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Summary. When light passes through media which are all at rest, the frequency is constant, and so, in the geometrical optics of these media, frequency is merely a passive parameter, with one value in one problem and another value in another. This is changed if the media are in relative motion, for then the frequency varies along a ray, and dispersion must be taken into account. Dispersion is usually small in practice, and one might be tempted to regard it as a troublesome complication, to be dealt with by some suitable approximation. But this would be a mistake from a theoretical standpoint, because it is dispersion that brings Hamiltonian dynamics into optics, and enables us to fit together, as two aspects of a single mathematical theory, physical theories seemingly distinct, namely, geometrical mechanics (particles and associated de Broglie waves) and geometrical optics (phase waves and [illegible] [illegible]).

In this paper Hamilton's method in geometrical optics, suitably [illegible], is used as a guide for relativistic geometrical optics in moving dispersive media. In mechanics it is natural to start with a particle, and develop secondarily the associated de Broglie waves. In optics, on the other hand, the natural datum is a [illegible] [illegible], expressing phase velocity in terms of frequency and direction of propagation, and so one starts with waves and develops secondarily the associated particles.

As for the physical validity of the theory here developed, no new assumption is [illegible] to [illegible] geometrical optics to the case of media

[illegible] without modification; all [illegible] to be [illegible] by a [illegible] [illegible] the [illegible] method of [illegible] such media [illegible] even by a Lorentz transformation. The results for accelerated media, on the other hand, must be regarded as tentative. The assumption is made that, in the local rest frame of the medium, phase velocity is given by the expression known for the medium when completely at rest.

T A B L E   O F   C O N T E N T S

GEOMETRICAL OPTICS IN MOVING DISPERSIVE MEDIA

1. Introduction

Geometrical optics, as ordinarily understood and as developed by Hamilton in particular, deals with the propagation of light in fixed media. The frequency of the light plays a minor part: it is a fixed parameter in any optical problem, since frequency is unchanged by passage through fixed media.

The special theory of relativity shows us how to treat media in uniform motion. We can reduce such a medium to rest by a Lorentz transformation, investigate the optical problem for this medium by the classical method, and then, by the inverse Lorentz transformation, restore the original frame which shows the medium in motion. Frequency is no longer a fixed parameter, for it changes when light enters a moving medium.

There is a snag not covered by the above plan, namely, when two media which meet one another with no intervening vacuum. There is then no frame in which both media are at rest. We may meet this difficulty by inserting a fictitious layer of vacuum between the media, a vacuum being essentially a "fixed medium" for any Galilean observer. This device may fail, however, for the passage of light from one medium to the other may be prevented by total reflection which keeps the light from entering the fictitious vacuum.

At best, this method of reducing each medium in turn to rest is mathematically sharp, and we are tempted to embark on a general and comprehensive theory for geometrical optics in moving media, the case of media in uniform motion being included as a special case, but with accelerated motions also included. That is what the present paper attempts to give. The method is essentially Hamilton's method in geometrical optics, now formulated in the four-dimensional space-time of Minkowski instead of in three-dimensional space, with frequency elevated from the subordinate role of constant parameter to be the fourth partner in the momentum-frequency tetrad, and, reciprocally, fourth partner in the space-time [illegible] tetrad of the photon.

How far is this mathematical theory physical? To what extent can it be used to predict the results of optical experiments performed with moving media?

We are familiar with the physical limitations of the geometrical optics of fixed media. The wave lengths involved must be very small relative to the dimensions of the apparatus (wavelengths [illegible]), and they must at the same time be large relative to the microscopic structure of the medium; the frequency must be small compared with the absorption frequencies of the medium. These limitations apply, presumably, in the case of moving media.

But there is another limitation, present in the case of fixed media but assumes more importance in the case of accelerated media, and that is the assumption that physical optics must come to geometrical optics before the latter can make physical predictions. Consider a fixed media. Geometrical optics is a self-contained theory requiring a metric function, the refractive index n characterizing the medium, depending on position for a inhomogeneous medium and on direction for an anisotropic medium. It is not the business of geometrical optics to say what form that function n takes; that is a datum to be supplied by physical optics. Likewise, in the case of moving media, a datum must be supplied.

In developing the geometrical optics of fixed media, Hamilton offered two alternative approaches corresponding respectively to corpuscular and wave theories. In the second method, he postulated a slowness-frequency equation of the form

$$\Omega(\sigma_1, \sigma_2, \sigma_3, \nu, x, y, z) = 0, \qquad (1.1)$$

where $(\sigma_1, \sigma_2, \sigma_3)$ is the slowness vector for waves, ν the frequency and (x, y, z) coordinates of position. This equation is the datum which physical optics must supply. For example, for an isotropic medium it reads

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - n^2/c^2 = 0, \qquad (1.2)$$

where c is the velocity in vacuo and n is the refractive index, which may be a constant, or, more generally, some specified function of position.

The formal passage from fixed to moving media is achieved simply as insert t as an eighth argument in (1.1), so that we have an equation of the form [illegible] (11)

[illegible] [illegible] [illegible] [illegible]. It may even happen that [illegible] is [illegible]. This means that the motion of the media is [illegible]. But once made it is the duty of physical [illegible] to supply the form of the function [illegible], are complicated by the fact that it depends not only on the properties that the medium would have if at rest, but also on the motion.

Ignorance of the forms of the function [illegible] which occur in nature does not prevent us from developing a general theory, whatever they may be. There are some physical situations to which the general method is applicable or may be applicable:

(a) A set of [illegible] media, separated by vacuum, the optical properties of each medium [illegible]. The media may be inhomogeneous and anisotropic.

(b) As in (a) but without the vacuum separations, the media sliding past one another in contact.

(c) Plasma in accelerated motion, [illegible].

We can feel quite confident about (a): nothing is involved but the [illegible] of media at rest plus the special theory of relativity. It is unlikely that any one would bother to test theoretical conclusions in (a) by experiment. There is no [illegible] of speculation in (b) and (c): the general theory here offers predictions which might be verified by experiment, or might not.

We would like to be able to follow [illegible] through a [illegible] tion of glass by the methods of geometrical optics, and we certainly could if the equation (1.1) was given to us in explicit form. This problem will not be treated in the present paper, but there is no harm in sketching an approach to it. Firstly, we would have to solve the elastic problem and find the stress in the glass. Secondly, we would write down the equation of form (1.1) expressing the optical properties of stressed glass at rest (it is optically anisotropic). Thirdly, we would assume that the [illegible] wave-velocity/frequency equation held in the local rest frame. Thus we would arrive at an equation of the form (1.2) (in which [illegible] would not actually appear if the angular velocity was constant).

When viewed in full generality, the mathematical structure of the geometrical optics of moving dispersive media is essentially the same as that of geometrical mechanics as I have developed it recently (Synge, 1954). But there are two differences, the first psychological, the second a matter of [illegible], trivial and yet of [illegible].

The first difference is that for mechanics the particle comes easily to our minds in mechanics, and the de Broglie waves associated with it are regarded as a secondary and rather surprising phenomenon. This means that geometrical mechanics is most naturally developed by means of Hamilton's first method of approach (Fermat's principle, principle of Maupertuis). But in optics we think first of waves, and photons from the photon concept, still more surprising to us. This indicates an approach through Hamilton's second method, the equation (1.5).

The second difference is that in mechanics we generally have in principle waves which travel faster than light in vacuo, whereas the speed of optical phase waves in transparent media is generally less than this fundamental velocity. Accordingly, the [illegible] situation is feasible for a particle, but impossible for a photon in matter. The notion particle-or-photon and the speed exceeds the fundamental velocity, that would be incompatible with relativity, since they might be used as signals.

On account of these differences it seems best to set up the geometrical optics of moving dielectric media as a theory in its own right and not as a corollary to relativistic mechanics, in spite of the fact that they are different aspects of a single general Hamiltonian theory.

The theory of the present paper is set in the special theory of relativity. The extension to general relativity would involve some formal complications but no essential change. There is however one point that should be mentioned. It is commonly stated in general relativity that the history of a light ray is a null geodesic. Does this mean that light rays a null geodesic in traverses a transparent [illegible] medium? That seems most unlikely. Rather, the null geodesic hypothesis, when applied inside a transparent medium, would seem to mean that, if we bore in the medium a hole having the form of a null geodesic, all matter being removed from the hole, then light will travel through this hole without bumping into the walls, provided the ray is aimed properly at the beginning. In fact, the null geodesic hypothesis, whether outside medium at point P, refers to [illegible] propagation in some [cf. Synge (1937)]. The treatment of [illegible] in moving media relativity given by Pauli and McConnell (1931) is in accordance with the theory of the present paper, but their treatment of [illegible] is not, since it is based on the null geodesic hypothesis for the propagation of light in a medium.

