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ASPECTS OF THE CURRENT ALGEBRA APPROACH

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I. INTRODUCTION.

The past two years has seen the very rapid development in particle physics of the subject known as the current algebra approach or simply current algebras. The central idea in this development was first stated explicitly by Gell-Mann[1) in his paper on the octet model of unitary symmetry, though it is contained implicitly in his earlier paper with Levy[2) on the partially conserved axial vector current hypothesis. Briefly this idea is that physical currents (i.e. currents which enter in physical theories) may form Lie algebras or more simply physical currents may satisfy among themselves certain well defined commutation relations. Further elaboration of this proposal by Gell-Mann and a number of collaborators is to be found in several subsequent publications[3).

The present spate of activity in this area seems to take

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its origin from two more or less complementary contributions which appeared at the beginning of 1965. The first of these is contained in the papers of B.W. Lee[4] and of Dashen and Gell-Mann[5] in which it is shown that many of the results of a higher symmetry (in this case SU(6)) can be obtained from a straightforward application of the current algebra approach. This was a very valuable discovery and while some of the arguments used in these papers were later shown to have quite drastic implications[6], the general philosophy of this work, namely to use current algebras to derive physically interesting results, was widely accepted and applied.

The second crucial contribution was that of Fubini and Furlan[7] who proposed an elegant method of deriving exact sum rules from certain current commutation relations. The importance of this method is that it enables one to fill certain gaps which remained in the more approximate treatment of Lee and of Dashen and Gell-Mann. This work of Lee, Dashen and Gell-Mann and of Fubini and Furlan together provided the impetus for the present extensive work in current algebras.

The aim of this report is to give a fairly simple account of and commentary on the early development of the current algebra approach. We first outline what is meant by this approach and how it is used to obtain results of physical interest. Secondly we comment on two of the major ingredients of the approach namely the currents and their commutation relations. Under the heading "Getting Results" are described the procedures of Lee, Dashen and Gell-Mann and of Fubini and Furlan for deriving physical results. Finally we mention some other applications and describe briefly some recent developments.

II. OUTLINE OF THE CURRENT ALGEBRA APPROACH.

Among the objects used to describe physical phenomena in quantum field theoretic terms a prominent place is held by a class of operators known as the currents. These are certain local operators whose matrix elements enter the amplitudes for various physical processes. As examples we might cite the electromagnetic and weak currents which play a central role in the description of electromagnetic and weak interactions or the pion current which is the operator which appears on the right hand side of the equation for the pion field operator. We denote by $J_A^{\alpha}(x,t)$ the current operators we are concerned with, A being the Lorentz index and α being the internal quantum number index. (Instead of speaking of currents here we might generalize the discussion and speak of local operators in general; anything said concerning currents can be applied, mutatis mutandis, to all local operators of the theory.)

Given the current operators $J_A^{\alpha}(x,t)$, we now make the basic assumption that the equal time commutator of at least some pairs of these are linear combinations of the current operators themselves, plus possibly some other measurable local operators. We assume, therefore, a relation of the form

$$\left[J_A^{\alpha}(x,t), J_B^{\beta}(y,t)\right] = \delta^3(x-y)\sum K(A\alpha, B\beta; C\gamma) J_C^{\gamma}(x,t) + \text{"other measurable local operators"} \quad (1)$$

where the $K(A\alpha, B\beta; C\gamma)$ are known constants which we might call structure constants. We see from this relation that if the "other measurable local operators" are absent, our assumption is equivalent to saying that the currents, which are finite in number, form a finite Lie algebra. This is the or igin of the term current algebra. Hereafter we shall assume that these "other measurable local operators" are indeed absent; this is not an essential assumption and we make it only in the interests of simplicity.

Now with regard to current operators in general, the physically meaningful things are their matrix elements between physical states. These matrix elements are measurable in various strong, electromagnetic and weak processes. The value of a relation such as (1) is that it offers the possibility of calculating these matrix elements theoretically, for this relation is nothing but an infinite set of coupled equations for the matrix elements of the currents $J_A^\alpha(x,t)$. This is seen by taking its matrix element between any pair of physical states $|i\rangle$ and $|j\rangle$ and inserting a sum over a complete set of states between the operators on the left hand side. In this way we get

$$\sum_n \Big[\langle j | J_A^\alpha(x,t) | n\rangle\langle n | J_B^\beta(y,t) | i\rangle - \langle j | J_B^\beta(y,t) | n\rangle\langle n | J_A^\alpha(x,t) | i\rangle \Big]$$

$$= \delta^3(x-y) \sum_{C\gamma} K(A\alpha, B\beta; C\gamma)\, \langle j | J_C^\gamma(x,t) | i\rangle \,, \quad (2)$$

and as $|i\rangle$ and $|j\rangle$ range over all physical states we obtain an infinite set of coupled equations for the matrix elements of the currents.

The important task now is to devise means of solving these equations in various cases of interest. This question we shall take up in due course. Before that we would like to discuss in more detail the basic commutation relations expressed in (1).

III. THE CURRENTS AND THE COMMUTATION RELATIONS.

The idea that physical current operators might satisfy a commutation relation like that given in (1) was first suggested by Gell-Mann[1]. His argument was as follows. In a Lagrangian quantum field theory where there exists an exact symmetry under a Lie group of transformations, the infinitesimal generators of this group are given as the space integrals of the time components of the divergenceless vector (or, in general, tensor) currents which arise naturally from the symmetry in accordance with Noether's theorem[8]. Thus if we denote by I_α the generators of such a group we have

$$I_\alpha = \int d^3x \, V_{4,\alpha}(x,t) \qquad (3)$$

where $V_{\mu,\alpha}(x,t)$ is the divergenceless vector (or tensor) current arising from the symmetry.

Now the infinitesimal generators of a Lie group form a Lie algebra i.e. the set of operators $\{I_\alpha\}$ is closed under commutation. We have therefore the commutation relations

$$\left[I_\alpha , I_\beta \right] = i \, C_{\alpha\beta\gamma} \, I_\gamma , \qquad (4)$$

where $C_{\alpha\beta\gamma}$ are the structure constants of the group. Furthermore if the group in question is assumed to be an internal symmetry group the divergenceless currents $V_{\mu,\alpha}(x,t)$ transform as members of the regular representation of the group i.e.

$$\left[I_\alpha , V_{\mu,\beta} (x,t) \right] = i \, C_{\alpha\beta\gamma} V_{\mu,\gamma} (x,t) \, . \qquad (5)$$

If we now wish to speculate about the commutator of $V_{4,\alpha}(x,t)$ and $V_{4,\beta}(y,t)$ we note that it must vanish for $x \neq y$ in accordance with microcausality and if we assume that at $x = y$ it is not more singular than a δ-function then eqs.(3) and (5) tell us that we must have

$$\left[V_{4,\alpha} (x,t) \, , V_{4,\beta} (y,t) \right] = - \delta^3(x-y) \, C_{\alpha\beta\gamma} \, V_{4,\gamma}(x,t) \, . \qquad (6)$$

This was the first equal time current commutation relation and all other such relations are written down in analogy with it.

We wish to make the following points about this commutation relation and about current commutation relations in general.

1. Although eq.(6) was suggested by a certain internal symmetry the existence of such a symmetry is not necessary in order that this relation be true. On the contrary, given a field theory involving Dirac fields $\psi_i(x)$ of number N equal to the dimension of some representation of the Lie group G and defining the quantities

$$Q_\alpha(x,t) = \overline{\Psi}(x,t) \, \gamma_4 \, t_\alpha \, \Psi(x,t) \quad ,$$

where

$$\Psi(x,t) = \begin{pmatrix} \psi_1(x,t) \\ \psi_2(x,t) \\ \\ \psi_N(x,t) \end{pmatrix} ,$$

and t_α are the matrices of the N-dimensional representation of G, we find that the operators $Q_\alpha(x,t)$ satisfy the commutation relations of the Lie algebra of the group G, provided only that the $\psi_i(x,t)$ satisfy the ordinary anticommutation relations of fermi fields and without any assumption of invariance under the group G. This illustrates the fact that a Lie group may through a current algebra play an important role without at all being an invariance group of the underlying theory. Herein lies a possibility of understanding the various higher symmetry groups which have been under discussion for some time.

