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Chaotic strings in a near Penrose limit of $\mathbf{AdS}_5 \times T^{1,1}$

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Abstract

We study chaotic motions of a classical string in a near Penrose limit of $\mathrm{AdS}_5 \times T^{1,1}$. It is known that chaotic solutions appear on $R \times T^{1,1}$, depending on initial conditions. It may be interesting to ask whether the chaos persists even in Penrose limits or not. In this paper, we show that sub-leading corrections in a Penrose limit provide an unstable separatrix, so that chaotic motions are generated as a consequence of collapsed Kolmogorov-Arnold-Moser (KAM) tori. Our analysis is based on deriving a reduced system composed of two degrees of freedom by supposing a winding string ansatz. Then, we provide support for the existence of chaos by computing Poincaré sections. In comparison to the $\mathrm{AdS}_5 \times T^{1,1}$ case, we argue that no chaos lives in a near Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$, as expected from the classical integrability of the parent system.

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1 Introduction

AdS/CFT dualities [1–3] are one of the most important subjects in string theory. The study of them diverges and continues to influence various fields including cosmology, nuclear physics, condensed matter physics, and recently non-linear dynamics. The most well-studied example is the duality between type IIB string theory on AdS$_5\times$S^5 and the $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills (SYM) theory in four dimensions. A recent progress is the discovery of an integrable structure behind this duality [4]. The integrability has played an important role in checking the duality in non-BPS regions.

In connection with the integrable structure, type IIB string theory on AdS$_5\times$S^5 is classically integrable in the sense that the Lax pair exists [5]. Apart from this integrable example, there are many non-integrable AdS/CFT dualities in which chaotic string solutions appear. For example, when the internal space is given by a Sasaki-Einstein manifold like $T^{1,1}$ [6] and $Y^{p,q}$ [7], the string world-sheet theory exhibits the chaotic behavior (For other examples, see [8–11]).

On the other hand, apart from the fundamental strings, chaotic motions of D0-branes in the BFSS matrix model [12] and the BMN matrix model [13] have been shown in [14] and [15], respectively[1]. Thanks to the mass-deformation, the BMN matrix model was robustly discussed by computing Poincaré sections, which explicitly exhibit chaos. It would be nice to consider a gravitational (or string theoretical) interpretation of the chaotic behavior of D0-branes. It may be related to a non-linear dynamical generation of space-time, fast scrambling of black hole [18] and the inequality proposed in [19]. In fact, a fast thermalization in the BMN matrix model is discussed in [20].

In this paper, we are concerned with non-integrable AdS/CFT dualities. As a particular example, we will concentrate on the $\mathrm{AdS}_5 \times T^{1,1}$ case, where the existence of chaos has been confirmed both numerically [6] and analytically [7]. The chaos appears basically because the classical string action on $R \times T^{1,1}$ contains a double pendulum as a subsystem. It may be interesting to ask whether the chaos persists even in Penrose limits [21] or not. The leading part in the limits gives rise to a free massive world-sheet theory. Then the sub-leading correction can be regarded as a small perturbation, but it is not so simple because quartic-order terms of canonical momenta are contained. Nevertheless, it is still possible to employ the standard procedure. Our analysis is based on deriving a reduced system composed of two degrees of freedom by supposing a winding string ansatz. Then, we provide support for the existence of chaos by computing Poincaré sections. In comparison to the $\mathrm{AdS}_5 \times T^{1,1}$ case, we argue that no chaos lives in a near Penrose limit of $\mathrm{AdS}_5{\times}\mathrm{S}^5$, as expected from the classical integrability of the parent system.

The organization of this paper is as follows. In section 2, we consider a Penrose limit of $\mathrm{AdS}_5 \times T^{1,1}$ including the sub-leading corrections. In section 3, the bosonic light-cone Hamiltonian is derived on the near pp-wave background. The sub-leading corrections induce interaction terms in the system. In section 4, we show that chaotic string solutions exist in the resulting Hamiltonian system by computing Poincaré sections. In section 5, we revisit a near Penrose limit of $\mathrm{AdS}_5{\times}\mathrm{S}^5$ and argue that no chaos appears. Section 6 is devoted to conclusion and discussion.

[1] For earlier works on chaos in classical (deformed) Yang-Mills theories, see [16, 17].

2 A near Penrose limit of $\mathbf{AdS}_5 \times T^{1,1}$

In this section we will consider a Penrose limit of the $\mathrm{AdS}_5 \times T^{1,1}$ background, including the sub-leading corrections. First of all, the metric of $\mathrm{AdS}_5 \times T^{1,1}$ is introduced in Sec. 2.1. Then we consider a Penrose limit of this background in Sec. 2.2.

2.1 The metric of $\mathbf{AdS}_5 \times T^{1,1}$

Let us introduce the metric of the $\mathrm{AdS}_5 \times T^{1,1}$ background. The internal compact space $T^{1,1}$ is a five-dimensional Sasaki-Einstein manifold. The $T^{1,1}$ geometry is obtained as a base space of conifold (which is a Calabi-Yau three-fold) [22]. The $\mathrm{AdS}_5 \times T^{1,1}$ background is obtained as the near-horizon limit of a stack of N D3-branes sitting at the tip of the conifold and the resulting geometry is considered as the gravity dual for an $\mathcal{N} = 1$ superconformal field theory in four dimensions [23].

