On the thermodynamic stability of rotating black holes in higher dimensions — a comparison of thermodynamic ensembles

Brian P. Dolan
Department of Mathematics, Heriot-Watt University
Colin Maclaurin Building, Riccarton, Edinburgh, EH14 4AS, U.K.

Maxwell Institute for Mathematical Sciences, Edinburgh, U.K.

Email: `B.P.Dolan@hw.ac.uk`

January 14, 2014

Abstract

Thermodynamic potentials relevant to the micro-canonical, the canonical and the grand-canonical ensembles, associated with rotating black holes in D-dimensions, are analysed and compared.

Such black holes are known to be thermodynamically unstable, but the instability is a consequence of a subtle interplay between specific heats and the moments of inertia and it manifests itself differently in the different ensembles. A simple relation between the product of the specific heat and the determinant of the moment of inertia in both the canonical and the grand-canonical ensembles is derived.

Myers-Perry black holes in arbitrary dimension are studied in detail. All temperature extrema in the micro-canonical ensemble are determined and classified. The specific heat and the moment of inertia tensor are evaluated in both the canonical and the grand-canonical ensembles in any dimension. All zeros and poles of the specific heats, as a function of the angular momenta, are determined and the eigenvalues of the isentropic moment of inertia tensor are studied and classified.

It is further shown that many of the thermodynamic properties of a Myers-Perry black hole in $D-2$ dimensions can be obtained from those of a black hole in D dimensions by sending one of the angular momenta to infinity.

PACS nos: 04.60.-m; 04.70.Dy

1 Introduction

An immediate consequence of Hawking's famous result that Schwarzschild black holes in four dimensions have a temperature that is inversely proportional to their mass [1] is that any such black hole in isolation is thermodynamically unstable, due to a negative specific heat. They can however be stabilised by putting them in a box with thermal walls [2] or by introducing a negative cosmological constant of sufficient magnitude [3]. While the specific heat of a Schwarzschild black hole can be rendered positive by making it rotate sufficiently fast this does not stabilise the black hole as the moment of inertia tensor then becomes negative, maintaining the instability.

The generalisation of the Kerr metric to a class of rotating black holes in D-dimensions, found by Myers and Perry [4], provides an arena for testing these ideas in a more general context. In 4-dimensions there is a maximum angular momentum that a rotating black hole can sustain, corresponding to an extremal black hole with vanishing Hawking temperature, but in higher dimensions this is not the case. There is more than one angular momentum in $D > 4$, corresponding to the fact that the rank of $SO(D-2)$ is greater than one for $D > 4$, and some, but not all, of the angular momenta can become arbitrarily large — the phenomenon of ultra-spinning black holes [4]. Infinite momentum does not however imply infinite angular velocity, rather the corresponding angular velocity vanishes as an angular momentum diverges — the infinite angular momentum is due to a singularity in the moment of inertia of the black hole and is not due to infinite angular velocity.

It was suggested some time ago that there should be a link between the thermodynamic properties of black holes, in particular the second law of thermodynamics, and dynamical instability for $D > 4$, [5]. Stability of Myers-Perry black holes was analysed in [6] and an extensive literature on the subject of the thermodynamic and dynamical instability of rotating black holes in higher dimensions has since emerged [7]-[19]. In particular it has been shown, with a very general argument utilising only the Smarr relation and the first law, that all asymptotically flat electrically neutral solutions of the vacuum Einstein equations in D-dimensions are thermodynamically unstable, [15]. Nevertheless it is still instructive to examine the details of thermodynamic stability in the different ensembles and in specific cases.

The thermodynamic quantities of interest are the mass (internal energy),[1]

[1]When there is a non-zero cosmological constant it is argued in [20] that the mass

the entropy, the temperature, the angular momenta and the angular velocities. With Newton's constant $G_N = 1$ and $c = 1$ these can all be given dimensions of length to some power:

Mass, M	$D-3$
Entropy, S (area)	$D-2$
Angular momenta, J^i	$D-2$
Temperature, T	-1
Angular velocity, Ω_i	-1

It is thus natural to consider M, S, and J^i to be extensive while T and Ω_i are intensive, and this classification will be adopted here.[2]

In §2.4 a general relation between the canonical and the grand-canonical ensembles, for electrically neutral rotating black holes, is derived. For the canonical ensemble the Hessian, $\partial^2 U$, of the internal energy $U(J^i, S)$ is shown to have determinant

$$\det(\partial^2 U) = \frac{1}{\beta C_J \det \mathcal{I}_T} \tag{1}$$

where C_J is the specific heat at constant angular momentum and $\mathcal{I}_T$ is the isothermal moment of inertia tensor. For the grand-canonical ensemble, on the other hand, the Hessian $\partial^2 G$ of the Gibbs free energy $G(\Omega_i, T)$ is shown to satisfy

$$\det(-\partial^2 G) = \beta C_\Omega \det \mathcal{I}_S, \tag{2}$$

where C_Ω is the specific heat at constant angular velocity and $\mathcal{I}_S$ is the isentropic moment of inertia tensor. Standard thermodynamics arguments then imply that

$$C_J \det \mathcal{I}_T = C_\Omega \det \mathcal{I}_S, \tag{3}$$

which is one of our main results.

We then compare the thermodynamics of Myers-Perry black holes in the different ensembles. In one of the first papers on the stability of Myers-Perry black holes [6] it was observed that, as one of the angular momenta is

is more correctly thought of as the enthalpy rather than the internal energy. These are of course the same for zero pressure and in this work we make no distinction between enthalpy and internal energy.

[2] Dimensional arguments do not single electric charge out as an extensive variable so easily, nevertheless most workers would consider charge as extensive and the corresponding electric potential as intensive. The present work is restricted to the case of electrically neutral black holes.

increased keeping the others zero and the mass fixed (the micro-canonical ensemble), there is a minimum in the temperature. The authors suggested that this was a signal of an instability — that there should be dynamical negative modes leading to a more stable solution of Einstein's equations, but with less symmetry. This gave further support, beyond the non-rotating case studied in [5], to the idea that thermodynamic and dynamical instability of black holes in higher dimensions are intimately related. Hints of the instability can be seen in the micro-canonical ensemble in which the entropy, which is a monotonically decreasing concave function of angular momenta, at constant mass, until the temperature hits a minimum at which point the entropy has an inflection point and becomes concave in at least one direction [8].

Thermodynamic instability manifests itself in different ways in the various ensembles. In the grand-canonical ensemble the Gibbs free energy $G(\Omega_i, T)$ is considered as a function of intensive variables and thermodynamic stability requires that G be a totally concave function of its arguments [21]. The particular cases of asymptotically flat Kerr and Myers-Perry black holes were investigated in [10] for $4 \leq D \leq 6$ and it was shown that the specific heat C_J is negative when all angular momenta vanish, but can become positive when some of the angular momenta become large enough. However when the angular momenta are large enough for the specific heat to be positive the isothermal moment of inertia, all of whose eigenvalues are positive for zero rotation, has at least one negative eigenvalue — there is thus always an instability. One of the results of the present work is to extend the explicit analysis of [10] to all D and show that the same phenomenon persists.

The relationship between the micro-canonical and the grand-canonical ensembles was studied in [16] and in §3.4.4 we extend this analysis further and derive a number of relations between the temperature, the specific heat at constant angular velocity C_Ω, and the isentropic moment of inertia tensor $\mathcal{I}_S$ for Myers-Perry black holes in any dimension. We show that there is a branched hypersurface in angular momentum space where βC_Ω (with $\beta = \frac{1}{T}$) develops a pole and that this is the same hypersurface as the one on which the temperature is minimised in the micro-canonical ensemble. This hypersurface can be obtained from the extremal $T = 0$ hypersurface by analytically continuing $(J^i)^2$ to $-(J^i)^2$, keeping the entropy constant. There is yet another significant hypersurface, one with a number of branches on which βC_Ω vanishes, and on this hypersurface the isentropic moment of inertia tensor develops an infinite eigenvalue, in the form of a pole. This pole exactly cancels the zero in βC_Ω in the Hessian of the Gibbs free energy. The branches of this hypersurface divide the space of angular momenta into separate regions determined by the signature of $\mathcal{I}_S$.

A by product of our analysis is that the thermodynamic properties of a Myers-Perry black hole in $D-2m$ dimensions, in the micro or the grand-canonical ensemble, can be obtained from those of a Myers-Perry black hole in D dimensions by sending m of the angular momenta to infinity in the latter.

In §2 the thermodynamics of rotating black holes in the different ensembles are analysed and equation (3) derived along with other relations between the various thermodynamic quantities. In §3 Myers-Perry black holes are studied and it is shown explicitly how the specific heats and moments of inertia conspire to satisfy the general relations of §2. The results are summarised in §4 and some technical results required in the analysis are relegated to five appendices.

2 Thermodynamics of rotating black holes

Rotating black holes in D space-time dimensions must be treated slightly differently for even and odd D because the rotation group $SO(D-1)$ has different characterisations of angular momenta in the even and odd dimensional cases. The Cartan sub-algebra has dimension $\frac{D-2}{2}$ for even D and $\frac{D-1}{2}$ for odd D so a general state of rotation is specified by $\frac{D-2}{2}$ independent angular momenta in even D and $\frac{D-1}{2}$ in odd D. Let $N = \left\lfloor \frac{D-1}{2} \right\rfloor$, the integral part of $\frac{D-1}{2}$, be the dimension of the Cartan sub-algebra of $SO(D-1)$, then there are N independent angular momenta J^i, $i = 1, \dots, N$. It is notationally convenient to introduce a parameter $\epsilon = \big(1 + (-1)^D\big)/2$ in terms of which

$$N = \frac{D-1-\epsilon}{2}. \tag{4}$$

In the micro-canonical ensemble the energy is fixed and we chose as thermodynamic control parameters the extensive quantities, J^i and M, with the entropy $S(J^i, M)$ being the thermodynamic potential, which is convenient for differentiation keeping M fixed. In the canonical ensemble the energy is allowed to fluctuate and the internal energy

$$U(J, S) = M \tag{5}$$

is used as the thermodynamic potential. In the grand-canonical ensemble all extensive variables are allowed to fluctuate and the intensive variables are used as control parameters, the relevant thermodynamic potential is then the Gibbs free energy

$$G(\Omega, T) = U - TS - \Omega_i J^i. \tag{6}$$

The Gibbs free energy is related to the Euclidean formulation, since the Euclidean action I_E is related to the mass by a Legendre transform

$$TI_E = M - TS - \Omega_i J^i \qquad \Rightarrow \qquad TI_E = G(\Omega, T). \tag{7}$$

2.1 Micro-canonical ensemble

For completeness we summarise in this sub-section some of the results pertaining to the micro-canonical analysis in [18].

With the entropy expressed as a function of J^i and M, $S(J, M)$, we have

$$\left.\frac{\partial S}{\partial J^i}\right|_M = -\beta\Omega_i, \qquad \left.\frac{\partial S}{\partial M}\right|_{J^i} = \beta, \tag{8}$$

where $\beta = \frac{1}{T}$. These thermodynamic quantities have a geometrical interpretation in the Euclidean formulation of the black hole, where demanding absence of a conical singularity requires periodicity in imaginary time, $\tau = it$, with τ identified with $\tau + \beta$, and periodicity in imaginary angle [18], $\varphi_i = i\phi_i$, with φ_i identified with $\varphi - \beta\Omega_i$. Thus the 1-form dS determines the size of the (τ, φ_i) torus,

$$\tau dM + \varphi_i dJ^i \sim \tau dM + \varphi_i dJ^i + dS. \tag{9}$$

For thermodynamic stability the entropy should be purely concave [21], which requires that the Hessian $H_{AB} = -\frac{\partial^2 S}{\partial \tilde{x}^A \partial \tilde{x}^B}$, where $\tilde{x}^A = (J^i, M)$, must be a positive definite matrix. The identity

$$\det\big(H_{AB}\big) = -\frac{1}{(D-3)MT}\det\big(H_{ij}\big), \tag{10}$$

for asymptotically flat black holes, is derived in [15], where H_{ij} is the $N \times N$ matrix

$$H_{ij} = -\left(\frac{\partial^2 S}{\partial J^i \partial J^j}\right)_M. \tag{11}$$

One can immediately conclude that such black holes can never be thermodynamically stable in $D \geq 4$, since $\det(H_{AB}) > 0$ requires $\det(H_{ij}) < 0$, hence at least one eigenvalue of H_{ij} would have to be negative and S cannot be a concave function.