(In practice it may be necessary to assume invariance under some internal symmetry group in order to derive useful results but this is not always the case. For instance the well-known Adler-Weisberger[9a,b] calculation does not assume isospin invariance.)

2. In deriving commutation relations for currents other than those which arise as a result of invariance under a Lie group we generally appeal to a model. For theories wherein the fundamental internal symmetry is isospin, we postulate a model in which the Lagrangian contains only a fundamental doublet field, while for unitary symmetry, SU(3), we take a model whose Lagrangian is constructed from a fundamental quark or triplet

field. In such theories we can construct currents of the form

$$J_A^{\alpha}(x) = \bar{\psi}(x,t)\,\Gamma_A\, t^{\alpha}\,\psi(x,t) \quad , \qquad (7)$$

where the Γ_A stand for the 16 members of the Dirac algebra and the t^{α} are the matrices of the fundamental representation of the internal symmetry group plus the unit matrix, being $(\frac{\tau_\alpha}{2}, 1)$ for the isospin group and $(\frac{\lambda_\alpha}{2}, 1)$ for the SU(3) group.

The equal-time commutation relations of currents $J_A^{\alpha}(x,t)$ may now be deduced provided only that the equal-time commutation relations of the fields $\psi(x,t)$ are known. If, as is usually assumed, these are simply the ordinary anticommutation relations of fermi fields, we obtain the equal-time commutation relation

$$[J_A^{\alpha}(x,t), J_B^{\beta}(y,t)] = \delta^3(x-y)\psi^{+}(x,t)[\gamma_4\Gamma_A t^{\alpha}, \gamma_4\Gamma_B t^{\beta}]\psi(x,t) \,. \qquad (8)$$

Then, using the identity

$$[\gamma_4\Gamma_A t^{\alpha}, \gamma_4\Gamma_B t^{\beta}] = \tfrac{1}{2}[\gamma_4\Gamma_A, \gamma_4\Gamma_B]\{t^{\alpha}, t^{\beta}\} + \tfrac{1}{2}\{\gamma_4\Gamma_A, \gamma_4\Gamma_B\}[t^{\alpha}, t^{\beta}]$$

and the fact that the Γ_A and t^{α} are algebras, the right hand side of eq.(8) can be expressed as a linear combination of the currents, thus yielding a relation of the form given in eq.(1). The point of this discussion is to show that there exist realistic models in which current commutation relations such as those postulated in section 2 are satisfied.

In connection with the derivation of eq.(8) it seems worthwhile drawing attention to two points. The first is that the assumption of ordinary fermi field anticommutation relations for $\psi(x,t)$ is not necessary. It is sufficient, for instance, if $\psi(x,t)$ satisfies

$$[\psi_a(x,t), \psi_b^+(y,t)\,\psi_c(y,t)] = \delta_{ab}\,\delta^3(x-y)\,\psi_c(x,t) \qquad (9)$$

and this relation is much more general than the ordinary anticommutation relations (it includes them as a special case). Relations such as (9) form the basis of the subject of parastatistics. From a physical viewpoint what this implies is that, as far as current commutation relations go, the quanta of the fundamental fields, if such exist, need not be fermions, but rather particles satisfying a more general type of statistics. This is a welcome feature since it allows one more easily to construct the normal hadron states as bound states of the fundamental particles[10].

The second point about the commutation relation (8) is that in general the assumption that the singularity involved is no worse than a delta-function is not justified. Specifically we have the so-called Schwinger terms [11]. These terms arise because of the highly singular nature of currents defined as bilinear products of the field quantities taken at the same space-time point. If instead of eq.(7) we define the currents $J_A^{\alpha}(x,t)$ as

$$J_A^{\alpha}(x,t) = \lim_{\varepsilon\to 0} \bar{\psi}(x-\varepsilon, t)\, \Gamma_A\, t^{\alpha}\, \psi(x+\varepsilon, t) \quad , \qquad (10)$$

we obtain the commutation relation

$$[J_A^{\alpha}(x,t), J_B^{\beta}(y,t)] = \delta^3(x-y)\lim_{\varepsilon\to 0} \psi^{+}(x-\tfrac{\varepsilon}{2},t)[\gamma_4\Gamma_A t^{\alpha}, \gamma_4\Gamma_B t^{\beta}] \times \psi(x+\tfrac{\varepsilon}{2},t)$$

$$+ \lim_{\varepsilon\to 0}\frac{\varepsilon}{2}(\vec{\nabla}\delta^3(x-y))\psi^{+}(x-\tfrac{\varepsilon}{2},t)\{\gamma_4\Gamma_A t^{\alpha},\gamma_4\Gamma_B t^{\beta}\} \times \psi(x+\tfrac{\varepsilon}{2},t) \quad , \qquad (11)$$

where $\lim_{\varepsilon\to 0}$ is understood to be averaged over all space directions and higher powers in ε are assumed to vanish in the limit. The second term on the right hand side is known as the Schwinger term.

Generally this term cannot be assumed to vanish. One way of avoiding the difficulties it presents is to use, not the current operators themselves, but their integrals over all space. If this is done the Schwinger term plays no role, as can be seen by integrating eq.(11) over $\vec{x}$ and $\vec{y}$. Another suggestion is that of Bouchiat and Meyer[12], who noticed that in the quark model the Schwinger term is symmetric against the simultaneous interchange of space and internal symmetry indices. Their advice is to start with a combination which is odd against this interchange and again the Schwinger terms will be absent. In the general situation however the presence of the Schwinger terms must be allowed for.

IV. GETTING SOLUTIONS.

We return now to the problem of finding solutions of the infinite set of equations represented by an equal time current commutation relation - eq.(2). In general a complete solution of these equations is out of the question. Accordingly we must devise means of reducing the problem to tractable proportions. This means doing one of two things:

(a) reducing the number of equations somehow to a finite number

or (b) devising some method of calculating the sum over the infinite set of states on the left hand side of (2).

Of these two approaches (a) is that adopted by Lee, Dashen and Gell-Mann while (b) is the one followed by Fubini and Furlan.

(a) The Lee-Dashen-Gell-Mann Procedure.[4,5]

A considerable simplification of our problem results if we start from the commutation relations for the space integrals of the currents rather than those for the currents themselves. Thus if we define

$$I_A^{\alpha} = \int d^3x \, J_A^{\alpha}(x,t) \tag{12}$$

then from (1), (assuming the other measurable operators are absent), we obtain for the I_A^{α} the commutation relations

$$\left[I_A^{\alpha}, I_B^{\beta} \right] = \sum_{C\gamma} K(A\alpha, B\beta; C\gamma) I_C^{\gamma} \quad . \tag{13}$$

These relations have the familiar look of the defining relations of a Lie algebra, such as those which we meet in connection with the ordinary space-time and internal symmetry groups. There is one important difference, however, and it is that, unlike the case of exact symmetry groups, the elements of this Lie algebra are not all time independent - only those which are the space integrals of the time components of divergenceless currents are. The time dependent elements do not commute with the Hamiltonian and so this is not the Lie algebra of an invariance group of the theory. In view of this we must consider the integration in eq.(12) as being done in a chosen Lorentz frame and all succeeding operations with the algebra (13) are then performed in this same frame.

The I_A^α are independent of the spatial coordinates and so they commute with the spatial momentum operators. This has the result that when deducing the consequences of eq.(13) by taking its matrix elements between physical states, it is necessary to consider only states of one fixed definite momentum. In eq.(2) on the other hand, a similar procedure would still require states of all spatial momentum to be included.

Lee, Dashen and Gell-Mann then made the assumption that the infinite set of equations got by taking the matrix elements of eq.(13) between all possible states may be approximated by setting all but a finite number of these matrix elements equal to zero, the non-zero elements being between single particle states only. It is to be hoped that the finite set of equations so obtained may yield a solution which is reasonably close to the exact one. A priori there is no way of judging how justified this approximation may be; however it would seem a reasonable hypothesis if the dominant matrix elements of a current are those between nearby states.