The metric of $\mathrm{AdS}_5 \times T^{1,1}$ is given by

$$
\begin{aligned}
ds^2 &= R^2(ds^2_{\mathrm{AdS}_5} + ds^2_{T^{1,1}})\,, && (2.1)\\
ds^2_{\mathrm{AdS}_5} &= -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\,, && (2.2)\\
ds^2_{T^{1,1}} &= \frac{1}{9}\left[d\psi + \cos\theta_1\, d\phi_1 + \cos\theta_2\, d\phi_2\right]^2 + \frac{1}{6}\sum_{i=1}^{2}\left[d\theta_i^2 + \sin^2\theta_i\, d\phi_i^2\right]\,. && (2.3)
\end{aligned}
$$

Here R is the radius of AdS_5. The isometry is $SU(2)_{\mathrm{A}} \times SU(2)_{\mathrm{B}} \times U(1)_{\mathrm{R}}$. Note here that $T^{1,1}$ is a homogeneous space and can be represented by the following coset[2]:

$$
T^{1,1} = \frac{SU(2)_{\mathrm{A}} \times SU(2)_{\mathrm{B}} \times U(1)_{\mathrm{R}}}{U(1)_{\mathrm{A}} \times U(1)_{\mathrm{B}}}\,. \qquad (2.4)
$$

Although the full Green-Schwarz string action has not been constructed yet, one may employ the bosonic part. In the following, we will concentrate on the bosonic part and consider classical string solutions moving on $R \times T^{1,1}$.

[2] In some references, the coset is said to be

$$
\frac{SU(2)_{\mathrm{A}} \times SU(2)_{\mathrm{B}}}{U(1)}\,.
$$

However, this coset does not lead to the correct metric, as argued in the original paper [22]. The coset in (2.4) can reproduce the metric correctly and even three-parameter deformations of $T^{1,1}$ [24] as shown in [25].

2.2 A Penrose limit of $\mathrm{AdS}_5 \times T^{1,1}$

It is known that classical strings moving on $R\times T^{1,1}$ exhibit random motions i.e., chaos. Now we would like to consider a question, "Can one observe chaos even in a near Penrose limit?" The answer is yes, as we will show later. Let us here introduce a near pp-wave geometry of $\mathrm{AdS}_5 \times T^{1,1}$ by including the sub-leading corrections in taking a Penrose limit. The leading part of the pp-wave geometry was originally discussed in [26–28].

To take a Penrose limit, a null-geodesic has to be picked up at first. Among the geodesics, we focus upon the $\psi + \phi_1 + \phi_2$ direction in $T^{1,1}$. Then the light-cone coordinates $\tilde{x}^\pm$ and new angle variables Φ_i $(i = 1, 2)$ are introduced as[3]

$$\tilde{x}^+ \equiv t\,, \quad \tilde{x}^- \equiv -t + \frac{1}{3}\left(\psi + \phi_1 + \phi_2\right)\,, \quad \Phi_1 \equiv \phi_1 - t\,, \quad \Phi_2 \equiv \phi_2 - t\,. \tag{2.5}$$

Then let us rescale the above coordinates by R as follows:

$$\tilde{x}^+ = x^+\,, \quad \tilde{x}^- = \frac{x^-}{R^2}\,, \quad \rho = \frac{r}{R}\,, \quad \theta_i = \sqrt{6}\,\frac{r_i}{R}\,. \tag{2.6}$$

Finally, the $R \to \infty$ limit is taken. This is the Penrose limit we consider.

After all, the resulting metric is given by

$$\begin{aligned}
ds^2 &= {ds_0}^2 + \frac{1}{R^2}\,{ds_2}^2 + \mathcal{O}\left(\frac{1}{R^4}\right)\,, \\
{ds_0}^2 &= 2dx^+dx^- - \left(r^2 + {r_1}^2 + {r_2}^2\right)\left(dx^+\right)^2 + dr^2 + r^2 d{\Omega_3}^2 \\
&\qquad + d{r_1}^2 + {r_1}^2 d{\Phi_1}^2 + d{r_2}^2 + {r_2}^2 d{\Phi_2}^2\,, \\
{ds_2}^2 &= \left(-\frac{1}{3}r^4 + 2{r_1}^2{r_2}^2\right)\left(dx^+\right)^2 - 2\left({r_1}^2 + {r_2}^2\right)dx^+dx^- + \left(dx^-\right)^2 + \frac{1}{3}r^4 d{\Omega_3}^2 \\
&\qquad + {r_1}^2\left(-{r_1}^2 + 2{r_2}^2\right)dx^+d\Phi_1 + {r_2}^2\left(-{r_2}^2 + 2{r_1}^2\right)dx^+d\Phi_2 - 2{r_1}^2dx^-d\Phi_1 \\
&\qquad - 2{r_2}^2dx^-d\Phi_2 + 2{r_1}^2{r_2}^2 d\Phi_1 d\Phi_2 - {r_1}^4 d{\Phi_1}^2 - {r_2}^4 d{\Phi_2}^2\,.
\end{aligned} \tag{2.7}$$

The leading part ds_0^2 is nothing but the familiar maximally supersymmetric pp-wave background [31]. As a matter of course, the sub-leading part ds_2^2 is different from that of $\mathrm{AdS}_5\times\mathrm{S}^5$. The sub-leading part ds_2^2 plays an important role in our later argument and indeed leads to chaotic string motions.

