

Title	The Deformed Two-Dimensional Black Hole
Creators	Teschner, J.
Date	1999
Citation	Teschner, J. (1999) The Deformed Two-Dimensional Black Hole.
URL	<a href="https://dair.dias.ie/id/eprint/35/">https://dair.dias.ie/id/eprint/35/</a>
DOI	DIAS-STP-99-02

# THE DEFORMED TWO-DIMENSIONAL BLACK HOLE

J. TESCHNER

ABSTRACT. A deformation of the wave equation on a two-dimensional black hole is considered as a toy-model for possible gravitational or stringy nonlocal effects. The deformed wave-equation allows for an initial-value problem despite being nonlocal. The singularity present in the classical geometry is resolved by the deformation, so that propagation of a wave-packet can be continued through the classically singular region, ultimately reaching another asymptotically “flat” region.

## 1. INTRODUCTION

It is an important open question whether a notion of locality can be maintained in fundamental theories involving gravity such as e.g. string theory. This problem appears to be closely related to the question whether (pseudo-) Riemannian geometry still makes sense down to arbitrarily small distances.

Arguments have been put forward that indicate the impossibility of measuring arbitrarily small distances when combining quantum theory and general relativity, see e.g. [DFR][FGR]. If Riemannian geometry can not be measured then it becomes questionable whether it is a useful concept for the description of physical situations in which space-time uncertainties are relevant.

The status of locality in string theory is unclear due to the absence of a fundamental, background-independent formulation. There are hints indicating that strings cannot probe arbitrarily small distances due to quantum fluctuations of their shape, which are related to the fact that effective actions for the fields corresponding to the low energy states of the string are generically<sup>1</sup> highly nonlocal. More recent developments have exhibited on the one hand D-branes as probes that sometimes are able to resolve smaller distance scales than strings [DKPS]<sup>2</sup>, but have on the other hand discovered new hints towards fundamental nonlocalities from non-perturbative dualities such as the dualities between strings on Anti-de Sitter (AdS) spaces and conformal field theories conjectured by Maldacena (see e.g. [BDHM]).

Having available more general frameworks for the formulation of field theories that take into account fundamental space-time uncertainties or nonlocal effects may be an important ingredient for the further development of gravitational or string theories. One rather general framework that has been proposed in this context is given by the noncommutative geometry [Co], where non-commutativity of the operator algebra supposed to describe position measurements prevents localization to arbitrarily small distances. One thereby naturally obtains space-time uncertainties as e.g. in the explicit example proposed in [DFR].

---

<sup>1</sup>if  $l_s/l_R$  is not very small, where  $l_s$  is the string length and  $l_R$  denotes the characteristic curvature radius

<sup>2</sup>The situation for  $l_s/l_R \sim \mathcal{O}(1)$  it is not clear to the author. It has been proposed in [D] that D0-branes behave like *point*-particles in some regime with  $l_s/l_R \sim \mathcal{O}(1)$  so that the metric probed by these objects would define the pseudo-Riemannian structure of a background. On the other hand it appears to the author that the description of D-brane effective actions in terms of noncommutative geometry that was found in some specific backgrounds (reviewed in [D]) should be generic rather than exceptional.

There is a generalization of the concept of geodesic distance for noncommutative spaces [Co][FG], which is based on the observation that the geodesic distance on a Riemannian manifold  $\mathcal{M}$  can be reconstructed from the *algebraic* data  $(\mathcal{A}, \mathcal{H}, \Delta)$ , where  $\mathcal{A}$  is the algebra of (say) smooth, bounded functions on  $\mathcal{M}$ ,  $\mathcal{H}$  may be taken as the space  $L^2(\mathcal{M}, dv)$  on which  $\mathcal{A}$  acts as multiplication operators and  $\Delta$  is the Laplacian on  $\mathcal{M}$  considered as a self-adjoint operator on  $\mathcal{H}$ . Physically this simply means that the geodesic distance can be reconstructed from the quantum mechanics of point particles on  $\mathcal{M}$ . This observation leads to a natural definition of geodesic distance for more general choices of data  $(\mathcal{A}, \mathcal{H}, \Delta)$ , in which the algebra  $\mathcal{A}$  may be noncommutative. A possibility that does not seem to have attracted much attention is the case in which  $\mathcal{A}$  is still commutative, but  $\Delta$  is replaced by some other, maybe nonlocal, self-adjoint operator on  $\mathcal{H}$ .

Unfortunately, an analogous generalization in the case of Minkowskian signature metrics does not seem to exist presently (to the author's knowledge). Physically one might guess that reconstructing the Riemannian metric from the quantum mechanics on  $\mathcal{M}$  should be replaced by a study of propagation of wave-packets according to the covariant wave equation. This suggests that it should ultimately be possible to define generalizations of pseudo-Riemannian structures from data such as  $(\mathcal{A}, \mathcal{H}, \square)$ , where now  $\square$  is some generalization of the covariant wave operator. It is in this spirit that deformations of the covariant wave equation will be considered as defining a deformed pseudo-Riemannian geometry.

Alternatively, one may simply take the point of view that nonlocal deformations of the covariant wave equation are one way to parametrize certain modifications of the propagation of fields on a manifold due to gravitational/stringy nonlocal effects.

The present paper will describe an example of a nonlocal deformation of the wave equation in the geometry of a two-dimensional black hole. In this example the generalization appears exclusively in the nonlocality of the deformed covariant wave operator, the underlying algebra of functions is *commutative*. However, the specific deformation studied has its roots in a *noncommutative* quantum deformation of the algebra of functions on  $SL(2, \mathbb{R})$  and was found by generalizing the relation between  $SL(2, \mathbb{R})$  and the two-dimensional black hole that exists classically (e.g. [Wi][DVV]), as will be further discussed elsewhere. By similar constructions one can alternatively obtain *noncommutative* deformations of Euclidean/Minkowskian  $AdS_3$ , the BTZ-black hole, and the two-dimensional euclidean black hole.