On the basis of the above approximation, if we denote by $|i(p)\rangle$ i=1,2,...N the states between which the operators I_A^α are assumed to have non-vanishing matrix elements (p is the spatial momentum) and if the matrix element of the operator I_A^α between two such states is written as

$$\langle j(p')|\ I_A^\alpha |i(p)\rangle = \delta^3(p-p')\left(\mathcal{G}_A^\alpha(p)\right)_{ji} ,$$

then the equations to be solved are

$$\sum_{j=i}^{N}\left[\left(\mathcal{G}_A^\alpha(p)\right)_{ij}\left(\mathcal{G}_B^\beta(p)\right)_{jk} - \left(\mathcal{G}_B^\beta(p)\right)_{ij}\left(\mathcal{G}_A^\alpha(p)\right)_{jk}\right]$$
$$= \sum_{C\gamma} K(A\alpha, B\beta; C\gamma)\left(\mathcal{G}_C^\gamma(p)\right)_{ik} , \quad (14)$$

i,k = 1,2,... N.

From the relation (14) it is clear that the present procedure amounts to nothing other than finding finite dimensional representations of the Lie algebra defined by eq.(13). For if we can find a set of quantities $\left(\mathcal{G}_A^\alpha(p)\right)_{ij}$, i,j = 1,2,...N, which satisfy eq.(14), then we have found a set of finite N x N matrices which satisfy the same commutation relations as the I_A^α and such a set of matrices is by definition a finite dimensional representation of the Lie algebra of the I_A^α. This point was first noted by Okubo[13] and in view of it it is easy to see why this procedure reproduces the higher symmetry results insofar as they refer to the current operators.[4,5] For this purpose all one has to do is to select those currents whose space integrals satisfy the commutation relations of the relevant higher symmetry group (in Lee's case SU(6)) and then take the matrix elements of this Lie algebra using the states which constitute the basis of an appropriate representation of the group. The emergence of the higher symmetry results is then automatic.

There are two major difficulties with the above described procedure. The first is that the approximation assumption upon which it rests is strictly unjustified except in the trivial case when the Lie algebra, eq.(13), is that of an exact symmetry group of the underlying theory. Coleman[6] has shown that if A is the space integral of a local operator $\mathcal{a}(x,t)$ such that all matrix elements of the form <m| A | s> , where | s > represents a single particle state and | m > a multiparticle one, vanish, then A commutes with the Hamiltonian. An assumption of this kind is an essential ingredient of the Lee-Dashen-Gell-Mann procedure and so this procedure is equivalent to assuming that the Lie algebra eq. (13) is that of an exact symmetry group. This means that strictly the method cannot be applied in cases where the Lie algebra (13) is that of a higher symmetry group which is not an invariance group of the Hamiltonian. This is a serious blow.

In addition to this difficulty of principle it transpires that the method gives consistent results generally only for special values of the spatial momentum, such as zero spatial momentum or infinite spatial momentum[14,15]. The reason for this is that the operators I_A^{α} are not simply the generators of the Lie algebra eq.(13); they are also by definition tensor operators under the Poincaré group. In obtaining a "solution" of the Lie algebra therefore, we need to respect the Poincaré group properties of its generators. Now it is known that representations of the Poincaré group for finite non-zero spatial momentum are all infinite in dimension. Thus for finite non-zero momentum states it will in general be impossible to find a solution of the Lie algebra eq.(13) using only a finite number of states.

Despite these difficulties we feel that the present method as a general approach to higher symmetries is a valuable one. At least it is preferable to the ordinary one where one explicitly assumes the existence of a higher symmetry from the outset. Here one realises clearly that the higher symmetry results can at best be only approximate. This approach also enables one to treat the various representations of a higher symmetry group on a more or less equal footing, to distinguish clearly between zero and non-zero momentum states and to discover the role of orbital angular momentum in higher symmetry groups[15].

An Example.

We consider a simple concrete example in order to illustrate this method. The space integrals of the axial vector isovector current A_i^{α}, the axial vector isoscalar current A_i^{o} and the vector isovector current V_4^{α} satisfy the commutation relations of the Lie algebra of the SU(4) group. The relation involving the commutator of the A_i^{α} with A_j^{β} is

$$[A_i^{\alpha}, A_j^{\beta}] = i\,\delta_{ij}\,\varepsilon_{\alpha\beta\gamma}\,V_4^{\gamma} + i\,\delta_{\alpha\beta}\,\varepsilon_{ijk}\,A_k^{o}, \qquad (15)$$

and there are similar expressions for the commutators of all other pairs of the set. We wish now to determine the matrix elements of these operators between zero momentum states of the nucleon. We denote these states by $|\sigma,s\rangle$ $s,\sigma = 1,2$ where σ is the isospin and s the ordinary spin index. By definition we have

$$\langle \sigma',s' \,|\, V_4^{\alpha} \,|\sigma,s\rangle = u^{+}(\sigma',s')\,\frac{\tau_\alpha}{2}\,u(\sigma,s), \qquad (16)$$

$$\langle \sigma',s' \,|\, A_i^{\alpha} \,|\, \sigma,s\rangle = g_A\, u^{+}(\sigma',s')\,\sigma_i \frac{\tau_\alpha}{2}\,u(\sigma,s), \qquad (17)$$

and

$$\langle \sigma',s' \,|\, A_i^{o} \,|\, \sigma,s\rangle = g\, u^{+}(\sigma',s')\,\frac{\sigma_i}{2}\,u(\sigma,s). \qquad (18)$$

Here $u(\sigma,s)$ is a constant spinor denoting zero momentum states of the nucleon and the above manner of writing the matrix elements is simply a concise way of writing the Wigner-Eckart theorem without introducing Clebsch-Gordan coefficients. g_A and g are reduced matrix elements and the fact that the reduced matrix element in (16) is unity is because the V_o^α are the generators of the conserved isospin group.

We now take the matrix element of eq.(15) between zero momentum states of the nucleon and, in the sum over the complete set of states inserted on the left hand side, we retain only the contributions from these same nucleon states. Then with the definitions eqs.(16) to (18) we get

$$g_A^2 \sum_{\sigma'' s''} [u^+(\sigma',s')\, \sigma_i \frac{\tau_\alpha}{2}\, u(\sigma'',s'')\, u^+(\sigma'',s'')\, \sigma_j \frac{\tau_\beta}{2}\, u(\sigma,s)$$

$$- u^+(\sigma',s')\, \sigma_j \frac{\tau_\beta}{2}\, u(\sigma'',s'')\, u^+(\sigma'',s'')\, \sigma_i \frac{\tau_\alpha}{2}\, u(\sigma,s)]$$

$$= i\delta_{ij}\, \varepsilon_{\alpha\beta\gamma}\, u^+(\sigma',s') \frac{\tau_\gamma}{2}\, u(\sigma,s) + ig\delta_{\alpha\beta}\, \varepsilon_{ijk}\, u^+(\sigma',s') \frac{\sigma_k}{2}\, u(\sigma,s). \tag{19}$$

Upon summing over σ'' and s'' and using the identity

$$\sigma_i\tau_\alpha\sigma_j\tau_\beta - \sigma_j\tau_\beta\sigma_i\tau_\alpha = \tfrac{1}{2}\{\sigma_i,\sigma_j\}[\tau_\alpha,\tau_\beta] + \tfrac{1}{2}[\sigma_i,\sigma_j]\{\tau_\alpha,\tau_\beta\}$$

the left hand side of this equation becomes

$$g_A^2\, u^+(\sigma',s')\, [i\, \delta_{ij}\, \varepsilon_{\alpha\beta\gamma} \frac{\tau_\gamma}{2} + i\, \delta_{\alpha\beta}\, \varepsilon_{ijk} \frac{\sigma_k}{2}]\, u(\sigma,s) \quad .$$

When we then compare the two sides we find the relations

$$g_A{}^2 = 1 , \quad g_A{}^2 = g . \qquad (20)$$

These relations represent the "solution" by the Lee-Dashen-Gell-Mann procedure of the algebra of currents in the present instance.

Physically the quantity g_A is the strength of the axial vector isovector current between zero momentum nucleon states. This quantity can be measured in neutron β-decay and it is known that $|g_A| = 1.18$. (We assume that the bare axial vector constant is equal to the bare vector constant.) We might conclude from this and eq.(20) that our approximation procedure is quite good in this instance. This is a delusion. Our approximation is in fact quite bad because we know that the coupling of the nucleon to the N*(1238) (3,3) resonance through the axial vector current is quite large[16] (of the order of 0.6 to 0.8 in the above units) and yet we have neglected it completely.

