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Relativistic baryons in the Skyrme model revisited

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Abstract

Starting from the static Skyrme model baryon wavefunctions in their helicity eigenstates, we generalize the wavefunctions to the non-static and relativistic regime. A new representation for gamma matrices in the SU(2) collective space is constructed and the corresponding Dirac equation is obtained. Finally, we comment on possible applications of our results to the calculation of matrix elements of baryonic currents and the corresponding form factors in the relativistic case.

1. Introduction

The Skyrme ($SU(2) \times SU(2)$ chiral) model [1] has attracted renewed interest as a simplified description of baryons in the large N_c limit of QCD. The static properties of baryons in the Skyrme model (skyrmions) have been studied a long time ago [2,3]. Some properties of slowly moving skyrmions have been studied in various publications [4,5].
In [6], Sakai and Sugimoto derive the Skyrme model as the $3+1$ dimensional pion effective theory descending from the dynamics of the gauge fields living on the flavor $D8$-branes that probe the $D4$-brane geometry generated by a stack of color branes. The basic idea is to treat baryons in holographic theories as solitons in an effective theory of mesons which at low energies reduces to a non-linear sigma model with broken chiral symmetry. Further restriction to two flavors only, i.e. $SU(2)$ flavor symmetry, results in a vanishing Wess-Zumino term. This yields precisely the above-mentioned Skyrme model.
The purpose of this letter is to calculate non-static baryon wave functions in their helicity eigenstates and generalize the results of [2] to the relativistic case. This will be presented in a form compatible with the Dirac spinor representation that eventually will allow us to consider relativistic baryons by boosting in any direction. Here, we are disregarding relativistic corrections to the non-relativistic Lagrangian obtained by quantizing slowly moving baryons via the moduli space approximation method. Related works discussing corrections to the Lagrangian of the collective states can be found in [7,8,9]. The approach discussed in this note provides a relativistic description of the baryon wavefunctions that allows us to correctly decompose the matrix elements of baryonic currents and properly define the associated form factors.

2. From Skyrmions to Instantons (and back)

In this section we briefly review the Skyrme theory and its link to instantons and holography. We follow closely the previous works [10,11]. Skyrmions are soliton solutions of a nonlinear effective field theory of pions. The model is nonrenormalizable and holographic models [6,12,13,14,15] provide a UV completion. The action of the Skyrme model is given by

$$S = \int d^4x \left(\frac{f_\pi^2}{4} \operatorname{tr}\left(U^{-1}\partial_\mu U\right)^2 + \frac{1}{32e^2} \operatorname{tr}\left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U\right]^2 \right) \tag{2.1}$$

where the pion fields $\pi(x^\mu)$ are encoded in the SU(2) valued Skyrme field,

$$U(x^\mu) = e^{2i\pi(x^\mu)/f_\pi}. \tag{2.2}$$

The Skyrmion is a static solution that goes to a constant value at infinity which compactifies the space to S^3, and a topological charge $n_B \in \pi_3(SU(2)) = \mathbb{Z}$ is identified with the baryon number,

$$n_B = -\frac{1}{24\pi^2} \int \epsilon_{ijk} \mathrm{Tr}\left(R_i R_j R_k\right) d^3x, \tag{2.3}$$

where $R_i = \partial_i U\, U^{-1}$. It is worth noting that the Skyrmion field does not saturate the BPS bound,

$$E_{\text{Skyrmion}} \geq 12\pi^2 |n_B|, \tag{2.4}$$

although it approaches the bound for $n_B \to \infty$.
In a seminal publication, Atiyah and Manton [10] showed that (static) skyrmions can be approximated remarkably well by generating fields $U_{\text{inst.}}$ analytically from the holonomy of a SU(2) Yang-Mills instanton A_I on $\mathbb{R}^4$, i.e.,

$$U_{\text{inst.}}(\vec{x}) = \pm \mathcal{P}\text{exp} \int_{-\infty}^{+\infty} A_4(\vec{x}, x_4) dx_4, \tag{2.5}$$