The material which follows was presented in Seminar lectures at the Dublin Institute for Advanced Studies in 1954, in lectures at the St. Andrews Mathematical Colloquium in 1955, and, in brief form, in lectures at several places in the United States and Canada in 1956.

2. Geometry of a Lorentz Continuum.

We shall use the flat space-time of Minkowski with coordinates x_r ($x_4 = ict$) such that the fundamental form is $dx_r dx_r$. Latin suffixes take the values 1, 2, 3, 4, Greek suffixes the values 1, 2, 3, and the summation convention is understood for both, unless it is indicated to the contrary. We note that

$$dx_r dx_r > 0 \quad \text{for spacelike } dx_r$$

$$dx_r dx_r = 0 \quad \text{for null } dx_r \qquad (2.1)$$

$$ds^2 = -dx_r dx_r > 0 \quad \text{for timelike } dx_r;$$

the element of proper time is $dT = ds/c$;

The motion of a homogeneous medium may be described by giving the world lines of the particles. Let $\lambda_r = dx_r/ds$ be the unit tangent vector to the world line at any event; then the moving medium is described kinematically by the unit vector field

$$\lambda_r = \lambda_r(x), \qquad \lambda_r \lambda_r = -1. \tag{3.1}$$

This is the 4-velocity of the medium. The 3-velocity v_ρ is related to the 4-velocity by

$$\lambda_\rho = \gamma v_\rho / c, \qquad \lambda_4 = i\gamma, \qquad \gamma = (1 - v^2/c^2)^{-\frac{1}{2}}, \qquad v^2 = v_\rho v_\rho. \tag{3.2}$$

At any event there is a local rest frame, say $S^{(0)}$, and we shall use (0) to indicate components relative to it. We have

$$\lambda_\rho^{(0)} = 0, \qquad \lambda_4^{(0)} = i. \tag{3.3}$$

If $F(x)$ is any function of the space-time coordinates and C a constant, the equation

$$F(x) = C \tag{3.4}$$

defines a 3-space. Its intersection with x_4 = constant is a 2-space which changes with t, and so we may regard (3.4) as a wave, which is the history of a moving wave. Note that the 3-space is an absolute thing, but the 2-space (the wave) depends on the frame of reference. It is easy to see (cf. Synge, 1956, p. [illegible]) that the velocity v_w of the wave (normal to it) is

$$n_\rho = -\frac{F_{,\rho}}{(F_{,\sigma} F_{,\sigma})^{1/2}} , \qquad (1.5)$$

the commas indicating partial derivatives. For the speed u of the ripple we have

$$u^2 = -c^2 \frac{F_{,4}^2}{F_{,\rho} F_{,\rho}} , \qquad (1.7)$$

It is immediately obvious that $u \geq c$ if $F_{,\rho} F_{,\rho} \leq 0$, i.e. if $F_{,\rho}$ is timelike, and that $u \leq c$ if $F_{,\rho}$ is spacelike.

Now u^2 as in (1.7) has not an invariant meaning, but $u^{(0)}$ has, this being the speed of the ripple in the local rest frame of the medium. We easily verify that this invariant is

$$u^{(0)2} = c^2 \frac{(\lambda_\rho F_{,\rho})^2}{F_{,\sigma} F_{,\sigma} + (\lambda_\sigma F_{,\sigma})^2} , \qquad (1.8)$$

and this is an invariant expression which reduces to (1.7) when we substitute from (1.6).

If we give a continuous range of values to the constant C in (1.1), we get ∞^1 [illegible], ordered by the value of C (parameter). This we may call a lamination of space-time.

In making out a distinction between a set of these ripples (see next section) and a lamination, we note that the function $F(x)$ which defines a lamination is by no means unique. For, if Φ is any monotonic function, then the lamination (1.1) is equally well described by the equation

$$[illegible]$$ [illegible]

where

$$[illegible]$$ [illegible]

4. <u>Phase [illegible] [illegible]</u>

Let O (Fig. 1) be the (timelike) world line of a source which emits light [illegible]

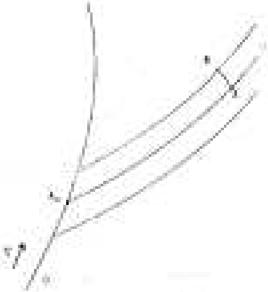


Fig. 1

[illegible] emitted by a source with world line O.

of period T, measured in the proper time of T. The phase angle θ of [illegible] is

$$\theta = 2\pi\tau / T,$$ (3.1)

where τ is proper time on T. The phase is [illegible], equivalently, the [illegible] pair (cos θ, sin θ).

We think of the phase angle as propagating out from T according to some law not yet specified, with the result that space-time is filled with phase fronts on each of which the phase angle θ is constant. This means that to any given event x_p there corresponds a value of θ, so that we can write $\theta = \theta(x)$; it is convenient to introduce the phase function $\Phi(x)$ defined as

$$\Phi(x) = [illegible]$$ (3.2)

[illegible] if x_p is [illegible], then τ is the proper time on T of that event x_p where the phase front [illegible] through [illegible].

[illegible] values to Φ in (3.2) [illegible] a set of phase fronts [illegible] between a definite value of θ [illegible] to each front [illegible], the function $\Phi(x)$ is physically significant, and we cannot [illegible] a transformation [illegible] (3.2) [illegible] its significance.

Note that the phase fronts have a natural order, [illegible] of increasing Φ, or equivalently decreasing θ.

The formula (3.5) [illegible] the phase velocity v_p; (3.7) gives the phase [illegible] w; (3.8) gives the phase [illegible] $x^{(\mu)}$ in the local rest frame. We recall that

$= > 0$ if $F_{,x} F_{,r} < 0$ ($F_{,r}$ [illegible]),

$= 0$ if $F_{,x} F_{,r} = 0$ ($F_{,r}$ null), (5.5)

$= < 0$ if $F_{,x} F_{,r} > 0$ ($F_{,r}$ [illegible]).

[illegible] these also hold for $u^{(r)}$. The second case corresponds to a [illegible] [illegible] in the [illegible] [illegible] in [illegible] [illegible].

If we pass from an event A (Fig. 2) to an event B on the [illegible] [illegible] with the same phase, we have

$$\int_A^B F_{,r}\, dx_r = [illegible]$$

[illegible] we consider the [illegible] [illegible] of [illegible] [illegible] [illegible] or short wave length), for we wish to replace this equation by an approximate [illegible] [illegible]

$$F_{,r}\, dx_r = [illegible] . \qquad (5.6)$$

dx_r being the displacement AB.

This last equation is [illegible] true for finite displacements dx_r [illegible] $\phi(x)$ is a linear function, so that $F_{,rs} = 0$ and $F_{,r}$ are constants. [illegible], [illegible] in [illegible] (5.6) [illegible] [illegible] the [illegible] is very small [illegible] dx_r is very small and $F_{,r}$ very [illegible], or F is approximately linear in the [illegible] of the [illegible] dx_r. The [illegible] [illegible] [illegible] [illegible] to an [illegible] [illegible] of [illegible] in [illegible] [illegible], and we [illegible] [illegible] [illegible] [illegible].

[illegible] (5.6), we [illegible] (5.5) [illegible] [illegible].

$$F_{\gamma P}\, d\Omega_P \;[illegible]\; \Sigma. \qquad (1.4)$$

Note the minus sign here. Had we defined $F(x)$ with the opposite sign in (1.3), we would have got [illegible] instead of [illegible]. But to discuss all cases, we must choose a definition of $F(x)$ and stick to it. (1.4) seems to be the most convenient, adapted to later work.

Let us now read (1.4) [illegible], taking $F_{\gamma P}$ to be in turn timelike, null and spacelike. In the first two cases we shall take $F_{\gamma P}$ pointing into the future (it is easy to make the necessary changes if it points into the past).

(i) $F_{\gamma P}$ timelike, future-pointing.

A timelike $F_{\gamma P}$ implies [illegible], and Σ is the phase front corresponding to $F_{\gamma P}$. Then (1.4) implies that $d\Omega_P$ and $F_{\gamma P}$ point to the same side of Σ (Fig. 2).

(ii) $F_{\gamma P}$ null, future-pointing.

Now [illegible]. The phase front Σ becomes the null cone, $F_{\gamma P}$ being the null vector on the line of contact. By (1.4) $d\Omega_P$ points out from each side of Σ on which the future null cone lies (Fig. 3).

(iii) $F_{\gamma P}$ spacelike [illegible].

A spacelike $F_{\gamma P}$ implies [illegible]. By (1.4) $d\Omega_P$ and $F_{\gamma P}$ point to the same side of Σ (Fig. 4).

The above results are easily understood by using spatial shapes of surfaces and contours.