From a mathematical viewpoint also our approximation proves to have quite drastic consequences. We notice that eq.(20) gives $g = 1$ where g is the reduced matrix element of the operator $A_i{}^o$ between nucleon states. Now this operator is, in the model we are using, the spin part of the angular momentum i.e. the total angular momentum J_i is given by

$$J_i = L_i + S_i \quad , \tag{21}$$

where L_i is the orbital angular momentum and S_i the spin angular momentum, is equal to $A_i^{\,o}$. Since the nucleon has spin $\frac{1}{2}$ we have

$$< \sigma' s' \mid J_i \mid \sigma, s > \;=\; u^{+}(\sigma', s') \frac{\sigma_i}{2} u(\sigma, s) \;, \tag{22}$$

and so from eq.(18) with $g = 1$ and eqs.(21) and (22) we find

$$< \sigma' s' \mid L_i \mid \sigma, s > \;=\; 0 \;.$$

In other words our approximation forces the nucleon to have orbital angular momentum equal to zero.

This is not, as it might appear, a trivial point. One might think, for instance, that since $\mid \sigma, s >$ represents a nucleon state at rest, its orbital angular momentum ought to be zero. However, in the present theory, a one-nucleon state is not necessarily a one-particle state; it is more likely to be a composite of three or more fundamental particles corresponding to some mixture of representations of $SU_{I,S}(4)$, the group which merges the spin and the isospin, and in that case the orbital angular momentum will not be zero.

Thus our method of solution has forced the zero momentum nucleon states, which at first constitute a (2,2) representation of the group $SU_I(2) \times R_3(J_i = L_i + S_i)$, that is the direct product of the isospin group and the group of spatial rotations, to be a (2,2,1) representation of the group $SU_I(2) \times SU_S(2) \times R_3(L_i)$ where $SU_S(2)$ is the spin angular momentum group and $R_3(L_i)$ is the orbital angular momentum group. Even more, these states are forced to be a (4,1) representation of the group $SU_{I,S}(4) \times R_3(L_i)$; this is seen from the fact that, in the approximation used, the sum of the squares of all 15 operators of the algebra is diagonal and a multiple of the unit matrix and since this sum is the first Casimir operator of the group $SU_{I,S}(4)$ it follows by Schur's lemma that the states used must constitute an irreducible representation of $SU_{I,S}(4)$. Hence, although we did not start out with any assumptions about $SU_{I,S}(4)$ invariance or multiplet structure, the approximation method employed imposes on the states used that they constitute a representation of the group $SU_{I,S}(4) \times R_3(L_i)$.

This conclusion is not at all surprising. It is in fact simply a verification of the discussion given above after eq.(14). A situation such as it imposes will always be possible for zero momentum states provided only that the states chosen have the correct spin isospin complexion. It is clear, however, that this kind of circumstance can at best be only an approximation to the real one; for zero momentum states, while they are eigen-states of the total angular momentum, are not generally eigen-states of the orbital and spin angular momenta separately.

From this example one sees how the current algebra approach works in conjunction with the Lee-Dashen-Gell-Mann procedure. The limitations of the approximation are clearly seen and it is to overcome these limitations that we turn now to a second technique.[7]

(b) The Fubini-Furlan Technique.

The essential difference between the Fubini-Furlan technique and that outlined above is that the former does not truncate the sum over the complete set of states on the left hand side of eq. (2) but rather expresses it in terms of physically accessible quantities, namely integrals over cross-sections or known matrix elements. The method is analogous to that used by some authors to calculate electromagnetic mass differences[17] and is as follows.

One begins, as above, from the integrated form of the commutation relations, eq. (13). One then takes those relations where

(i) $I_B^{\beta} = (I_A^{\alpha})^+$,

(ii) I_A^{α} is the space integral of the time component of a current

$$I_A^{\alpha} = \int d^3x \, J_{4A}^{\alpha}(x,t) \ ,$$

(iii) the divergence of $J_{\mu A}^{\alpha}$ is a measurable local operator

$$\partial_\mu J_{\mu A}^{\alpha}(x,t) = k \, O_A^{\alpha}(x) \ , \quad k \text{ a constant}$$

and

(iv) the two states between which one takes the matrix element of the commutation relation are one and the same.

One sees that this method has a more restricted range of application than the previous one (because of the requirements (i) to (iv)) but it is much more satisfactory because it is much more exact.

Using (i) and (iv) above, we see that on this occasion our matrix element equation is

$$\sum_{j(p'')} \Big[< i(p')| I_A^{\alpha} | j(p'') > < j(p'')| (I_A^{\alpha})^{+} | i(p)> - < i(p')| (I_A^{\alpha})^{+} | j(p'') > < j(p'')| I_A^{\alpha} | i(p) > \Big]$$

$$= \sum_{C_\gamma} K(A\alpha, (A\alpha)^{+}; C\gamma) < i(p')| I_C^{\gamma} | i(p) > \ . \qquad (23)$$

In this equation the matrix element on the right hand side is always known exactly because, on account of the way I_A^{α} and I_B^{β} were chosen [condition (i) above], the operator I_C^{γ} turns out to be either the charge, the baryon number or the strangeness operator or some linear combination of these three. This side of the equation poses no problems therefore. On the left hand side the sum over the states $\{ j(p'') \}$ will usually receive contributions from one or more single particle states - we call these discrete states - and then from two or more particle states - we call these the continuum states because the mass of such states has a continuum of values. For simplicity we assume that there is only one discrete state contributing and we call it $j_D(p'')$. We next separate the contribution of this state from that of the continuum states and this latter is then expressed as an integral over certain cross sections or known matrix elements. The net result is that eq.(23) becomes a sum rule for the matrix element of I_A^α between the state $|i(p)\rangle$ and the discrete state $|j_D(p'')\rangle$.

We remark here that the continuum contribution to the sum rule thus obtained has a straightforward interpretation in terms of broken symmetry ideas. If the algebra of the I_A^α were the Lie algebra of an exact symmetry group the I_A^α would all commute with the Hamiltonian; this would have the result that all matrix elements between single particle and multiparticle states would vanish and so the continuum contribution to the above sum rule would vanish. This continuum contribution is therefore a measure of the amount by which the symmetry, represented by the algebra of the I_A^{α}, is broken.

In order to calculate this continuum contribution we begin by considering the matrix element of I_A^{α} between the state $|i(p)>$ and a general continuum state $|j_C(p')>$. In view of the specification of the I_A^{α} given above [condition (ii)] we have

$$< j_C(p')| I_A^{\alpha} | i(p) > = < j_C(p)| \int J_{4A}^{\alpha}(x,t)\, d^3x | i(p) > ,$$

which, upon using the relations

$$i \dot{I}_A^{\alpha} = [I_A^{\alpha}, H] , \text{ [H is the Hamiltonian]}$$

and

$$i\dot{I}_A^{\alpha} = -\int d^3x \frac{\partial J_{4A}^{\alpha}}{\partial x_4} = -\int d^3x\, \partial_{\mu} J_{\mu A}^{\alpha}(x,t) ,$$

we can write

$$< j_C(p')| I_A^{\alpha} | i(p)> = \frac{< j_C(p')| \int d^3x\, \partial_{\mu} J_{\mu A}^{\alpha}(x,t) | i(p)>}{E_{j_C} - E_i} , \tag{24}$$

where E_{j_C} and E_i are the energies of the states $| j_C(p) >$ $|i(p) >$ respectively. (In cases where the continuum states are not separated from the discrete ones because of the presense of massless particles - photons - the denominator on the right hand side of eq.(24) may need to be defined more carefully; we shall ignore this difficulty in the interests of simplicity.) One sees clearly that the "inelastic" matrix elements vanish when the current $J_{\mu A}{}^{\alpha}(x,t)$ is divergenceless. When $J_{\mu A}{}^{\alpha}(x,t)$ is not divergenceless we replace its divergence by $k\, O_A{}^{\alpha}(x,t)$ in accordance with condition (iv) above and since this is assumed to be a known local operator its matrix elements can be inserted in the sum rule to deduce the value of the matrix element $< j_D(p') | I_A{}^{\alpha} | j(p) >$.