where $\mathcal{P}$ denotes path ordering, $\vec{x} \in \mathbb{R}^3$, and x_4 denotes time. In holographic models [12,16], x_4 denotes the holographic (radial) direction. Then one can show that the instanton number precisely agrees with the baryon number n_B defined above and that, for a suitable choice of scale, the skyrmions are approximated very well (the energies usually lie within a percent of the corresponding numerical Skyrme solution).
Another significant extention of the Skyrme model is the Hidden Local Symmetry approach [17]. The Skyrme model is a theory of pions that lacks the presence of other mesons as it is expected in the large N_c limit of QCD. In the Hidden Local Symmetry model, the diagonal $SU(2)$ subgroup of the chiral symmetry $SU(2)_L \times SU(2)_R$ is gauged and gives rise to vector mesons.
In holographic models, the effective action on the probe brane typically leads to a Yang-Mills-Chern-Simons theory in a curved five-dimensional background space-time whose solitons are identified with the baryons of the theory. However, since the soliton solution is at present unknown, one usually argues that when the Chern-Simons coupling and hence the size of the of the soliton are sufficiently small, it is possible to invoke the flat space Yang-Mills instanton approximation. The resulting 3+1-dimensional effective theory, obtained via a Kaluza-Klein expansion of the holographic direction, contains the Skyrme field and an infinite tower of massive vector mesons. Such holographic theories are a realization of hidden local symmetry where the couplings of the vector mesons are fixed by the Kaluza-Klein mechanism. This extended Skyrme model reduces to the standard Skyrme model upon neglecting all the massive vector meson modes. In the next section we review aspects of quantization of the extended holographic Skyrme model which includes hidden local symmetry and vector mesons, since it has a stronger phenomenological appeal.

3. SU(2) collective coordinates

We begin by reviewing the moduli space approximation method to quantize slowly moving solitons. As usual [2,17,12], the fundamental idea is to approximate the slowly moving soliton by a classical soliton solution where the $SU(2)$ moduli are replaced by time-dependent collective coordinates,

$$A_M(t, x^M) = V A_M^{cl}(x^M, X^M(t)) V^{-1} - iV\partial_M V^{-1}, \tag{3.1}$$

where $V = V(t, x^M)$ is an $SU(2)$ element and X^M, $M = \{1, \dots, 4\}$, represents the position of the soliton in the spatial $\mathbb{R}^4$. Introducing the collective coordinates $\boldsymbol{a}(t) = a_4(t) + i a_a(t)\tau^a$ as a point in S^3 representing the $SU(2)$ orientation, where $a_I = (a_1, a_2, a_3, a_4)$ is a unit vector in $\mathbb{R}^4$, we note $V(t,x) \to \boldsymbol{a}(t)$ as $x_4 \to \infty$. The classical BPST instanton configuration [12] is given by

$$A^{cl}_M(x^M) = -i\,\frac{\xi^2}{\xi^2+\rho^2} g\partial_M g^{-1}\ ,\ g(x^M) = \frac{(x^4 - X^4) - i(\vec{x}-\vec{X})\cdot\vec{\tau}}{\xi}. \tag{3.2}$$

Let us consider the part of the non-relativistic Hamiltonian related to the collective coordinates obtained from the instanton quantization,

$$H = \sum_{I=1}^{4}\left(-\frac{1}{2m}\frac{\partial^2}{\partial(\rho a_I)^2} + \frac{1}{2}m\omega_\rho^2\rho^2 a_I^2\right) = \frac{1}{2m}\left(\frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho) + \frac{1}{\rho^3}(\nabla^2_{S^3})\right), \tag{3.3}$$

where ρ is the radius and m is the mass of the instanton. The eigenstates (wavefunctions) are factorized into radial and spherical harmonics components. On the S^3, they are scalar spherical harmonics and are known to be homogenous polynomials

$$T^{(l)}(a_I) = C_{I_1\cdots I_l}\, a_{I_1}\cdots a_{I_l}\ , \tag{3.4}$$

where $C_{I_1\cdots I_l}$ is a traceless (in any 2 indexes) symmetric tensor of rank l. They satisfy