The paper starts with a presentation of some results on wave propagation and scattering in the case of the classical two-dimensional black hole. Some important results had been obtained in [DVV], but for the purpose of comparison with the deformed case it appeared to be necessary to complete (e.g. by the solution of the Cauchy problem) and generalize (to arbitrary mass) the discussion therein.

The following section then carries out a study of the deformed case along similar lines as in the classical case. In order to focus on the physically interesting aspects and to keep the discussion brief, the basic technical results, being analogous to the classical case, are only announced. One should note however that their proof requires methods rather different from the well-known techniques that can be used in the classical case, so a full account of the technicalities will be given elsewhere.

## 2. THE TWO-DIMENSIONAL BLACK HOLE

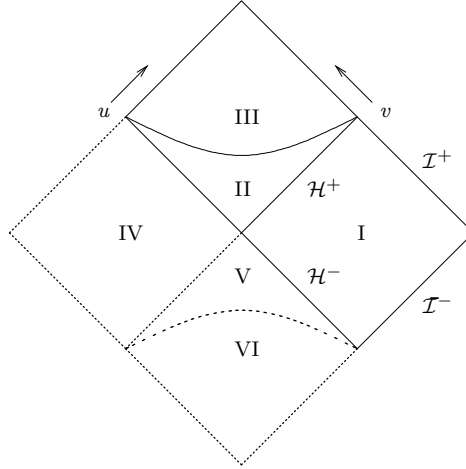
A two-dimensional analogue of the black hole <sup>3</sup> can be found as solution of the equations of motion for the dilaton gravity theory defined by the action

$$(1) \quad S = \int d^2x \sqrt{G} e^\Phi \left( R + (\nabla\Phi)^2 - 8 \right).$$

It is given by the following expressions for metric and dilaton field:

$$(2) \quad ds^2 = \frac{dudv}{1-uv}, \quad \phi = \log(1-uv).$$

One should note that the metric has all the characteristic features of a black hole: There is a horizon at  $uv = 0$  and a curvature singularity at  $uv = 1$ . The following figure shows the Penrose diagram of the fully extended geometry supplemented by regions behind the singularities (regions III and VI):



In the present paper only regions I, II and III will be considered, which is why the rest is represented by dotted lines. One may consider the metric in regions I and II as an idealization of a black hole formed by gravitational collapse if one imposes the boundary condition that there is no flux of matter from regions IV and V into II and I respectively, cf. [Un].

The corresponding wave operator reads

$$(3) \quad \begin{aligned} \square &= -\frac{1}{2e^\Phi \sqrt{-g}} \partial_\mu e^\Phi g^{\mu\nu} \sqrt{-g} \partial_\nu \\ &= -\frac{1}{2} (\partial_u(1-uv)\partial_v + \partial_v(1-uv)\partial_u). \end{aligned}$$

**2.1. Solutions to the wave equation in region I.** The wave equation to be solved reads

$$(4) \quad \square f = \left( m^2 - \frac{1}{4} \right) f.$$

Splitting off  $\frac{1}{4}$  on the right hand side is necessary for the mass  $m$  to be the mass as defined by an asymptotic observer.

In order to solve the wave-equation (4) it is useful to introduce variables  $r = \log(-uv)$ ,  $t = \log(-u/v)$  in which the covariant wave operator  $\square$  takes the form

$$(5) \quad \square = e^{-r} \partial_r (1 + e^r) \partial_r - (e^{-r} + 1) \partial_t^2.$$

<sup>3</sup>See e.g. [Wi][DVV] for more extensive discussions

The operator  $\Delta$  can furthermore be brought into the form of a one-dimensional Schrödinger operator by considering the field  $g = e^{\Phi/2} f = \sqrt{1+e^r} f$  instead of  $f$ . The wave equation for  $g$  then reads

$$(\partial_t^2 + \Delta')g = 0, \quad \Delta' = -\partial_r^2 + V(r), \quad V(r) = \frac{e^r}{4(1+e^r)^2} + m^2 \frac{e^r}{1+e^r}.$$

Any solution which at some fixed time  $t$  allows expansion of  $g(r, t)$  and  $\dot{g}(r, t) \equiv \partial_t g(r, t)$  into generalized eigenfunctions of the Schrödinger operator  $\Delta'$  can then be written in the form

$$g(r, t) = \int d\mu(\omega) (e^{-i\omega t} W_\omega(r) + e^{+i\omega t} \bar{W}_\omega(r)) \quad \text{with} \quad \Delta_h W_\omega(r) = \omega^2 W_\omega(r).$$

The eigenvalue equation for  $W_\omega(r)$  is brought into the form of a hypergeometric differential equation by  $W_\omega(r) = (-x)^{i\omega} (1-x)^{-1/2} F_\omega(x)$ ,  $x = -e^r$ . One has two linearly independent solutions:

$$U_k(r) = N_k e^{-i\omega r} (1+e^r)^{\frac{1}{2}} F\left(\frac{1}{2} + i(k-\omega), \frac{1}{2} - i(k+\omega), 1 - 2i\omega, -e^r\right)$$

and its complex conjugate  $V_k(r) \equiv \bar{U}_k(r)$ , where  $k$  is fixed by the mass-shell condition  $\omega^2 - k^2 = m^2$ , and  $N_k$  is a normalization factor to be fixed below. The asymptotic behavior for  $r \rightarrow \infty$  corresponding to large spacelike distance from the black hole is given by plane waves:

$$(6) \quad U_k(x) \sim N_k (B_+(k) e^{-ikr} + B_-(k) e^{ikr}), \quad B_\pm(k) = \frac{\Gamma(1-2i\omega)\Gamma(\mp 2ik)}{\Gamma^2(\frac{1}{2} - i(\omega \pm k))}.$$

