Ind} = \text{Ind}_H^G$ and

$$\text{Sym}^{i_1, i_2, \dots, i_k}(\chi) = \prod_j \text{Sym}^{i_j}(\chi), \quad \text{Alt}^{i_1, i_2, \dots, i_k}(\chi) = \prod_j \text{Alt}^{i_j}(\chi), \quad (79)$$

we find

$[G : H] = 1 :$

$$\begin{aligned} \text{SymInd}(\chi) &= \text{Ind}(\chi) = \chi, \\ \text{AltInd}(\chi) &= \text{Ind}(\chi) = \chi, \end{aligned} \quad (80a)$$

$[G : H] = 2 :$

$$\begin{aligned} \text{SymInd}(\chi) &= \text{Sym}^2 \text{Ind}(\chi) - \text{Ind} \text{Sym}^2(\chi), \\ \text{AltInd}(\chi) &= \text{Alt}^2 \text{Ind}(\chi) - \text{Ind} \text{Alt}^2(\chi), \end{aligned} \quad (80b)$$

$[G : H] = 3 :$

$$\begin{aligned} \text{SymInd}(\chi) &= \text{Sym}^3 \text{Ind}(\chi) - \text{Ind} \text{Sym}^3(\chi) \\ &\quad - \text{Ind} \text{Sym}^2(\chi) \text{Ind}(\chi) + \text{Ind} \text{Sym}^{2,1}(\chi), \\ \text{AltInd}(\chi) &= \text{Alt}^3 \text{Ind}(\chi) - \text{Ind} \text{Alt}^3(\chi) \\ &\quad - \text{Ind} \text{Alt}^2(\chi) \text{Ind}(\chi) + \text{Ind} \text{Alt}^{2,1}(\chi), \end{aligned} \quad (80c)$$

$[G : H] = 4 :$

$$\begin{aligned}
\text{SymInd}(\chi) &= \text{Sym}^4 \text{Ind}(\chi) - \text{Sym}^2 \text{Ind} \text{Sym}^2(\chi) - \text{Sym}^2 \text{Ind}(\chi) \text{Ind} \text{Sym}^2(\chi) \\
&\quad + \text{Ind} \text{Sym}^{2,2}(\chi) - \text{Ind} \text{Sym}^{2,1,1}(\chi) - \text{Ind} \text{Sym}^3(\chi) \text{Ind}(\chi) \\
&\quad + \text{Ind} \text{Sym}^{2,1}(\chi) \text{Ind}(\chi) + \text{Ind} \text{Sym}^2(\chi) \text{Ind} \text{Sym}^2(\chi), \quad (80d) \\
\text{AltInd}(\chi) &= \text{Alt}^4 \text{Ind}(\chi) + \text{Alt}^2 \text{Ind} \text{Alt}^2(\chi) - \text{Alt}^2 \text{Ind}(\chi) \text{Ind} \text{Alt}^2(\chi) \\
&\quad + \text{Ind} \text{Alt}^{2,2}(\chi) - \text{Ind} \text{Alt}^{2,1,1}(\chi) - \text{Ind} \text{Alt}^3(\chi) \text{Ind}(\chi) \\
&\quad + \text{Ind} \text{Alt}^{2,1}(\chi) \text{Ind}(\chi),
\end{aligned}$$