$$\nabla^2_{S^3} T^{(l)} = -l(l+2)T^{(l)}\ , \tag{3.5}$$

The dimension of the tensor C and therefore the dimension of the space of spherical harmonics of degree l, $\mathbb{H}_l$ is $(l+1)^2$. The space $\mathbb{H}_l$ is a representation of the rotation group $SO(4)$ and corresponds to the $(S_{l/2}, S_{l/2})$ representation of $(SU(2)\times SU(2))/\mathbb{Z}_2 \simeq SO(4)$. $S_{l/2}$ denotes the spin $l/2$ representation of $SU(2)$, with $\dim S_{l/2} = l+1$. Under a group rotation $\boldsymbol{a}$ transforms as

$$\boldsymbol{a} \to g_I\, \boldsymbol{a}\, g_J\ ,\quad g_{I,J} \in SU(2)_{I,J}\ . \tag{3.6}$$

where $SU(2)_I$ and $SU(2)_J$ are identified with the isospin rotation and the spatial rotation, respectively. In order to shed some light on this physical interpretation, we note from equation (3.1) that $SU(2)_I$ isospin rotation of the gauge fields corresponds to

$$V \to g_I V\ ,\quad g_I \in SU(2)_I\ . \tag{3.7}$$

Under spatial $SO(3)$ rotation, the coordinates change according to

$$x^i \to x'^i = \Lambda^i_j x^j. \tag{3.8}$$

Hence,

$$g(x) \to g(\Lambda x) = u^{-1}g(x)u\ ,\ \ u \in SU(2), \tag{3.9}$$

where we used the fact that there is a homomorphism between $SO(3)$ and $SU(2)$ groups and every $SO(3)$ transformation Λ in $\mathbb{R}^3$ corresponds to a $SU(2)$ transformation u in the space of anti-hermitian matrices $i\vec{x}.\vec{\tau}$ as in (3.9). Under spatial rotation Λ,

$$\begin{aligned} A^{cl\,i}(x) \to A'^{cl\,i}(x') &= g(\Lambda x)\Lambda^i_j\partial^j g(\Lambda x)^{-1} = u^{-1}g(x)u\Lambda^i_j\partial^j u^{-1}g(x)^{-1}u \\ &= \Lambda^i_j u^{-1}g(x)\partial^j g(x)^{-1}u. \end{aligned} \tag{3.10}$$

where in the last line we made use of the fact that the u matrix does not depend on x .With respect to the moduli, the gauge field transforms as

$$\begin{aligned} A^i(t,x) &\to A'^i(t,x') = \Lambda^i_j A^j(t,x) = \Lambda^i_j (V A^{cl\,j}(x) V^{-1} - iV\partial^j V^{-1}) \\ = &\quad V' A'^{cl\,i}(x') V'^{-1} - iV'\Lambda^i_j \partial^j V'^{-1} = \Lambda^i_j (V' u^{-1} g(x) \partial^j g(x)^{-1} u V'^{-1} - iV' \partial^j V'^{-1}) \\ = &\quad \Lambda^i_j (V' u^{-1} A^{cl\,j}(x) u V'^{-1} - iV'\partial^j V'^{-1}). \end{aligned} \tag{3.11}$$

where in the first line it should be noted that A^i transforms as a vector and in the second and third lines we used (3.9) and (3.10). Comparing the first and third lines of eq. (3.11), we find that

$$V'(t,x')u^{-1} = V(x). \tag{3.12}$$

Identifying $u = g_J \in SU(2)_J$, we find that V transforms under spatial rotation $SO(3) \simeq SU(2)_J$,

$$V \to V' = V g_J \ , \quad g_J \in SU(2)_J \ . \tag{3.13}$$

3.1. Non-relativistic baryons

The instanton collective state is quantized considering slowly moving instantons. Therefore it is related to non-relativistic baryons. We will extend this analysis to relativistic baryons by simply treating them as static instantons boosted in a given direction, disregarding further relativistic corrections. First, we relate the spherical harmonics tensors to static nucleons. The lowest states are at $l = 1$ and the tensors become linear in a_I coordinates. They correspond to states with spin and isospin 1/2 and we identify them with protons and neutrons. In spinorial notation we write the particle states as

$$|N,h\rangle = \chi^N \otimes \chi_h =: \chi^N_h , \tag{3.14}$$

where $N = \{p,n\}$, $h = \{+,-\}$ and

$$\chi^p = \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \chi^n = \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.15}$$