2.2 Canonical ensemble

The canonical ensemble uses the internal energy $U(J^i, S)$ as thermodynamic potential, depending on extensive arguments that are the Legendre transforms of Ω_i and T, and for black holes U is identified with the ADM mass, at least in the asymptotically flat case. Stability requires that U be a totally convex function of its arguments. Let $x^A = (J^i, S)$, with $A = 1, \ldots, N+1$, $x^i = J^i$ and $x^{N+1} = S$, then

$$T = \left.\frac{\partial U}{\partial S}\right|_J, \qquad \Omega_i = \left.\frac{\partial U}{\partial J^i}\right|_S \tag{12}$$

and $U_{AB} = \frac{\partial^2 U}{\partial x^A \partial x^B}$ must be a positive matrix. Explicitly

$$U_{AB} = \begin{pmatrix} \frac{\partial^2 U}{\partial J^i \partial J^j} & \frac{\partial^2 U}{\partial J^i \partial S} \\ \frac{\partial^2 U}{\partial S \partial J^j} & \frac{\partial^2 U}{\partial S^2} \end{pmatrix} = \begin{pmatrix} \left(\mathcal{I}_S{}^{-1}\right)_{ij} & \zeta_i \\ \zeta_j & (\beta C_J)^{-1} \end{pmatrix}, \tag{13}$$

where $\beta = \frac{1}{T}$; the symmetric matrix

$$\mathcal{I}_S{}^{ij} = \left.\frac{\partial J^i}{\partial \Omega_j}\right|_S \tag{14}$$

is the isentropic moment of inertia tensor; C_J is the specific heat at constant J,

$$C_J = \left(\frac{\partial U}{\partial T}\right)_J = T\left(\frac{\partial S}{\partial T}\right)_J \tag{15}$$

and

$$\zeta_i = \left.\frac{\partial \Omega_i}{\partial S}\right|_J = \left.\frac{\partial T}{\partial J^i}\right|_S \tag{16}$$

(this last equation is a Maxwell relation).

A necessary, but not sufficient, condition for stability is thus

$$\det(\partial^2 U) = \frac{\det\left(\mathcal{I}_S{}^{-1} - \beta C_J \zeta \zeta^T\right)}{\beta C_J} > 0. \tag{17}$$

2.3 Grand-canonical ensemble

In the grand-canonical ensemble stability requires that $G(\Omega, T)$ be a concave function. Let $y_A = (\Omega_i, T)$, with $y_i = \Omega_i$ and $y_{N+1} = T$, then

$$S = -\left.\frac{\partial G}{\partial T}\right|_\Omega, \qquad J^i = -\left.\frac{\partial G}{\partial \Omega_i}\right|_T \tag{18}$$

and $G^{AB} = \frac{\partial^2 G}{\partial y_A \partial y_B}$ must be a negative matrix. Explicitly

$$- G^{AB} = \begin{pmatrix} -\frac{\partial^2 G}{\partial \Omega_i \partial \Omega_j} & -\frac{\partial^2 G}{\partial \Omega_i \partial T} \\ -\frac{\partial^2 G}{\partial T \partial \Omega_j} & -\frac{\partial^2 G}{\partial T^2} \end{pmatrix} = \begin{pmatrix} \mathcal{I}_T{}^{ij} & \eta^i \\ \eta^j & \beta C_\Omega \end{pmatrix}, \tag{19}$$

where the symmetric matrix

$$\mathcal{I}_T^{ij} = \left. \frac{\partial J^i}{\partial \Omega_j} \right|_T \tag{20}$$

is the isothermal moment of inertia tensor; C_Ω is the specific heat at constant Ω,

$$C_\Omega = T \left(\frac{\partial S}{\partial T} \right)_\Omega = -T \left(\frac{\partial^2 G}{\partial T^2} \right)_\Omega \tag{21}$$

and

$$\eta^i = \left. \frac{\partial J^i}{\partial T} \right|_\Omega = \left. \frac{\partial S}{\partial \Omega_i} \right|_T . \tag{22}$$

A necessary, but not sufficient, condition for stability is thus

$$\det(-\partial^2 G) = \beta C_\Omega \det\bigl(\mathcal{I}_T - (\beta C_\Omega)^{-1} \eta \eta^T\bigr) > 0. \tag{23}$$

2.4 Relation between the canonical and the grand-canonical ensembles

The canonical and the grand-canonical ensembles are of course related. An immediate consequence of the Legendre transform (6) is that

$$- G^{AB} = (U^{-1})^{AB} \tag{24}$$

and this has important consequences for the individual components.

A relation between the specific heats was derived in [10],

$$\begin{aligned} \left. \frac{\partial S}{\partial T} \right|_\Omega &= \left. \frac{\partial S}{\partial T} \right|_J + \left. \frac{\partial S}{\partial J^j} \right|_T \left. \frac{\partial J^j}{\partial T} \right|_\Omega = \left. \frac{\partial S}{\partial T} \right|_J + \left. \frac{\partial S}{\partial \Omega_i} \right|_T \left. \frac{\partial \Omega_i}{\partial J^j} \right|_T \left. \frac{\partial J^j}{\partial T} \right|_\Omega & (25) \\ &= \left. \frac{\partial S}{\partial T} \right|_J + \bigl(\mathcal{I}_T^{-1}\bigr)_{ij} \eta^i \eta^j & (26) \\ \Rightarrow \qquad & \beta C_\Omega = \beta C_J + \bigl(\mathcal{I}_T^{-1}\bigr)_{ij} \eta^i \eta^j , & (27) \end{aligned}$$

where the Maxwell relation (22) has been used.

Similar manipulations can be used to relate the isentropic and isothermal moment of inertia tensors,

$$\mathcal{I}_T{}^{ij} = \mathcal{I}_S{}^{ij} + (\beta C_\Omega)^{-1}\eta^i\eta^j. \tag{28}$$

Or equivalently,

$$\left(\mathcal{I}_S^{-1}\right)_{ij} = \left(\mathcal{I}_T^{-1}\right)_{ij} + \beta C_J \zeta^i\zeta^j. \tag{29}$$

The stability conditions (17) and (23) can thus be expressed as

$$\det(\partial^2 U) = \frac{1}{\beta C_J \det(\mathcal{I}_T)} > 0 \tag{30}$$

and

$$\det(-\partial^2 G) = \beta C_\Omega \det(\mathcal{I}_S) > 0. \tag{31}$$

Equation (24) now gives the identity

$$\beta C_J \det(\mathcal{I}_T) = \beta C_\Omega \det(\mathcal{I}_S). \tag{32}$$

A new instability would be expected to develop every time one of the eigenvalues of $-\partial^A\partial^B G$ changes from a positive to a negative value, either by going through zero or infinity. In general this might be expected to happen on a hypersurface on which $\det(-\partial^2 G)$ is either zero or infinity, but we shall see that, at least in the case of Myers-Perry black holes, there are some subtle cancellations so that the change is not reflected in the determinant.

The form of the Hessians (13) and (19) can be simplified by using a Legendre transform on the scalar variable, respectively S and T. In the canonical ensemble let $x^{A'} = (J^i, T)$ and

$$F(J^i, T) = U - TS. \tag{33}$$

Then the co-ordinate transformation matrix is

$$\frac{\partial x^{A'}}{\partial x^B} = \begin{pmatrix} \delta^i{}_j & 0 \\ \zeta_j & (\beta C_J)^{-1} \end{pmatrix} \tag{34}$$

with inverse

$$\frac{\partial x^{A}}{\partial x^{B'}} = \begin{pmatrix} \delta^i{}_j & 0 \\ -\beta C_J\zeta_j & \beta C_J \end{pmatrix}. \tag{35}$$

So, in (J^i, T) co-ordinates,

$$U_{A'B'} = U_{CD} \frac{\partial x^C}{\partial x^{A'}} \frac{\partial x^D}{\partial x^{B'}} = \begin{pmatrix} \left(\mathcal{I}_T^{-1}\right)_{ij} & 0 \\ 0 & \beta C_J \end{pmatrix}. \tag{36}$$

is partially diagonalised.[3]

Similarly, in the canonical ensemble, we can transform from $y_A = (\Omega_i, T)$ to $y_{A'} = (\Omega_i, S)$ to get

$$- G^{A'B'} = -G^{CD} \frac{\partial x^{A'}}{\partial x^C} \frac{\partial x^{B'}}{\partial x^D} = \begin{pmatrix} \mathcal{I}_S^{ij} & 0 \\ 0 & (\beta C_\Omega)^{-1} \end{pmatrix}. \tag{37}$$

Note that, although the canonical ensemble implicitly involves $\mathcal{I}_S^{-1}$, its stability properties are most easily seen using $\mathcal{I}_T^{-1}$ in (36) and, while the grand-canonical ensemble implicitly involves $\mathcal{I}_T$, its stability properties are most easily studied using $\mathcal{I}_S$ in (37).

3 Myers-Perry black holes

Myers-Perry black holes in D space-time dimensions are assumed to have an event horizon which has the topology of a $(D-2)$-dimensional sphere. This can be described in terms of Cartesian co-ordinates x_a in $\mathbf{R}^{D-1}$ by

$$\sum_{a=1}^{D-1} x_a^2 = 1, \tag{38}$$

and we can write this as

$$\sum_{i=1}^{N} \rho_i^2 + \epsilon y^2 = 1, \tag{39}$$

where $x_{2i-1} + i x_{2i} = \rho_i e^{i\phi_i}$, $i = 1, \ldots, N$, are complex co-ordinates for both the even and odd cases while $y = x_{D-1}$ is only present for even D.

Then ρ_i, ϕ_i and y are co-ordinates that can be used to parameterise the sphere and, for the black hole, J^i are angular momenta in the (x_{2i-1}, x_{2i})-plane.[4]

[3] Of course $U_{A'B'} \neq \partial^2 F / \partial x^{A'} \partial x^{B'}$. The stability properties of the canonical ensemble are determined by the signature of the Hessian U_{AB}, which is the same as that of $U_{A'B'}$. The matrix $\partial^2 F / \partial x^{A'} \partial x^{B'}$ has a different signature.

[4] Hereinafter we shall not distinguish between upper and lower indices, $J_i = J^i$.

The Myers-Perry line element can be expressed as

$$
\begin{aligned}
ds^2 &= -W dt^2 + \frac{2\mu}{U}\left(W dt - \sum_{i=1}^{N} a_i \rho_i^2 d\phi_i\right)^2 \\
&\quad + \left(\frac{U}{Z-2\mu}\right) dr^2 + \epsilon\, r^2 dy^2 + \sum_{i=1}^{N}(r^2+a_i^2)(d\rho_i^2+\rho_i^2 d\phi_i^2),
\end{aligned}
$$

where the functions W, Z and U are

$$
\begin{aligned}
W &= \epsilon y^2 + \sum_{i=1}^{N} \rho_i^2 \\
Z &= \frac{1}{r^{2-\epsilon}} \prod_{i=1}^{N} (r^2 + a_i^2) \\
U &= Z\left(1 - \sum_{i=1}^{N} \frac{a_i^2 \rho_i^2}{r^2 + a_i^2}\right).
\end{aligned}
\tag{40}
$$

The a_i are rotation parameters in the (x_{2i-1}, x_{2i})-plane and μ is a mass parameter. We use units in which the D-dimensional Newton's constant and the speed of light are set to one.