[3] Here the light-cone convention is slightly different from the one in [26–28]. Our convention follows the work [29, 30] in which the sub-leading corrections are discussed in a near Penrose limit of $\mathrm{AdS}_5\times\mathrm{S}^5$. The present choice in (2.5) is convenient to deal with the sub-leading part.

3 Hamiltonian of a near pp-wave string

In this section, we will derive the light-cone Hamiltonian of a string moving on the near pp-wave background (2.7). Our derivation follows the procedure developed in [29, 30] for the $\mathrm{AdS}_5\times\mathrm{S}^5$ case, though we employ only the bosonic part.

We first work on a general background and solve the constraint conditions. Then the metric (2.7) is substituted into the resulting expression and the light-cone Hamiltonian we consider is derived.

3.1 A light-cone string on a general background

Let us consider a general background with the metric $g_{\mu\nu}$ ($\mu,\nu=+,-,1\ldots,8$) that satisfies the following conditions

$$g_{+I}=g_{-I}=0 \quad (I=1,\ldots,8)\,.$$

In addition, we suppose that the dilaton is constant and the NS-NS two-form is zero.

The bosonic part of the classical string action is given by

$$\mathcal{S}_{\mathrm{B}}=\int d\tau d\sigma\,\mathcal{L}=\frac{1}{2}\int d\tau d\sigma\, h^{ab}\partial_a x^\mu \partial_b x^\nu g_{\mu\nu}\,. \tag{3.1}$$

The string world-sheet is parametrized by τ and σ and the dynamical variables $x^\mu(\tau,\sigma)$ describe the string dynamics. The quantity h^{ab} ($a,b=\tau,\sigma$) is defined as

$$h^{ab}\equiv\sqrt{-\gamma}\,\gamma^{ab}\,,\qquad \gamma\equiv\det(\gamma_{ab})\,,$$

where γ_{ab} is the world-sheet metric.

Then the canonical momenta p_μ are introduced as usual:

$$p_\mu=\frac{\partial\mathcal{L}}{\partial\left(\partial_\tau x^\mu\right)}=h^{\tau a}\partial_a x^\nu g_{\mu\nu}\,. \tag{3.2}$$

Solving (3.2) in terms of $\dot{x}^\mu$ leads to the relation:

$$\dot{x}^\mu=\frac{1}{h^{\tau\tau}}g^{\mu\nu}p_\nu-\frac{h^{\tau\sigma}}{h^{\tau\tau}}x'^\mu\,. \tag{3.3}$$

Here the following notations have been introduced:

$$\dot{x}^\mu\equiv\partial_\tau x^\mu\,,\qquad x'^\mu\equiv\partial_\sigma x^\mu\,.$$

The equation of motion for h_{ab} provides constraint conditions, by which the energy-momentum tensor T^{ab} is forced to vanish:

$$T^{ab} = h^{ac}h^{bd}\partial_c x^\mu \partial_d x^\nu g_{\mu\nu} - \frac{1}{2}h^{ab}h^{cd}\partial_c x^\mu \partial_d x^\nu g_{\mu\nu} = 0\,. \qquad (3.4)$$

By making use of (3.3), $\dot{x}$ can be removed from the expression (3.4). Then the constraints in (3.4) can be written in terms of p_μ and x'^μ like

$$p_\mu p_\nu g^{\mu\nu} + x'^\mu x'^\nu g_{\mu\nu} = 0\,, \qquad (3.5)$$

$$p_\mu x'^\mu = 0\,. \qquad (3.6)$$

Let us here impose the light-cone gauge,

$$x^+ = \tau\,, \qquad p_- = \text{constant}\,.$$

Then the light-cone Hamiltonian $\mathcal{H}_{\rm lc}$ is defined as

$$\mathcal{H}_{\rm lc} \equiv -p_+\,.$$

With (3.5) and (3.6), x'^- and $\mathcal{H}_{\rm lc}$ can be expressed in terms of x^I and p_I:

$$x'^- = -\frac{x'^I p_I}{p_-}\,, \qquad (3.7)$$

$$\mathcal{H}_{\rm lc} = -\frac{p_- g^{+-}}{g^{++}} - \frac{1}{g^{++}}\sqrt{{p_-}^2 g - g^{++}\left(g_{--}\left(\frac{p_I x'^I}{{p_-}^2}\right)^2 + p_I p_J g^{IJ} + x'^I x'^J g_{IJ}\right)}\,, \qquad (3.8)$$

where the following quantity has been introduced:

$$g = (g^{+-})^2 - g^{++}g^{--}\,.$$

Note that in the above derivation we have assumed that $g_{--} \neq 0$. When $g_{--} = 0$ like the usual pp-wave metric (i.e., only the leading part), the light-cone Hamiltonian becomes

$$\mathcal{H}_{\rm lc} = -\frac{p_- g^{--}}{2g^{+-}} - \frac{1}{2g^{+-}p_-}\left(p_I p_J g^{IJ} + x'^I x'^J g_{IJ}\right)\,. \qquad (3.9)$$

For a given metric, the expressions of $\mathcal{H}_{\rm lc}$ in (3.8) and (3.9) are very useful.

3.2 The Hamiltonian in the near pp-wave limit of $\mathbf{AdS}_5 \times T^{1,1}$

For later argument, let us explicitly write down the light-cone Hamiltonian on the near pp-wave background (2.7).