The isospin I_3 and spin J_3 operators in this representation read

$$I_a = \frac{i}{2}\left(a_4 \frac{\partial}{\partial a_a} - a_a \frac{\partial}{\partial a_4} - \epsilon_{abc}\, a_b \frac{\partial}{\partial a_c} \right) \ , \quad J_a = \frac{i}{2}\left(-a_4 \frac{\partial}{\partial a_a} + a_a \frac{\partial}{\partial a_4} - \epsilon_{abc}\, a_b \frac{\partial}{\partial a_c} \right), \tag{3.16}$$

and their eigenstates are given by

$$\begin{aligned} |p,+\rangle &= \frac{a_1 + ia_2}{\pi} \quad , \quad |p,-\rangle = -\frac{i}{\pi}(a_4 - ia_3), \\ |n,+\rangle &= \frac{i(a_4 + ia_3)}{\pi} \quad , \quad |n,-\rangle = -\frac{1}{\pi}(a_1 - ia_2). \end{aligned} \tag{3.17}$$

One of the objectives of the subsequent section is to generalize these expressions to the relativistic case.

3.2. Nucleon relativistic wavefunctions

Initially, we will restrict the discussion to proton and neutron states separately and disregard the isospin information. As mentioned before relativistic approach presented here does not involve any correction to the non-relativistic effective Lagrangian of the instanton collective modes. We consider the static baryons as spin half particles living in Minkowski space and boost them in the $\vec{p}$ direction, but there is no explicit knowledge of the other generators of Lorentz symmetry. Regardless of its limitations at high energies, we consider this approach to be relevant for calculating the electromagnetic current vacuum expectation values at low energy where baryons have different momenta (inelastic scattering) and their helicity states are defined differently under the $\vec{p}.\vec{\sigma}$ operator. In such a way, we obtain a description equivalent to Dirac spinor states for the baryon wavefunctions that allows us to calculate the matrix elements of baryonic currents and define the corresponding form factors. We introduce a relativistic spinor for a fermion with a given momentum $\vec{p}$,

$$u(p,h) = \frac{1}{\sqrt{2E}} \begin{pmatrix} f\chi_h \\ \frac{\vec{p}\cdot\vec{\sigma}}{f}\chi_h \end{pmatrix} \quad , \text{ with } f = \sqrt{E+m_B}. \tag{3.18}$$

Here χ_h is a helicity eigenstate, which means $\vec{p}\cdot\vec{\sigma}\chi_h = h|\vec{p}|\chi_h$ where $h = \pm 1$. The action of the operator $\vec{p}\cdot\vec{\sigma}$ on these states is given by

$$\vec{p}\cdot\vec{\sigma}\,\chi_h = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \chi_h = h|p|\chi_h. \tag{3.19}$$

This yields the eigenstates

$$\chi_+(\vec{p}) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}|+p_3)}} \begin{pmatrix} |\vec{p}|+p_3 \\ p_1+ip_2 \end{pmatrix} \quad , \quad \chi_-(\vec{p}) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}|+p_3)}} \begin{pmatrix} -p_1+ip_2 \\ |\vec{p}|+p_3 \end{pmatrix}. \tag{3.20}$$

Notice that $\chi_h^\dagger\chi_h = 1$ and $|\vec{p}| = \sqrt{p_1^2+p_2^2+p_3^3}$. If we choose $\vec{p} = p_3\hat{z}$, we recover σ_3 eigenstates

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.21}$$

Hence we expect the proton and neutron linear ($l = 1$) spherical harmonic tensors (equivalent to $\chi_h(\vec{p})$ helicity eigenstates) to be

$$\begin{aligned}
\chi_+^p(a_i,\vec{p}) &= \frac{(|\vec{p}|+p_3)(a_1+ia_2) - i(p_1+ip_2)(a_4-ia_3)}{\pi\sqrt{2|\vec{p}|(|\vec{p}|+p_3)}}, \\
\chi_-^p(a_i,\vec{p}) &= \frac{(-p_1+ip_2)(a_1+ia_2) - i(p_3+p)(a_4-ia_3)}{\pi\sqrt{2p(p+p_3)}}, \\
\chi_+^n(a_i,\vec{p}) &= \frac{(|\vec{p}|+p_3)i(a_4+ia_3) - (p_1+ip_2)(a_1-ia_2)}{\pi\sqrt{2|\vec{p}|(|\vec{p}|+p_3)}}, \\
\chi_-^n(a_i,\vec{p}) &= \frac{(-p_1+ip_2)i(a_4+ia_3) - (p_3+p)(a_1-ia_2)}{\pi\sqrt{2p(p+p_3)}}.
\end{aligned} \tag{3.22}$$