There is an event horizon at r_h, the largest root of $Z - 2\mu = 0$, so

$$
\mu = \frac{1}{2r_h^{2-\epsilon}} \prod_{i=1}^{N} (r_h^2 + a_i^2), \tag{41}
$$

and the area of the event horizon is

$$
\mathcal{A}_h = \frac{\varpi}{r_h^{1-\epsilon}} \prod_{i=1}^{N} (r_h^2 + a_i^2)\,, \tag{42}
$$

Where ϖ is is the volume of the round unit $(D-2)$-sphere,

$$
\varpi = \frac{2\pi^{\frac{(D-1)}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}\,. \tag{43}
$$

The Bekenstein-Hawking entropy is

$$
S = \frac{\varpi}{4r_h^{1-\epsilon}} \prod_{i=1}^{N} (r_h^2 + a_i^2) \tag{44}
$$

and the Hawking temperature is

$$T = \frac{1}{4\pi r_h}\left(D - 3 - 2\sum_{i=1}^{N}\frac{a_i^2}{r_h^2 + a_i^2}\right). \tag{45}$$

The angular momenta, the entropy and the ADM mass, M, of the black hole are related to each other, and to the metric parameters, via

$$J^i = \frac{2Ma_i}{D-2} = \frac{\mu\,\varpi a_i}{4\pi}, \qquad M = \frac{(D-2)\varpi\mu}{8\pi} = \frac{(D-2)S}{4\pi r_h}\,, \tag{46}$$

while the angular velocities are

$$\Omega_i = \frac{a_i}{(r_h^2 + a_i^2)}. \tag{47}$$

3.1 Micro-canonical ensemble

The micro-canonical ensemble was developed for Myers-Perry black holes in [12], [15] and [18]. In particular the Hessian (11) was evaluated explicitly in [18] and is reproduced in appendix A,

$$H_{ij} = \frac{(D-2)}{2r_h^2TM}\left\{\frac{(1-j_i^2)}{(1+j_i^2)^2}\delta_{ij} + \frac{\omega_i\omega_j}{\pi r_h T}\left(\frac{1}{1+j_i^2} + \frac{1}{1+j_j^2} - \frac{1}{2} + \frac{2}{t}\Omega^2\right)\right\}, \tag{48}$$

where $j_i = \frac{2\pi J_i}{S}$, $\omega_i = \frac{j_i}{1+j_i^2}$ are dimensionless angular velocities, and $\Omega^2 = \sum_{i=1}^{N}\omega_i^2$.

Black hole thermodynamics in $D > 4$ has a subtle relation with dynamical instability. It was noted in [6] that, for $D \geq 6$, the temperature of a Myers-Perry black hole, with only one $J^i \neq 0$, has a minimum as the spin increases at fixed mass. Taking $J^1 \neq 0$ and $J^i = 0$ for $i = 2, \ldots, N$ the minimum is at $\frac{a_1^2}{r_h^2} = \frac{D-3}{D-5}$ and in [6] it was suggested that this minimum signals the onset of a dynamical instability for a rotating black hole. Thermodynamic functions are thus giving hints of possible dynamical instability and this was studied in [15] and [18], where some special cases of non-zero spin were analysed. These authors studied the matrix (48) in the symmetric cases were the non-zero a_i are all equal,

$$a_1 = \ldots = a_n = a \neq 0, \qquad a_i = 0 \quad \text{for} \quad i = n, \ldots, N. \tag{49}$$

The entropy and the temperature decrease as the angular momenta are increased, at constant M, until the temperature reaches a minimum, and at precisely that point the matrix H_{ij} develops a zero eigenvalue, signalling the fact that the entropy ceases to be concave in that direction. The temperature has a minimum for configurations of the form (49) with

$$\frac{a^2}{r_h^2} = \frac{D-3}{D-3-2n}. \tag{50}$$

In particular $n = 1$ gives the original expression in [6], and the two special cases $n = 1$ and $n = N$ were considered in [18], while (50) for general n appeared in [16].

In appendix A all extrema of the temperature, in the micro-canonical ensemble with fixed mass,

$$\left.\frac{\partial T}{\partial J^i}\right|_{M,\mathbf{J}_*} = 0, \tag{51}$$

are found and classified. For finite J^i they are all of the same from as (49),

$$J^1 = \cdots = J^n = J_*, \qquad J^{n+1} = \cdots = J^N = 0, \tag{52}$$

(up to permutations of the J^i) with

$$j_*^2 = \frac{D-3}{D-3-2n,} \tag{53}$$

where $j_* = \frac{2\pi J_*}{S}$ are dimensionless angular momentum, in units of entropy.

The value of the temperature at the extrema (53) is

$$T_* = \frac{1}{4\pi r_h}\frac{(D-3)(D-3-2n)}{(D-3-n)}. \tag{54}$$

The temperature is a maximum, $T_{max} = \frac{D-3}{4\pi r_h}$, for non-rotating Schwarzschild-Tangherlini black holes ($n = 0$). For finite J^i the stationary points $\mathbf{J}_*$ are saddle points with minima along the directions $\mathbf{J}_* = (J_*, \cdots, J_*, 0, \ldots, 0)$ satisfying (53) and maxima in the directions orthogonal to these. At the same time H_{ij} in equation (48) develops a single zero eigenvalue at $\mathbf{J}_*$, in the direction $\mathbf{J}_*$, indicating an inflection point in that direction.[5] There are

[5]It is argued in [12] that this point of inflection is not in itself necessarily a sign of dynamical instability: it can indicate a zero mode, taking one Myers-Perry solution into another, rather than a negative mode dynamical instability.

also $(n-1)$ negative eigenvalues of H_{ij} at $\mathbf{J}_*$, (137), indicating convexity of the entropy in these directions with associated thermodynamic instabilities, while the entropy is concave in all other finite directions. At all stationary points of T, $\det(H_{ij})$ vanishes.

When some number m of the J^i are allowed to become infinite equations (53) and (54) are modified to

$$j_*^2 = \frac{D-3-2m}{D-3-2m-2n}$$

$$T_* = \frac{1}{4\pi r_h}\frac{(D-3-2m)(D-3-2m-2n)}{(D-3-2m-n)}$$

respectively. Indeed in many of the following formulae the thermodynamic properties of a Myers-Perry black hole in D dimensions, with m angular momenta sent to infinity, are seen to be the same as those of a $D-2m$ dimensional black hole with all angular momenta finite and the moment of inertia tensor, which has zero eigenvalues in the infinite directions, suitably truncated (a caveat to this statement is that we must restrict to $m < \frac{D-3}{4}$, equation (110)). This can be seen in the formulae in the appendix, though C_J and $\det(\mathcal{I}_T)$ are exceptions and so thermodynamic dimensional reduction using this limit does not work in the canonical ensemble. In this sense lower dimensional black holes can be obtained by starting from large D and sending more and more of the J_i to infinity.

3.2 Heat capacities

The heat capacity at constant J is derived in appendix B. It can be expressed fairly concisely by using the functions

$$\Sigma_n^{\pm} = \sum_{i=1}^{N} \frac{j_i^2}{(1 \pm j_i^2)^n}. \tag{55}$$

The specific heat at constant J is then

$$C_J = \frac{4\pi r_h M t}{\left[t^2 - (D-2)\left(t - 4\Sigma_2^+\right)\right]}, \tag{56}$$

where

$$t = D - 3 - 2\Sigma_1^+ = 4\pi r_h T. \tag{57}$$

Equation (56) generalises the formulae for the specific cases $D = 4$, 5 and 6, given in [10], to arbitrary D.

As is well known the specific is negative for $J^i = 0$,

$$C_{J=0} = -4\pi r_h M, \tag{58}$$

but can be positive for non-zero J^i.

The specific heat at constant J^i (the canonical ensemble) is related to the specific heat at constant Ω_i (the grand-canonical ensemble) by equation (27). Alternatively C_Ω can be evaluated directly for a general D without knowing the moment of inertia explicitly. The details are left to appendix C and here we just quote the result,

$$C_\Omega = -\frac{4\pi r_h M t\big(D-2+2\Sigma_1^-\big)}{(D-2)(D-3+2\Sigma_1^-)}\,, \tag{59}$$

which generalises the $D = 4$ result of [11] to $D \geq 4$.

To simplify some later formulae It will be convenient to define, in analogy with (57),

$$\overline{t} = D - 3 + 2\sum_{i=1}^{N} \frac{j_i^2}{1 - j_i^2} = D - 3 + 2\Sigma_1^-,$$

in terms of which

$$C_\Omega = -\frac{4\pi r_h M\big(D-2+2\Sigma_1^-\big)t}{(D-2)\overline{t}}\,,$$

Note the signs: in this notation

$$\Sigma_1^-(j^2) = -\Sigma_1^+(-j^2)\,, \qquad \overline{t}(j^2) = t(-j^2)\,. \tag{60}$$

There is a curious parallel between the singularities of βC_Ω, where $\overline{t} = 0$, and extremal black holes for which t, and hence T, vanishes. Since $\overline{t}(j^2) = t(-j^2)$ these are related by mapping $(J^i)^2 \to -(J^i)^2$, keeping the entropy constant.

3.3 Moment of inertia tensor

The isothermal moment of inertia tensor,

$$\mathcal{I}_T^{ij} = \left(\frac{\partial J^i}{\partial \Omega_j}\right)_T, \tag{61}$$

is derived in appendix D. It is

$$\mathcal{I}_T^{ij} = \frac{2Mr_h^2}{D-2}\left\{\frac{(1+j_i^2)^2}{(1-j_i^2)}\delta_{ij} - 2j_i j_j\left(1+\frac{4}{\bar{t}(1-j_i^2)(1-j_j^2)}\right)\right\}. \tag{62}$$

Equation (62) generalises the formulae for the particular cases $D = 4$, 5 and 6 derived in [10].

The isentropic moment of inertia tensor

$$\mathcal{I}_S^{ij} = \left(\frac{\partial J^i}{\partial \Omega_j}\right)_S \tag{63}$$

was given for general D in [22]. A derivation is outlined in appendix E and it has the form

$$\mathcal{I}_S^{ij} = \frac{2Mr_h^2}{(D-2)}\left\{\frac{(1+j_i^2)^2}{(1-j_i^2)}\delta_{ij} - \frac{2j_i j_j}{\left(D-2+2\Sigma_1^-\right)}\frac{(1+j_i^2)(1+j_j^2)}{(1-j_i^2)(1-j_j^2)}\right\}. \tag{64}$$

The determinant of the isentropic moment of inertia tensor is

$$\det \mathcal{I}_S = \left(\frac{2Mr_h^2}{D-2}\right)^N \frac{(D-2)}{\left(D-2+2\Sigma_1^-\right)}\prod_{i=1}^{N}\frac{(1+j_i^2)^2}{(1-j_i^2)}, \tag{65}$$

the factor $D-2+2\Sigma_1^-$ in the denominator of (65) exactly cancels the same factor in the numerator of (59).

3.4 Stability analysis in the canonical and grand-canonical ensembles

In this section we examine the thermodynamic stability of Myers-Perry black holes in the canonical and the grand-canonical ensembles, using the formulae of sections §3.2 and §3.3. We first summarise the well known case of $D = 4$ and the results of [10] for $D = 5$ and 6, before going on to describe the situation for general D.

3.4.1 D=4

The case $D = 4$ is well known, but is included here for completeness. In four-dimensions $N = 1$ and there is only one J. The temperature is

$$T = \frac{1}{4\pi r_h}\left(\frac{1-j^2}{1+j^2}\right) \tag{66}$$

so we must restrict to $0 \leq j \leq 1$, with $j = 1$ being extremal. C_J evaluates to

$$C_J = -2\pi r_h^2 \left(\frac{(1-j^2)(1+j^2)^2}{1-6j^2-3j^4} \right) \tag{67}$$

which is positive for $\frac{2}{\sqrt{3}} - 1 < j^2 < 1$, (in terms of J and M,

$$\frac{J^2}{M^4} = \frac{4j^2}{(1+j^2)^2}, \tag{68}$$

and these limits corresponding to $2\sqrt{3} - 3 < \frac{J^2}{M^4} < 1$).