By substituting the pp-wave metric (2.7) into the formula (3.8) , the light-cone Hamiltonian $\mathcal{H}_{\rm lc}$ is given by

$$
\mathcal{H}_{\rm lc} = \mathcal{H}_0 + \frac{1}{R^2}\mathcal{H}_{\rm int} + \mathcal{O}\left(\frac{1}{R^4}\right)\,, \tag{3.10}
$$

$$
\begin{aligned}
\mathcal{H}_0 = \frac{1}{2}\Biggl(& {p_r}^2 + {p_{r_1}}^2 + {p_{r_2}}^2 + \frac{{p_{\Phi_1}}^2}{{r_1}^2} + \frac{{p_{\Phi_2}}^2}{{r_2}^2} \\
& + r^2 + {r_1}^2 + {r_2}^2 + r'^2 + {r_1}'^2 + {r_2}'^2 + {r_1}^2{\Phi_1}'^2 + {r_2}^2{\Phi_2}'^2 \Biggr)\,,
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
\mathcal{H}_{\rm int} = -\frac{1}{8}\Biggl(& {p_r}^2 + {p_{r_1}}^2 + {p_{r_2}}^2 + \frac{{p_{\Phi_1}}^2}{{r_1}^2} + \frac{{p_{\Phi_2}}^2}{{r_2}^2} - r^2 + {r_1}^2 + {r_2}^2 + r'^2 + {r_1}'^2 + {r_2}'^2 \\
& + {r_1}^2{\Phi_1}'^2 + {r_2}^2{\Phi_2}'^2 \Biggr)^2 + \frac{1}{2}\left(p_{\Phi_1} - p_{\Phi_2}\right)^2 - \frac{1}{2}\left({r_1}^2{\Phi_1}' - {r_2}^2{\Phi_2}'\right)^2 \\
& + \frac{1}{6}r^4 + \frac{1}{2}\left({r_1}^4 + {r_2}^4\right) + \frac{1}{2}\left(p_r r' + p_{r_1}{r_1}' + p_{r_2}{r_2}' + p_{\Phi_1}{\Phi_1}' + p_{\Phi_2}{\Phi_2}'\right)^2\,.
\end{aligned} \tag{3.12}
$$

Here we have set $p_- = 1$ and a constant term has been dropped off. In addition, we have ignored the terms concerned with $d\Omega_3^2$ for simplicity. In our later argument, we are not interested in the angular part of AdS_5 . In fact, the terms with $d\Omega_3^2$ can be dropped off by supposing that a constant position is taken on the S^3 . In the following, we will not consider the higher-order terms with $\mathcal{O}(1/R^4)$ as well. Note also that p_{Φ_1} and p_{Φ_2} are constants of motion.

The resulting system (3.10) can be regarded as a sum of simple harmonic oscillators in $\mathcal{H}_0$ and a small perturbation by $\mathcal{H}_{\rm int}$. Hence it seems likely that the system is simple, but this is not the case actually. The interaction Hamiltonian $\mathcal{H}_{\rm int}$ contains four-order terms of canonical momenta and hence the Hamiltonian dynamics is quite intricate. Thus the behavior of classical trajectories is far from obvious and it is worth to study it.

In the next section, we will consider the Hamiltonian dynamics with (3.10) and show that chaotic string solutions are contained.

4 Chaos in a near Penrose limit of $\mathrm{AdS}_5 \times T^{1,1}$

In this section, we show chaotic string motions in the near pp-wave background (2.7). There are some standard methods to display chaotic motions (For an introductory book, see [32]). Here we compute Poincaré sections[4]. As evidence of chaos, the resulting sections show random motions with some islands.

We study classical trajectories of a string moving on the near pp-wave background (2.7) . The light-cone Hamiltonian (3.10) is very intricate and hence it is helpful to impose an ansatz to make the system much simpler. In addition, the string world-sheet is two-dimensional. Hence it is convenient to perform a dimensional reduction to one dimension by supposing a string wrapping on Cartan directions.

Concretely, we consider two cases of a winding string. The one is that all of the motions are confined into the $T^{1,1}$ geometry. The other is that the radial direction of AdS_5 is included in the motions. In the following, we will investigate each of them.

4.1 Chaos in the $T^{1,1}$ directions

The first ansatz we consider is the following:

$$\begin{aligned} &r = 0\,, \quad p_r = 0\,, \quad r_1 = r_1(\tau)\,, \quad p_{r_1} = p_{r_1}(\tau)\,, \quad r_2 = r_2(\tau)\,, \quad p_{r_2} = p_{r_2}(\tau)\,, \\ &\qquad \Phi_1 = \alpha_1\,\sigma\,, \quad p_{\Phi_1} = 0\,, \quad \Phi_2 = \alpha_2\,\sigma\,, \quad p_{\Phi_2} = 0\,. \end{aligned} \tag{4.1}$$

Here α_i $(i = 1, 2)$ is an integer due to the periodicity of Φ_i . This ansatz describes a string moving only in the $T^{1,1}$ geometry. Note that the spatial direction of the string world-sheet is wrapped on Cartan directions.