The most immediate application of the foregoing is in the case when the currents in question are members of the octets of vector and axial vector currents which arise in connection with the unitary symmetry SU(3). Since this symmetry is not exact the divergences of the vector currents are not all zero and one does not expect the axial vector currents to vanish anyway. In each case it is possible to give an expression for $\partial_\mu J_{\mu A}{}^{\alpha}$ in terms of other measurable local operators and so we can apply the above theory. The relevant expressions for $\partial_\mu J_{\mu A}{}^{\alpha}(x,t)$ are as follows:

1. $J_{\mu A}^{\alpha}(x,t) = V_\mu^{o(\pm)}(x,t)$, the strangeness conserving vector current,

   $\partial_\mu V_\mu^{o(\pm)}(x,t) = \pm\, i\, e\, A_\mu(x)\, V_\mu^{o(\pm)}(x,t)$, $A(x)$ the electromagnetic field;

2. $J_{\mu A}^{\alpha}(x,t) = V_\mu^{1(\pm)}(x,t)$, the strangeness changing vector current,

   $\partial_\mu V_\mu^{1(\pm)}(x,t) = c_\kappa\, \kappa^{(\pm)}(x)$, where $\kappa^{(\pm)}(x)$ is the kappa field and c_κ its leptonic decay constant;

3. $J^{\alpha}_{\mu A}(x,t) = A^{o(\pm)}_{\mu}(x,t)$, the strangeness conserving axial vector current,

$\partial_{\mu} A^{o(\pm)}_{\mu}(x,t) = c_{\pi}\, \pi^{(\pm)}(x,t)$ where $\pi^{(\pm)}(x,t)$ is the pion field and c_{π} the pion decay constant;

4. $J^{\alpha}_{\mu A}(x,t) = A^{1(\pm)}_{\mu}(x,t)$, the strangeness changing axial vector current,

$\partial_{\mu} A^{1(\pm)}(x,t) = c_K\, K^{(\pm)}(x,t)$ where $K^{(\pm)}(x,t)$ is the kaon field and c_K the K_{l_2} decay constant.

These expressions for the divergences enable us to compute the contribution of the continuum terms in a manner which we shall now illustrate. There is no need to take these relations literally as operator relations which arise from an underlying field theory; rather, they should be thought of as definitions of the respective fields in accordance with the Haag-Nishijima theorem[18].

As an example we take the case when $J^{\alpha}_{\mu A}(x,t)$ is the strangeness conserving axial vector current to illustrate how the present method works.

__An Example:__ __The Adler-Weisberger[9a,b] calculation.__

On choosing $I^{\alpha}_{A} = \int A_4(x,t)\, d^3x = Q_5^{\,+}$ our commutation relation becomes

$$\left[Q_5^{\,+}, Q_5^{\,-}\right] = 2\, I_3 \,, \qquad (25)$$

where I_3 is the third component of isospin. Taking the matrix element of this relation between the proton states $|P_s(p)\rangle$ and $|P_{s'}(p')\rangle$ - s and s' being spin indices and p and p' the momenta - we obtain for the right hand side the expression $\delta_{s's}\,\delta^3(p'-p)$; the first term on the left hand side receives a contribution from neutron states and with the definition

$$< N_{s'}(p')|\, Q_5^-\,|\, P_s(p) > \;=\; \delta^3(p'-p)\, g_A \frac{M}{E_p} \bar{u}_N^{(s')}(p')\gamma_4\gamma_5 u_P^{(s)}(p)$$

this contribution is equal to

$$\delta_{s's}\,\delta^3(p'-p)\, g_A^{\,2}(1-\frac{M^2}{E_p^{\,2}})$$

M being the nucleon mass and E_p its energy. Then if we denote by $|\,\alpha\,(p'')>$ a generic continuum state contributing to the sum rule we have

$$\delta_{s's}\,\delta^3(p'-p) \;=\; g_A^{\,2}(1-\frac{M^2}{E^2})\;\delta_{s's}\,\delta^3(p'-p)$$
$$+\sum_{\alpha(p'')}\Big[< P_{s'}(p')|\, Q_5^+\,|\,\alpha(p'')><\alpha(p'')|\, Q_5^-\,|\, P_s(p)> \;-\; Q_5^+\leftrightarrow Q_5^-\Big]\,. \tag{26}$$

Now using eq.(24) with $I_A^{\,\alpha} = Q_5^{\pm}$ and the partially conserved axial current relation

$$\partial_\mu\, A_\mu^{\,o(\pm)}(x,t) \;=\; c_\pi\, \pi^{\pm}(x,t)\quad, \tag{27}$$

we find

$$< \alpha(p'') | Q_5^{\pm} | P_s(p) > \; = \; \frac{c_\pi}{E_\alpha - E_p} < \alpha(p'') | \int \pi^{\pm}(x,0)\, d^3x \, | P_s(p) > ,$$

and a little manipulation with the right hand side here yields

$$< \alpha(p'') | Q_5^{\pm} | P_s(p) >$$

$$= \frac{c_\pi}{(E_\alpha - E_p)^2 - m_\pi^{\;2}} \frac{(2\pi)^3 \, \delta^3(p'' - p)}{(E_\alpha - E_p)} < \alpha(p'') | J_\pi^{\pm}(0) | P_s(p) > , \tag{28}$$

where $J_\pi^{\pm}(x)$ is the pion current defined by the relation

$$J_\pi^{\pm}(x) = (\Box - m_\pi^{\;2})\, \pi^{\pm}(x) .$$

Next we write the summation $\sum\limits_{\alpha(p'')}$ in eq.(26) as

$$\sum_{\alpha(p'')} = \int d^3p'' \int dW \sum_{\alpha_{int}} \delta(W - M_\alpha) , \tag{29}$$

where $\sum\limits_{\alpha_{int}}$ denotes summation over the internal variables of the state α , W is its total energy in the centre of mass system and M_α is its mass. [Notice from eq.(28) the spatial momentum of α is equal to that of the proton state $P_s(p)$]. Then on setting $s = s'$ in eq.(26) and taking $\frac{1}{2}\sum\limits_s$ we obtain, using eqs.(28) and (29)

$$1 = g_A^{\;2}(1 - \frac{M^2}{E_p^{\;2}}) + \int dW \sum_{\alpha_{int}} \delta(W - M_\alpha) \frac{c_\pi^{\;2}}{[(E_\alpha - E_p)^2 - m_\pi^{\;2}]^2} \frac{(2\pi)^6}{(E_\alpha - E_p)^2}$$

$$\times \; \frac{1}{2} \sum_s \Big[| < \alpha(p) | J_\pi^{-}(0) | P_s(p) > |^2 - | < \alpha(p) | J_\pi^{+}(0) | P_s(p) > |^2 \Big] \tag{30}$$

Quantities $F_{\alpha s}{}^{\pm}$ and $K^{\pm}$ are now defined by the relations

$$<\alpha(p)|\, J_\pi{}^{\pm}(0)\,|\, P_s(p)> \;=\; \frac{1}{(2\pi)^3}\left(\frac{M}{E_p}\,\frac{M_\alpha}{E_\alpha}\right)^{\frac{1}{2}} F_{\alpha s}{}^{\pm} \;, \qquad (31,a)$$

and

$$\frac{1}{2}\sum_{s}\sum_{\alpha_{int}} \delta(W-M_\alpha)\,|\,F_{\alpha s}{}^{\pm}|^2 \;=\; K^{\pm}[\,W,\,(p_\alpha-p)^2] \;, \qquad (31,b)$$

where p_α is the four momentum of the state α - $p_\alpha = (\vec{p}, E_\alpha)$.
Then eq.(30) takes the concise form

$$1 \;=\; g_A{}^2\left(1-\frac{M^2}{E_p{}^2}\right)$$
$$+\int dW\,\frac{c_\pi{}^2}{[(E_\alpha-E_p)^2-m_\pi{}^2]^2}\,\frac{1}{(E_\alpha-E_p)^2}\,\frac{MW}{E_pE_\alpha}\Big\{K^-[W,(p_\alpha-p)^2]$$
$$-\,K^+[W,(p_\alpha-p)^2]\Big\}\;. \qquad (32)$$

This relation is to be looked upon as a sum rule for the quantity g_A.