Fig. 5 illustrates a procedure which has proved [illegible]. [illegible] is a source in vacuo, emitting phase fronts. The adjacent [illegible] of the same phase are drawn, the arrows being marked to [illegible]

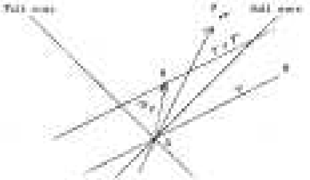


Fig. 1

Tangent vector T_{μ} is timelike (group speed) < 1 and points into the future.

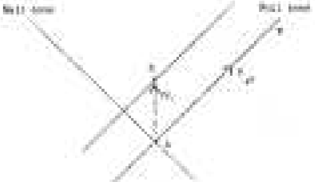


Fig. 2

Tangent vector T_{μ} is null (group speed = 1) and points into the future.

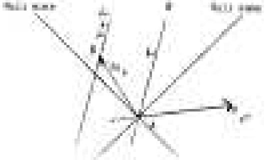


Fig. 4

Time [illegible]

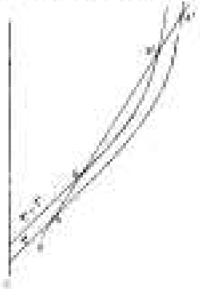


Fig. 5

[illegible]

passes through a medium, in which the phase speed is less than c, [illegible] is the world line of an observer. The two events give their [illegible] to the experimenter. The first one (x , x) occurs in the natural time-order, but the others (x' , x') in the opposite order. The natural time-order is reversed. For this to happen, it is necessary that the phase 3-waves must be read in the frame of the observer. But there is nothing strange about this, for any wave which travels with speed less than c can be induced to halt, or indeed made to go backwards, by merely changing the frame of reference.

Let us now consider the frequency of a set of phase 3-waves as measured by (a) any Galilean observer and (b) by an observer carried along by a particle of the medium.

In the first case the observer's world line is parallel to the T-axis. Let it cut the phase 3-waves T , $T + 1$ at events A , B respectively. If dt is the increment in t in passing from A to B, then dt is the period of the waves and the frequency is $\nu = 1/dt$. By (5.4) we have

$$F_{,4}\, ic\, dt = -1 , \tag{5.5}$$

and so the frequency is given by

$$\nu = -ic F_{,4} . \tag{5.6}$$

To find the frequency $\nu^{(0)}$ relative to the medium we set $dx_\mu = \lambda_\mu ds$ as in (5.4), we find

$$F_{,\mu} \lambda_\mu ds = -1 \tag{5.7}$$

for the passage from 3-wave T to 3-wave $T + 1$. Hence

$$\nu^{(0)} = 1/ds = -F_{,\mu} \lambda_\mu . \tag{5.8}$$

In the normal case (as in Fig. 5) [illegible] is positive and (3.5) gives a [illegible] frequency. In the abnormal case [illegible] is negative and we get a negative frequency. A negative frequency means that the phase waves are received by the observer (in particular, by a particle of the medium) in the order opposite to that in which they were emitted.

4. [illegible]

The partial derivatives $W_{,r}$ of the phase function form a covariant object of fundamental [illegible] in the theory, and it is convenient to have [illegible] notation which [illegible] it as a [illegible] without reference to the fact that it is a gradient. We write

$$\sigma_r = W_{,r} \qquad (4.1)$$

and call σ_r the [illegible] (or briefly the [illegible]) for reasons indicated by the following formulae. By (3.5) we have

$$\sigma_4 = [illegible] \qquad (4.2)$$

and by (3.2), (3.7)

[illegible] (4.3)

For [illegible] [illegible] the [illegible] σ and [illegible] have the same direction and the covariant of ϖ_μ is ψ/c. Thus ϖ_μ measures the slowness of the phase 3-wave.

The 4-vector ϖ_μ is the relativistic generalization of Hamilton's slowness 3-vector; to within a constant factor, ϖ_μ is the propagation vector of modern physics, but Hamilton's terminology is more suggestive of its nature.

Let us write (3.5) and (3.6) in terms of the slowness 4-vector:

$$\psi^{(+)} = c^2 \frac{(\lambda_\rho \varpi_\rho)^2}{\varpi_\rho \varpi_\rho + (\lambda_\rho \varpi_\rho)^2}, \qquad \psi^{(-)} = -\lambda_\rho \varpi_\rho. \quad (3.8)$$

We note that a positive value of $\psi^{(-)}$ implies that, in the local rest frame, ϖ_μ points in the direction of propagation of the phase 3-wave and not in the opposite direction; and further, since λ_μ is necessarily timelike, a positive $\psi^{(-)}$ implies that ϖ_μ points into the future (inside or outside the null cone) in the local rest frame (for which $\lambda_\rho = 0$, $\lambda_4 = i$).

4. The slowness-frequency equation.

So far our work has been purely kinematical and has involved no hypothesis as to the optical character of the medium. We now set up a very general hypothesis in the form of an equation

$$\Omega(\varpi, x) = 0, \quad (4.1)$$

connecting the 4-vector ϖ_μ with the coordinates x_μ. This slowness-frequency equation expresses completely, so far as geometrical optics is concerned, the optical [illegible] of the [illegible] to [illegible].

The [illegible] equation may be interpreted in several ways. If it is solved for σ_4, so that we write

$$[illegible] \qquad (5.2)$$

it expresses the frequency of plane waves in terms of their direction [illegible], position and time. On the other hand, if it is solved for [illegible], we then obtain

$$[illegible] \qquad (5.3)$$

it expresses, by (4.2), the speed of the plane waves in terms of the direction of propagation, frequency, position and time.

For general mathematical arguments it is best to use the general form (5.1); (5.2) links the present theory with familiar Hamiltonian dynamics [illegible]; (5.3) connects the theory with physical ideas, the speed of propagation being expressed in terms of direction of propagation and frequency.

In a [illegible] function is

$$[illegible] \qquad (5.4)$$

[illegible] the factor [illegible] is introduced merely as a [illegible]. [illegible] the *[illegible]*, and the *[illegible]* [illegible].

For an [illegible] medium at rest we have

$$[illegible] \qquad (5.5)$$

By (4.2) this is equivalent to [illegible]. The refractive index [illegible] is in general a function of [illegible] (to allow for dispersion) and a function of x_1, x_2, x_3 (to allow

[illegible]

[illegible] we have

$$[illegible]$$

$$[illegible] \qquad (5.4)$$

$$[illegible]$$

[illegible]

6. [illegible]

Let there be any number of media, [illegible]

[illegible]

---

* [illegible]

[illegible] (5.1) [illegible] (5.2) [illegible]

[illegible]

$$[illegible] \qquad (5.1)$$

[illegible]

Let μ_p be the [illegible]. For an isotropic medium we have

$$[illegible] \qquad (5.2)$$

[illegible]

$$[illegible] \qquad (5.3)$$

[illegible] We denote them by $\lambda_p^{(r)}$, [illegible]

$$[illegible] \qquad (5.4)$$

[illegible] Thus, for a general [illegible] we have

$$[illegible] \qquad (5.5)$$

Here $v^{(r)}$ and μ_r are constants for the medium and $n_{\mu\nu}$ are functions of $U_\rho k_\rho$ to allow for dispersion. In such a case Ω is independent of x_μ; but for the sake of generality [illegible] of media not covered it is to be regarded as a [illegible] function of x_μ.

To follow the propagation of light through media as considered above we need the concept of rays and also the laws of reflection and refraction. These will be treated in Sections 8 and 10 respectively.

7. Isotropic fluid medium in general motion.

[illegible] attempt to set up a frequency-slowness relation for the [illegible] motion of a crystalline medium or even of a solid isotropic medium, such as glass, but we can make a reasonable hypothesis for the case of an isotropic fluid medium.

We shall assume a basic relation between the phase speed $v^{(s)}$ and the frequency $\nu^{(s)}$, both measured in the local rest frame. [illegible] It is convenient to introduce the refractive index n by the definition

$$n = c/v^{(s)}. \qquad (7.1)$$

Then by (4.?)

$$n^2 - 1 = \frac{k_\rho k_\rho}{(U_\mu k_\mu)^2}, \qquad \nu^{(s)} = -U_\mu k_\mu. \qquad (7.2)$$

The following equation then holds:

$$2\Omega(x, k) = k_\rho k_\rho - (n^2 - 1)(U_\mu k_\mu)^2 = 0. \qquad (7.3)$$

This is the general dispersion-frequency equation for a fluid in general motion provided we specify a knowledge of n as a function of $\nu^{(r)}$ and $\rho(x)$, the proper density of the fluid; we write

$$n = n(\sigma_r \mu^r, \rho) . \qquad (7.4)$$

Note that Ω involves σ_r explicitly and also implicitly through n; it involves x_r through $\mu_r(x)$ and also through $\rho(x)$, these functions being assumed given.

The equation (7.1) has the same form as [illegible] (?.?), the difference being that we previously thought of μ_r as constant (uniform motion), and now we think of it as variable (non-uniform motion).