It can then be checked that  $\Delta'$  is essentially self-adjoint in  $L^2(\mathbb{R})$ : There are no square-integrable eigenfunctions of  $\Delta'$ , so the deficiency indices vanish. It can furthermore be shown that the set of generalized eigenfunctions  $\{U_k(x); k \in \mathbb{R}_+\} \cup \{V_k(x); k \in \mathbb{R}_+\}$  constitutes a plane wave basis for  $L^2(\mathbb{R})$ . The normalization  $N_k$  is finally given by

$$\int_{\mathbb{R}} dr \bar{U}_{k_2}(r) U_{k_1}(r) = 2\pi \delta(k_2 - k_1), \quad \int_{\mathbb{R}} dr \bar{U}_{k_2}(r) V_{k_1}(r) = 0,$$

if the normalization  $N_k$  is chosen as

$$N_k = \frac{1}{B_+(k)} = \frac{\Gamma^2(\frac{1}{2} - i(k+\omega))}{\Gamma(1-2i\omega)\Gamma(-2ik)},$$

corresponding to normalizing the ‘‘incoming’’ plane wave in (6) to unity.

This yields existence and uniqueness of a solution to the Cauchy-problem for suitable subspaces  $\mathcal{S} \subset L^2(\mathbb{R})$  of test-functions, where  $\mathcal{S}$  could be for example the usual Schwartz space of  $L^2(\mathbb{R})$ . It takes the form

$$(7) \quad g(r, t) = \int_0^\infty dk \left( e^{-i\omega t} (a_k U_k(r) + b_k V_k(r)) + e^{i\omega t} (\bar{a}_k \bar{U}_k(r) + \bar{b}_k \bar{V}_k(r)) \right),$$

where the coefficients are given in terms of the values  $g(r, t)$ ,  $\dot{g}(r, t)$  at fixed time  $t$  via

$$(8) \quad \begin{aligned} a_k &= a_k[g, \dot{g}] = \frac{1}{4\pi} \int_{\mathbb{R}} dr e^{i\omega t} \bar{U}_k(r) \left( g(r, t) + \frac{i}{\omega} \dot{g}(r, t) \right) \\ b_k &= b_k[g, \dot{g}] = \frac{1}{4\pi} \int_{\mathbb{R}} dr e^{i\omega t} \bar{V}_k(r) \left( g(r, t) - \frac{i}{\omega} \dot{g}(r, t) \right). \end{aligned}$$

In view of a similar phenomenon that will be found in the deformed case it may be worthwhile noting that expansion into eigenfunction of  $\Delta'$  is possible for considerably more general choices of

subspaces  $\mathcal{T}$ ,  $\mathcal{S} \subset \mathcal{T} \subset L^2(\mathbb{R})$ <sup>4</sup>. Moreover, the expression (7) still makes sense for  $a_k = a_k[g, \dot{g}]$ ,  $b_k = b_k[g, \dot{g}]$  corresponding to initial data  $g, \dot{g}$  in  $\mathcal{T}$ . But the resulting function  $g(r, t)$  will then generically not be differentiable; it can therefore be considered as a solution of the wave-equation only in the distributional sense. This phenomenon is familiar from the simple case  $\partial_t^2 f = \partial_x^2 f$ : One may consider  $f(x - t)$  to be a distributional solution of  $\partial_t^2 f = \partial_x^2 f$  even if  $f$  is not differentiable.

**2.2. Scattering in the black hole geometry.** By the method of stationary phase it is possible to show that for  $t \rightarrow -\infty$  any solution (7) is asymptotic to a function  $g^{\text{as}}(r, t)$  in the sense that

$$\lim_{t \rightarrow -\infty} \int_{\mathbb{R}} dr |g(r, t) - g^{\text{as}}(r, t)|^2 = 0.$$

The function  $g^{\text{as}}(r, t)$  is expressed in terms of  $a_k, b_k$  as follows

$$g^{\text{as}} = g_1^{\text{as}} + g_2^{\text{as}} \quad \begin{aligned} g_1^{\text{as}}(r, t) &= \int_0^\infty dk (e^{-i\omega(t-r)} b_k \bar{N}_k + e^{i\omega(t-r)} \bar{b}_k N_k) \\ g_2^{\text{as}}(r, t) &= \int_0^\infty dk (e^{-i(\omega t + kr)} (a_k + b_k) + e^{i(\omega t + kr)} (a_k + b_k)). \end{aligned}$$

The functions  $g_1^{\text{as}}$  ( $g_2^{\text{as}}$ ) describe right- (left-) moving wave-packets coming in from  $\mathcal{H}_-$  ( $\mathcal{I}_-$ ).

However, the right-moving plane waves at  $\mathcal{H}_-$  represent an inflow from region  $V$  into region  $I$ . In order to be consistent with the interpretation of regions I/II as being an idealization of a black hole formed by gravitational collapse, it is necessary to impose the boundary condition of vanishing  $g_1^{\text{as}}$  corresponding to  $b_k \equiv 0$ .