In order to evaluate this sum rule we need to relate the quantities $K^{\pm}[W,(p_\alpha-p)^2]$ to some known functions. This is done as follows. The total cross sections for $\pi^{\pm}$-proton scattering are given by

$$\sigma^{\pm}(W)\times\text{flux} \;=\; (2\pi)^{10}\tfrac{1}{2}\sum_{s}\sum_{\alpha}\frac{|<\alpha(p_\alpha)|\,J_\pi^{\pm}(0)\,|\,P_s(p)>|^2}{(2\pi)^3\,2k_o}\,\delta^4(p+k-p_\alpha)$$

$$=\frac{\pi}{k_o}(2\pi)^6\,\tfrac{1}{2}\sum_{s}\sum_{\alpha_{int}}\;|<\alpha(p_\alpha)|\,J_\pi^{\pm}(0)\,|\,P_s(p)>|^2\;\delta(E+k_o-E_\alpha)$$

$$=\frac{\pi}{k_o}\,\frac{M}{E_p}\,K^{\pm}(W,-m_\pi{}^2)\;,$$

$$(33)$$

this last line coming from a specialisation to the centre of momentum system so that $E_p + k = W$, $E_\alpha = M_\alpha$. Since in this system the flux is

$$\frac{|k|}{k_o} + \frac{|k|}{E_p} = \frac{|k|W}{k_o E_p}$$

while

$$k_o = \frac{1}{2W}(W^2 + m_\pi^2 - M^2) \quad ,$$

we obtain for $K^{\pm}(W, -m_\pi^2)$ the expression

$$K^{\pm}[W, -m_\pi^2] = \frac{\sqrt{k_o^2 - m_\pi^2}\, W}{\pi M} \quad \sigma^{\pm}(W) \quad . \qquad (34)$$

Thus when the second invariant variable $(p_\alpha - p)^2$ takes on the value $-m_\pi^2$ the functions $K^{\pm}[W, (p_\alpha - p)^2]$ in (32) are simply related to the total $\pi^{\pm} - p$ cross-sections. In order to use this information it is necessary to devise some means of making $(p_\alpha - p)^2$ equal to $-m_\pi^2$ in eq.(32). This turns out not to be possible, for on account of the way that sum rule was derived we have $(p_\alpha - p)^2 = -(E_p - E_\alpha)^2 = -(\sqrt{p^2 + M^2} - \sqrt{p^2 + W^2})^2$ and this cannot take on the fixed value $-m_\pi^2$, because p is fixed but W ranges over all values from $M + m_\pi$ to infinity in the course of the integration.

But if $(p_\alpha - p)^2$ cannot be made to take on its physical value $-m_\pi^2$ it can be made to take on the fairly nearby value of 0. This is because the quantity p is a parameter in our sum rule and may be set to any value we wish. From the above expressions for $(p_\alpha - p)^2$ we see that when $p \to \infty$ this quantity becomes zero. If we now take the lim. $p \to \infty$ in the sum rule, eq.(32), assume that this limiting process can be interchanged with the integration over W and use the relation

$$(E_\alpha - E_p)^2 = \frac{(M_\alpha^{\,2} - M^2)^2}{(E_\alpha + E_p)^2} = \frac{(W^2 - M^2)^2}{(E_\alpha + E_p)^2} ,$$

we find

$$1 = g_A^{\,2} + \left(\frac{c_\pi}{m_\pi^{\,2}}\right)^2 \int dW \, \frac{4MW}{(W^2 - M^2)} \left[K^-(W,0) - K^+(W,0)\right] , \quad (35)$$

and so, using (34)

$$1 = g_A^{\,2} + \frac{2}{\pi}\left(\frac{c_\pi}{m_\pi^{\,2}}\right) \int dW \, \frac{W}{W^2 - M^2} \left[\sigma_o^{\,-}(W) - \sigma_o^{\,+}(W)\right] , \quad (36)$$

where $\sigma_o^{\,\pm}(W)$ is the $\pi^\pm - p$ total cross-section for zero mass pions. Eq.(36) is the Adler-Weisberger sum rule[9a,b].

The constant $\frac{c_\pi}{m_\pi^{\,2}}$ appearing in eq.(36) can be obtained from the π_{l_2} decay rate Γ_{l_2}. Because of the PCAC assumption, eq.(27),

$$\Gamma_{l_2}^{\,\pi} = \left(\frac{c_\pi}{m_\pi^{\,2}}\right)^2 \frac{G_A^{\,o\,2} \, m_l^{\,2} \, m_\pi}{8\pi} \left(1 - \frac{m_l^{\,2}}{m_\pi^{\,2}}\right)^2 , \quad (37)$$

where m_l denotes the mass of the lepton (m_μ or m_e) and $G_A{}^o$ is the bare coupling constant of the strangeness conserving axial vector current. From the value $\Gamma^\pi_{\mu_2} = 3.84 \times 10^7$ sec.$^{-1}$ [19]) we find

$$\left(\frac{c_\pi}{m_\pi{}^2}\right)^2 G_A{}^{o2} = 1.969 \times 10^{-12} M^{-2} \quad , \qquad (38)$$

M being the nucleon mass.

The integral over the total cross sections has been computed by Adler and Weisberger. They find [9 a,b])

$$\int dW \frac{W}{W^2 - M^2} [\sigma_o{}^-(W) - \sigma_o{}^+(W)] = -37.01 \frac{1}{M^2} \quad \text{(Weisberger)} \ , \quad (39a)$$

$$= -50.33 \frac{1}{M^2} \quad \text{(Adler)} \ , \quad (39b)$$

the difference coming from the fact that Adler made certain corrections for the continuation in the pion mass. On substituting eqs.(38) and (39) into eq.(36) we obtain

$$G_A{}^{o2} - (G_A{}^o\, g_A)^2 = -0.729 \times 10^{-10} \frac{1}{M^2} \quad \text{(Weisberger)}, \quad (40a)$$

$$= -0.991 \times 10^{-10} \frac{1}{M^2} \quad \text{(Adler)}. \quad (40b)$$

The quantity $G_A{}^o\, g_A$ appearing here is nothing but the observed (renormalized) axial vector coupling constant for neutron decay.

Experimentally its value is

$$G_A = G_A^{\,o}\, g_A = -1.18 \;\; G_V(n \to p) \;\; ^{20)}, \qquad (41)$$

with $$G_V(n \to p) = 1.0045 \,\frac{10^{-5}}{M^2} \;\; ^{21)} \;. \qquad (42)$$

Eqs.(40),(41) and (42) then yield the result

$$\left| G_A^{\,o} \right| = 0.82 \times \frac{10^{-5}}{M^2} \;\; \text{(Weisberger)}, \qquad (43a)$$

$$= 0.64 \times \frac{10^{-5}}{M^2} \;\; \text{(Adler)} \;. \qquad (43b)$$

In view of the uncertainties involved in calculating the integral over the total cross sections, eq.(39), we may consider these results as supporting the commonly held view that in the bare Lagrangian the vector and axial vector constants for strangeness conserving processes are the same. Alternatively, if we make this hypothesis, i.e. $G_A^{\,o} = G_V^{\,o} = G\,(n \to p)$, - this second equality resulting from the conserved vector current hypothesis - then eq.(40) enables us to predict $|g_A|$, the absolute value of the axial vector renormalization in neutron decay. The answer is

$$|g_A| = 1.31 \qquad \text{(Weisberger)}, \qquad (44a)$$

$$= 1.41 \qquad \text{(Adler)}, \qquad (44b)$$

to be compared with the experimental value 1.18.

[The present treatment differs from that of Adler and Weisberger in that these authors express the quantity c_π/m_π^2 in terms of g_A and the strong pion-nucleon coupling constant $g_{\pi N}$ ($g^L_{\pi N}$ / 4π = 14.6) by means of the Goldberger - Treiman relation [22]

$$\frac{c_\pi}{m_\pi^2} = -\frac{2 g_A M}{g_\pi} . \qquad (45)$$

We have chosen not to do this as we believe that the pion decay rate gives a more direct measure of this constant. If we had used eq.(45) we would have found

$$|g_A| = 1.16 \qquad \text{(Weisberger)}, \qquad (46a)$$

$$= 1.24 \qquad \text{(Adler)} .] \qquad (46b)$$

This, then, is an example of the Fubini-Furlan technique in action. It is seen that at least in the present case current commutation relations, when applied without drastic approximations can yield answers in good agreement with experiment.