Then the ansatz (4.1) reduces the free part (3.11) and the interaction part (3.12) into the following forms:

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{2}\Big[p_{r_1}^2 + p_{r_2}^2 + (1+\alpha_1^2)\,r_1^2 + (1+\alpha_2^2)\,r_2^2\Big]\,, \\ \mathcal{H}_{\rm int} &= -\frac{1}{8}\Big[p_{r_1}^2 + p_{r_2}^2 + (1+\alpha_1^2)\,r_1^2 + (1+\alpha_2^2)\,r_2^2\Big]^2 \\ &\quad -\frac{1}{2}\left(\alpha_1\,r_1^2 - \alpha_2\,r_2^2\right)^2 + \frac{1}{2}\left(r_1^4 + r_2^4\right)\,, \end{aligned} \tag{4.2}$$

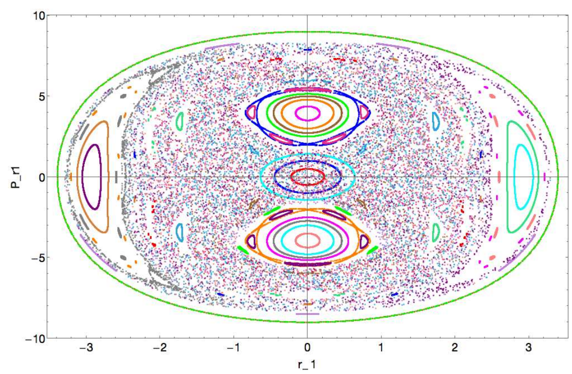
[4] One may think of that Lyapunov exponents may be computed as well. However, it seems quite difficult to compute them in the present case as we will explain later.

respectively. Now the dynamical variables of this system depend only on τ , and it is simple enough to compute Poincaré sections.

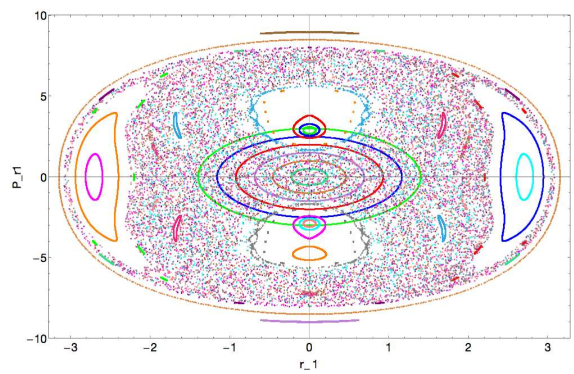
In the following, we will provide numerical results to support the existence of chaos even in the near Penrose limit.

Poincaré section

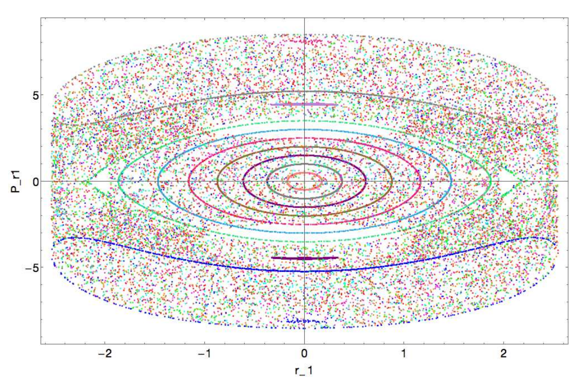
Poincaré sections are plotted for $E = 1.0$, 5.0 and 10 [Figs. 1 (a)-(c)]. The sections are taken at $r_2 = 0$ with $p_{r_2} > 0$. The AdS radius R and the winding numbers α_1 and α_2 are set to $R = 5.0$, $\alpha_1 = 2.0$ and $\alpha_2 = 1.0$, respectively. The results clearly show that chaotic motions appear in each energy level, and indicate that the near Penrose limit of $\mathrm{AdS}_5 \times T^{1,1}$ is also non-integrable.



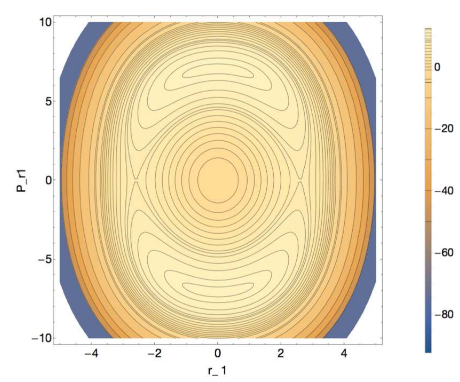
(a) Poincaré section with $E = 1.0$



(b) Poincaré section with $E = 5.0$



(c) Poincaré section with $E = 10$



(d) Energy contours in the r_1-p_{r_1} phase space

Figure 1: Poincaré section with the ansatz (4.1).

It is worth mentioning the qualitative behavior of the Poincaré sections. The energy contours in the r_1-p_{r_1} phase space at $r_2 = p_{r_2} = 0$ are drawn in Fig. 1 (d). A point is that it has a ring-like structure around the origin. In Fig. 1 (a) chaotic motions have already appeared at $E = 1.0$ together with islands and islets [Kolmogorov-Arnold-Moser (KAM) tori [33–35]]. The location of islands is understood from the energy contours. When $E = 5.0$, three islands collide each other and form a series of tori around the origin [Fig. 1 (b)]. This position corresponds to the stable point around the origin in Fig. 1 (d). When $E = 10$, the centered tori grow up at last [Fig. 1 (c)].

Note that chaotic motions overlap with the tori in all of the sections. This is a peculiarity coming from the quartic terms of canonical momenta. Actually, we are not sure whether the Poincaré section we took here is suitable or not, though it should be enough to see the existence of chaos. There may be a possibility that another appropriate section can be chosen, for example, by imposing an additional condition to take the slice. For example, by taking a Poincaré section at $r_2 = 0$ with $0 < p_{r_2} < 5.2$, the overlap between chaotic trajectories and the KAM tori vanishes as plotted in Fig. 2.