The isothermal moment of inertia is

$$\mathcal{I}_T := \frac{r_h^3}{2}(1-6j^2-3j^4), \tag{69}$$

which is only positive when C_J is negative. Indeed

$$\beta C_J \mathcal{I}_T = -4\pi^2 r_h^6 (1+j^2)^3, \tag{70}$$

clearly illustrating that Kerr metrics are thermodynamically unstable in the grand-canonical ensemble for all values of the angular momentum: when the specific heat is positive the moment of inertia is negative and vice-versa. Note that the pole in C_J exactly cancels a zero in $\mathcal{I}_T$ — a phenomenon that we shall see persists for all D.

Equation (32) immediately shows that an instability must be present in the canonical ensemble, though the full story is a little simpler there. Explicitly

$$\beta C_\Omega = -8\pi^2 r_h^3, \qquad \mathcal{I}_S = \frac{r_h^3}{2}(1+j^2)^3, \tag{71}$$

and indeed $\beta C_J \mathcal{I}_T = \beta C_\Omega \mathcal{I}_S$ as it should be, even though the instability can shift between the specific heat and the moment of inertia in the former case while it always resides in the specific heat in the latter, the isentropic moment of inertia always being positive.

3.4.2 D=5

In five-dimensions the Hawking temperature is

$$T = \frac{1}{2\pi r_h} \frac{1 - j_1^2 j_2^2}{(1+j_1^2)(1+j_2^2)}, \tag{72}$$

so $j_1^2 j_2^2 \leq 1$, with the locus of extremal black holes being the hyperbolae $j_1^2 j_2^2 = 1$.

The specific heat at constant J and the isothermal moment of inertia tensor are easily determined from the general formulae in §2, with $N = 2$, but the explicit forms are not illuminating and we shall resort to a graphical representation. The specific heat is positive in the region of the $j_1 - j_2$ plane indicated in figure 1: it diverges on the boundary of the red inner region and vanishes on the outer hyperbolae (the latter being the $T = 0$ curve); and is positive in the yellow region enclosed by the curves.

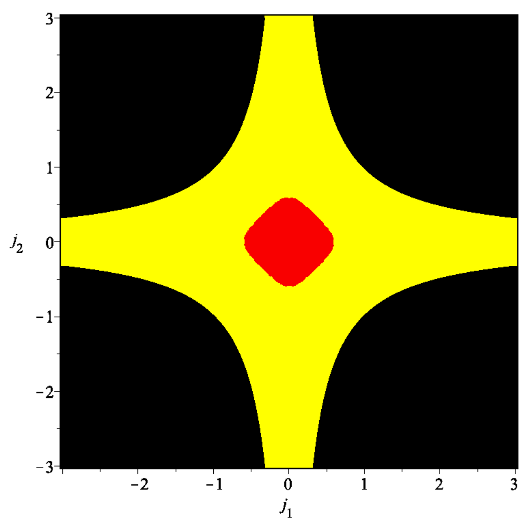

Figure 1: C_J for 5-dimensions in the $j_1 - j_2$ plane. It is negative in the red region, positive in the yellow region and diverges on the boundary between these two regions. C_J vanishes on the outer hyperbolae, because that is the $T = 0$ locus, and is negative in the black region, where $T < 0$. (This figure is essentially the same as one in [10], using slightly different variables.)

The eigenvalues of the isothermal moment of inertia tensor are plotted in figure 2. Both eigenvalues are positive for small j_i, and one is always positive, but the other vanishes on the same curve that bounds the red region in figure 1 and is negative outside this region. The innermost surface on which the moment of inertia tensor develops a negative eigenvalue is termed the *ultra-spinning surface* in reference [18] and it was shown there that there is no ultra-spinning surface in the micro-canonical ensemble for a singly spinning

Myers-Perry black hole $D = 5$. In contrast we see here that there is an ultra-spinning surface for $\mathcal{I}_T$ in the grand-canonical ensemble — the concept of an ultra-spinning surface depends on the ensemble used.

Thermodynamic instability can nevertheless be seen directly from

$$C_J \det \mathcal{I}_T = -\frac{3r_h^{11}\pi^4}{32}(1+j_1^2)^4(1+j_2^2)^4 < 0. \tag{73}$$

$\mathcal{I}_T$ has a negative eigenvalue when C_J is positive and when $\mathcal{I}_T$ has two positive eigenvalues, $C_J < 0$. Hence $C_J \det \mathcal{I}_T$ is always negative for any black-hole, and so these black-holes are thermodynamically unstable for any choice of angular-momenta with positive temperature.

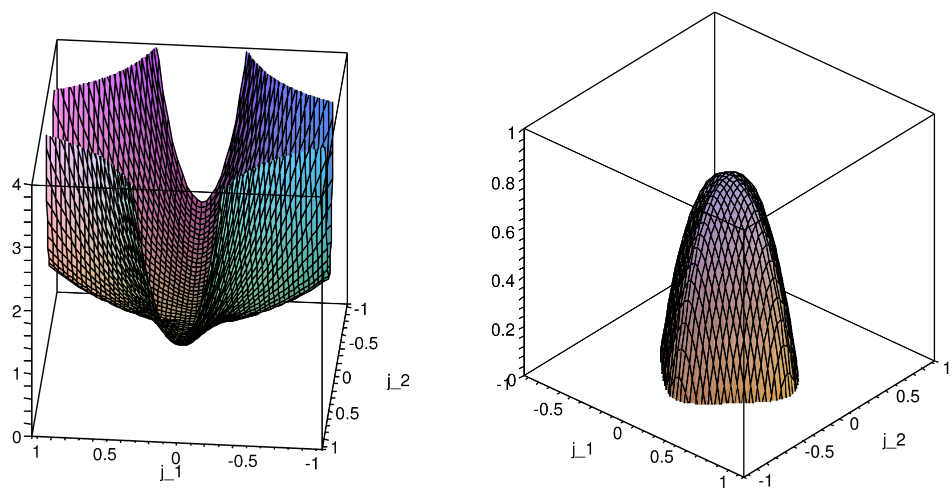

Figure 2: Eigenvalues of the isothermal moment of inertia tensor, $\mathcal{I}_T^{ij}$, for $D = 5$. The boundary of the central peak in the right-hand figure, the zero locus of this eigenvalue, coincides with the boundary between the red and yellow regions of C_J in figure 1

3.4.3 D=6

In $6-D$ the temperature

$$T = \frac{1}{4\pi r_h}\frac{(3+j_1^2+j_2^2-j_1^2 j_2^2)}{(1+j_1^2)(1+j_2^2)} \tag{74}$$

again vanishes on hyperbolae in the j_1-j_2 plane.

The specific heat at constant J looks a little more complicated than in $5D$, but only because some of the hyperbolae overlap. Figure 3 displays similar information to figure 1, but for $D = 6$ — this figure is essentially the same as one in [10] and is reproduced here for comparison with figure 4.

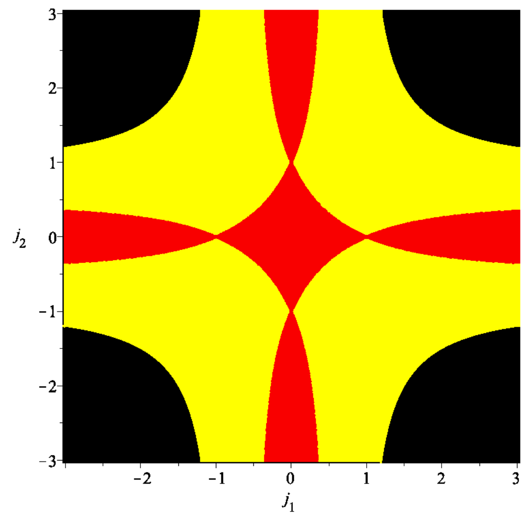

Figure 3: C_J in 6-dimensions, with the same colour coding as in figure 1. (This figure is essentially the same as one in [10], using slightly different variables.)

The regions in the j_1-j_2 plane where the eigenvalues of the isothermal moment of inertia tensor take positive values are plotted in figure 4. The yellow cross-shaped shaded region in the left hand graph of figure, where one of the eigenvalues of $\mathcal{I}_T$ is positive, exactly co-incides with the inner red region of C_J in figure 3, thus the product of these two quantities is always negative. The determinant

$$\beta C_J \det \mathcal{I}_T = -\frac{32}{27}\pi^5 r_h^{15} \left(\frac{(1+j_1^2)^5(1+j_2^2)^5}{3 - j_1^2 - j_2^2 - j_1^2 j_2^2} \right), \tag{75}$$

is thus negative in the central yellow region of the other eigenvalue of $\mathcal{I}_T$, indicated in the right hand picture in figure 4, but it is positive in the red region of the right hand figure, where both eigenvalues of $\mathcal{I}_T$ are negative and $C_J > 0$. While the determinant itself can positive there is no regime in which C_J and both eigenvalues of $\mathcal{I}_T$ are simultaneously positive. The grand-canonical ensemble for six-dimensional Myers-Perry black holes is thus everywhere unstable.

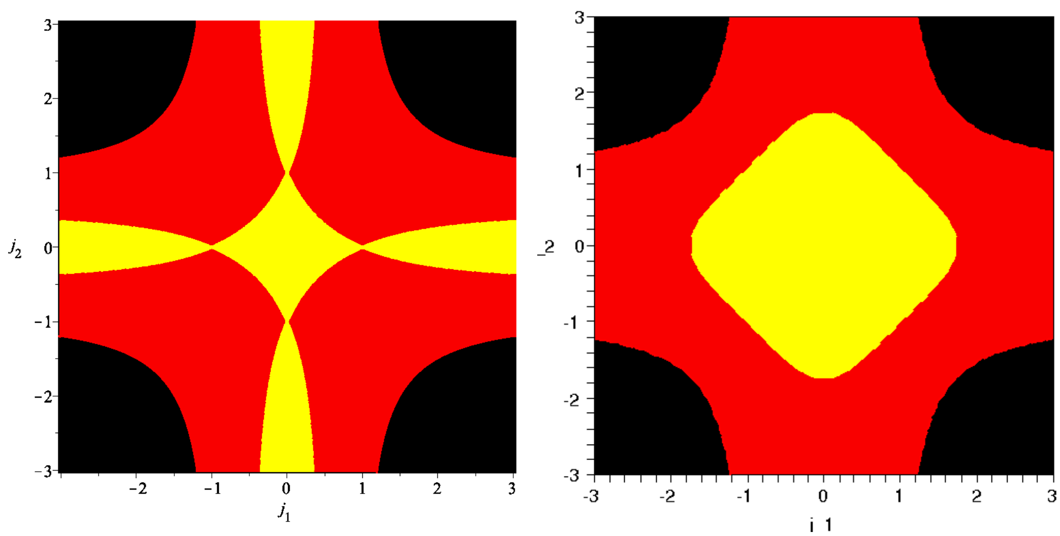

Figure 4: Eigenvalues of the isothermal moment of inertia tensor, $\mathcal{I}_T^{ij}$, for $D = 6$ (same colour coding as in fig 1).

3.4.4 General $D \geq 7$

In §3.4.1—3.4.3 the stability properties of Myers-Perry black holes in the canonical ensemble were analysed in terms of C_J and isothermal moment of inertia, associated to the canonical ensemble through (30). We focus in this section on the grand-canonical ensemble, partly because the canonical ensemble has already been analysed (albeit only for $D = 4, 5, 6$) but primarily because it is algebraically somewhat simpler than the canonical ensemble. The general principles of §2.4 ensure that the stability properties are the same: since $\partial_A \partial_B U$ and $-\partial^A \partial^B G$ are inverses of each other their signature is the same.