As a particular case of (7.4) we may take the Sellmeier formula (cf. R. W. Wood (1934), p. 455)

$$n^2 - 1 = \sum_k \frac{A_k \nu^2}{\nu^2 - \nu_k^2} = \sum_k \frac{A_k \nu_k^2}{\nu_k^2 - \nu^2} , \qquad (7.5)$$

where A_k and ν_k (the absorption frequencies for the medium at rest) are to be regarded as known functions of the density ρ. Then for a Sellmeier medium the dispersion-frequency equation is

$$2\Omega(\sigma, x) = \sigma^r \sigma_r - (\sigma_r \mu^r)^2 \sum_k \frac{A_k \nu_k^2}{\nu_k^2 - (\sigma_r \mu^r)^2} = 0 \qquad (7.6)$$

This formula becomes physically invalid as we approach the absorption frequencies. For relatively small frequencies the approximate form is

$$2\Omega(\sigma, x) = \sigma^r \sigma_r + (n^2 - 1)(\sigma_r \mu^r)^2 ,$$

$$n^2 - 1 = A(\rho) + B(\rho)(\sigma_r \mu^r)^2 . \qquad (7.7)$$

5. Hamilton's partial differential equation. Characteristics or rays.

[illegible]

$$\Omega(\sigma_r, x) = 0 \qquad (5.1)$$

becomes a partial differential equation of the first order for the phase function $\sigma(x)$. This is Hamilton's partial differential equation and we shall refer to it as HPDE. It is essentially the same as that equation which, in dynamics, is called the Hamilton-Jacobi equation.

We are concerned with the problem of solving the HPDE subject to certain initial conditions, the σ assigned on a given 3-space. The method of solving this problem by means of characteristics is well known, but the argument will be presented here because it is desirable to have it before us in the present notation and in a form which shows us that the method [illegible].

If $\Omega(\sigma_r, x)$ is any function of the eight quantities σ_r, x_r (not necessarily satisfying (5.1)), the ordinary differential equations

$$\frac{dx_1}{\partial\Omega/\partial\sigma_1} = \dots = \frac{dx_4}{\partial\Omega/\partial\sigma_4} = \frac{d\sigma_1}{-\partial\Omega/\partial x_1} = \dots = \frac{d\sigma_4}{-\partial\Omega/\partial x_4} \qquad (5.2)$$

define a congruence of curves in the 8-dimensional (σ_r, x)-space, one curve passing through any assigned point in that space. These equations may also be written

$$\frac{dx_r}{dw} = \frac{\partial\Omega}{\partial\sigma_r}, \qquad \frac{d\sigma_r}{dw} = -\frac{\partial\Omega}{\partial x_r}, \qquad (5.3)$$

w being a parameter suitably chosen along each curve. It is evident that (5.3) imply

$$\frac{d\Omega}{dw} = 0. \qquad (5.4)$$

Instead of thinking in terms of (γ', x)-space, let us [illegible]. The [illegible] (3.1) or (3.3) [illegible] as a set of world lines with associated vector fields σ_μ [illegible] each such world line with the associated σ_μ is a [illegible] [illegible] being determined [illegible] σ_μ, [illegible] the function $\Omega(\sigma, x)$ has been assigned. [illegible] Ω [illegible] the only difference being in the parameter λ.

To ensure that the condition for a timelike [illegible] be

$$[illegible] \leq 0 \qquad (3.5)$$

(This is an essential relativistic condition when we come to identify the rays with the world lines of photons, [illegible] would have a signal speed greater than [illegible].)

We now turn to the [illegible].

$$\Omega(\sigma, x) = 0, \qquad \sigma_\mu = \xi_{\mu}, \qquad (3.6)$$

and seek a solution [illegible] [illegible] [illegible] [illegible]. [illegible] [illegible] [illegible]

The [illegible] is to choose σ_μ and [illegible] to satisfy

$$\sigma_\mu \delta x_\mu = [illegible], \qquad \Omega(\sigma, x) = 0, \qquad (3.7)$$

for every displacement in [illegible], since there are three degrees of freedom in such displacement; we have four equations [illegible] for four unknowns. [illegible] the number of equations is eight, it may happen that no such vector σ_μ can be found. To investigate this, we fix our attention on some event x_0 [illegible] a $(n-1)$-space in which ϖ_p are coordinates. The first of (4.7) determines the orthogonal projection of ϖ_p on the tangent flat of Π (and $\bar{\Pi}$), and the second places the extremity of ϖ_p on a certain $(n-1)$-space $\Omega = 0$. As one sees (4.7) is, and only if, the orthogonal projection of $\Omega = 0$ on Π contains the assigned orthogonal projection of ϖ_p on $\bar{\Pi}$. And if a solution exists, there may be more than one.

Rather than spend on the question of the possible insolubility of (4.7) and of multiple solutions, we shall assume that a solution exists and that it is unique.

With the initial values of ϖ_p on Π given by (4.7), we proceed to draw the rays by (4.1). In general these rays form a congruence filling a portion of space; but there is an exceptional case. If, on Π, we have

$$\frac{\partial \Omega}{\partial \varpi_p} n_p = 0 \qquad (4.8)$$

(we have taken the equation of Π to be $f = 0$). Then the rays do not leave Π, but we suppose that this does not occur.

We have thus a congruence of rays emanating from Π. At any point x_p in the domain filled by them we define $F(x)$ by

$$F(x) = F(x') + \int_{x'}^{x} \varpi_p \, dx_p , \qquad (4.9)$$

where x'_p is the point where the ray through x_p meets Π, and the integral is taken along the ray. Then $F(x)$ has the assigned values on Π, and we see at once that $F(x)$ satisfies the initial conditions.

We have now to show that $F(x)$ satisfies the PDE, and to do this we vary x_p, obtaining

$$[illegible] \qquad ([illegible])$$

[illegible]

$$[illegible] \qquad ([illegible])$$

[illegible]

$$[illegible] \qquad ([illegible])$$

[illegible]

$$[illegible] \qquad ([illegible])$$

[illegible]

[illegible]

[illegible]

$$[illegible] \qquad ([illegible])$$

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

normal to the [illegible] and

$$[illegible] \qquad (5.7)$$

Thus (5.5) reads

$$[illegible] \qquad (5.8)$$

the partial differentiation being carried out holding θ_p fixed. (Note that for fixed θ_p and ν_p (5.7) is an equation connecting κ and ω.) But a general formula for [illegible] speed u is

$$[illegible] \qquad (5.9)$$

where u is the phase speed corresponding to frequency ω, and so (5.8) tells us that the component of the ray velocity normal to the phase front [illegible] to the phase speed. The [illegible] proof of this fundamental result is a variant of the method [illegible] (1965, p. [illegible]).

Although the [illegible] field in general [illegible] is covered by the above comment, it is interesting to examine it directly. By (5.5) we have

$$[illegible] \qquad (5.10)$$

where κ^2 is the derivative with respect to [illegible]. Substitution in (5.8) gives the ray velocity. In the [illegible] we have [illegible], and so by (5.5)

$$v^{(g)} = \pm c \frac{cq^{(r)}}{Q^{(r)}_4 (c^2 + \ldots)}$$

$$= c^2 \frac{-q^{(r)}_4}{\ldots} \qquad (5.[illegible])$$

$$= v^{(r)} \frac{c^2}{c^2 + \ldots}$$

We observe that the ray velocity and the group velocity have the same direction (or possibly opposite directions) in the local rest frame, and we have

$$\frac{c}{v^{(r)}} = \frac{c^2 \ldots}{c} \ldots \quad [illegible]$$

(5.[illegible])

The right hand side is a well known form for w/g, equivalent to (5.5), and thus we verify in this particular case the more general result established above.

This brief discussion of group velocity has been included here in order to link the ideas of the present paper to ideas current in physics. But, having made this connection, one may very well forget about group velocity and think instead of ray velocity, in which it appears to be subsumed. In the theory of waves at rest group velocity seems an artificial concept. Variation of the frequency plays no role in ordinary geometrical optics. But in the present theory variation of frequency is inherent, since in the stationary principle (5.14) for rays, frequency is one of the quantities to be varied. Thus the essential idea of group velocity is built into the present theory.

The inequality (5.5) is the condition that the group velocity should not exceed c.

[illegible] [illegible]

[illegible] the following statement: [illegible]

[illegible] (25.1)

where [illegible]

This statement gives to the photon [illegible] the role of the particle in [illegible] with momentum and energy. [illegible]

The components [illegible] in (25.1) have the dimensions of energy. [illegible]

[illegible] (25.2)

where [illegible] the energy. [illegible] (25.1) we have

[illegible] (25.3)

where [illegible]

[illegible]

[illegible]

[illegible]

[illegible] (25.5)

so that we recognize it as an action integral. The integration along any world line [illegible]

$$A = [illegible] \int [illegible] \, dx_\rho \; = \; [illegible] \, [F] \, , \qquad (12.4)$$

where F [illegible] is the phase function. If the world line leads from an event on a certain phase [illegible] to an event on the next phase [illegible] with the same [illegible] [illegible] we have $[F] = [illegible]$ and therefore

$$A = h \qquad (12.5)$$

In words, <u>the action between adjacent phase [illegible] of the same phase is equal to Planck's constant.</u> (Cf. the [illegible] of [illegible] quantization, Synge (1954), p.12?)