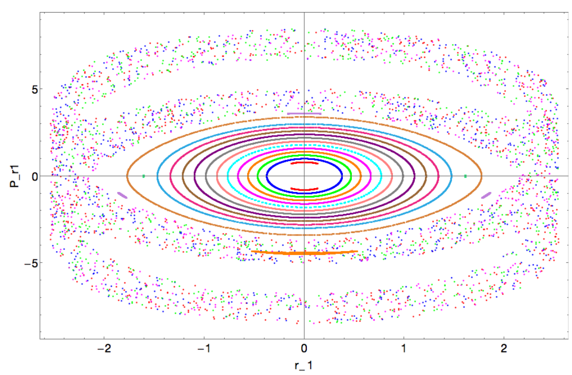


Figure 2: Improved Poincaré section with $E = 10$.

Finally, it is worth mentioning about Lyapunov spectra. The existence of the quartic terms of canonical momenta also makes it very difficult to compute them because the convergence of the exponents would become worse owing to it. In particular, this is the case even for the largest Lyapunov exponent. So far, no satisfactory result has been obtained.

4.2 No chaos in the radial direction of AdS_5

In the previous subsection, we have observed chaotic motions associated with the chaos in $T^{1,1}$. As the next question, it may be interesting to ask the radial direction r coming from the AdS_5 part may exhibit chaotic motions depending on initial conditions. In fact, the dynamics of r is affected by the motions of the other $T^{1,1}$ variables and hence the answer would not be far from obvious.

To answer this question, let us consider the following ansatz including the r-direction:

$$
\begin{aligned}
&r = r(\tau)\,, \quad p_r = p_r(\tau)\,, \quad r_1 = r_1(\tau)\,, \quad p_{r_1} = p_{r_1}(\tau)\,, \quad r_2 = 0\,, \quad p_{r_2} = 0\,, \\
&\Phi_1 = \alpha_1\,\sigma\,, \quad p_{\Phi_1} = 0\,, \quad \Phi_2 = \alpha_2\,\sigma\,, \quad p_{\Phi_2} = 0\,.
\end{aligned} \tag{4.3}
$$

Here α_i $(i = 1, 2)$ are winding numbers again.

The ansatz (4.3) simplifies the Hamiltonian (3.11) and (3.12) as

$$
\begin{aligned}
\mathcal{H}_0 &= \frac{1}{2}\Big[p_{r_1}^2 + p_{r_2}^2 + r^2 + (1+\alpha_1^2)\,r_1^2\Big]\,, \\
\mathcal{H}_{\rm int} &= -\frac{1}{8}\Big[p_r^2 + p_{r_1}^2 - r^2 + (1+\alpha_1^2)\,r_1^2\Big]^2 + \frac{1}{6}r^4 + \frac{1}{2}\left(1-\alpha_1^2\right)\,r_1^4\,.
\end{aligned} \tag{4.4}
$$

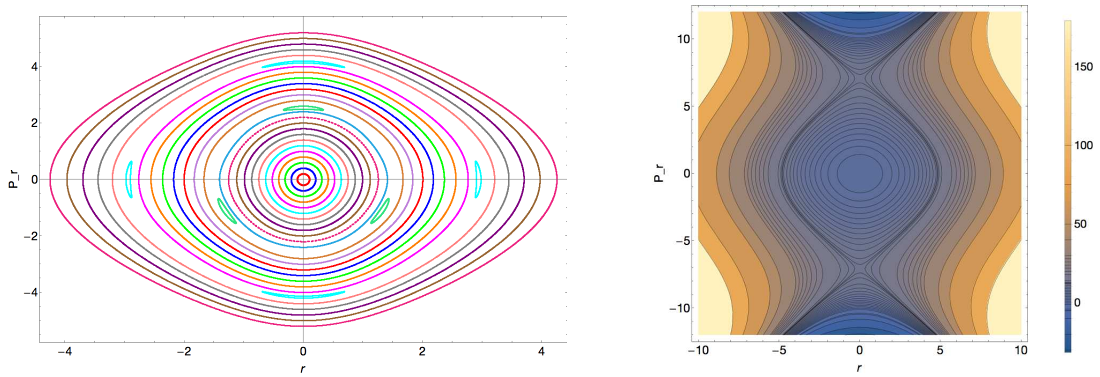
Note here that α_2 does not appear.

A Poincaré section at $r_1 = 0$ with $p_{r_1} > 0$ is plotted for $E = 10$ with $R = \alpha_1 = 5.0$ [Fig. 3 (a)]. Energy contours at $r_1 = p_{r_1} = 0$ are drawn in Fig. 3 (b). Figure 3 (a) indicates that the KAM tori are not destroyed and there exist no chaotic motions for the r-direction. This is just an example, but we have obtained similar results for the other energy levels as far as we have tried. Thus, though we will not present a bunch of the plots, we have succeeded to give support for the classical integrability for the r-direction.

The classical integrability should be associated with the integrability of AdS_5 , but we have not obtained an analytical confirmation for this integrability. It may be a good direction to try to reveal it.

Finally, it should be remarked that the plot in Fig. 3 (b) shows that the energy is not bounded for large values of p_r and unbounded trajectories may appear. But the unbounded motions should not be confused with the onset of chaos[5].

[5] A simple exponential growth is not chaos. In general, the definition of chaos requires the finiteness of trajectories.