The Adler-Weisberger relation: some comments.

While the broad outlines of the foregoing discussion are generally satisfactory, there are at least two points of detail which one might consider in need of further examination. These are the questions (1) of taking the limit $\vec{p} \to \infty$ and (2) of extrapolating to zero pion mass.

(1) The first problem concerning the infinite momentum limit is whether the limit $\vec{p} \to \infty$ can be interchanged with the integration over W in eq.(32). The answer is that it can if the integral is uniformly convergent. Unfortunately we are not in a position to say whether this is in fact the case. The resulting integral is of course convergent if the Pomeranchuk theorem is valid but this alone does not justify the interchange. Adler[9a] has shown, however, that the assumption that this interchange is justified is equivalent to the assumption that a certain pion-nucleon scattering amplitude continued to zero pion mass obeys an unsubtracted dispersion relation and it seems almost certain that this latter assumption is correct. Thus interchanging the limit $\vec{p} \to \infty$ and the integration over W is almost certainly allowed.

With regard to the limit $\vec{p} \to \infty$ itself, we recall that our sole purpose in taking it was so as to be able to express our sum rule eq.(32) in terms of integrals over $\pi^{\pm} - p$ total cross-sections; the $\vec{p} \to \infty$ limit is therefore a strategic device. There have been a number of attempts, notably by Fubini and Furlan[7] and Dashen and Gell-Mann,[23] to raise this $\vec{p} \to \infty$ limit almost to the rank of a principle; at the moment, however, it seems best to regard this limit as a question of strategy rather than of principle.

Another aspect of the infinite momentum limit is that it has been shown[24] that the chiral SU(2) x SU(2) algebra upon which the Adler-Weisberger calculation rests [see eq.(25)], is equivalent to the algebra of the collinear SU(2) x SU(2) group when operating on single particle states of infinite momentum. This fact explains how these two apparently different groups give the same results in the limit of infinite momentum states and it has also been used to interpret the physical values of the axial vector coupling constant and the magnetic moments, as indications that the infinite momentum nucleon states constitute a definite mixture of irreducible representations of collinear SU(2) x SU(2).[25]

(2) As we have seen, the Adler-Weisberger sum rule gives the strangeness conserving axial vector renormalization in terms of integrals of total cross-sections for $\pi^{\pm} - p$ scattering where the mass of the external pion has been continued to zero. How this continuation affects the cross-sections is something we cannot know; the effect is assumed to be small and certain model calculations support this view[9a] but ultimately we have no real proofs one way or the other.

In view of this we might now raise the following question. The above calculation does not assume that the pion mass is zero and the zero pion mass limit arises simply from putting equal to zero the invariant square of the momentum transfer $(p - p_\alpha)^2$ which normally is equal to $-m_\pi^2$. But what if the pion mass were actually zero? Would we then have an exact sum rule? Initially this idea runs into difficulties, for if we set m_π equal to zero everywhere in eq.(36) the second term on the right hand side becomes infinite. However, this difficulty is spurious rather than real because when the pion mass is exactly zero one cannot use the PCAC relation eq.(27), and so the above method of deriving the sum rule is inapplicable. On the other hand it is possible in the zero mass case to postulate instead of eq.(27) the relation

$$\partial_\mu A_\mu^{0\pm}(x) = d_\pi \, j_\pi^{\pm}(x) \quad , \qquad (47)$$

where $j_\pi^{\pm}(x)$ is the pion current. (You can get this relation in a gradient coupling model, for example, though it should be remarked that it seems difficult to get the required current commutation relations in the same model.) With this relation a procedure analogous to the above yields again the sum rule eq.(36) with c_π / m_π^2 replaced by d_π and furthermore the sum rule yields the same value of $|g_A|$ as before, for due to the Goldberger - Treiman relation, eq.(45), d_π is numerically equal to c_π / m_π^2. Thus the answer to our question is that the Adler-Weisberger sum rule would be exact in a world with pions having zero mass, provided that the relations given in eq.(47) were valid.

V. OTHER APPLICATIONS.

Applications of the current algebra approach are now very numerous and only a selection can be referred to here. For those applications which use the Lee-Dashen-Gell-Mann procedure to deduce strengths of currents, magnetic moments and mass relations the reader is referred to references and the papers cited therein. The "goodness" of these results varies widely. Applications which employ the Fubini-Furlan technique seem on the other hand to yield better results. We now discuss some of the latter.

(a) Renormalization of Strangeness Changing Axial Current.

In order to explain the importance of the calculation of the renormalization of the strangeness changing axial current we recall that the weak interaction Lagrangian for semileptonic processes is given by

$$\frac{1}{\sqrt{2}}\left[G_V^{o} V_\mu^{o(+)} + G_A^{o} A_\mu^{o(+)} + G_V^{1} V_\mu^{1(+)} + G_A^{1} A_\mu^{1(+)}\right] l_\mu + \text{h.c.} \qquad (48)$$

Here $V_\mu^{o(+)}$ $(V_\mu^{1(+)})$ is the positively charged strangeness conserving (strangeness changing) vector current and G_V^{o} (G_V^{1})

is its bare coupling constant; $A_\mu^{o(+)}$ ($A_\mu^{(+)}$) is the positively charged strangeness conserving (strangeness changing) axial vector current and G_A^o (G_A^1) is its bare coupling constant and l_μ is the usual weak current of leptons,

$$l_\mu = \bar{e}(x)\,\gamma_\mu(1+\gamma_5)\,\nu_e(x) + \bar{\mu}(x)\,\gamma_\mu(1+\gamma_5)\,\nu_\mu(x) .$$

As a result of the Adler-Weisberger calculation it is now possible to equate G_V^o and G_A^o as explained above. We would next like to make a similar analysis in the case of G_V^1 and G_A^1. With regard to these constants it was first hoped that all G's in (48) could be chosen equal - this was known as the universal Fermi interaction. However, when that theory is applied in a straightforward fashion (neglecting all renormalization effects) the rates for the leptonic decays of strange particles come out about twenty times too large on the average; the effective coupling constants for these decays must therefore be much smaller than G_V^o - of the order of $\frac{1}{5} G_V^o$ or so. The problem is now to decide whence comes this diminution of the effective strangeness changing coupling constants:

(a) Is it due solely to strong interaction renormalization effects with the bare constants chosen equal to those for strangeness conserving processes ? or

(b) Does the diminution arise primarily because the bare strangeness coupling constants are much smaller than their strangeness conserving counterparts with strong interaction renormalization effects playing only a minor role ?

The alternative (b) is that adopted by Cabibbo [27] in his theory of weak interactions. He postulated a weak interaction of the form (48) with

$$G_V^0 = G_A^0 = G \cos\theta \ ; \quad G_V^1 = G_A^1 = G \sin\theta \ ,$$

$$\tan\theta \simeq 0.26 \ ,$$

G being the muon decay constant. This theory gives quite a good account of semileptonic decays [28]. Let us now examine the consistency of alternative (a) using the algebra of currents.

We begin by defining the operators $S_5^{\pm}$ as

$$S_5^{\pm} = \int d^3x \, A_4^{1(\pm)}(x) \ . \qquad (49)$$

$S_5^{\pm}$ are the analogues of $Q_5^{\pm}$ introduced in eq.(25); S_5^+ and S_5^- are assumed to satisfy the commutation relation

$$[S_5^+, S_5^-] = Y + Q \ , \qquad (50)$$

where Y is the hypercharge and Q the electric charge. (We abstract this relation from the quark model.) We now take the matrix element of this relation between single neutron states, notice that the only single particle contribution on the left hand side comes from the Σ^- state, use the PCAC relation

$$\partial_\mu A_\mu^{1(\pm)}(x) = c_K K^{\pm}(x) , \tag{51}$$

where $K^{\pm}(x)$ is the kaon field operator and generally proceed precisely as in the Adler-Weisberger calculation. The result is the sum rule[29]

$$g_A^2(\Sigma^- \to n) = 1 + \frac{2}{\pi}\left(\frac{c_K}{m_K^2}\right)^2 \int dW \frac{W}{W^2 - M^2}\left[\sigma_o^{K^+n}(W) - \sigma_o^{K^-n}(W)\right] , \tag{52}$$

where notation is similar to that used in the Adler-Weisberger sum rule. In principle this relation enables us to compute G_A^1, the bare strangeness changing axial vector coupling constant.