The scattering problem for a wave-packet with asymptotic form  $g^{\text{in}}$  consists therefore in determining the late-time asymptotics  $g^{\text{out}}(r, t)$  defined by

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}} dr |g(r, t) - g^{\text{out}}(r, t)|^2 = 0.$$

for  $f(r, t)$  subject to the boundary condition  $b_k = 0$ . It is given by  $g^{\text{out}}(r, t) = g_1^{\text{out}}(r, t) + g_2^{\text{out}}(r, t)$ :

$$\begin{aligned} g_1^{\text{out}}(r, t) &= \int_m^\infty d\omega (e^{-i\omega(t+r)} T_k a_k + e^{i\omega(t+r)} \bar{T}_k \bar{a}_k) \\ g_2^{\text{out}}(r, t) &= \int_0^\infty dk (e^{-i(\omega t + kr)} R_k a_k + e^{i(\omega t + kr)} \bar{R}_k \bar{a}_k), \end{aligned}$$

where  $g_1^{\text{out}}(r, t)$  describes the matter that falls through the future horizon  $\mathcal{H}_+$ , whereas  $g_2^{\text{out}}(r, t)$  represents the part that escapes towards space-like infinity. The corresponding ‘‘transmission’’ amplitude  $T_k$  and ‘‘reflection amplitude  $R_k$  are respectively given by

$$T_k = \frac{\omega}{k} N_k = \frac{\Gamma^2(\frac{1}{2} - i(k + \omega))}{\Gamma(-2i\omega)\Gamma(1 - 2ik)} \quad R_k = \frac{N_k}{N_{-k}} = \frac{\Gamma^2(\frac{1}{2} - i(k + \omega)) \Gamma(+2ik)}{\Gamma^2(\frac{1}{2} + i(k - \omega)) \Gamma(-2ik)}$$

Note that information is conserved:  $|T_k|^2 + |R_k|^2 = 1$ .

<sup>4</sup>see the first two sections of [Be] for a lucid discussion of the conditions  $\mathcal{T}$  has to satisfy in order to allow expansion of *any*  $g \in \mathcal{T}$  into generalized eigenfunctions

**2.3. Continuation into region II.** The continuation of the wave-packet (7) into region II is trivial when expressing the modes  $e^{-i\omega t}U_k(r)$  in terms of  $u, v$ -coordinates:

$$(9) \quad e^{-i\omega t}U_k(r) \equiv \mathcal{U}_k(u, v) = u^{-2i\omega} F\left(\frac{1}{2} + i(k - \omega), \frac{1}{2} - i(k + \omega), 1 - 2i\omega, uv\right),$$

where  $\mathcal{U}_k(u, v)$  is analytic in  $v$  near the horizon  $v = 0$ . The expression

$$f(u, v) = \int_m^\infty d\omega \left( \mathcal{U}_k(u, v) T_k a_k + \bar{\mathcal{U}}_k(u, v) \bar{T}_k \bar{a}_k \right)$$

therefore defines a wave-packet  $f$  in the union of regions I and II. This continuation becomes unique by imposing the condition of vanishing on the boundary between regions IV and II, which is motivated by arguments analogous to those that motivated vanishing on  $\mathcal{H}_-$ .

However, the singularity at  $uv = 1$  prevents further continuation into region III. Technically this follows from the fact that the modes  $\mathcal{U}_k(u, v)$  develop a singularity of the form  $\log(1 - uv)$ . Predictability of the evolution of the wave-packet  $f$  breaks down at the singularity since there is no unique way of defining  $\log(x)$  for negative  $x$ .

### 3. PROPAGATION AND SCATTERING IN THE DEFORMED BLACK HOLE

The deformation of the covariant wave-equation that will be considered takes the form

$$(10) \quad \square_h f = \left\{ m^2 - \frac{1}{4} \right\}_h f, \quad \left\{ m^2 - \frac{1}{4} \right\}_h \equiv \left( \frac{\sinh^2(\pi h m)}{\sin^2(\pi h)} - \frac{\sin^2\left(\frac{\pi h}{2}\right)}{\sin^2(\pi h)} \right),$$

where  $h \in (0, 1)$  is the deformation parameter and the differential operator  $\square$  that appeared in the classical case (5) has been replaced by the following *finite difference* operator:

$$\begin{aligned} \square_h &= e^{-r} D_r (1 + e^r) D_r - (e^{-r} + 1) D_t^2 \\ D_r &\equiv \frac{\delta_r^+ - \delta_r^-}{2i \sin(\pi h)}, & \delta_r^\pm f(r, t) &= f(r \pm \pi i h, t), \\ D_t &\equiv \frac{\delta_t^+ - \delta_t^-}{2i \sin(\pi h)}, & \delta_t^\pm f(r, t) &= f(r, t \pm \pi i h). \end{aligned}$$

One obviously recovers the classical counterparts in the limit  $h \rightarrow 0$ . What appears to be unusual about this kind of deformation is the appearance of *imaginary* shifts of the arguments  $r, t$ . In order for operators such as  $\square_h$  to be defined as operators on spaces of functions of *real* variables  $r, t$  one needs an unambiguous prescription to extend the functions defined for real values of the arguments to the relevant strips in the complex plane. The most natural such extension seems to be given by requiring analyticity in the strip

$$\mathcal{S} = \left\{ (r, t) \in \mathbb{C}^2; |\Im(r)| < 2\pi h, \quad |\Im(t)| < 2\pi h \right\},$$

and existence of the limits

$$f(r \pm 2\pi i h, t) = \lim_{\epsilon \rightarrow 0} f(r \pm 2\pi i h \mp i\epsilon, t), \quad f(r, t \pm 2\pi i h) = \lim_{\epsilon \rightarrow 0} f(r, t \pm 2\pi i h \mp i\epsilon), \quad \epsilon > 0$$

for almost any  $r, t \in \mathbb{R}$ . This choice can alternatively be justified by considering the reduction from the quantized space  $SL_q(2, \mathbb{R})$ .