(a) Poincaré section with $E = 10$ (b) Energy contours in the r-p_r phase space

Figure 3: Poincaré section with the ansatz (4.3).

5 A near Penrose limit of $\mathrm{AdS}_5\times\mathrm{S}^5$ revisited

So far, we have considered the $\mathrm{AdS}_5 \times T^{1,1}$ case. Let us here revisit a near Penrose limit of $\mathrm{AdS}_5\times\mathrm{S}^5$. As a matter of course, the $\mathrm{AdS}_5\times\mathrm{S}^5$ geometry is known as an integrable background. However, it would be interesting to ask whether a near pp-wave limit of $\mathrm{AdS}_5\times\mathrm{S}^5$ is still integrable or not. This is because the interaction Hamiltonian contains the quartic terms of canonical momenta and it does not seem that the classical integrability is so obvious.

First of all, let us introduce the metric of $\mathrm{AdS}_5\times\mathrm{S}^5$ with the global coordinates:

$$ds^2 = R^2(ds^2_{\mathrm{AdS}_5} + ds^2_{S^5})\,, \tag{5.1}$$

$$ds^2_{\mathrm{AdS}_5} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\,, \tag{5.2}$$

$$ds^2_{S^5} = \cos^2\theta\, d\phi^2 + d\theta^2 + \sin^2\theta\, d{\Omega'}_3^2\,. \tag{5.3}$$

Here R is the curvature radius of the AdS_5 and S^5 . It is convenient to perform the coordinate transformations from ρ and θ to $\tilde{z}$ and $\tilde{y}$ through the relations:

$$\cosh\rho = \frac{1+\tilde{z}^2/4}{1-\tilde{z}^2/4}\,, \qquad \cos\theta = \frac{1-\tilde{y}^2/4}{1+\tilde{y}^2/4}\,. \tag{5.4}$$

Then the metric is rewritten as

$$ds^2_{\mathrm{AdS}_5} = -\left(\frac{1+\tilde{z}^2/4}{1-\tilde{z}^2/4}\right)^2 dt^2 + \left(\frac{1-\tilde{y}^2/4}{1+\tilde{y}^2/4}\right)^2 d\phi^2 + \frac{d\tilde{z}^2+\tilde{z}^2 d\Omega_3^2}{(1-\tilde{z}^2/4)^2} + \frac{d\tilde{y}^2+\tilde{y}^2 d{\Omega'}_3^2}{(1+\tilde{y}^2/4)^2}\,. \tag{5.5}$$

In the metric (5.5) , the $SO(4)\times SO(4)$ isometry is manifest.

Next, by following the work [29], the light-cone coordinates are introduced as

$$\tilde{x}^+ = t\,, \qquad \tilde{x}^- = -t + \phi\,. \tag{5.6}$$

After rescaling the coordinates as

$$\tilde{x}^+ = x^+\,, \quad \tilde{x}^- = \frac{x^-}{R^2}\,, \quad \tilde{z} = \frac{z}{R}\,, \quad \tilde{y} = \frac{y}{R}\,, \tag{5.7}$$

the $R \to \infty$ limit is taken. The resulting metric is given by

$$ds^2 = {ds_0}^2 + \frac{1}{R^2}{ds_2}^2 + \mathcal{O}\left(\frac{1}{R^4}\right)\,, \tag{5.8}$$

$${ds_0}^2 = 2dx^+dx^- - (z^2+y^2)(dx^+)^2 + dz^2 + z^2 {d\Omega_3}^2 + dy^2 + y^2 {d\Omega'_3}^2\,, \tag{5.9}$$

$$\begin{aligned} {ds_2}^2 &= -2y^2dx^+dx^- + \frac{1}{2}(y^4 - z^4)(dx^+)^2 + (dx^-)^2 \\ &\quad + \frac{1}{2}z^2\left(dz^2 + z^2{d\Omega_3}^2\right) - \frac{1}{2}y^2\left(dy^2 + y^2{d\Omega'_3}^2\right)\,. \end{aligned} \tag{5.10}$$

This metric with the sub-leading corrections was originally discussed in [29].

Now it is an easy task to derive the light-cone Hamiltonian $\mathcal{H}_{\rm lc}$ on the background (5.8) by making use of (3.8). After setting $p_- = 1$ and dropping a constant term, we obtain the Hamiltonian:

$$\mathcal{H}_{\rm lc} = \mathcal{H}_0 + \frac{1}{R^2}\mathcal{H}_{\rm int} + \mathcal{O}\left(\frac{1}{R^4}\right)\,, \tag{5.11}$$

$$\mathcal{H}_0 = \frac{1}{2}\left((p_A)^2 + ({x'}^A)^2 + (x^A)^2\right)\,, \tag{5.12}$$

$$\begin{aligned} \mathcal{H}_{\rm int} &= \frac{1}{4}\left(z^2({p_y}^2 + {y'}^2 + 2{z'}^2) - y^2({p_z}^2 + {z'}^2 + 2{y'}^2)\right) \\ &\quad + \frac{1}{8}\left(((x^A)^2)^2 - ((p_A)^2 + ({x'}^A)^2)^2\right) + \frac{1}{2}(p_A {x'}^A)^2\,, \end{aligned} \tag{5.13}$$

where $x^A = (z, y)$ and $p_A = (p_z, p_y)$. Here we have assumed that a constant position is taken in each of two S^3's, and dropped the terms concerned with ${d\Omega_3}^2$ and ${d\Omega'_3}^2$, as we did in Sec. 3. In addition, we will not consider the higher-order terms with $\mathcal{O}(1/R^4)$.