One may raise the question: Has a photon a proper mass? Such a question is of course meaningless without a definition of proper mass. It is natural, perhaps, to take the definition appropriate to a material particle, for which the proper mass m is given in terms of energy E and 3-momentum p_ρ by

$$m^2 c^4 = E^2 - c^2 p_\rho p_\rho \qquad (12.6)$$

Then we substitute on the right from (12.1) we get

$$m^2 c^4 = [illegible] = -c^2 \, dP_\alpha \, dP_\alpha \qquad (12.7)$$

For a wave like above, [illegible] (cf. (12.1)). But in the ordinary case of a homogeneous medium, the [illegible] or equivalently dP_ρ is spacelike, it makes m^2 negative and m imaginary.

It may not be useful to speak of the proper mass of a photon at all. The should be sign for a definition which makes $n > 0$. For a resonant [illegible] and for a medium in which $n < 0$, we have [illegible] to change the sign in our definition, so that

$$n = [illegible] \qquad (10.8)$$

11. Emission of a photon by an atom. [illegible]

Let us try to form a picture of the emission of a photon by an atom in terms of geometrical optics.

First we recall the argument of Section 6. Given a [illegible] with [illegible] [illegible], we sought to draw rays, and for that we had to solve (6.7), viz.

$$[illegible] \qquad (11.1)$$

where [illegible] on [illegible]. If these equations can be solved and the [illegible] (6.6) is [illegible], we get rays emanating from [illegible] and filling a 4-dimensional region of space-time.

Suppose now that [illegible] is a [illegible] instead of a [illegible]. Then the equations (11.1) are effectively three in number (instead of four), and so [illegible] we can solve these equations to get ∞^1 values of [illegible] at each event on [illegible]. Thus we have ∞^4 rays altogether, enough to fill a 4-dimensional region of space-time as before.

If [illegible] is a world line (1-space), then we take only two equations in (11.1) and take ∞^2 values of [illegible] at each event on [illegible]. There are ∞^1 events on Σ, and so again we get rays adequate to fill a 4-dimensional region of space-time. If this world line Γ is regarded as the world line of an emitting atom (the frequency of emission defining F on it), we see space-time filled with rays. We do not think of a photon on each of these rays, but only one photon for the whole process. The rays constitute the possible histories of this one photon, and the associated values of $\hbar k_\mu$ the possible momentum-energy directions. (We can see in this picture a reflection of Heisenberg's uncertainty principle. The [illegible] frequency is assigned and the initial time completely undetermined, and the exact position, momentum and energy of the photon have that degree of indeterminacy expressed by the fact that, although we have a definite pattern of rays in space-time, we do not know which one the photon has chosen to traverse until it shows its presence by colliding with matter.)

If k_μ is the 4-velocity of the source or atom and ν_0 the assigned frequency of emission, then we have as in (11.1) the equations

$$k_\mu \lambda_\mu = -\nu_0 , \qquad [illegible] \quad (11.1)$$

to be satisfied by k_μ on the world line of the source, and the totality of rays is to be found by solving (4.2) subject to these initial conditions. The phase function F is then given by (5.2) with [illegible].

Interference phenomena can be treated only crudely in geometrical optics. If, by means of two pin-holes or otherwise, we offer two paths to the rays, we get brightness if the actions for the two rays differ by Nh and darkness if they differ by (N + ½)h, N being an integer, these two cases corresponding to differences of phase [illegible] of amounts 2πN and 2πN + π, respectively.

In terms of probability, we may say that, in the first case, it is very probable that the photon will choose one or other of the two rays and in the second case that it is very improbable that it will choose either.

Suppose we know of a photon only one event in its history. This we may call a [illegible] event. The possible values of σ_p are subject only to $\Omega(\sigma, x) = 0$. (In fact, we have (11.2) with the first equation deleted.) This means that there are ∞^3 initial values of σ_p and we would expect, as in the cases considered above, a set of rays filling a 4-dimensional region of space-time. Let us examine this question, taking an isotropic medium with $\Omega(\sigma, x)$ as in (9.3).

Writing $\xi_p = \partial\Omega/\partial\sigma_p$, [illegible] we have

$$\xi_{\mu} = [illegible]$$

(11.4)

Let us use the local rest frame, so that (9.3) becomes

$$2\Omega(\sigma, x) = \sigma_p\sigma_p + (n^2-1)\sigma_4^2 = 0, \qquad (11.5)$$

and (11.4) becomes

$$\xi_{\rho} = \sigma_{\rho}, \qquad \xi_4 = (1-n)\sigma_4 + [illegible]\sigma_4^2. \qquad (11.6)$$

Then (11.5) gives

$$\xi_{\rho}\xi_{\rho} + (n^2-1)\sigma_4^2 = 0, \qquad (11.7)$$

where σ_4 is to be regarded as a function of ξ_4 determined by the second of (11.6), in which σ_4 occurs explicitly and also implicitly in n and n'.

It appears the rays fill a 4-dimensional region of space-time, obtained by joining the origin of ξ_p [illegible] to the points on the [illegible] (11.7). But if the

the medium is non-dispersive. Then $\xi' = \xi$, ξ being a function of ρ only and hence a function of position only. The second of (11.13) reduces to

$$\xi_4 = (1 + a)\,\varpi_4 , \qquad (11.7)$$

and (11.6) reads

$$\xi_\mu \xi_\mu + a^{-1} \xi_4^2 = 0 . \qquad (11.8)$$

This is a cone, flattened within the null cone (as it should be to make the speed of the photon less than c) if $a > 0$. The fact that it is a cone means that the rays from a point-event in a non-dispersive isotropic medium fill only a 3-dimensional region in space-time. This is of course true in particular for a vacuum, the cone being then the null cone ($\xi_\mu \xi_\mu = 0$).

12. The laws of reflection and refraction.

Let S be the history of a moving surface separating media M', M. When light passes from M' into M (refraction) or is reflected back into M', a discontinuity occurs, and the variational equation (3.11) leads at once to the equation

$$(\varpi_r - \varpi'_r)\,\delta x_r = 0 \qquad (12.1)$$

for all displacements δx_r in S, ϖ'_r and ϖ_r being the slowness-frequency 4-vectors in M' and M respectively at the point of reflection or refraction, (equivalently).

$$\sigma_p = \sigma'_p + k N_p , \qquad (11.2)$$

where N_p is the unit normal to Σ, pointing from W' into W, and k is an undetermined factor. This is the law of reflection and refraction. It tells us that the increment in the slowness-frequency 4-vector is normal to the history of the surface of separation.

Regarding σ'_p as given, and also the media in which the reflection or refraction occurs, we have in (11.2) four equations for the five unknowns σ_p, k; for refraction a fifth equation is supplied by the slowness-frequency equation of W,

$$\Omega(\sigma, x) = 0 , \qquad (11.3)$$

while for reflection we are to use the slowness-frequency equation for W', say

$$\Omega'(\sigma, x) = 0 , \qquad (11.4)$$

which must be satisfied not only by σ'_p but also by σ_p.

To investigate refraction, we must see whether (11.2) and (11.3) possess solutions. If they do, we then (still) to investigate whether the refracted ray velocity passes into W. To investigate reflection, we must do the same, using (11.2) and (11.4).

In view of this need we try to simplify the algebra as far as possible by choice of the frame of reference. Suppose, as in general, that the speed of the surface of separation is less than c, the 4-vector N_p is spacelike. Let us take the x_1-axis in its direction, so that $N_1 = 1$, $N_2 = N_3 = N_4 = 0$. Then (11.2) gives

$$\sigma_1 = \sigma'_1 + k , \quad \sigma_2 = \sigma'_2 , \quad \sigma_3 = \sigma'_3 , \quad \sigma_4 = \sigma'_4 . \qquad (11.5)$$

[illegible]

[illegible] may be written

$$[illegible] \qquad (11.6)$$

[illegible]

Let us [illegible]

$$[illegible] \qquad (11.7)$$

where [illegible]

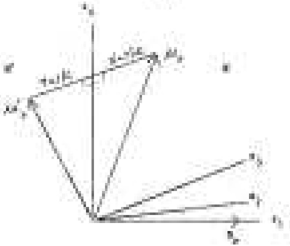
$$[illegible] = 0 \qquad (11.8)$$

[illegible]

$$[illegible] \qquad (11.9)$$

[illegible] plane frame of reference for a pair of media (Fig. 6). [illegible]

$$[illegible] \qquad (11.10)$$



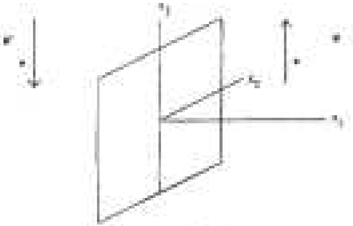


Fig. [illegible]

Standard frame of reference for a pair of [illegible].