**3.1. Preliminaries.** It is useful to write (10) in the form

$$(D_t^2 + \Delta_h) f = 0, \quad \Delta_h = -\frac{1}{1 + e^r} D_r (1 + e^r) D_r + \left\{ m^2 - \frac{1}{4} \right\}_h \frac{e^r}{1 + e^r},$$

or in terms of  $g = (1 + e^r)^{\frac{1}{2}} f$

$$(D_t^2 + \Delta'_h) g = 0, \quad \Delta'_h = -\frac{1}{\sqrt{(1 + e^r)}} D_r (1 + e^r) D_r \frac{1}{\sqrt{(1 + e^r)}} + \left\{ m^2 - \frac{1}{4} \right\}_h \frac{e^r}{1 + e^r}.$$

As in the classical case one may try to construct a representation for the general solution by means of an eigenfunction expansion for  $\Delta_h$ .<sup>5</sup> If

$$f(r, t) = \int d\mu(\omega) a_\omega(t) W_\omega(r) \quad \text{with} \quad \Delta_h W_\omega(r) = [\omega]_h^2 W_\omega(r), \quad [z]_h \equiv \frac{\sinh(\pi h z)}{\sin(\pi h)},$$

then  $a_\omega(t)$  will be determined as a solution of  $D_t^2 a_\omega(t) = -[\omega]_h^2 a_\omega(t)$ . The most general solution of the latter equation that has the required analyticity is given by

$$a_\omega(t) = \sum_{m \in \mathbb{Z}} (A_m e^{-i(\omega + i\kappa m)t} + B_m e^{i(\omega + i\kappa m)t}).$$

Note that the contributions from  $m \neq 0$  blow up for  $t \rightarrow \infty$  or  $t \rightarrow -\infty$ . They correspond to the “runaway”-solutions that typically cause problems for higher-derivative equations of motion. However, in the present case one has not any problem to simply throw away the “badly-behaved” solutions with  $n \neq 0$ . Together with the eigenfunction decomposition of  $\Delta_h$  one will thereby obtain a perfectly well-defined initial value problem:

**3.2. Eigenfunction expansion for  $\Delta_h$ .**  $\Delta_h$  will be considered as an operator defined on the dense domain  $\mathcal{D} \subset L^2(\mathbb{R}, d\lambda(r))$ ,  $d\lambda(r) = dr(1 + e^r)$  which consists of functions that allow extension to a function holomorphic in the strip  $\{r \in \mathbb{C}; |\Im(r)| < 2\pi h\}$  and satisfy

$$\sup_{|\eta| < 2\pi h} \int_{\mathbb{R}} d\lambda(r) |f(r + i\eta)|^2 < \infty.$$

**Theorem 1.** *The operator  $\Delta_h$  is essentially self-adjoint. There exists an expansion into generalized eigenfunctions of  $\Delta_h$ .*

**Theorem 2.** *The following set  $\mathfrak{B}$  constitutes a basis of generalized eigenfunctions for  $\Delta_h$ :*

$$\mathfrak{B} = \{U_{h,k}; k \in \mathbb{R}_+\} \cup \{V_{h,k}; k \in \mathbb{R}_+\},$$

where the eigenvalue  $\omega^2$  is given in terms of the parameter  $k$  by the  $h$ -mass-shell relation

$$(11) \quad [\omega]_h^2 - [\kappa]_h^2 = [m]_h^2$$

and  $U_{h,k}(r)$  is given in terms of the  $q$ -hypergeometric functions introduced in the Appendix (eqn. (18)) as

$$U_{h,k}(r) = N_{h,k}^+ e^{-i\omega r} F_h\left(\frac{1}{2} + i(k - \omega), \frac{1}{2} - i(k + \omega), 1 - 2i\omega, -e^r\right),$$

whereas  $V_{h,k}(r) = \bar{U}_{h,k}(r)$ , the complex conjugate of  $U_{h,k}(r)$ .

**Theorem 3.** *The functions  $U_{h,k}(r)$ ,  $\bar{U}_{h,k}(r)$  are orthonormalized according to*

$$\int_{\mathbb{R}} d\lambda(r) \bar{U}_{h,k_2}(r) U_{h,k_1}(r) = 2\pi \delta(k_2 - k_1), \quad \int_{\mathbb{R}} d\lambda(r) \bar{V}_{h,k_2}(r) U_{h,k_1}(r) = 0,$$

if the normalization  $N_{h,k}$  is chosen as

$$N_{h,k} = \frac{\Gamma_h^2\left(\frac{1}{2} - i(k + \omega)\right)}{\Gamma_h(1 - 2i\omega)\Gamma_h(-2ik)}.$$

<sup>5</sup>In this case it is technically more convenient to consider  $\Delta_h$  instead of  $\Delta'_h$



The normalization is as in the classical case such that the “incoming wave” in the  $(r \rightarrow \infty)$ -asymptotics is normalized to one:

$$U_{h,k}(r) \sim e^{-\frac{1}{2}r} \left( e^{-ikr} + \frac{N_{h,k}}{N_{h,-k}} e^{ikr} \right).$$

Although these theorems are precise analogues of the corresponding statements in the undeformed case, their proof is quite different. To the authors knowledge these are the first nontrivial results on spectral analysis of finite difference operators of this type.