One necessary condition for thermodynamic stability is

$$\beta C_\Omega = -\frac{16\pi^2 M r_h^2}{(D-2)} \frac{\left(D - 2 + 2\Sigma_1^-\right)}{\left(D - 3 + 2\Sigma_1^-\right)} > 0, \tag{76}$$

in particular βC_Ω is negative for non-rotating black holes with all $j_i = 0$. More generally we must examine the condition

$$\frac{\left(D-2+2\Sigma_1^-\right)}{\left(D-3+2\Sigma_1^-\right)} < 0. \tag{77}$$

In terms of the variables

$$x_i = \frac{j_i^2}{1-j_i^2}, \tag{78}$$

(77) is a simple ratio of linear functions

$$\frac{\left(D-2+2\sum_i^N x_i\right)}{\left(D-3+2\sum_i^N x_i\right)} < 0, \tag{79}$$

and positivity of βC_Ω requires that

$$-\frac{D-2}{2} < \sum_{i=1}^N x_i < -\frac{D-3}{2}, \tag{80}$$

i.e. the x's are constrained to lie between two hyperplanes in x-space, which never intersect for finite x_i. However x_i diverges when j_i^2 passes through 1, and this description pushes some subtleties around $j_i^2 = 1$ out to infinity.

So we consider instead the condition

$$\frac{\left(D-2+2\Sigma_1^-\right)\prod_i^N(1-j_i^2)}{\left(D-3+2\Sigma_1^-\right)\prod_i^N(1-j_i^2)} < 0. \tag{81}$$

This ratio can only change sign either across the hypersurface

$$\left(D-2+2\Sigma_1^-\right)\prod_i^N(1-j_i^2) = 0, \tag{82}$$

where it has a zero, or across the hypersurface

$$\left(D-3+2\Sigma_1^-\right)\prod_i^N(1-j_i^2) = 0, \tag{83}$$

where it has a pole. Both these hypersurfaces are of the form

$$\mathcal{C}_{D,s} := (D-s)\prod_i^N(1-j_i^2) + 2\sum_{k=1}^N\left(j_k^2\prod_{i\neq k}^N(1-j_i^2)\right) = 0, \tag{84}$$

with $s = 2$ or 3. If any $j_i^2 = 1$, for example if $j_1^2 = 1$, then

$$\mathcal{C}_{D,s} = 2\prod_{i=2}^{N}(1 - j_i^2) = 0, \tag{85}$$

and at least one other j_i^2 must be one, the remaining j_i's, $N-2$ of them, are arbitrary. Indeed the hypersurfaces $\mathcal{C}_{D,2}$ and $\mathcal{C}_{D,3}$ intersect on a manifold of co-dimension two, which is actually a flat $\mathbf{R}^{N-2}$ in j-space). In $D = 7$ for example, the relevant hypersurfaces are $\mathcal{C}_{7,3}$ and $\mathcal{C}_{7,2}$, while in $D = 8$ they are $\mathcal{C}_{8,3}$ and $\mathcal{C}_{8,2}$. These various hypersurfaces are shown in figures 5.

We can determine whether or not βC_Ω changes sign across these hypersurfaces by following it out along rays from the origin in specific directions. For example in the direction $j_1 = j$, $j_2 = \cdots = j_N = 0$,

$$\frac{\mathcal{C}_{D,2}}{\mathcal{C}_{D,3}} = \frac{D-2-(D-4)j^2}{D-3-(D-5)j^2}, \tag{86}$$

which is negative between $\mathcal{C}_{D,2} = 0$ and $\mathcal{C}_{D,3} = 0$ where

$$\frac{D-2}{D-4} < j^2 < \frac{D-3}{D-5}. \tag{87}$$

Thus βC_Ω does indeed change sign when it crosses either of the hypersurface $\mathcal{C}_{D,2} = 0$ or $\mathcal{C}_{D,3} = 0$ in this direction (in this specific direction each hypersurface has only one branch, and so is only crossed once). We note in passing that the hypersurface $\mathcal{C}_{D,3}$, on which $j^2 = \frac{D-3}{D-3-5}$, coincides with the surface on which T is minimised in the micro-canonical ensemble, equation (53) with $n = 1$.

We now examine the singularities in the isentropic moment of inertia tensor. The determinant of the Hessian in the grand-canonical ensemble (31) is derived in appendix E, equation (177),

$$\beta C_\Omega \det \mathcal{I}_S = -\frac{8\pi^2(D-2)}{(D-3+2\Sigma_1^-)}\left(\frac{2Mr_h^2}{D-2}\right)^{N+1}\prod_{i=1}^{N}\frac{(1+j_i^2)^2}{(1-j_i^2)}, \tag{88}$$

for Myers-Perry black holes. Thus the factor $D-2+\Sigma_1^-$ in the numerator of C_Ω, giving rise to a zero in the specific heat, is cancelled by a similar factor in the denominator of $\det(\mathcal{I}_S)$. For a better understanding of the thermodynamic structure we need to examine the eigenvalues of $\mathcal{I}_S$. It is

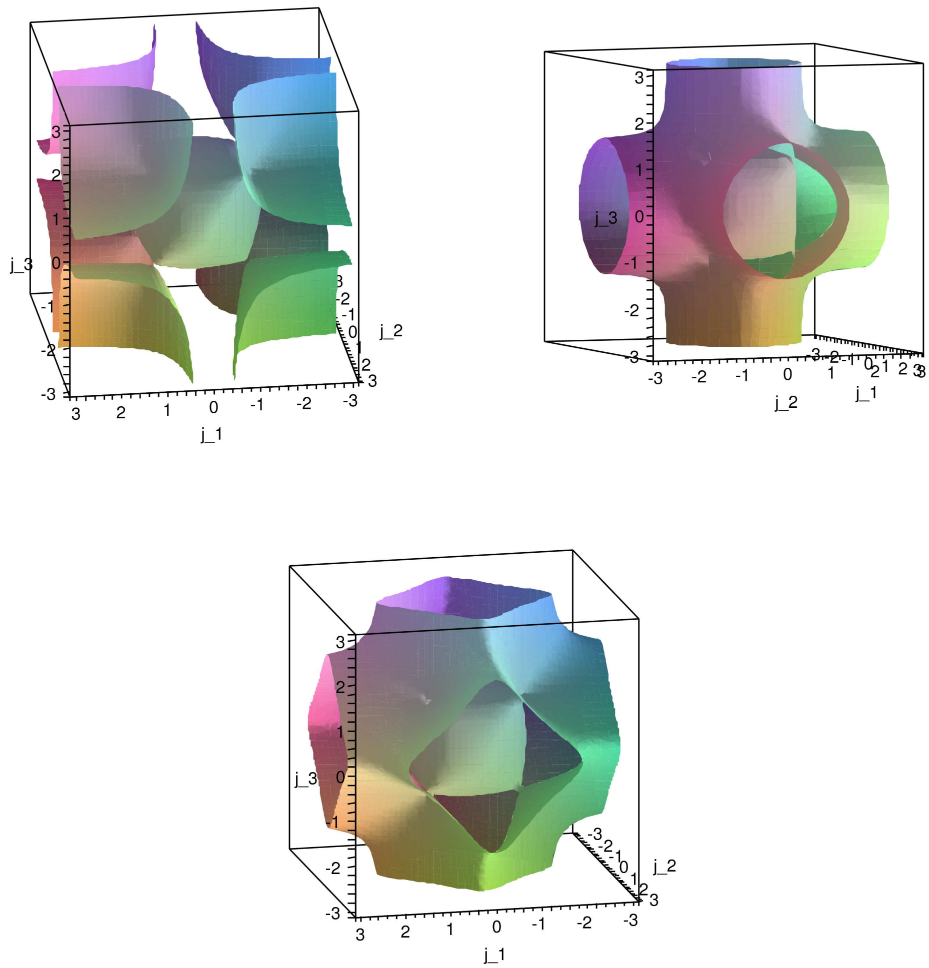

Figure 5: Hypersurfaces on which βC_Ω changes sign in 7 and 8 dimensions. Top-left is $\mathcal{C}_{7,3} = 0$, on which βC_Ω diverges in $D = 7$; top-right figure is $\mathcal{C}_{8,2} = 0$, on which βC_Ω vanishes in $D = 8$; the bottom figure is both $\mathcal{C}_{7,2} = 0$ and $\mathcal{C}_{8,3} = 0$, on which βC_Ω vanishes in $D = 7$ and diverges in $D = 8$.

actually more convenient to examine ${\mathcal{I}_S}^{-1}$ rather than $\mathcal{I}_S$, as it has the sightly simpler form (173),

$$\left(\mathcal{I}_S^{-1}\right)_{ij} = \left(\frac{\partial \Omega_i}{\partial J_j}\right)_S = \frac{(D-2)}{2Mr_h^2}\left\{\frac{(1-j_i^2)}{(1+j_i^2)^2}\delta_{ij} + \frac{2j_ij_j}{(D-2)(1+j_i^2)(1+j_j^2)}\right\}. \tag{89}$$

Focusing first on the determinant, stability requires

$$\det \mathcal{I}_S{}^{-1} = \left(\frac{D-2}{2Mr_h^2}\right)^N \frac{(D-2+2\Sigma_1^-)}{(D-2)} \prod_{i=1}^{N} \frac{(1-j_i^2)}{(1+j_i^2)^2} > 0. \tag{90}$$

Of course positivity, while necessary for stability, is not sufficient, (90) is satisfied when there are an even number of negative eigenvalues, but we do know that $\det \mathcal{I}_S{}^{-1}$ can only change sign when $\mathcal{C}_{D,2} = 0$.

To understand the eigenvalue structure in more detail consider first the two cases $D = 7$ and $D = 8$. The relevant surfaces are shown in figure 5. Each surface $\mathcal{C}_{D,2}$ consists of two branches, on which at least one eigenvalue of (89) must vanish, touching at the symmetric point $j_1^2 = j_2^2 = j_3^2 = 1$ where two eigenvalues vanish and the third is positive. These two surfaces divide the parameter space into three regions. All three eigenvalues are positive in the interior region, inside the inner surface that is visible through the holes in the outer surface, because they are positive at the origin where $\mathcal{I}_S{}^{-1}$ is a positive multiple of the identity matrix.

We can determine how many negative eigenvalues there are in the intermediate region between the two surfaces simply by checking the number at any one point in the region, there must be the same number at any other point in the region as none can change sign unless we cross one of the surfaces. Similarly we can find the number in the exterior region outside both surfaces.

For the region between the two surfaces we need merely set $j_1 = j_2 = 0$ and choose $j_3^2 = j^2$ large enough to ensure that we are outside the interior region. Then $\left(\frac{2Mr_h^2}{D-2}\right)\mathcal{I}_S{}^{-1}$ in (89) is diagonal and the eigenvalues are easily read off as

$$1, \qquad 1 \qquad \text{and} \qquad \frac{(D-2)-(D-4)j^2}{(D-2)(1+j^2)^2}, \tag{91}$$

Hence there is one negative eigenvalue if $j^2 > \frac{D-2}{D-4}$, with $j^2 = \frac{D-2}{D-4}$ marking the boundary of the interior region in the j_3-direction.

For the region exterior to both the surfaces we can set $j_1^2 = j_2^2 = j_3^2 = j^2$, with j large enough to ensure that we are in the exterior region. Now (89) shows that that the eigenvalues of $\left(\frac{r_h S}{2\pi}\right)\mathcal{I}_S{}^{-1}$ are

$$\frac{(1-j^2)}{(1+j^2)^2}, \qquad \frac{(1-j^2)}{(1+j^2)^2} \qquad \text{and} \qquad \frac{(D-2)-(D-8)j^2}{(D-2)(1+j^2)^2}. \tag{92}$$

Hence there are two degenerate negative eigenvalue if $j^2 > 1$ and always one other positive eigenvalue for $D = 7$ or 8. We have thus shown that, for every point in the interior region, $\mathcal{I}_S{}^{-1}$ has three positive eigenvalues, every point in the intermediate region has two positive eigenvalues and one negative one while every point in the exterior regions has two negative eigenvalues and one positive one. Since βC_Ω vanishes on the same surfaces, and is negative in the interior region since it is negative at the origin, we see that the canonical ensemble is never stable in 7 or 8 dimensions.