The expression above is not valid for $p_1 = p_2 = 0$ and $p_3 = -p$ and in that case we recall that helicity changes sign when momentum changes in sign to the opposite direction. This means that

$$\chi^N_h(a_i, -\vec{p}) \equiv \chi^N_{-h}(a_i, \vec{p}), \tag{3.23}$$

up to a complex phase. After some algebra we can verify that (3.22) are eigenfunctions of $\vec{p}.\vec{J}$ operators (3.16):

$$\vec{p}.\vec{J}\,\chi^N_+(a_i, \vec{p}) = +\frac{1}{2}|\vec{p}|\,\chi^N_+(a_i, \vec{p}) \quad \text{and} \quad \vec{p}.\vec{J}\,\chi^N_-(a_i, \vec{p}) = -\frac{1}{2}|\vec{p}|\,\chi^N_-(a_i, \vec{p}). \tag{3.24}$$

In the static case the nucleon wavefunctions (3.17), (3.22) represent 2-spinor helicity states, cf. (3.15). Therefore the complete relativistic nucleon SU(2) wavefunctions are 2-spinors containing as components the helicity states $\chi^N_\pm(a_i, \vec{p})$ (SU(2) wavefunctions); thus these spinors are equivalent to Dirac 4-spinors, cf. (3.18). The nucleon SU(2) wavefunction becomes

$$u_N(a_i, \vec{p}, +) = \frac{1}{\sqrt{2E}} \begin{pmatrix} f\chi^N_+(a_i, \vec{p}) \\ \frac{|\vec{p}|}{f}\chi^N_+(a_i, \vec{p}) \end{pmatrix} \quad , \quad u_N(a_i, \vec{p}, -) = \frac{1}{\sqrt{2E}} \begin{pmatrix} f\chi^N_-(a_i, \vec{p}) \\ -\frac{|\vec{p}|}{f}\chi^N_-(a_i, \vec{p}) \end{pmatrix}. \tag{3.25}$$

The explicit dependence on $a_I \in S^3$ is not directly observable but rather encodes the spin/isospin in this representation. The integration in S^3 moduli recovers the 4D Dirac spinor with appropriate isospin, as we will show in the next subsection.

3.3. Dirac equation and spin sum

In order to work with relativistic SU(2) wavefunctions instead of Dirac spinorial notation, we define the substitutes of gamma matrices in the SU(2) collective space by simply replacing the Pauli matrices σ^i with $2J^i$ operators (3.16). Hence, the new 2×2 gamma matrices are

$$\gamma^0 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \gamma^i = -i \begin{pmatrix} 0 & 2J^i \\ -2J^i & 0 \end{pmatrix}. \tag{3.26}$$

Such operators act only on spin and we will disregard the isospin index for now. Upon such substitution we can verify the validity of the Dirac equation:

$$(i\not{p} + m_B)u_N(a_i, \vec{p}, h) = 0. \tag{3.27}$$

Since J^i operators have real eigenvalues and behave like Pauli matrices, we define their operation to the left by transpose conjugation:

$$\psi^\dagger_h(\vec{p}.\vec{J}) = (\vec{p}.\vec{J}\psi_h)^\dagger = |\vec{p}|\frac{h}{2}\psi^\dagger_h \tag{3.28}$$

and using $\overline{u}_N(a_i, \vec{p}, h) = u^\dagger_N(a_i, \vec{p}, h)i\gamma^0$ we get the second Dirac equation:

$$\overline{u}_N(a_i, \vec{p}, h)(i\not{p} + m_B) = 0. \tag{3.29}$$

As mentioned before, the spinor normalization is given by an integration of the a_i moduli,

$$\overline{u}(\vec{p}, h')u(\vec{p}, h) = \int_{S^3} \overline{u}(a_i, \vec{p}, h')u(a_i, \vec{p}, h) \tag{3.30}$$