**3.3. Initial value problem.** The results just given allow one to write any solution as

$$(12) \quad f(r, t) = \int_0^\infty dk \left( e^{-i\omega t} (a_k U_{h,k}(r) + b_k V_{h,k}) + e^{i\omega t} (\bar{a}_k \bar{U}_{h,k}(r) + \bar{b}_k \bar{V}_{h,k}) \right),$$

where the coefficients are given in terms of the values  $f(r, t)$ ,  $\dot{f}(r, t) \equiv \partial_t f(r, t)$  at fixed time  $t$  via

$$(13) \quad \begin{aligned} a_k &= a_k[f, \dot{f}] = \frac{1}{4\pi} \int_{\mathbb{R}} d\lambda(r) e^{i\omega t} \bar{U}_{h,k}(r) \left( f(r, t) + \frac{i}{\omega} \partial_t f(r, t) \right), \\ b_k &= b_k[f, \dot{f}] = \frac{1}{4\pi} \int_{\mathbb{R}} d\lambda(r) e^{i\omega t} \bar{V}_{h,k}(r) \left( f(r, t) - \frac{i}{\omega} \partial_t f(r, t) \right). \end{aligned}$$

However, it can be shown that the expression (12) remains sensible for much more general choices of the coefficients  $a_k$ ,  $b_k$  than those provided by (13) for solutions  $f(r, t)$ . Stated differently, general choice of  $a_k$ ,  $b_k$  in (12) will not yield  $f$  that have the analyticity properties necessary to be solutions in the strict sense, but only in the distributional sense.<sup>6</sup>

On the other hand one may observe that the function  $f(r, t)$  given by (12) can alternatively be characterized as a solution of a *second order differential* equation w.r.t. time of the form

$$(14) \quad (\partial_t^2 + \mathcal{D}_h) f = 0, \quad (\mathcal{D}_h f)(r, t) = \int_{\mathbb{R}} dr' \mathcal{K}_h(r, r') f(r', t)$$

where the kernel  $\mathcal{K}_h(r, r')$  is given by

$$(15) \quad d_h(r, r') = \frac{1}{2\pi} \int_0^\infty dk \omega^2 (U_{h,k}(r) \bar{U}_{h,k}(r') + V_{h,k}(r) \bar{V}_{h,k}(r')),$$

and  $\omega$  has to be expressed in terms of  $k$  by means of the h-mass-shell relation. The corresponding expression  $d(r, r')$  in the classical case is of course simply equal to  $(-\partial_r^2 + V(r))\delta(r - r')$ . The fact that  $d(r, r')$  is supported on the diagonal can e.g. be found by extending the  $k$ -integration in the classical analogue of (15) to the full axis and closing the contour in the upper (lower) half-plane depending on the sign of  $r - r'$ . This will no longer be possible in the deformed case since  $\omega$  as function of  $k$  has logarithmic and square-root branch cuts.  $\mathcal{D}_h$  is therefore most likely a genuinely nonlocal operator.

To summarize: The deformed wave equation supplemented with the condition of absence of “runaway”-solutions is equivalent to the nonlocal evolution equation (14), which manifestly has a well-posed initial value problem.

---

<sup>6</sup>What appears to be at work is a generalization of the Paley-Wiener theorems relating analyticity of functions on a strip to exponential decay properties of their Fourier transforms.

**3.4. Scattering in region I.** At this point it becomes possible to carry through a discussion of scattering in region I of the deformed black hole in complete analogy to the undeformed case. It basically amounts to adding subscript “h” at the appropriate places.

First of all it turns out that the boundary condition of vanishing on the past horizon  $\mathcal{H}_-$  again corresponds to  $b_k \equiv 0$  in (14). Then one finds

**Theorem 4.** *The asymptotics  $g^{in}$ ,  $g^{out}$  for  $t \rightarrow \mp\infty$  defined by*

$$\lim_{t \rightarrow \mp\infty} \int_{\mathbb{R}} dr |g(r, t) - g^{in/out}(r, t)|^2 = 0$$

is explicitly given by

$$\begin{aligned} g^{in}(r, t) &= \int_0^\infty dk \left( e^{-i(\omega t + kr)} a_k + e^{i(\omega t + kr)} \bar{a}_k \right), \\ g_1^{out}(r, t) &= \int_m^\infty d\omega \left( e^{-i\omega(t+r)} T_{h,k} a_k + e^{+i\omega(t+r)} \bar{T}_{h,k} \bar{a}_k \right), \\ g^{out}(r, t) &= g_1^{out}(r, t) + g_2^{out}(r, t), \\ g_2^{out}(r, t) &= \int_0^\infty dk \left( e^{-i(\omega t + kr)} R_{h,k} a_k + e^{+i(\omega t - kr)} \bar{R}_{h,k} \bar{a}_k \right), \end{aligned}$$

where the reflection and transmission coefficients are given by

$$\begin{aligned} T_{h,k} &= \frac{[2\omega]_h}{[2k]_h} N_{h,k} = \frac{\Gamma_h^2\left(\frac{1}{2} - i(k + \omega)\right)}{\Gamma_h(-2i\omega)\Gamma_h(1 - 2ik)} \\ R_{h,k} &= \frac{N_{h,k}}{N_{h,-k}} = \frac{\Gamma_h^2\left(\frac{1}{2} - i(k + \omega)\right) \Gamma_h(+2ik)}{\Gamma_h^2\left(\frac{1}{2} + i(k - \omega)\right) \Gamma_h(-2ik)} \end{aligned}$$

It is noteworthy that one has *ordinary plane waves* in the asymptotic regions! This can be understood by noting that for scales in  $r$ ,  $t$ -space that are large compared to  $h$  and  $\omega$ ,  $k$ ,  $m$  small compared to  $h$  one may approximate the deformed wave equation by the undeformed one. The asymptotic observer will see the effect of deformation only by analyzing high frequencies of the reflected waves.

Finally, one may again check that information is preserved:  $|T_{h,k}|^2 + |R_{h,k}|^2 = 1$ .