The above analysis is easily extended to $D > 8$. We only need determine the signs of the eigenvalues of (89) in special directions $j_1 = \cdots j_n = j$, $j_{n+1} = \cdots j_N = 0$, and this gives the signs in each of regions separated by the roots of

$$\mathcal{C}_{D,2} = \left[D-2-(D-2-2n)j^2\right](1-j^2)^{n-1} = 0. \tag{93}$$

The number of regions in any specific direction is determined by the number of roots, with $j^2 > 0$, and the greatest number is when $n = N$: there are then $N-1$ such roots and the different branches of $\mathcal{C}_{D,2} = 0$ divide j-space into N regions.

The form of (89) in these directions is

$$\left(\frac{2Mr_h^2}{D-2}\right)\mathcal{I}_S{}^{-1} = \begin{pmatrix} \frac{1-j^2}{(1+j^2)^2}\mathbf{1}_{n\times n} + \frac{2}{D-2}\frac{j^2}{(j^2+1)^2}\mathbf{Q}_{n\times n} & 0 \\ 0 & \mathbf{1}_{(N-n)\times(N-n)} \end{pmatrix}, \tag{94}$$

where $\mathbf{1}_{d\times d}$ are $d \times d$ identity matrices and $\mathbf{Q}_{n\times n}$ is the $n \times n$ matrix whose entries are all one.[6] There are $N-n$ eigenvalues $+1$ and the remaining eigenvectors $\mathbf{V} = (V_1, \ldots, V_n, 0 \ldots, 0)^t$ and eigenvalues λ are determined by

$$\frac{1-j^2}{(1+j^2)^2}V_i + \frac{2}{(D-2)}\frac{j^2}{(1+j^2)^2}\sum_{k=1}^{n} V_k = \lambda V_i. \tag{95}$$

There are two possibilities:

[6] We use the same notation as [18].

1. $\sum_{k=1}^{n} V_k \neq 0$: this requires $V_1 = \cdots = V_n$ which implies that, for $i = 1, \ldots, n$,

$$\lambda = \frac{1-j^2}{(1+j^2)^2} + \frac{2n}{(D-2)}\frac{j^2}{(1+j^2)^2} = \frac{D-2-(D-2-2n)j^2}{(D-2)(1+j^2)^2}. \quad (96)$$

   For $1 \leq n < N$, this configuration returns a negative eigenvalue for $j^2 > \frac{D-2}{D-2-2n}$, while for $n = N = \frac{D-1-\epsilon}{2}$ this eigenvalue is positive for all values of j^2.

2. $\sum_{k=1}^{n} V_k = 0$: giving $\lambda = \frac{1-j^2}{(1+j^2)^2}$, and

$$\mathbf{V} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ -n \end{pmatrix}. \quad (97)$$

   This requires $n \geq 2$ and has degeneracy $n-1$. This gives n negative eigenvalues when $j^2 > 1$.

The overall picture is then that there are $N-1$ branches to the hypersurface $\mathcal{C}_{D,2}$ which divide j-space into N regions. All eigenvalues of ${\mathcal{I}_S}^{-1}$ are positive at the origin and at every point inside the first branch. Every time a branch is crossed by a ray emanating from the origin, one of the positive eigenvalues of ${\mathcal{I}_S}^{-1}$ changes sign and becomes negative until, in the outer region after all $N-1$ branches have been crossed, there are $N-1$ negative eigenvalues and one remaining positive one. The only region in which ${\mathcal{I}_S}^{-1}$, and hence $\mathcal{I}_S$, is a positive matrix is the innermost one. But we have already seen that βC_Ω is negative in the innermost region, hence the canonical ensemble is always unstable for any choice of metric parameters in any D.

4 Conclusions

We have compared the micro-canonical, the canonical and the grand-canonical ensembles in the thermodynamic description of asymptotically flat rotating black holes in arbitrary dimensions. These black holes are always thermodynamically unstable but the thermodynamic instability manifests itself differently in the different ensembles. There is however an elegant and simple relation between the specific heats and moment of inertia tensors in the canonical and the grand-canonical ensembles, given by equation (3),

$$C_J \det \mathcal{I}_T = C_\Omega \det \mathcal{I}_S. \tag{98}$$

The case of Myers-Perry black-holes has been analysed in detail and all extrema of the temperature in the micro-canonical ensemble have been found and classified and shown to correspond to inflection points of the entropy.

In the canonical ensemble it has been shown that, in D dimensions, the specific heat C_J in equation (56) vanishes when $T=0$ and changes sign on a hypersurface in angular momentum space given by

$$(4\pi r_h T)^2 = (D-2)\left(4\pi r_h T - 4\sum_{i=1}^{N} \frac{j_i^2}{(1+j_i^2)^2}\right) \tag{99}$$

(where $j_i = \frac{2\pi J^i}{S}$), on which it diverges. In the determinant of the Hessian this singularity in C_J is exactly cancelled by an equivalent zero in $\det(\mathcal{I}_T)$. There are also singularities in $\det(\mathcal{I}_T)$ when $\mathcal{C}_{D,3}$ in equation (83) vanishes.

In the grand-canonical ensemble C_Ω in equation (59) also vanishes when $T=0$ and has divergences, this time on the hypersurface defined by $\mathcal{C}_{D,3}=0$ rather than that given by (99). In addition C_Ω also has zeros on the hypersurface $\mathcal{C}_{D,2}=0$ in equation (82). In the determinant of the Hessian for the grand-canonical ensemble (31) the zeros of C_Ω are cancelled by corresponding poles in $\det(\mathcal{I}_S)$ on $\mathcal{C}_{D,2}=0$. The locus of these singular points of $\det(\mathcal{I}_S)$ corresponds to a branched hypersurface in angular momentum space which divides the space into N separate regions. Every time a branch of this hypersurface is crossed an eigenvalue of $\mathcal{I}_S$ changes sign and the moment of inertia tensor has different signature in the N separate regions. Only the region surrounding the origin in angular momentum space gives a positive definite moment of inertia tensor and this region corresponds precisely to the region where C_Ω is negative.

There is a curious relation between the hypersurface $\mathcal{C}_{D,3}=0$ on which both C_Ω and $\det(\mathcal{I}_T)$ diverge on the one hand and extremal $T=0$ Myers-Perry black holes on the other: the algebraic equations defining these two hypersurfaces are related by analytic continuation $(J^i)^2 \to -(J^i)^2$, with the entropy held constant.

Our analysis has also shown that, in the micro-canonical and the grand-canonical ensembles, many of the thermodynamic properties of Myers-Perry black holes in $D-2$ dimensions can be obtained from those of a black hole in D dimensions by letting one of the angular momenta in D dimensions tend to infinity, keeping the entropy constant.

The thermodynamic instabilities of Myers-Perry black holes thus have a very rich structure, beyond that of the ultra-spinning surface upon which the moment of inertia tensor develops its first negative eigenvalue.

An obvious direction for future work on this topic is to include a charge on the black hole and to introduce a cosmological constant to encompass the case of asymptotically anti-de Sitter rotating black holes. The latter should prove particularly interesting as the black holes will become thermodynamically stable when the magnitude of the cosmological constant is large enough and much could be learned by mapping out the boundary of the stability region.

A Temperature extrema and inflection points of the entropy

In this appendix we extend the study in [15] and [18] to find all isenthalpic (*i.e.* constant mass) extrema of T for Myers-Perry black holes, as the J^i are varied in asymptotically flat space-times.

At constant M equation (46) implies that

$$\left.\frac{\partial a_i}{\partial J^j}\right|_M = \frac{(D-2)}{2M}\delta_{ij} \,, \tag{100}$$

with, which the expression

$$M = \frac{(D-2)\varpi}{16\pi} r_h^{D-3} \prod_{i=1}^{N} \left(1 + j_i^2\right) \tag{101}$$

for the mass, gives

$$dM = 0 \qquad \Rightarrow \qquad \left.\frac{\partial r_h}{\partial J^i}\right|_M = -\frac{(D-2)}{4\pi T M}\Omega_i = -\frac{(D-2)}{tM}\,\omega_i \,, \tag{102}$$

where we have defined

$$t := 4\pi r_h T = \left(D - 3 - 2\sum_{i=1}^{N} \frac{j^2}{1+j_i^2}\right) \quad \text{and} \quad \omega_i = r_h \Omega_i = \frac{j_i}{1+j_i^2} \tag{103}$$

(both t and ω_i are invariant under $j_i \to \frac{1}{j_i}$).

Equations (100) and (102) together imply

$$\left.\frac{\partial j_i}{\partial J^j}\right|_M = \frac{(D-2)}{2r_h M}\left(\delta_{ij} + \frac{2}{t}\frac{j_i j_j}{(1+j_j^2)}\right). \tag{104}$$

We now have all the information we need to calculate $\left.\frac{\partial T}{\partial J^i}\right|_M$ from

$$T = \frac{1}{4\pi r_h}\left(D-3-2\sum_{i=1}^{N}\frac{j_i^2}{1+j_i^2}\right), \tag{105}$$

we find

$$\left.\frac{\partial T}{\partial J^i}\right|_M = \frac{(D-2)}{4\pi r_h^2 M}\left(1-\frac{2}{1+j_i^2}-\frac{4\,\Omega^2}{t}\right)\omega_i\,, \tag{106}$$

where $\Omega^2 := \sum_k \omega_k^2$.

For fixed M extrema of T occur for

$$\omega_i = 0 \;\Leftrightarrow\; j_i = 0 \qquad \text{or} \qquad 1-\frac{2}{1+j_i^2}-\frac{4\,\Omega^2}{t} = 0. \tag{107}$$

In particular any finite non-zero j_i are all equal at an extremum.

It is also possible that some of the j_i might tend to infinity. Suppose m of the j_i diverge as $j_i \approx \Lambda \to \infty$. Then, at fixed finite mass,

$$r_h \approx \Lambda^{-\frac{2m}{D-3}}, \tag{108}$$

and hence

$$\left.\frac{\partial T}{\partial J^i}\right|_M \approx \Lambda^{\frac{4m}{D-3}}\left(1-\frac{2}{1+j_i^2}-\frac{4\,\Omega^2}{t}\right)\omega_i\,. \tag{109}$$

For $j_i \approx \Lambda$, $\omega_i \approx \Lambda^{-1}$ so

$$\left.\frac{\partial T}{\partial J^i}\right|_M \approx \Lambda^{\frac{4m}{D-3}-1}, \tag{110}$$

which tends to zero for $\Lambda \to \infty$ provided $m < \frac{D-3}{4}$, which is possible for $D \geq 8$.

To keep the discussion general we shall suppose n of the j_i are finite and equal, m are infinite and $N-n-m$ are zero. Up to permutations of the j_i, extrema of T can only occur for configurations with

$$j_1 = \cdots = j_n = j > 0, \qquad j_{n+1} = \cdots = j_{N-m} = 0,$$
$$j_{N-m+1} = \cdots = j_N = \Lambda \to \infty \tag{111}$$

so we focus on the symmetric angular momentum configurations

$$\vec{j} = \lim_{\Lambda\to\infty}(\overbrace{j,\ldots,j}^{\overset{n}{\text{times}}},\overbrace{0,\ldots,0}^{\overset{N-m-n}{\text{times}}},\overbrace{\Lambda,\ldots,\Lambda}^{\overset{m}{\text{times}}}). \tag{112}$$

The extrema require j to satisfy the second equation in (107), $j = j_*$ with

$$j_*^2 = \frac{t + 4\Omega^2}{t - 4\Omega^2}. \tag{113}$$

At $\vec{j}_*$

$$\Omega^2 = \frac{n j_*^2}{(1+j_*^2)^2} \qquad \text{and} \qquad t = D - 3 - 2m - \frac{2n j_*^2}{(1+j_*^2)} \tag{114}$$

which gives the solution of (113) to be

$$j_*^2 = \frac{D-3-2m}{D-3-2m-2n}. \tag{115}$$

The temperature at these extrema is, from (105),

$$T_* = \frac{1}{4\pi r_h}\frac{(D-3-2m)(D-3-2m-2n)}{(D-3-2m-n)}. \tag{116}$$

Demanding $T_* \geq 0$ imposes the restriction

$$m + n \leq \frac{D-3}{2}. \tag{117}$$

More generally when the angular momenta are of the form (112), but not necessarily at j_*, the temperature is

$$T = \frac{1}{4\pi r_h}\frac{(D-3-2m) + (D-3-2m-2n)j^2}{(1+j^2)} \tag{118}$$

and vanishes for

$$j^2 = j_0^2 = \frac{D-3-2m}{2m+2n-(D-3)}, \tag{119}$$

which is only possible for $m + n \geq \frac{D-3}{2}$.