Poincaré section

The next task is to investigate numerically the dynamics of the Hamiltonian system with (5.11). In the following, we will compute a Poincaré section and provide support for the classical integrability of the system with (5.11).

To make the system simpler, let us take the following ansatz,

$$y = y(\tau)\,, \qquad p_y = p_y(\tau)\,, \qquad z = z(\tau)\,, \qquad p_z = p_z(\tau)\,. \tag{5.14}$$

With this ansatz, the light-cone Hamiltonian is simplified as

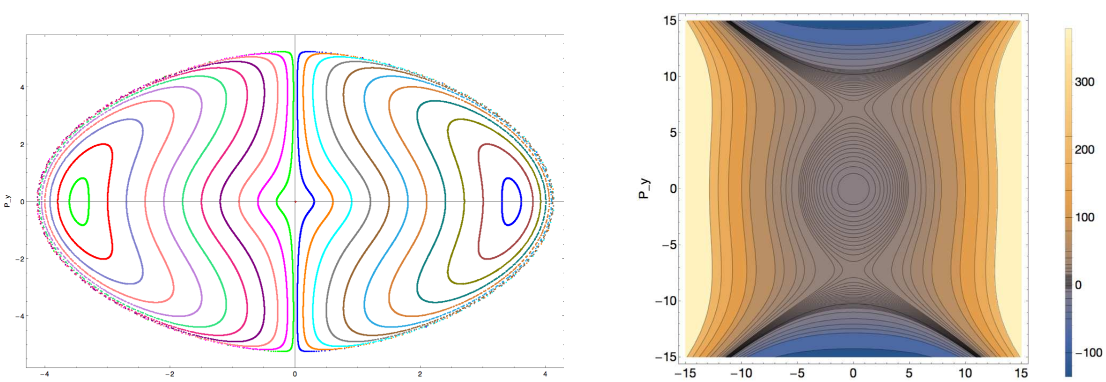
$$\mathcal{H}_{\rm lc} = \mathcal{H}_0 + \frac{1}{R^2}\mathcal{H}_{\rm int} + \mathcal{O}\left(\frac{1}{R^4}\right)\,, \tag{5.15}$$

$$\mathcal{H}_0 = \frac{1}{2}\left((p_A)^2 + (x^A)^2\right)\,, \tag{5.16}$$

$$\mathcal{H}_{\rm int} = \frac{1}{4}\left(z^2 {p_y}^2 - y^2 {p_z}^2\right) + \frac{1}{8}\left(\left((x^A)^2\right)^2 - \left((p_A)^2\right)^2\right)\,. \tag{5.17}$$

A Poincaré section is presented in Fig. 4 (a) , The section is taken at $z = 0$ with $p_z > 0$ and computed for $E = 10$ with $R = 5.0$. Energy contours are drawn in the y-p_y phase space with $z = p_z = 0$ [Fig. 4 (b)]. Figure 4 (a) shows that there are no chaotic motions at $E = 10$. Although the energy is sufficiently high, the KAM tori are not destroyed. Note that the plot in Fig. 4 (b) shows that the energy is not bounded for large values of p_y again. But, as in Sec. 2.2, the unbounded motions should not be interpreted as the onset of chaos.

Figure 4 (a) is just an example, but beautiful KAM tori continue to survive for other energy levels, as far as we have tried. Thus, though we will not present a bunch of the plots, we have obtained support for the classical integrability even in the near Penrose limit of $\mathrm{AdS}_5\times\mathrm{S}^5$.



(a) Poincaré section with $E = 10$ (b) Energy contours in the y-p_y phase space

Figure 4: Poincaré section with the ansatz (5.14).

This result should be related to the classical integrability of type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ [5]. Then, at least in principle, it would be possible to show the integrability by explicitly constructing an infinite number of conserved charges or the Lax pair. However, because of the quartic terms of canonical momenta, it seems quite difficult and hence our numerical support would be valuable.

6 Conclusion and Discussion

In this paper, we have considered chaotic motions of a classical string in a near Penrose limit of $\mathrm{AdS}_5 \times T^{1,1}$. We have shown that sub-leading corrections in a Penrose limit provide an unstable separatrix, so that chaotic motions are generated as a consequence of collapsed KAM tori. By deriving a reduced system composed of two degrees of freedom with a winding string ansatz, we have computed Poincaré sections and provided support for the existence of chaos. In addition, we have argued that no chaos appears in a near Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$, as expected from the classical integrability of the parent system.

There are some open problems associated with the chaos in the $\mathrm{AdS}_5 \times T^{1,1}$. A most important issue is to clarify what kind of gauge-theory operators correspond to the chaotic string solutions. We have studied here a near Penrose limit and hence the associated operators should be almost BPS. That is, a few impurities are inserted into the BPS vacuum operator. Hence it seems likely that the problem would now be much easier than the setup discussed in [6], because the associated composite operators are quite intricate in the case of the full geometry. However, we have no definite answer for the operators so far. We need to make more of an effort, For example, by following the argument for a near Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$ [29]. It may also be useful to try to figure out a fractal structure associated with the chaos. Probably, one may expect that the impurities would randomly be inserted in the vacuum operator.

We hope that our result would open up a new arena to check the AdS/CFT correspondence even for chaotic string solutions.

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