Working out the integrand,

$$\begin{aligned}
\overline{u}(a_i,\vec{p},h')u(a_i,\vec{p},h) &= \frac{1}{2E}(f\chi^*_{h'}(a_i,\vec{p}) \quad \tfrac{|\vec{p}|}{f}h'\chi^*_{h'}(a_i,\vec{p}))\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} f\chi^{a_i}_h(\vec{p}) \\ \frac{|\vec{p}|}{f}h\chi_h(a_i,\vec{p}) \end{pmatrix} \\
&= \frac{1}{2E}\left(f^2 - h'h\frac{|\vec{p}|^2}{f^2}\right)\chi^*_{h'}(a_i,\vec{p})\chi_h(a_i,\vec{p}) \qquad (3.31)
\end{aligned}$$

In order to integrate (3.31) over S^3 we write (a_1,a_2,a_3,a_4) in spherical coordinates:

$$\begin{aligned}
a_1 &= \sin(\theta_0)\sin(\theta_1)\sin(\theta_2), \\
a_2 &= \sin(\theta_0)\sin(\theta_1)\cos(\theta_2), \\
a_3 &= \sin(\theta_0)\cos(\theta_1), \\
a_4 &= \cos(\theta_0). \qquad (3.32)
\end{aligned}$$

The volume element is $d\Omega_3 = \sin^2\theta_0 \sin\theta_1 d\theta_0 d\theta_1 d\theta_2$. Using (3.22), the numerical integration over (3.31) turns out to be

$$\int_{S^3}\overline{u}(a_i,\vec{p},h')u(a_i,\vec{p},h) = \frac{1}{2E}\left((E+m_B) - h'h(E-m_B)\right)\int_{S^3}\chi^{a_i*}_{h'}(\vec{p})\chi^{a_i}_h(\vec{p}) = \frac{m_B}{E}\delta_{h'h}. \qquad (3.33)$$

The spin sum is given by

$$\sum_h u(\vec{p},h)\overline{u}(\vec{p},h) = \sum_h \int_{S^3} u(a_i,\vec{p},h)\overline{u}(a_i,\vec{p},h), \qquad (3.34)$$

where the integrand of (3.34) reads

$$\begin{aligned}
u(a_i,\vec{p},h)\overline{u}(a_i,\vec{p},h) &= \frac{1}{2E}\begin{pmatrix} f\chi^{a_i}_h(\vec{p}) \\ \frac{2\vec{J}.\vec{p}}{f}\chi^{a_i}_h(\vec{p}) \end{pmatrix}(f\chi^{a_i*}_h(\vec{p}) \quad -\tfrac{2\vec{J}.\vec{p}}{f}\chi^{a_i*}_h(\vec{p})) \\
&= \frac{1}{2E}\begin{pmatrix} E+m_B & -2\vec{J}.\vec{p} \\ 2\vec{J}.\vec{p} & -E+m_B \end{pmatrix}|\chi^{a_i}_h(\vec{p})|^2 \\
&= \frac{1}{2E}(-i\not{p}+m_B)|\chi^{a_i}_h(\vec{p})|^2. \qquad (3.35)
\end{aligned}$$

Integrating over (3.35) we get the spin sum (3.34),

$$\sum_h \int_{S^3} u(a_i,\vec{p},h)\overline{u}(a_i,\vec{p},h) = \frac{1}{2E}(-i\not{p}+m_B)\sum_h\int_{S^3}|\chi^{a_i}_h(\vec{p})|^2 = \frac{1}{E}(-i\not{p}+m_B). \quad (3.36)$$

3.4. Application: trace vevs

In [2], the authors calculate the proton isovector magnetic moment which originates from the vector current $J_V = J_L + J_R$ of $SU(2)\times SU(2)$. An important ingredient in the derivation is the calculation of the vev of $\mathrm{Tr}(\tau^i \boldsymbol{a}^{-1}\tau^3\boldsymbol{a})$ for static protons. Here, we extend this calculation to external moving protons, as required for finding baryon form factors, which we are presently investigating in a forthcoming publication [18]. In spherical coordinates the traces become