To analyse the nature of the extrema we need the second derivative of T. A straightforward but tedious calculation gives

$$\begin{aligned}\left.\frac{\partial^2 T}{\partial J^i \partial J^j}\right|_M &= \frac{(D-2)^2}{4\pi M^2 r_h^3}\left\{\left[\frac{4j_i^2}{(1+j_i^2)^2} - \frac{2\,\Omega^2}{t}\frac{(1-j_i^2)}{(1+j_i^2)} - \frac{1}{2}\right]\frac{\delta_{ij}}{1+j_i^2}\right. \\ &\quad - \left[\frac{4}{t}\left(\frac{1-j_i^2}{(1+j_i^2)^2} + \frac{1-j_j}{(1+j_j)^2} - \frac{1}{4}\right) + \frac{8\,\Omega^2}{t^2}\left(\frac{1}{1+j_i^2} + \frac{1}{1+j_j} + \frac{1}{2}\right)\right. \\ &\quad \left.\left. + \frac{8}{t^2}\left(\frac{2\,\Omega^4}{t} + \widetilde{\Sigma}\right)\right]\omega_i\omega_j\right\}, \end{aligned} \tag{120}$$

where

$$\widetilde{\Sigma} := \sum_k \frac{(1-j_k^2)j_k^2}{(1+j_k^2)^3}. \tag{121}$$

Sums like $\widetilde{\Sigma}$ crop up frequently in this analysis and it will prove convenient to define

$$\Sigma_n^\pm = \sum_{k=1}^N \frac{j_k^2}{(1\pm j_k^2)^n} \tag{122}$$

in terms of which

$$\widetilde{\Sigma} = 2\Sigma_3^+ - \Sigma_2^+. \tag{123}$$

We wish to determine the signs of the eigenvalues of (120) at $\vec{j}_*$. There are three cases to consider:

- At an extremum of the form (111), if either of the indices i and j is in the range $[N-n-m,\cdots,N-m]$ then $\left.\frac{\partial^2 T}{\partial J^i \partial J^j}\right|_{\mathbf{j}_*} = 0$ unless the other index is in the same range. When both i and j are in the range $[N-n-m,\cdots,N-m]$,

$$\begin{aligned}\left.\frac{\partial^2 T}{\partial J^i \partial J^j}\right|_{\mathbf{j}_*} &= -\frac{(D-2)^2}{4\pi M^2 r_{h,*}^3}\left.\left(\frac{2\,\Omega^2}{t} + \frac{1}{2}\right)\right|_{\mathbf{j}_*}\delta_{ij} \\ &= -\frac{(D-2)^2}{4\pi M^2 r_{h,*}^3}\frac{(D-3-2m-2n)}{2(D-3-2m-n)}\delta_{ij}\,. \end{aligned} \tag{124}$$

  The eigenvalues are all negative in these directions, corresponding to a maximum of T around J_*.

- If one of the indices i or j is in the range $[N-m+1, \ldots, N]$ and the other is in the range $[1, \ldots, N-m]$ then $\left.\frac{\partial^2 T}{\partial J^i \partial J^j}\right|_{\mathbf{j}_*} = 0$. If both are in the range $[N-m+1, \ldots, N]$ then

$$\left.\frac{\partial^2 T}{\partial J^i \partial J^j}\right|_{\mathbf{j}_*} \approx \frac{1}{r_{h,*}^3} \frac{1}{\Lambda^2} \approx \Lambda^{\frac{6m}{D-3}-2} \underset{\Lambda \to \infty}{\longrightarrow} 0 \tag{125}$$

since $m < \frac{D-3}{4}$.

- If both indices i and j are in the range $[1, n]$, then the nature of the extremum is determined by

$$\left.\frac{\partial^2 T}{\partial J^i \partial J^j}\right|_{\mathbf{j}_*} = \frac{(D-2)^2}{4\pi M^2 r_{h,*}^3} \left(A_* \delta_{ij} + B_* Q_{ij}\right), \tag{126}$$

where Q_{ij} is the $n \times n$ matrix whose entries are all unity and A_* and B_* are ratios of polynomials in j_*. Since Q_{ij} has $n-1$ zero eigenvalues and one eigenvalue equal to n, (126) has $n-1$ degenerate eigenvalues $\lambda_1 = A_*$ and one eigenvalue $\lambda_2 = A_* + nB_*$. Evaluating A_* and B_* gives

$$\lambda_1 = \frac{(D-3-2m)(D-3-2m-2n)^2}{4(D-3-2m-n)^3} > 0, \tag{127}$$

$$\lambda_2 = \frac{1}{4}. \tag{128}$$

Thus $\vec{J}_*$ is in general a saddle point, with T minimised in the directions $i = 1, \ldots, n$ and maximised in the directions $i = N-n-m, \ldots, N-m$.

A necessary condition for stability is that the eigenvalues of

$$H_{ij} = -\left.\frac{\partial^2 S}{\partial J^i \partial J_j}\right|_M \tag{129}$$

be positive, [15]. Using (102) and

$$S = \frac{\varpi}{4} r_h^{D-2} \prod_k (1 + j_k^2), \tag{130}$$

gives

$$\left.\frac{\partial S}{\partial J^i}\right|_M = -\frac{2}{t} \frac{j_i}{(1 + j_i^2)}. \tag{131}$$

The entropy is thus a monotonically decreasing function of each of the j_i as $|j_i|$ increases. It was shown in [18] that the extrema of the temperature in the two special cases $n = 1$ and $n = N$ (with $m = 0$) are inflection points of the entropy. We now show that this is a general property of all the extrema of T.

An inflection point corresponds to a zero eigenvalue of the matrix (129), so we first determine

$$\begin{aligned} H_{ij} &= -\left.\frac{\partial^2 S}{\partial J^i \partial J_j}\right|_M \qquad (132) \\ &= \frac{(D-2)}{2r_h^2 TM}\left\{\frac{1-j_i^2}{(1+j_i^2)^2}\delta_{ij} + \frac{4}{t}\left(\frac{1}{1+j_i^2} + \frac{1}{1+j_j^2} - \frac{1}{2} + \frac{2}{t}\Omega^2\right)\omega_i\omega_j\right\}, \end{aligned}$$

which agrees with equation (2.13) in [18], apart from some minor typos.

The eigenvalues at the symmetric points (112) are easily determined:

- If both indices i and j are in the range $[1, \ldots, n]$ we write

$$H_{ij} = \frac{(D-2)}{2r_h^2 MT}\Big(\mathcal{A}\delta_{ij} + \mathcal{B}Q_{ij}\Big) \qquad (133)$$

  and the eigenvalues of $\mathcal{A}\delta_{ij} + \mathcal{B}Q_{ij}$, with $i, j = 1, \ldots, n$, are

$$\lambda_1 = \mathcal{A} \qquad \text{and} \qquad \lambda_2 = \mathcal{A} + n\mathcal{B}, \qquad (134)$$

  where λ_1 has degeneracy $n-1$. These are

$$\begin{aligned} \lambda_1 &= \frac{(1-j^2)}{(1+j^2)^2}, \qquad (135) \\ \lambda_2 &= \frac{(D-3-2m)\big\{D-3-2m-(D-3-2m-2n)j^2\big\}}{\big\{D-3-2m+(D-3-2m-2n)j^2\big\}^2}. \end{aligned}$$

- If either of the indices i and j is in the range $[n+1, \cdots, N-m]$, then H_{ij} vanishes unless the other index is in the same range in which case $j_i = j_j = 0$ and

$$H_{ij} = \frac{(D-2)}{2r_h^2 TM}\delta_{ij}\,. \qquad (136)$$

- If either of the indices i and j is in the range $(N-m+1, \ldots, N)$ then $\lambda = 0$.

As observed in [18] the eigenvalues are all positive near $J^i = 0$ and the first negative eigenvalue is encountered for $n = 1$ when $j = \frac{D-3}{D-5}$, which is precisely the inflection point of [6] at the first temperature minimum. This hypersurface $j = \frac{D-3}{D-5}$ on which H_{ij} first develops a zero eigenvalue is the ultra-spinning surface of [18]. For $D = 5$ the ultra-spinning surface in the micro-canonical ensemble is not closed and j_1 can reach infinity when $j_2 = 0$, It is shown in §3.4.2 that the $D = 5$ ultra-spinning surface in the canonical ensemble is closed, see figure 2.

At temperature minima, where $j = j_*$, the eigenvalues (135) above evaluate to

$$
\begin{aligned}
\lambda_1\Big|_{j_*} &= -\frac{n}{2}\frac{(D-3-2m-2n)}{(D-3-2m-n)^2} \leq 0, \qquad (n-1) \text{ times;} \\
\lambda_2\Big|_{j_*} &= 0. \qquad (137)
\end{aligned}
$$

In particular there is always one zero eigenvalue, corresponding to an inflection point in the entropy in the direction of the associated eigenvector..

B Specific heat at constant angular momentum

To calculate the heat capacity at constant J

$$C_J = \frac{\partial M}{\partial T}\Big|_J , \qquad (138)$$

we first observe that (46) gives

$$J^i = \frac{2M}{D-2} a_i, \qquad (139)$$

so $J^i = const$ implies

$$\frac{da_i\big|_J}{a_i} = -\frac{dM\big|_J}{M}. \qquad (140)$$

Next, combining (41), (44) and (46), $j_i = \frac{a_i}{r_h} = \frac{2\pi J^i}{S}$ vary as

$$\frac{dj_i}{j_i}\Big|_J = -\frac{dS}{S}\Big|_J = \left(\frac{da_i}{a_i} - \frac{dr_h}{r_h}\right)\Big|_J = -\left(\frac{dM}{M} + \frac{dr_h}{r_h}\right)\Big|_J . \qquad (141)$$

But explicitly from (44)

$$\begin{aligned} \left.\frac{dS}{S}\right|_J &= (D-2)\left.\frac{dr_h}{r_h}\right|_J + 2\sum_{k=1}^N \left.\frac{j_k dj_k}{1+j_k^2}\right|_J \\ &= (D-2)\left.\frac{dr_h}{r_h}\right|_J - 2\left.\left(\frac{dM}{M}+\frac{dr_h}{r_h}\right)\right|_J \sum_k^N \frac{j_k^2}{1+j_k^2}. \end{aligned} \tag{142}$$

Equations (141) and (142) together now give

$$\left.\frac{\partial M}{\partial r_h}\right|_J = \left(\frac{t}{D-2-t}\right)\frac{M}{r_h}. \tag{143}$$

Similar manipulations on $T = \frac{t}{4\pi r_h}$ yield

$$\left.\frac{\partial T}{\partial r_h}\right|_J = \frac{1}{4\pi r_h^2}\left\{\frac{t^2-(D-2)\left(t-4\Sigma_2^+\right)}{D-2-t}\right\}, \tag{144}$$

and then (143) and (144) can be combined to give the specific heat in the canonical ensemble with fixed J,

$$C_J = \frac{16\pi^2 r_h^2 MT}{\left[t^2-(D-2)\left(t-4\Sigma_2^+\right)\right]}. \tag{145}$$

This can be expressed in terms of M and j_i by noting that

$$M = \left(\frac{(D-2)\varpi}{16\pi}\right) r_h^{D-3}\prod_{i=1}^N\left(1+j_i^2\right). \tag{146}$$

C Specific heat at constant angular velocity

The specific heat at constant Ω is straightforward to determine, using similar techniques to those of §3.2. In terms of j_i the entropy (44) is

$$S = \frac{\varpi r_h^{D-2}}{4}\prod_{i=1}^N(1+j_i^2), \tag{147}$$

$$T = \frac{t}{4\pi r_h} = \frac{(D-3-2\Sigma_1^+)}{4\pi r_h} \tag{148}$$

and

$$\Omega_i = \frac{j_i}{r_h(1+j_i^2)}. \tag{149}$$

The specific heat at constant angular velocity is defined as

$$C_\Omega = T\left(\frac{\partial S}{\partial T}\right)_\Omega. \tag{150}$$

From (149)

$$\Omega_i = const \qquad \Rightarrow \qquad dj_i|_\Omega = \left(\frac{1+j_i^2}{1-j_i^2}\right) j_i \frac{dr_h}{r_h}\bigg|_\Omega. \tag{151}$$

Using these it is straightforward to show that

$$\left.\frac{\partial T}{\partial r_h}\right|_\Omega = -\frac{(D-3+2\Sigma_1^-)}{4\pi r_h^2}, \tag{152}$$

and

$$\left.\frac{\partial S}{\partial r_h}\right|_\Omega = \frac{S}{r_h}(D-2+2\Sigma_1^-). \tag{153}$$

Combining these we immediately arrive at equation (59) in the text,

$$C_\Omega = -\frac{4\pi r_h T S(D-2+2\,\Sigma_1^-)}{(D-3+2\,\Sigma_1^-)}. \tag{154}$$

This generalises the $D=4$ case derived in [11] to arbitrary D.