$[G : H] = 5 :$

$$\begin{aligned}
\text{SymInd}(\chi) &= \text{Sym}^5 \text{Ind}(\chi) - \text{Ind} \text{Sym}^5(\chi) + 2 \text{Sym}^4 \text{Ind}(\chi) \text{Ind}(\chi) \\
&\quad - \text{Sym}^2 \text{Ind} \text{Sym}^2(\chi) \text{Ind}(\chi) - \text{Sym}^3 \text{Ind}(\chi) \text{Sym}^2 \text{Ind}(\chi) \\
&\quad - \text{Sym}^3 \text{Ind}(\chi) \text{Ind} \text{Sym}^2(\chi) - 9 \text{Ind} \text{Sym}^{4,1}(\chi) + \text{Ind} \text{Sym}^{3,2}(\chi) \\
&\quad + 19 \text{Ind} \text{Sym}^{3,1,1}(\chi) - \text{Ind} \text{Sym}^{2,2,1}(\chi) - 12 \text{Ind} \text{Sym}^{3,1}(\chi) \text{Ind}(\chi) \\
&\quad - 9 \text{Ind} \text{Sym}^{2,1,1,1}(\chi) + \text{Ind} \text{Sym}^{2,1,1}(\chi) \text{Ind}(\chi) \\
&\quad + 2 \text{Ind} \text{Sym}^{1,1,1,1}(\chi) \text{Ind}(\chi) + \text{Ind} \text{Sym}^3(\chi) \text{Ind} \text{Sym}^2(\chi) \\
&\quad + 6 \text{Ind} \text{Sym}^{2,2}(\chi) \text{Ind}(\chi), \quad (80e) \\
\text{AltInd}(\chi) &= \text{Alt}^5 \text{Ind}(\chi) - \text{Ind}(\text{Alt}^5(\chi) + 2 \text{Alt}^4 \text{Ind}(\chi) \text{Ind}(\chi) \\
&\quad + \text{Alt}^2 \text{Ind} \text{Alt}^2(\chi) \text{Ind}(\chi) - \text{Alt}^3 \text{Ind}(\chi) \text{Alt}^2 \text{Ind}(\chi) \\
&\quad - \text{Alt}^3 \text{Ind}(\chi) \text{Ind} \text{Alt}^2(\chi) - 9 \text{Ind} \text{Alt}^{4,1}(\chi) + \text{Ind} \text{Alt}^{3,2}(\chi) \\
&\quad + 19 \text{Ind} \text{Alt}^{3,1,1}(\chi) + 4 \text{Ind} \text{Alt}^{2,2,1}(\chi) - 12 \text{Ind} \text{Alt}^{3,1}(\chi) \text{Ind}(\chi) \\
&\quad - 9 \text{Ind} \text{Alt}^{2,1,1,1}(\chi) + \text{Ind} \text{Alt}^{2,1,1}(\chi) \text{Ind}(\chi) \\
&\quad + 2 \text{Ind} \text{Alt}^{1,1,1,1}(\chi) \text{Ind}(\chi) + \text{Ind} \text{Alt}^3(\chi) \text{Ind} \text{Alt}^2(\chi).
\end{aligned}$$

Note that the formula for SymInd and AltInd are exactly analogous for $[G : H] \leq 3$, but contain different coefficients for $[G : H] \geq 4$.

C Guide to the Data Files

The complete list of free actions is available at <http://www.stp.dias.ie/~vbrown/CICY/Quotients.ta>. Each action is contained in one of the 1695 files `Data/FreeQuotients/<CICY>-<Nr>.gap`, where `<CICY>` is the CICY number, and `<Nr>` is an arbitrary and non-consecutive labeling of different actions on the same CICY. The data files themselves are GAP records with, hopefully, descriptive keywords and can be read directly into GAP. As an example of how to use this information, the GAP script `Data/LoadAction.gap` takes this information and computes a basis for the invariant polynomials.