Upper diagram: space-time.

Lower diagram: space.

and (4.7) become

[illegible equations] (11.11)

[illegible] respectively for reflection and refraction, [illegible]

[illegible equations] (11.12)

[illegible]

[illegible equation] (11.13)

[illegible]

[illegible equation] (11.14)

[illegible]

[illegible equation] (11.15)

[illegible] frequency ν [illegible] (ν^2), [illegible] ν^2 (dispersion).

[illegible]

[illegible]

for a photon to pass from M' into M, and the condition for this is $v_1 > 0$ where, by (5.1),

$$[illegible]$$ (II.18)

This we are to calculate by differentiating the expression in the first line of (II.4.1). Assuming M non-dispersive for simplicity, we get

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$ (II.19)

Now assuming (II.1.1) satisfied, the first of (II.23) gives two real values of [illegible], one the negative of the other. One makes v_1 positive and the other makes it negative. This means that the satisfaction of (II.1.1) (or equivalently (II.4.1) or (II.25)) ensures the existence of a refracted ray proceeding from M' into M, and only one such ray. If dispersion is admitted, there is no such simple result in general.

Actually, the case of equality in (II.1.1) should be looked into, as then we have $v_1 = 0$. We may say that the condition for <u>total reflection</u> is

$$[illegible]$$ (II.20)

We recall that u_1' is the component of incident phase-velocity normal to the surface of separation (fixed in our standard frame) and u_2' is the component in the direction of motion of the fluid medium; V is the speed of either medium.

We can check (11.15) by reducing both media to rest, so that $v = 0$, $\beta = 1$, $u' = c/n'$, $\sin i = u_1'/u'$, where i is the angle of incidence. Then (11.20) becomes

$$n'^2 \sin^2 i + n^2 - n'^2 \leq 0 , \qquad (11.20)$$

which is immediately recognized as the elementary formula for total reflection.

To discuss reflection we use the second of (11.6) instead of the first, or, for isotropic media with the standard frame of reference, the second of (11.12) instead of the first. But σ'_μ satisfy the second of (11.11), and so the equation simplifies to

$$(\sigma'_1 + k)^2 - \sigma'^2_1 = 0 . \qquad (11.21)$$

This gives the two solutions

$$k = 0 , \qquad k = -2\sigma'_1 . \qquad (11.22)$$

The first is ruled out (it simply gives the incident ray). From the second we get for the reflected wave-number-frequency 4-vector σ''_μ

$$\sigma''_1 = -\sigma'_1 , \qquad \sigma''_2 = \sigma'_2 , \qquad \sigma''_3 = \sigma'_3 , \qquad \sigma''_4 = \sigma'_4 . \qquad (11.23)$$

This is the very simple law of reflection when the standard frame of reference is used.

Since the momentum-energy 4-vector of a photon is $p_\mu = h\sigma_\mu$, and this is changed abruptly by reflection or refraction, we conclude that a [illegible] impulse acts on the photon and (in view of the conservation of momentum and energy) that an equal and opposite [illegible] impulse acts on the mirror or medium. By (11.[illegible]) the [illegible] impulse on the photon is

$$I_{r} = \quad [illegible] \qquad (11.15)$$

[illegible] being determined by (11.[illegible]), (11.[illegible]) for reflection and by (11.[illegible]), (11.[illegible]) for refraction. For a fixed mirror this gives an impulse of magnitude [illegible] with no fourth component. For a pair of isotropic media referred to their standard frame the energy of a photon is unchanged by reflection or refraction, and the only component of momentum to be changed is that along the normal to the (fixed) surface of separation.

15. Connection with standard Hamiltonian dynamics.

The theory of this paper has been developed by methods very close to those used by Hamilton in his geometrical optics, and if I did not translate the results [illegible] to the more more familiar language of Hamiltonian dynamics the reader might feel [illegible] and that what we have developed is in fact the Hamiltonian dynamics of a photon. This is not a modification of Hamiltonian dynamics (e.g. with replacement of ordinary time by proper time). It is precisely Hamiltonian dynamics of the form familiar in Newtonian theory.

The best way to make the translation is by means of a dictionary of notation, one column containing symbols used in the present paper and another those ordinarily encountered in dynamics, with the [illegible] indicated. The simplicity of [illegible] for motion [illegible]

Present theory			Standard notation
Coordinates	[illegible]	[illegible]	[illegible]
Time	[illegible]	[illegible]	[illegible]
Momentum	[illegible]	[illegible]	[illegible]
Energy	[illegible]	[illegible]	[illegible]

Hamiltonian* [illegible]

Equations of motion [illegible]

[illegible] (II.4)

Variational principle

[illegible]

Note that in [illegible] and [illegible] the familiar Hamiltonian equations of motion and action principle are the exact transcriptions of the corresponding formulas in the present theory.

In standard (classical) theory [illegible] we are used to starting with a Lagrangian $L = L(q, q', t)$ where $q' = dq/dt$ and Hamilton's principle

---

* [illegible]

$$\delta \int L\, dt = 0 \,, \qquad (15.3)$$

We define the generalized momenta by

$$p_r = \partial L / \partial \dot{q}_r \,, \qquad (15.4)$$

and solve these equations for $\dot{q}_r$. We define the Hamiltonian by

$$H(p, q, t) = p_r \dot{q}_r - L(q, \dot{q}, t) \,. \qquad (15.5)$$

Then

$$L\, dt = p_r\, dq_r - H\, dt \,, \qquad (15.6)$$

and we establish the equivalence of Hamilton's principle in the form (15.3) and the action principle written second in (15.2).

On the other hand we might start with $H(p, q, t)$, define $\dot{q}_r$ by

$$\dot{q}_r = \partial H / \partial p_r \,, \qquad (15.7)$$

solve for p_r, and define L by

$$L(q, \dot{q}, t) = p_r \dot{q}_r - H(p, q, t) \,, \qquad (15.8)$$

which is simply (15.5) written a little differently.

This second plan is not the usual one in classical dynamics, but it is essentially the plan we must follow here if we are to obtain a Lagrangian for our quantum dynamics. In view of the [illegible], we can do this by the method of (15.7), (15.8), but it is better (having seen the connection with classical dynamics) to go back to our second method, which takes [illegible] (q', t) in a general form and not in the special form $[illegible]_r = [illegible]$.

The connection is contained in the equation (cf. [illegible] ([illegible]), p. [illegible]). I have written [illegible] instead of [illegible]

14. Determination of the [illegible] for an isotropic medium in general motion.

For an isotropic medium in general motion we have the [illegible]-frequency equation (9.x)

$$[illegible] \qquad (14.1)$$

where for the scalar product of any two four-vectors we write

$$[illegible] \qquad (14.2)$$

and where

$$[illegible] \qquad (14.3)$$

We shall denote by n' the partial derivative of n with respect to [illegible]. We recall that n is the refractive index, computed in the ordinary way for the local rest frame, and that in that frame the frequency is [illegible].

If we are not given the explicit form of the dependence of n on [illegible], we cannot here carry out explicitly the determination of the [illegible] function [illegible] as described in the preceding section. Even if we are given that form, the algebra may be very complicated. But we can at least analyse the problem with a view to explicit calculations in the next section for a medium which is only slightly dispersive. Here we proceed without approximation.

The basic equations are (11.2) and (11.x). We have that

$$[illegible] \qquad (14.4)$$

is to be solved for [illegible]. Multiplication by μ_p gives, since $\mu_p \mu_p = -1$,

$$[illegible] \qquad (14.5)$$

$$\theta \alpha_r = \frac{\partial \Omega}{\partial \sigma_r}, \qquad \Omega(\sigma, x) =$$

with

$$\alpha_r \alpha_r = -1 .$$

The plan is to solve the first of (13.?) (four
($\theta \alpha_r$, x_r) , substitute these values in the
as a function of (α_r , x_r) ; hence we obta
Substitution in the last of (13.9) gives

$$f = f(x, \alpha) .$$

Taking any timelike world line with unit tangen

$$f(x, \alpha)\, ds = -\theta^{-1} \sigma_r \frac{\partial \Omega}{\partial \sigma_r} ds =$$

and so we reconcile the motion principle $\delta \int$

or, by [illegible],

$$[illegible]$$

$$[illegible] \qquad (14.[illegible])$$

[illegible] functions [illegible] and [illegible] are known.