**3.5. Continuation into regions II/III.** A remarkable qualitative difference to the undeformed case shows up in considering the continuation of the wave-packet that passes through the horizon into regions II/III. To this aim one should again use the  $u$ ,  $v$ -coordinates. In terms of these one has

$$(16) \quad e^{-i\omega t} U_{h,k}(r) \equiv \mathcal{U}_{h,k}(u, v) = u^{-2i\omega} F_h\left(\frac{1}{2} + i(\kappa - \omega), \frac{1}{2} - i(\kappa + \omega), 1 - 2i\omega, uv\right),$$

which is continuously differentiable w.r.t.  $v$  on the future horizon  $v = 0$ .<sup>7</sup> Wave packets of these modes therefore have a well-defined continuation into region II. As in the classical case one gets a unique solution in region II by demanding vanishing on the boundary between regions II and IV. Explicitly it reads

$$(17) \quad f_{II}(u, v) = \int_0^\infty dk \left( \mathcal{U}_{h,k}(u, v) T_{h,k} a_k + \bar{\mathcal{U}}_{h,k}(u, v) \bar{T}_{h,k} \bar{a}_k \right).$$

But what appears to be remarkable is the fact that the modes  $\mathcal{U}_{h,k}(u, v)$  are nonsingular for any  $u, v > 0$ : The singularity has disappeared. In fact, the integral (18) that defines the q-hypergeometric function in (16) converges absolutely for any positive as well as negative values of  $uv$ . Using the variable  $\rho = \log(uv)$  in region II/III one finds that the singularity that classically

<sup>7</sup>Cf. Proposition 2.1. of the Appendix. Here it is important to restrict  $h$  to be in  $(0, 1)$

was at  $uv = 1$  resp.  $\rho = 0$  now has been resolved into a series of poles at  $\rho = i(nh + (m - 1))$ ,  $n, m = 1, 2, \dots$ . These poles approach the real axis in the classical limit  $h \rightarrow 0$ .

So what is the fate of matter fallen into the black hole in the deformed case? The further propagation of (17) in regions II/III can be described in terms of the time variable  $\tau = \log(u/v)$  the same way as was discussed in region I. The late time asymptotics of a wave-packet that has fallen into the black hole is then given by

$$f_{II}^{\text{out}}(\rho, \tau) = \int_0^\infty dk \left( \left( e^{-i(\tau\omega+k\rho)} S_{h,k}^+ + e^{+i(\tau\omega+k\rho)} \bar{S}_{h,k}^+ \right) + \left( e^{-i(\tau\omega-k\rho)} S_{h,k}^- + e^{+i(\tau\omega-k\rho)} \bar{S}_{h,k}^- \right) \right),$$

where  $S_{h,k}^+ = e^{-\frac{\pi i}{2}} e^{-\pi(\omega-k)}$        $S_{h,k}^- = e^{-\frac{\pi i}{2}} e^{-\pi(\omega+k)} R_{h,k}$ .

#### 4. CONCLUSIONS

In the author's opinion there are three main lessons to learn from the example studied in the present paper:

1. There are nonlocal evolution laws that may be interpreted as describing propagation of fields on deformed geometries which allow one to avoid some of the usual problems associated with nonlocalities in a natural way.
2. This way of deformation indeed provides a resolution of singularities present in the classical geometry, which allows one to propagate wave-packets through the region that classically was singular. Such deformations may therefore open ways to resolve the black hole information problem.
3. There are ways to describe some kinds of small-scale ‘‘fuzziness’’ or nonlocality that do not require non-commutativity of the underlying algebra of functions.

The example presented here is of course somewhat artificial in being distinguished by its simplicity and solvability. Its main value is to illustrate the above-mentioned qualitative points which one may expect to persist in considerably more general types of deformations.

More can be done along similar lines as presented here: First one may observe that the present discussion already contains important ingredients for studying the quantization of solutions of the deformed wave equation, with the aim of ultimately determining how the effect of deformation shows up in the spectrum of the Hawking-radiation.

Furthermore, it was mentioned in the introduction that the present model is just one case of a class of models that can be constructed and investigated along similar lines. In contrast to the present one however, the other models are all noncommutative deformations.

Of particular interest may be to study the deformation of  $SL(2, \mathbb{R}) \simeq ADS_3$  as a model for the possible nonlocality (e.g. [BDHM]) of string theory on backgrounds with  $AdS_3$ , similarly to what was recently proposed in [JR].

Finally it should be emphasized that the real task remains to find more concrete evidence on the small scale structure of space-time from the study of the full-fledged gravitational theories such as string theory.

#### 5. APPENDIX: $q$ -SPECIAL FUNCTIONS FOR $q = e^{\pi i h}$

**5.1.  $q$ -Gamma function.** The basic building block for the class of special functions to be considered is the the Double Gamma function introduced by Barnes [Ba]

**Definition 1.** The Double Gamma function is defined as

$$\log \Gamma_2(s|\omega_1, \omega_2) = \left( \frac{\partial}{\partial t} \sum_{n_1, n_2=0}^{\infty} (s + n_1\omega_1 + n_2\omega_2)^{-t} \right)_{t=0}.$$

**Definition 2.** The h-Gammafunction  $\Gamma_h$ :

$$\Gamma_h(s) = \frac{\Gamma_2(s|1, \kappa)}{\Gamma_2(1 + \kappa - s|1, \kappa)}, \quad \kappa = h^{-1},$$

**Proposition 1.** *Properties:*

1. *Functional relations:*

$$\Gamma_h(s+1) = 2 \sin(\pi h s) \Gamma_h(s) \quad \Gamma_h(s+\kappa) = 2 \sin(\pi s) \Gamma_h(s)$$