For example, let us look at the three-generation model studied in [13]:


```
[vbraun@volker-desktop Data]$ gap.sh FreeQuotients/7246-21.gap LoadAction.gap
```

```
#####          #####          #####          ###
#####          #####          #####          #####
#####          #####          #####          #####
#####          #####          #####          #####
#####          #####          #####          #####
#####          #####          #####          #####
#####          #####          #####          #####
#####          #####          #####          #####
#####          #####          #####          #####
#####          #####          #####          #####
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#####          #####          #####          #####
#####          #####          #####          #####
#####          #####          #####          #####
#####          #####          #####          #####

Information at: http://www.gap-system.org
Try 'help' for help. See also '?copyright' and '?authors'

Loading the library. Please be patient, this may take a while.
GAP4, Version: 4.4.12 of 17-Dec-2008, x86_64-unknown-linux-gnu-gcc
Components: small 2.1, small2 2.0, small3 2.0, small4 1.0, small5 1.0, small6 1.0, small7 1.0, small8 1.0, small9 1.0, small10 0.2,
            id2 3.0, id3 2.1, id4 1.0, id5 1.0, id6 1.0, id9 1.0, id10 0.1, trans 1.0, prim 2.1 loaded.
Packages:   AutPGP 1.4, GAPDoc 1.2, TomLib 1.1.4 loaded.

gap> FreeAction;
rec( CICY := rec( CICYmatrix := [ [ 0, 0, 1, 1, 1 ], [ 0, 0, 1, 1, 1 ], [ 1, 1, 0, 0, 1 ], [ 1, 1, 0, 0, 1 ] ], Pn := [ 3, 3, 3, 3 ],
G := Group([ (1,2,3,5)(4,10,7,12)(6,11,9,8), (1,8,4)(2,10,6)(3,11,7)(5,12,9) ]),
GProw := [ (1,10,4,7)(2,11,6,9)(3,12,5,8), (1,3,2)(4,5,6)(7,9,8)(10,11,12) ] -> [ (1,3,2,4), () ],
GPcol := [ (1,10,4,7)(2,11,6,9)(3,12,5,8), (1,3,2)(4,5,6)(7,9,8)(10,11,12) ] -> [ (1,2), () ], Prow := Group([ (1,3,2,4), () ])
, Pcol := Group([ (1,2), () ]), RowOrbit := [ [ 1, 2, 3, 4 ] ], RowOrbitFirst := [ 1 ],
RowOrbitFirstStabilizer := [ Group([ (1,2,3)(4,6,5)(7,8,9)(10,12,11) ])], ColOrbit := [ [ 1, 2 ], [ 3 ] ],
ColOrbitFirst := [ 1, 3 ], ColOrbitFirstStabilizer := [ Group([ (1,5,2,4,3,6)(7,11,8,10,9,12) ],
Group([ (1,10,4,7)(2,11,6,9)(3,12,5,8), (1,3,2)(4,5,6)(7,9,8)(10,11,12) ])], DistinctEqns := [ [ 1, 2 ], [ 3, 4 ], [ 5 ] ]
, Gcover := [ (1,10,4,7)(2,11,6,9)(3,12,5,8), (1,3,2)(4,5,6)(7,9,8)(10,11,12) ] -> [ (1,2,3,5)(4,10,7,12)(6,11,9,8),
(1,8,4)(2,10,6)(3,11,7)(5,12,9) ], K := Sym( [ ] ), TrivialTwists := [ Character( CharacterTable( Sym( [ ] ) ), [ 1 ] ) ]
, Field := CF(12), Ring := CF(12)[x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_10,x_11,x_12],
Coord := [ [ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_10, x_11, x_12 ] ],
CoordPn := [ [ x_1, x_2, x_3 ], [ x_4, x_5, x_6 ], [ x_7, x_8, x_9 ], [ x_10, x_11, x_12 ] ],
NontrivialConjClasses := [ (1,2,3)(4,6,5)(7,8,9)(10,12,11), (1,4)(2,6)(3,5)(7,10)(8,12)(9,11), (1,5,2,4,3,6)(7,11,8,10,9,12),
(1,7,4,10)(2,9,6,11)(3,8,5,12), (1,10,4,7)(2,11,6,9)(3,12,5,8) ], Num := 7246 ),
ChiGamma := [ Character( CharacterTable( Group([ (1,2,3)(4,6,5)(7,8,9)(10,12,11) ] ) ), [ 3, 0, 0 ] ) ],
Gamma := [ [ (1,10,4,7)(2,11,6,9)(3,12,5,8), (1,3,2)(4,5,6)(7,9,8)(10,11,12) ] ->
[ [ [ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ],
[ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ],
[ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
], [ [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, E(3)^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
[ 0, 0, E(3), 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, E(3)^2, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, E(3), 0, 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, E(3), 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 0, 0, E(3)^2, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 0, E(3), 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, E(3), 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, E(3)^2 ] ] ],
ChiRho := [ Character( CharacterTable( Group([ (1,5,2,4,3,6)(7,11,8,10,9,12) ] ) ), [ 2, -E(3), -E(3)^2, 2, -E(3)^2, -E(3) ] ),
Character( CharacterTable( Group([ (1,10,4,7)(2,11,6,9)(3,12,5,8), (1,3,2)(4,5,6)(7,9,8)(10,11,12) ] ) ), [ 1, 1, 1, 1, 1, 1 ]
) ] ],
Rho := [ [ (1,10,4,7)(2,11,6,9)(3,12,5,8), (1,3,2)(4,5,6)(7,9,8)(10,11,12) ] -> [ [ [ 0, 0, 1, 0 ], [ 0, 0, 0, 1 ], [ 1, 0, 0, 0 ],
[ 0, 1, 0, 0 ], [ [ 1, 0, 0, 0 ], [ 0, E(3), 0, 0 ], [ 0, 0, 1, 0 ], [ 0, 0, 0, E(3)^2 ] ] ],
[ (1,10,4,7)(2,11,6,9)(3,12,5,8), (1,3,2)(4,5,6)(7,9,8)(10,11,12) ] -> [ [ [ 1 ] ], [ [ 1 ] ] ] ],
Invariant := [ [ [ 0, 0, 0, 0 ], [ 0, 0, 0, 0, x_3*x_6*x_9*x_12 ],
[ 0, 0, 0, 0, x_2*x_6*x_8*x_12+x_2*x_6*x_9*x_11+x_3*x_5*x_8*x_12+x_3*x_5*x_9*x_11 ], [ 0, 0, 0, 0, x_2*x_5*x_8*x_11 ],
[ 0, 0, 0, 0, x_1*x_6*x_8*x_11+x_2*x_5*x_7*x_12+x_2*x_5*x_9*x_10+x_3*x_4*x_8*x_11 ],
[ 0, 0, 0, 0, x_1*x_6*x_7*x_12+x_1*x_6*x_9*x_10+x_3*x_4*x_7*x_12+x_3*x_4*x_9*x_10 ],
[ 0, 0, 0, 0, x_1*x_5*x_9*x_12+x_2*x_4*x_9*x_12+x_3*x_6*x_7*x_11+x_3*x_6*x_8*x_10 ],
[ 0, 0, 0, 0, x_1*x_5*x_7*x_11+x_1*x_5*x_8*x_10+x_2*x_4*x_7*x_11+x_2*x_4*x_8*x_10 ],
[ 0, 0, 0, 0, x_1*x_4*x_8*x_12+x_1*x_4*x_9*x_11+x_2*x_6*x_7*x_10+x_3*x_5*x_7*x_10 ], [ 0, 0, 0, 0, x_1*x_4*x_7*x_10 ],
[ 0, x_9*x_12, 0, x_3*x_6, 0 ], [ 0, x_7*x_11+x_8*x_10, 0, x_1*x_5+x_2*x_4, 0 ],
[ x_8*x_12+x_9*x_11, 0, x_2*x_6+x_3*x_5, 0, 0 ], [ x_7*x_10, 0, x_1*x_4, 0, 0 ] ] ) ] ] )
```