The [illegible] [illegible] (14.[illegible]) [illegible] (14.[illegible]) [illegible]

[illegible] (14.[illegible]) [illegible] (14.[illegible]). By (14.[illegible]) we have

$$[illegible] \qquad (14.[illegible])$$

Substituting in (14.[illegible]) and dividing by [illegible], we get

$$[illegible] \qquad (14.[illegible])$$

This equation has to be solved for [illegible] [illegible] (14.[illegible]). By (14.[illegible]) [illegible]

$$[illegible] \qquad (14.[illegible])$$

The case of a non-dispersive medium is simpler. For such a medium [illegible] and [illegible] disappears from (14.[illegible]), which reduces to

$$[illegible] \qquad (14.[illegible])$$

in which [illegible] is a function of [illegible] only. In fact, [illegible]

This is to be regarded as an equation for $(\sigma\mu)$, which appears not only explicitly, but also in q and q'. Let the solution be

$$(\sigma\mu) = G[\theta(\alpha\mu), \rho], \qquad (14.6)$$

a function of two variables, now supposed known. Then by (14.4)

$$\sigma_r = \theta\alpha_r + \mu_r(q\,G + \tfrac{2}{3}q'\,G^2); \qquad (14.7)$$

but by (14.5)

$$q\,G + \tfrac{2}{3}q'\,G^2 = \theta(\alpha\mu) - G, \qquad (14.8)$$

and so

$$\sigma_r = \theta\alpha_r + \mu_r[\theta(\alpha\mu) - G]. \qquad (14.9)$$

Hence

$$(\sigma\sigma) = -\theta^2 + 2\theta(\alpha\mu)[\theta(\alpha\mu) - G] - [\theta(\alpha\mu) - G]^2$$

$$= \theta^2[(\alpha\mu)^2 - 1] - G^2. \qquad (14.10)$$

Our instructions are to substitute for σ_r in (14.1), which is the second of (13.9), and we do this by substituting (14.6) and (14.10); we get

$$\theta^2[(\alpha\mu)^2 - 1] - (1+q)\,G^2 = 0. \qquad (14.11)$$

Now q is a function of G and ρ, and G is a function of θ, $(\alpha\mu)$ and ρ; thus (14.11) is an equation to determine θ as a function of $(\alpha\mu)$ and ρ; let the solution be

$$\theta = K[(\alpha\mu), \rho]. \qquad (14.12)$$

By the last of (13.9) combined with (14.4) we have

$$\theta f = -\sigma_r\frac{\partial\Omega}{\partial\sigma_r} = -[(\sigma\sigma) - q(\sigma\mu)^2 - \tfrac{4}{3}q'(\sigma\mu)^3]. \qquad (14.13)$$

15. [illegible] Function ([illegible]) for a special class of media

In the following formula [illegible], let $1/\lambda$ be large. Expanding and retaining only the leading terms, we get [cf. [illegible]]

$$\kappa^2 - 1 = A(\rho) + B(\rho)\,\nu^{[illegible]}, \qquad (15.1)$$

where

$$A(\rho) = \sum_m a_m, \qquad B(\rho) = \sum_m b_m / [illegible], \qquad (15.2)$$

When $B(\rho)$ is small, $B = 0$ gives a non-dispersive medium, and we may think of the formula (15.1) as referring to a slightly dispersive medium.

Let us apply the method of Section 14 to find the [illegible] function ([illegible]) of a medium for which (15.1) holds. The work applies in particular to a slightly dispersive medium (B small), but the approximations are [illegible], and it is clearer to accept (15.1) as an exact formula and make no approximations based on the smallness of B. Accordingly in the notation of Section 14 we have

$$[illegible] \qquad (15.3)$$

The [illegible] in [illegible], and [illegible] (15.3) [illegible] in B. [illegible] the calculation [illegible] implicitly.

To simplify the [illegible], we note that $[illegible]$ is an invariant, and its value may be obtained by taking the local rest frame; we have

$$[illegible] \qquad (15.4)$$

where

$$[illegible] \qquad (15.5)$$

non-dispersive medium, and the reason is not hard to see. A Lagrangian principle of the form $\delta \int f\,ds = 0$ can be expected to hold only if two arbitrary events (or at least events arbitrary between limits) can be joined by an extremal, or equivalently if the rays from an event fill a 4-dimensional region in space-time. That is not the case for a non-dispersive medium. Only a cone is filled, as indicated in (11.8). It is easy to reconcile (11.8), which uses the local rest frame, with the more general formula (14.18): in the local rest frame we have $\mu_\rho = 0$, $\mu_4 = 1$, and (14.18) may be written in the following equivalent forms:

$$-q\,\alpha_4^2 = 1 + q,$$

$$q\,\alpha_4^2 = (1+q)\,\alpha_r\,\alpha_r, \qquad (14.19)$$

$$(1+q)\,\alpha_\rho\,\alpha_\rho + \alpha_4^2 = 0,$$

$$n^2\,\alpha_\rho\,\alpha_\rho + \alpha_4^2 = 0,$$

the last of which is the same as (11.8) in a different notation.

We see that the Lagrangian or medium function does not exist for a non-dispersive medium, and even in the case of a dispersive medium its determination in explicit form is in general not possible. It is in fact better to discuss rays by means of the Hamiltonian equations (8.3) instead of trying to make use of the Euler-Lagrange equations (13.14): the equations (8.3) can be used for non-dispersive media as well as for dispersive media.

[illegible] (Hamiltonian). [illegible], for a [illegible] medium [illegible] index as in (25.[illegible]), the rays satisfy the stationary principle [illegible] with [illegible] as in (25.[illegible]), [illegible] being given by (25.[illegible]) with [illegible] as in (25.[illegible]).

The stationary principle may also be written

[illegible] (25.45)

If the medium is an isotropic medium, [illegible], then the stationary principle becomes

[illegible] (25.46)

If, further, the medium is homogeneous, so that [illegible] and [illegible] are constants, we have

[illegible] (25.47)

[illegible]. This stationary principle is not to be confused with Fermat's principle for which the frequency is held fixed [illegible]. [illegible] (25.45) is a Hamiltonian principle [illegible], whereas Fermat's principle is the analogue of Jacobi's stationary principle, usually written in dynamical theory as [illegible].

If we wish to use the Euler-Lagrange equations (2[illegible]) for the rays, we must be careful to make [illegible] homogeneous of degree unity in the [illegible] before taking the partial derivatives [illegible]. This homogeneity is obtained in (25.45) if, in [illegible] equal to the local rest frame [illegible] group speed). Then

$$[illegible] \qquad (15.6)$$

We have

$$[illegible] \qquad (15.7)$$

and if we define X by

$$[illegible] \qquad (15.8)$$

(14.16) gives the following quadratic equation for X:

$$[illegible] = 0 , \qquad (15.9)$$

Hence, selecting the positive root,

$$X = [illegible] \qquad (15.10)$$

Then, taking [illegible] to be positive, we have

$$[illegible] \qquad (15.11)$$

and (14.17) gives

$$[illegible] \qquad (15.12)$$

Remembering that [illegible] is given in terms of $(k_0 \mu)$ by (15.13), we have here the

[illegible] (15.3)

[illegible] [illegible]

[illegible] (15.4)

where [illegible] is a constant. [illegible] equations (8.5)

[illegible] (15.5)

[illegible]

[illegible] (15.6)

[illegible]

[illegible] (15.7)

[illegible]

substituting for k from (15.10), we may [illegible] of [illegible] by using for β^2 the following expression, equivalent to (15.[illegible])

$$\beta^2 = \frac{[illegible]}{[illegible]} , \qquad (15.15)$$

16. Waves in a rotating medium

On account of the [illegible] set up by rotation, we [illegible] discuss the passage of light through a cylinder of glass spinning about its axis [illegible] in the sense of Section 7. But we may reasonably assume that a rotating fluid [illegible] on the local rest frame, so that the phase-and-frequency equation is of the form (7.1).

Consider, then, a fluid (e.g. water) spinning about the x_3-axis with constant angular velocity ω, so that its velocity is

$$v_1 = -\omega x_2 , \quad v_2 = \omega x_1 , \quad v_3 = 0 , \qquad (16.1)$$
$$v^2 = \omega^2 r^2 , \quad r^2 = x_1^2 + x_2^2 .$$

Putting

$$\gamma = (1 - v^2/c^2)^{-\frac{1}{2}} = (1 - \omega^2 r^2 / c^2)^{-\frac{1}{2}} , \qquad (16.2)$$
$$u(r) = \omega \gamma_{0} / c ,$$

we have for the 4-velocity λ_μ of the medium

[illegible] (15.14)

[illegible] (15.15)

Thus

[illegible] = constant , (15.16)

which tells us a [illegible]

By (15.4) we have

[illegible] (15.17)

and so (15.14) may be written

[illegible] (15.18)

Note that the right hand side consists of constants and known functions of r.
We have also by (15.8)

[illegible] (15.19)

and so (15.20) gives a differential equation of the second order to determine r as a function of ξ :

[illegible] (15.21)

[illegible]

$$\sigma_1 = \text{constant}, \qquad \sigma_2 = \text{constant}, \qquad (16.8)$$

on account of which we may say that a *charge conserves the axial component of momentum and the energy* [cf. (16.4)].