$$\begin{aligned}
\mathrm{Tr}(\tau^1\boldsymbol{a}^{-1}\tau^3\boldsymbol{a}) &= 4\sin(\theta_0)\sin(\theta_1)(\cos(\theta_0)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_0)\sin(\theta_2)),\\
\mathrm{Tr}(\tau^2\boldsymbol{a}^{-1}\tau^3\boldsymbol{a}) &= 4\sin(\theta_0)\sin(\theta_1)(\cos(\theta_1)\cos(\theta_2)\sin(\theta_0)-\cos(\theta_1)\sin(\theta_2)),\\
\mathrm{Tr}(\tau^3\boldsymbol{a}^{-1}\tau^3\boldsymbol{a}) &= 2(\cos(\theta_0)^2+\cos(2\theta_1)\sin(\theta_0)^2).
\end{aligned} \tag{3.37}$$

Rewriting the proton states in this notation, we get

$$\langle B_X,\vec{p}\,',h'|\,\mathrm{Tr}(\tau^a\boldsymbol{a}^{-1}\tau^3\boldsymbol{a})|B_0,\vec{p},h\rangle = \int_{S^3} d\Omega^3\, \overline{u}_X(a_i,\vec{p}\,',h')\,\mathrm{Tr}(\tau^j\boldsymbol{a}^{-1}\tau^3\boldsymbol{a})u_0(a_i,\vec{p},h) \tag{3.38}$$

The S^3 integration above simplifies if we write the helicity states $\chi_{h'}(a_i,\vec{p}\,')$ and $\chi_h(a_i,\vec{p})$ as linear combination of the basis $\{\chi_+(a_i),\chi_-(a_i)\}$ of helicity states of z direction as in eq. (3.17). The integration is resummed to a linear combination of the following form (as in [2]):

$$\int_{S^3} d\Omega^3\chi^*_{h'}(a_i)\,\mathrm{Tr}(\tau^a\boldsymbol{a}^{-1}\tau^3\boldsymbol{a})\chi_h(a_i) = -\frac{2}{3}\sigma^a_{h'h} = -\frac{2}{3}\chi^\dagger_{h'}\sigma^a\chi_h \tag{3.39}$$

The helicity states can be written as:

$$\chi_h(a_i,\vec{p}) = \frac{(|\vec{p}|+p_3)\chi_h(a_i)+(hp_1+ip_2)\chi_{-h}(a_i)}{\sqrt{2|\vec{p}|(|\vec{p}|+p_3)}} \tag{3.40}$$

Thus we find,

$$\int_{S^3} d\Omega^3\,\overline{u}_X(a_i,\vec{p}\,',h')\,\mathrm{Tr}(\tau^j\boldsymbol{a}^{-1}\tau^3\boldsymbol{a})u_0(a_i,\vec{p},h) = \frac{-1}{3\sqrt{E_XE}}(ff_X-\tfrac{hh'|\vec{p}||\vec{p}\,'|}{ff_X})\chi^\dagger_{h'}(\vec{p}\,')\sigma^j\chi_h(\vec{p}) \tag{3.41}$$

where $f_X=\sqrt{E_X+m_{B_X}}$ and $\chi_{h'}(\vec{p}\,')$ and $\chi_h(\vec{p})$ are the helicity spinors (3.20). The equation above is part of the calculus of the electromagnetic proton form factors [6]. In the elastic case, and in the Breit frame with $\vec{p}=-\vec{p}\,'=p_3\hat{z}$, the expression (3.41) takes the form

$$-\frac{2}{3}(\delta_{h,h'}+\delta_{-h',h}\tfrac{m_B}{E})\sigma^j_{-h'h} = \begin{cases} \text{if } j=1: & -\frac{2}{3}\sigma^j_{-h'h}\\ \text{if } j=2: & -\frac{2}{3}\sigma^j_{-h'h}\\ \text{if } j=3: & -\frac{2}{3}\frac{m_B}{E}\sigma^j_{-h'h}\end{cases} \tag{3.42}$$

where we used the substitution (3.23). This amounts to a correction in the direction of movement $\hat{z}$ w.r.t. the static case [2], cf. (3.39). The expressions (3.41, 3.42) will be important in the general calculation of non-elastic form factors that we will present in a forthcoming publication [18], where we study generalized form factors and structure functions for holographic baryons in the Sakai-Sugimoto model.

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