The CICY configuration matrix is recorded as

$$\begin{aligned} \vec{d} &= \text{FreeAction.CICY.Pn}, \\ c &= \text{FreeAction.CICY.CICYmatrix}, \\ \vec{\delta} &= \text{List}(\text{FreeAction.CICY.DistinctEqns}, \text{Size}) \end{aligned} \tag{81}$$

and the CICY group, see Definition 1, is

$$\begin{aligned} C &= (d_i, c_{ij}, \delta_j)_{i=1..n, j=1..m}, \\ \tilde{G} &= \text{Source}(\text{FreeAction.CICY.Gcover}), \\ \pi_r &= \text{FreeAction.CICY.GProw}, \\ \pi_c &= \text{FreeAction.CICY.GPcol}. \end{aligned} \tag{82}$$

Note that the group we are working with is always the (generalized) Schur cover for the π -representation. The freely acting group on the Calabi-Yau threefold is

$$G = \text{FreeAction.CICY.G}, = \text{Image}(\text{FreeAction.CICY.Gcover}). \tag{83}$$

To entirely specify the CICY group action, we only need to specify two (linear) π -representations of \tilde{G} acting on the homogeneous coordinates and the polynomials. These are

$$\begin{aligned} \gamma &= \text{FreeAction.Gamma}, \\ \rho &= \text{FreeAction.Rho}. \end{aligned} \tag{84}$$

This is how the data file records the CICY group representation $(C, G, \pi_r, \gamma, \pi_c, \rho)$, see Definition 3. Finally, a set of generators¹⁷ for the invariant polynomials is stored in `FreeAction.Invariant`.

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¹⁷Note that no attempt is made to return linearly independent generators.

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