D Isothermal moment of inertia

To calculate the isothermal moment of inertia tensor

$$\mathcal{I}_T^{ij} = \left(\frac{\partial J^i}{\partial \Omega_j}\right)_T \tag{155}$$

in asymptotically flat Myers-Perry space-times our starting point is again

$$T = \frac{D-3-2\Sigma_1^+}{4\pi r_h}, \tag{156}$$

from which we find

$$dr_h|_T = -\frac{1}{\pi T}\,\frac{j_i dj_i}{(1+j_i^2)^2}\bigg|_T\,. \tag{157}$$

Now use this in (46), re-written using (41) in the form

$$J^i = \frac{\varpi r_h^{D-2}}{8\pi}\prod_{k=1}^{N}(1+j_k^2)j_i, \tag{158}$$

to deduce that

$$\frac{\partial J^i}{\partial j_k}\bigg|_T = \frac{2Mr_h}{D-2}\left(\delta_{ik} + 2\left[\frac{(D-3-2\,\Sigma_1^+)(1+j_k^2)-2(D-2)}{(D-3-2\,\Sigma_1^+)(1+j_k^2)^2}\right] j_i j_k\right). \tag{159}$$

Similar manipulations applied to (47) produce

$$\frac{\partial \Omega_j}{\partial j_k}\bigg|_T = \frac{1}{r_h}\left\{\frac{(1-j_j)}{(1+j_j^2}\delta_{jk} + \frac{4j_j j_k}{(D-3-2\Sigma_1^+)(1+j_j)(1+j_k^2)^2}\right\}, \tag{160}$$

the inverse of which is

$$\frac{\partial j_k}{\partial \Omega_j}\bigg|_T = r_h\left\{\frac{(1+j_k^2)^2}{(1-j_k^2)}\delta_{kj} - \frac{4(1+j_k^2)}{(t+4\,\overline{\Sigma})(1-j_j^2)} j_k j_j\right\}. \tag{161}$$

where

$$\overline{\Sigma} := \sum_{i=1}^{N}\frac{j_i^2}{1-j_i^4} = \frac{1}{2}(\Sigma_1^+ + \Sigma_1^-). \tag{162}$$

Equations (159)–(161) are now easily combined to give the symmetric isothermal moment of inertia tensor

$$\mathcal{I}_T^{ij} = \frac{2Mr_h^2}{D-2}\left\{\frac{(1+j_i^2)^2}{(1-j_i^2)}\delta_{ij} - 2j_i j_j\left(1 + \frac{4}{\overline{t}(1-j_i^2)(1-j_j^2)}\right)\right\}, \tag{163}$$

where

$$\overline{t} := D-3+2\,\Sigma_1^-. \tag{164}$$

This is equation (62) in the text.

The determinant of $\mathcal{I}_T$ can be evaluated by observing that the components of matrix are of the form

$$\mathcal{I}_T^{ij} = \frac{2Mr_h^2}{D-2}\big(A_i\delta_{ij} - B_iB_j - C_iC_j\big) \tag{165}$$

with

$$A_i = \frac{(1+j_i^2)^2}{(1-j_i^2)}, \qquad B_i = \sqrt{2}j_i \qquad \text{and} \qquad C_i = \sqrt{\frac{8}{\overline{t}}}\frac{j_i}{(1-j_i^2)}. \tag{166}$$

The determinant of (165) has a compact expression, because the off-diagonal entries factorise,

$$\begin{aligned} \det(\mathcal{I}_T) \ &= \\ \left(\frac{2Mr_h^2}{D-2}\right)^N \left(\prod_{i=1}^N A_i\right) & \left\{1 - \sum_{i=1}^N \frac{\left(B_i^2 + C_i^2\right)}{A_i} + \left(\sum_{i=1}^N \frac{B_i^2}{A_i}\right)\left(\sum_{j=1}^N \frac{C_j^2}{A_j}\right) - \left(\sum_{i=1}^N \frac{B_iC_i}{A_i}\right)^2\right\}. \end{aligned} \tag{167}$$

A little manipulation, using (166), then yields

$$\det \mathcal{I}_T = -\left(\frac{2Mr_h^2}{D-2}\right)^N \frac{\left[t^2 - (D-2)(t - 4\Sigma_2^+)\right]}{\overline{t}} \prod_{i=1}^N \frac{(1+j_i^2)^2}{(1-j_i^2)}, \tag{168}$$

with $\overline{t}$ defined in (164). This, together with the expression for C_J in (56), leads to the compact expression

$$\beta C_J \det \mathcal{I}_T = -\frac{8\pi^2(D-2)}{(D-3+2\Sigma_1^-)}\left(\frac{2Mr_h^2}{D-2}\right)^{N+1} \prod_{i=1}^N \frac{(1+j_i^2)^2}{(1-j_i^2)}. \tag{169}$$

E Isentropic moment of inertia

To calculate the isentropic moment of inertia in asymptotically flat Myers-Perry space-times, again re-write (44) as

$$S = \frac{\varpi r_h^{D-2}}{4}\prod_{i=1}^N (1+j_i^2) \tag{170}$$

from which

$$dr_h\big|_S = -\frac{2r_h}{D-2}\sum_{k=1}^N \frac{j_k\, dj_k}{1+j_k^2}\bigg|_S, \tag{171}$$

with $dJ_i\big|_S = \frac{S}{2\pi} dj_i\big|_S$. Then

$$\Omega_i = \frac{j_i}{r_h(1+j_i^2)} \tag{172}$$

yields

$$\left(\mathcal{I}_S^{-1}\right)_{ij} = \left(\frac{\partial \Omega_i}{\partial J_j}\right)_S = \frac{2\pi}{r_h S}\left\{\frac{(1-j_i^2)}{(1+j_i^2)^2}\delta_{ij} + \frac{2j_i j_j}{(D-2)(1+j_i^2)(1+j_j^2)}\right\}, \tag{173}$$

which was reported in [22]. Equation (173) is easily inverted to give

$$\mathcal{I}_S^{ij} = \frac{2Mr_h^2}{D-2}\left\{\frac{(1+j_i^2)^2}{(1-j_i^2)}\delta_{ij} - \frac{2j_i j_j}{(D-2+2\Sigma_1^-)}\frac{(1+j_i^2)(1+j_j^2)}{(1-j_i^2)(1-j_j^2)}\right\}. \tag{174}$$

Similar manipulations to those of appendix D, with

$$A_i = \frac{(1+j_i^2)^2}{(1-j_i^2)}, \qquad B_i = \sqrt{\frac{2}{t}}\frac{(1+j_i^2)}{(1-j_i^2)}j_i \qquad \text{and} \qquad C_i = 0, \tag{175}$$

reveal that

$$\det \mathcal{I}_S = \left(\frac{2Mr_h^2}{D-2}\right)^N \frac{(D-2)}{(D-2+2\Sigma_1^-)}\prod_{i=1}^N \frac{(1+j_i^2)^2}{(1-j_i^2)}. \tag{176}$$

Combining this with C_Ω in (59) leads to the same expression as (169),

$$\beta C_\Omega \det \mathcal{I}_S = -\frac{8\pi^2(D-2)}{(D-3+2\Sigma_1^-)}\left(\frac{2Mr_h^2}{D-2}\right)^{N+1}\prod_{i=1}^N \frac{(1+j_i^2)^2}{(1-j_i^2)}, \tag{177}$$

so (3) is indeed satisfied.

References

[1] S.W. Hawking, *Nature* **248** (1974) 30;
S.W. Hawking, *Comm. Math. Phys.* **43** (1975) 199;
S.W. Hawking, *Phys. Rev.* **D13** (1976) 191.

[2] J.W. York Jr., *Phys. Rev.* **D 33** (1986) 2092.

[3] S.W. Hawking and D.N. Page, *Comm. Math. Phys.* **87** (1983) 577.

[4] R.C. Myers and M.J. Perry, *Ann. Phys.* **172** (1986) 304.

[5] R. Gregory and R. Laflamme, *Phys. Rev. Lett.* **70** (1993) 2837, [hep-th/9301052]; *Nuc. Phys.* **B428** (1994) 399, [hep-th/9404071].

[6] R. Emparan and R. Myers, *JHEP* **0309** (2003) 025, [hep-th/0308056].

[7] H.K. Kunduri, J. Lucietti and H.S. Reall, *Phys. Rev.* **D74** (2006) 084021, [hep-th/0606076].

[8] R. Emparan, T. Harmark, V. Niarchos, N.A. Obers and M.J. Rodriguez, *JHEP* **0710** (2007) 110 [arXiv:0708.2181 [hep-th]].

[9] R. Emparan and H.S. Reall, *Living Rev. Rel.* **11:6** (2008), [arXiv:0801.3471].

[10] R. Monteiro, M.J. Perry and J.E. Santos, *Phys. Rev.* **D80** (2009) 024041, [arXiv:0903.3256 [gr-qc]].

[11] R. Monteiro, M.J. Perry and J.E. Santos, *Phys. Rev.* **D81** (2010) 024001, [arXiv:0903.2334 [gr-qc]].

[12] O.J.C. Dias, P. Figueras, R. Monteiro, J.E. Santos and R. Emparan, *Phys. Rev.* **D80** (2009) 111701, [arXiv:0907.2248 [hep-th]].

[13] D. Astefanesei, R.B. Mann, M. J. Rodriguez and C. Stelea, *Class. Quantum Grav.* **27** (2010) 165004, [arXiv:0909.3852[hep-th]].

[14] M. Shibata and H. Yoshino, *Phys. Rev.* **D81** (2010) 021501, [arXiv:0912.3606[gr-qc]].

[15] O.J.C. Dias, P. Figueras, R. Monteiro, H.S. Reall and J.E. Santos, *JHEP* **1005** (2010) 076, [arXiv:1001.4527 [hep-th]].

[16] *Thermodynamic instability of doubly spinning black objects*, D. Astefanesei, M.J. Rodriguez and S. Theisen [arXiv:1003.2421[hep-th]].

[17] M. Shibata and H. Yoshino, *Phys. Rev.* **D81** (2010) 104035, [arXiv:1004.4970 [gr-qc]].

[18] O.J.C. Dias, P. Figueras, R. Monteiro and J.E. Santos, *Phys. Rev.* **D82** (2010) 104025, [arXiv:1006.1904[hep-th]].

[19] O.J.C. Dias, R. Monteiro and J.E. Santos, *JHEP* **1108** (2011) 139, [arXiv:1106.4554 [hep-th]].

[20] D. Kastor, S. Ray and J. Traschen, *Class.Quant.Grav.* **26** (2009) 195011, [arXiv:0904.2765].

[21] H.B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, Wiley (2006).

[22] B.P. Dolan, *Compressibility of Myers-Perry black holes*, [arXiv:1308.5403].