2. *Zeros at  $s = 1 + \kappa + n + m\kappa$ , Poles at  $s = s_{n,m} = -n - m\kappa$ ,  $n, m = 0, 1, 2, \dots$*

$$\lim_{s \rightarrow s_{mn}} s \Gamma_h(s) = \frac{1}{2\pi b} \frac{(-)^{n+m+mn}}{[n]_h! [m]_h!_{h^{-1}}} \quad [n]_h! = \prod_{r=1}^n (q^r - q^{-r})$$

3. *Asymptotics: For  $|s| \rightarrow \infty$  in any sector not containing the real line one has*

$$\log \Gamma_h(s) \sim \mp \frac{\pi i h}{2} (s^2 - s(1 + \kappa)) + \mathcal{O}(s^{-1}) \quad \text{for } \pm \Im(s) > 0$$

Proof: [Sh]

**Definition 3.** The q-hypergeometric function will be defined as

$$(18) \quad F_h(\alpha, \beta; \gamma; z) = \frac{\Gamma_h(\gamma)}{\Gamma_h(\alpha)\Gamma_h(\beta)} \int_{-i\infty}^{i\infty} ds \frac{(-z)^s \Gamma_h(\alpha+s)\Gamma_h(\beta+s)}{\sin(\pi s) \Gamma_h(\gamma+s)\Gamma_h(1+s)},$$

where the contour is to the right of the poles at  $s = -\alpha - n - m\kappa$   $s = -\beta - n - m\kappa$  and to the left of the poles at  $s = n + m\kappa$   $s = 1 + \kappa - \gamma + n + m\kappa$ ,  $n, m = 0, 1, 2, \dots$

This definition of a q-hypergeometric function is closely related to the one first given in [NU].

**Proposition 2.** *Properties:*

1. *Asymptotic behavior for  $x \rightarrow 0$*

$$F_h(\alpha, \beta, \gamma; z) = 1 + \mathcal{O}(z) + \frac{\Gamma_h(1 + \kappa + \alpha - \gamma)\Gamma_h(1 + \kappa + \alpha - \gamma)\Gamma_h(\gamma)}{\Gamma_h(\alpha)\Gamma_h(\beta)\Gamma_h(2 + 2\kappa - \gamma)} (-z)^{1+\kappa-\gamma} (1 + \mathcal{O}(z))$$

2. *Asymptotic behavior for  $x \rightarrow -\infty$*

$$F_h(\alpha, \beta, \gamma; z) = \frac{\Gamma_h(\gamma)\Gamma_h(\beta - \alpha)}{\Gamma_h(\beta)\Gamma_h(\gamma - \alpha)} (-z)^{-\alpha} (1 + \mathcal{O}(z^{-1})) + \frac{\Gamma_h(\gamma)\Gamma_h(\alpha - \beta)}{\Gamma_h(\alpha)\Gamma_h(\gamma - \beta)} (-z)^{-\beta} (1 + \mathcal{O}(z^{-1}))$$

## REFERENCES

- [DFR] S. Doplicher, K. Fredenhagen, J.E. Roberts: The quantum structure of spacetime of the Planck scale and quantum fields. Commun. Math. Phys. **172**(1995) 187-220
- [FGR] J. Fröhlich, O. Grandjean, A. Recknagel: Supersymmetric quantum theory, non-commutative geometry, and gravitation, in: Proceedings of the Les Houches Summer School 1995, Session LXIV, North-Holland, Elsevier (1998)
- [DKPS] M.R. Douglas, D. Kabat, P. Pouliot, S.H. Shenker: D-branes and Short Distances in String Theory, Nucl.Phys.B **485**(1997) 85-127
- [D] M.R. Douglas: Two Lectures on D-Geometry and Noncommutative Geometry, hep-th/9901146
- [BDHM] T. Banks, M.R. Douglas, G.T. Horowitz, E. Martinec: AdS Dynamics from Conformal Field Theory, hep-th/9808016
- [Co] A. Connes: Noncommutative geometry. San Diego, CA: Academic Press (1994)
- [FG] J. Froehlich, K. Gawedzki: Conformal field theory and geometry of strings. In: Feldman, J. (ed.) et al., Mathematical quantum theory I: Field theory and many body theory. Proceedings of the Canadian Mathematical Society annual seminar, Vancouver, August 4-14, 1993. Providence, RI: American Mathematical Society, CRM Proc. Lect. Notes. 7, (1994) 57-97

- [Wi] E. Witten: On string theory and black holes, *Phys.Rev.D***44** (1991) 314-324.
- [DVV] R. Dijkgraaf, H. and E. Verlinde: String propagation in a black hole geometry, *Nucl. Phys.* **B371** (1992) 269-314
- [Un] W. Unruh: Notes on black hole evaporation, *Phys.Rev.D***14** (1976) 870
- [Be] J. Bernstein: On the support of Plancherel measure, *J. Geom. Phys.* **5** (1988) 663-710 .
- [JR] A. Jevicki, S. Ramgoolam: Non-commutative gravity from the ADS-CFT correspondence, hep-th/9902059
- [Ba] E.W. Barnes: Theory of the double gamma function, *Phil. Trans. Roy. Soc. A* **196** (1901) 265-388
- [Sh] T. Shintani: On a Kronecker limit formula for real quadratic fields, *J. Fac. Sci. Univ. Tokyo Sect.1A* **24**(1977)167-199
- [NU] M. Nishizawa, K. Ueno: Integral solutions of q-difference equations of the hypergeometric type with  $|q| = 1$ , q-alg/9612014

SCHOOL FOR THEORETICAL PHYSICS, DUBLIN INSTITUTE FOR ADVANCED STUDIES, 10 BURLINGTON ROAD,  
DUBLIN 4, IRELAND

*E-mail address:* `teschner@stp.dias.ie`