Let us define P and $\bar{P}$ by

$$P = \sigma_1 x_1 + \sigma_2 x_2, \qquad \bar{P} = \sigma_2 x_1 - \sigma_1 x_2 ; \qquad (16.9)$$

then, by (16.7),

$$\sigma_1 \mu_1 + \sigma_2 \mu_2 = E P, \qquad \sigma_2 \mu_1 - \sigma_1 \mu_2 = -E \bar{P}, \qquad (16.10)$$

[illegible]

Also by (16.1) with $r^2 = x_1^2 + x_2^2$

$$\mu_{1,2} = -k' x_1 x_2 / r, \qquad \mu_{1,1} = -k - k' x_2^2 / r,$$
$$\mu_{2,1} = k + k' x_1^2 / r, \qquad \mu_{2,2} = k' x_1 x_2 / r, \qquad (16.11)$$

so that

$$x_1 \mu_{1,1} + x_2 \mu_{1,2} = -(k + k' r) x_2,$$
$$x_1 \mu_{2,1} + x_2 \mu_{2,2} = (k + k' r) x_1, \qquad (16.12)$$

and

$$x_2 \mu_{1,2} - x_2 \mu_{2,1} = -k x_1,$$
$$x_1 \mu_{1,2} - x_1 \mu_{2,1} = -k x_2 . \qquad (16.13)$$

By now differentiating P and $\bar{P}$ as given in (16.9), substituting for the derivatives of $x_1, x_2, \sigma_1, \sigma_2$ from (16.6), and making use of (16.10), (16.11), (16.12), this shows or

$$[illegible] \qquad (16.29)$$

Thus ... has been found as a function of ξ. From (16.15), then (16.29) gives ... as a function of ξ. Thus the path of the photon is determined. As for time-dependence, we have by (16.8)

$$[illegible] \qquad (16.30)$$

Since it is easy to get a first integral of (16.30) (multiply by $d(\xi^2)/d\sigma_1$), the complete determination of the path is essentially reduced to quadratures.

The formulae are much simplified if the ray starts from the axis of rotation, the angular momentum of the photon about that axis being then zero. Let us take as initial conditions, for $\xi = 0$,

$$[illegible] \qquad (16.31)$$

θ_0 means the inclination of the initial ray to the axis of rotation. By (16.2), (16.14) we have

$$[illegible] \qquad (16.32)$$

$$[illegible] \qquad \tau_{\zeta} = [illegible] \qquad [illegible] \qquad (16.31)$$

Hence, by (16.17), we have

$$[illegible] \qquad (16.32)$$

and σ_{ζ} is a constant, [illegible] from η to ζ. [illegible]

[illegible] (16.37) [illegible]. From (16.1) we find

$$[illegible] \qquad (16.33)$$

and if we define a real dimensionless constant K by

$$K = [illegible] \qquad (16.34)$$

our equation for ψ becomes

$$[illegible] \qquad (16.35)$$

or, by (16.32),

$$[illegible] \qquad (16.36)$$

Everything on the [illegible] is a constant except r.

We now introduce the cylindrical angle θ by

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad \tan\theta = x_2/x_1, \qquad (16.37)$$

and obtain by (16.8)

$$[illegible] \qquad (16.38)$$

Hence

$$[illegible] \qquad (15.41)$$

so that

$$[illegible] \qquad (15.42)$$

If we choose the axis so that [illegible] initially for the ray under discussion. This gives us the angular [illegible], note that it is proportional to x_3.

So far all calculations are exact. Let us now approximate taking [illegible] small. Expanding the [illegible] in (15.20) we get

$$[illegible] \qquad (15.43)$$

where

$$[illegible] \qquad (15.44)$$

Hence

$$[illegible] \qquad (15.45)$$

or

$$[illegible] \qquad (15.46)$$

to the order ω^2/c^2 inclusive. The first term corresponds to rectilinear propagation; the second term indicates a curvature of the ray away from the axis of rotation. To the same order [illegible] gives for the rate of twisting of the ray

$$[illegible] \qquad (15.47)$$

and hence

$$[illegible] \qquad (15.48)$$

and the equation (16.10) for [illegible] simplifies to

$$\frac{d}{dx_1}(k\,r^2) = -1 - \frac{\alpha r_1^2}{r^2}\left\{1 + \frac{[illegible]}{1 - \omega^2 r^2 / c^2}\right\}, \qquad (16.11)$$

Applying (16.6), we get

$$\tan^2\theta_1 = [illegible], \qquad \frac{[illegible]}{[illegible]} = \frac{1}{2}\sec\theta_1 . \qquad (16.14)$$

and so (16.11) may be written

$$\frac{d}{dx_1}(k\,r^2) = \frac{1}{2}\sec^2\theta_1 - 1 + \frac{[illegible]\sec^2\theta_1}{r^2\,(1 - \omega^2 r^2 / c^2)} . \qquad (16.15)$$

Multiplying by $4(r^2)/dx_1$ and integrating, we get

$$\left(\frac{d}{dx_1}(k\,r^2)\right)^2 = r^2\left(\frac{1}{r^2}\sec^2\theta_1 - 1\right) - \frac{c^2\,[illegible]\sec^2\theta_1}{r^2\,\omega^2}\log\left(1 - \omega^2 r^2 / c^2\right),$$

$$(16.16)$$

which, if we define θ by

$$\tan\theta = dr / dx_1 , \qquad (16.17)$$

may be expressed as

$$\tan^2\theta = \frac{1}{2}\sec^2\theta_1 - 1 - \frac{c^2\,[illegible]\sec^2\theta_1}{r^2\,\omega^2\,r^2}\log\left(1 - \omega^2 r^2 / c^2\right) . \qquad (16.18)$$

As for [illegible] and x_1, we substitute from (16.17) and (16.18) in (16.[illegible]) and (16.[illegible]), obtaining

$$\frac{dx}{dx_1} = [illegible]\,\frac{\sec\theta_1}{1 - \omega^2 r^2 / c^2}, \qquad (16.19)$$

$$\frac{1}{2}\frac{dx_4}{dx_1} = \frac{1}{2}\sec\theta_1\left(1 + \frac{1}{1 - \omega^2 r^2 / c^2}\right) . \qquad (16.20)$$

momenta are ϖ_p, the "time" is χ, the usual Hamiltonian function is Ω; this Hamiltonian is independent of the "time", and so (17.1) possess the first integral

$$\Omega(\varpi_p, x) = \text{const.} \qquad (17.2)$$

This constant does not have to vanish.

If $\Omega(\varpi, x) = 0$ happens to be the slowness-frequency equation of some optical problem, then the solutions of (17.1) provide us with the rays, as previously discussed. But these rays are included in the wider class of solutions of (17.1), obtained as one might with arbitrary initial values of x_p, ϖ_p, and subject to $\Omega(\varpi, x) = 0$. This wider viewpoint may be of advantage in discussing systems of rays, because it brings us more closely in touch with classical theory. For example, we may think in terms of a phase-space of 2N dimensions, in which the coordinates are x_p, ϖ_p, χ, Ω, and then we recognize

$$\oint (\varpi_p dx_p - \Omega d\chi) \qquad (17.3)$$

as that relative integral invariant which is usually written

$$\oint (y_p dx_p - H dt) . \qquad (17.4)$$

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[illegible] (16.[illegible]),

$$[illegible] \qquad (16.[illegible])$$

We recall that the argument of the present section [illegible] the propagation of light in an isotropic medium rotating about an axis with constant angular velocity, the [illegible] and of [illegible] refractive index is a constant. [illegible] (16.[illegible]) [illegible] and finally [illegible] small.

17. [illegible]

By the [illegible] of Section [illegible] a photon appears as a [illegible] system with three degrees of freedom [illegible] coordinates x_r, momenta ϖ_r [illegible] and ϖ_4 [illegible] energy [illegible]. But there is another way of looking at [illegible].*

Let $\Omega(\varpi, x)$ be any given function of the eight variables ϖ_ρ, x_ρ. [illegible] the equations

$$\frac{dx_\rho}{d\lambda} = \frac{\partial\Omega}{\partial\varpi_\rho}, \qquad \frac{d\varpi_\rho}{d\lambda} = -\frac{\partial\Omega}{\partial x_\rho}, \qquad (17.1)$$

[illegible] equations of motion in canonical form for a [illegible] system with four degrees of freedom, the coordinates are x_ρ, the

* Suggested by Professor [illegible].