From $K_{\mu 2}$ decay we obtain for $(c_K/m_K^2)^2$ the relation

$$\left(\frac{c_K}{m_K^2}\right)^2 = \frac{\Gamma(K \to \mu + \bar{\nu}_\mu)}{\frac{1}{8\pi}(G_A^1)^2 m_K m_\mu^2 \left(1 - \frac{m_\mu^2}{m_K^2}\right)^2} ,$$

where $\Gamma(K \to \mu + \bar{\nu}_\mu)$ is the $K_{\mu 2}$ decay rate. On inserting for this quantity the experimental value[33] $\Gamma_{expt}(K \to \mu + \bar{\nu}_\mu) = 5.11 \times 10^7$ sec.$^{-1}$, we find

$$\frac{2}{\pi}\left(\frac{c_K}{m_K^2}\right)^2 (G_A^1)^2 = \frac{0.945}{M^2} \times 10^{-12} , \quad M = \text{proton mass},$$

whereupon the sum rule in eq.(52) multiplied by $(G_A{}^1)^2$ becomes

$$[G_A{}^1\ g_A(\Sigma^- \to n)]^2 = (G_A{}^1)^2 + \frac{0.945 \times 10^{-12}}{M^2} \int \frac{W\,dW}{W^2 - M^2} [\sigma_o^{K^+n}(W) - \sigma_o^{K^-n}(W)] \, . \qquad (53)$$

The evaluation of the integral in this relation is beset with difficulties. In the first place the cross-sections, since they involve neutrons, are not very well known. Secondly, the continuation of the mass of the external kaon to zero is likely to have more serious consequences than did the similar continuation of the pion mass in the previous calculation. Lastly, the sum rule this time receives contributions from the $I = 1$, $Y = 0$, $\Sigma\pi$ and $\Lambda\pi$ states below threshold and these contributions have to be included in some way, (by putting in the $Y_1^*(1405)$ resonance for example). In view of these uncertainties we are unlikely to obtain very accurate information from the present relation.

Estimates of this integral are available in the work of a number of authors. These are

$$\int \frac{W\,dW}{W^2 - M^2} \left[\sigma_o^{K^+n}(W) - \sigma_o^{K^-n}(W) \right] = -23 \text{ mb} \quad {}^{29)}$$

$$= -22.5 \text{ mb} \quad {}^{32)}$$

$$= -34 \text{ mb} \quad {}^{30)} \, .$$

On using these values in eq.(53) together with the relation

$$1 \text{ mb} = \frac{2.28}{M^2}, \quad (M = \text{proton mass})$$

we find for $G_A^{\,1\,2}$ the equations

$$(G_A^{\,1})^2 = [G_A^{\,2}\; g_A(\Sigma^- \to n)]^2 + \frac{10^{-10}}{M^4}\left\{\begin{matrix} 0.49 \\ 0.48 \\ 0.73 \end{matrix}\right\}. \quad (54)$$

Now the decay rate for $\Sigma^- \to n + e^- + \bar{\nu}_e$ as computed from the weak interaction given in eq.(48) is

$$\Gamma(\Sigma \to n + e^- + \bar{\nu}_e) = 3.27 \times 10^{18}\, M^4\{1.03\, [G_V^{\,1}\; g_V\, (\Sigma^- \to n)]^2 + 3.04\, [G_A^{\,1}\; g_A\, (\Sigma^- \to n)]^2\} \text{ sec}^{-1} \quad (55)$$

where $g_V(\Sigma^- \to n)$ is the renormalization factor for the vector current and the various numbers appearing are the phase space factors. The experimental decay rate is [34]

$$\Gamma_{expt}\, (\Sigma^- \to n + e^- + \bar{\nu}_e) = 8.075 \times 10^6 \text{ sec}^{-1},$$

from which we deduce the inequality

$$[G_A^{\,2}\; g_A(\Sigma^- \to n)]^2\, M^4 \leq 0.81 \times 10^{-12}.$$

Consequently we obtain for $G_A^{\,1}$, the bare coupling constant for the strangeness changing axial current, the upper limit

$$(G_A^{\,2})^2\, M^4 \leq \left[\left\{\begin{matrix} 0.49 \\ 0.48 \\ 0.73 \end{matrix}\right\} + 0.0081\right] \times 10^{-10}. \quad (56)$$

The outcome of this calculation is then the following: since the bare strangeness conserving vector coupling constant $G_V{}^o$ has a value of about $1 \times 10^{-5}\ M^{-2}$, it is scarcely possible to maintain the equality of all bare coupling constants in the weak interaction Lagrangian eq.(48). This conclusion is predicated on the assumption that all the ingredients of the above calculation (algebra of currents and PCAC) are correct and also that the uncertainties in the evaluation of the sum rule are not large. This is, of course, a lot to assume. However, one can say that the indications are that the bare coupling constants for strangeness changing semileptonic decays are smaller than their strangeness conserving counterparts, but the amount of the diminution (i.e. the magnitude of the bare Cabibbo angle) cannot be stated with any degree of confidence.

(b) Pion-Pion and Pion-Kaon Scattering.

The commutation relations given in eqs.(25) and (50) if valid should yield information about $\pi - \pi$ and $K - \pi$ scattering. If, for example, one takes the relation (25) between π^+ states and uses the Fubini-Furlan-Adler-Weisberger (FFAW) technique, one finds the sum rule

$$2 = \frac{2}{\pi}\left(\frac{c_\pi}{m_\pi{}^2}\right)^2 \int \frac{w\,dw}{w^2 - M_\pi{}^2}\,[\sigma_o{}^{\pi^-\pi^+}(w) - \sigma_o{}^{\pi^+\pi^+}(w)] \quad (57)$$

while the same relation taken between K^+ states leads to the sum rule

$$1 = \frac{2}{\pi}\left(\frac{c_\pi}{m_K^2}\right)^2 \int \frac{w\,dw}{w^2 - M_K^2}\left[\sigma_o^{\pi^- K^+}(w) - \sigma_o^{\pi^+ K^+}(w)\right] . \quad (58)$$

Now although there is no direct pion-pion or kaon-pion scattering data available there is sufficient information on these interactions to make some comparison of the sum rules (57) and (58) with experiment.

The first thing to remark is that the contribution from the second term in the integral is likely to be very small in both sum rules; these terms involve $\pi - \pi$ scattering in the $I = 2$ state and $K - \pi$ scattering in the $I = 3/2$ state and cross-sections in both these channels seem quite small in the low energy region. The signs of the right hand sides are thus likely to be positive in agreement with the sum rule.

If we now evaluate the right hand sides of these sum rules by putting in the low energy resonant contributions - the $J^P = 1^- \rho^o$ (750) and $J^P = 2^+ f^o$ (1254) in (57) and the $J^P = 1^-$ K* (891) and $J^P = 2^+$ K** (1415) in (58) - we find that we are far from satisfying the sum rules. If, as is likely, the higher partial wave contributions are not very large this means that there must be sizable s-wave cross sections in both cases and/or large non-resonant contributions in the 1^- and 2^+ partial waves.

(c) Baryon Magnetic Moments.

A number of different methods of deducing magnetic moment sum rules have been proposed in connection with the algebra of currents. The first was due to Lee[4)] who defined the magnetic moment operator $\mathcal{M}_i^{\alpha}$ as

$$\mathcal{M}_i^{\alpha} = \tfrac{1}{2}\, \varepsilon_{ijk} \int x_j\, V_k^{\alpha}(x)\, d^3x \qquad (59)$$

where $V_k^{\alpha}(x)$ is the octet vector current. By taking the commutator of this operator with $A_1^{\alpha} = \int d^3x\, A_1^{\alpha}(x)$ where $A_1^{\alpha}(x)$ is the octet axial vector current and using the Lee-Dashen-Gell-Mann procedure one can deduce the SU(6) result $\mu_p/\mu_n = -3/2$ [35)] where μ_p and μ_n are the magnetic moments of the proton and the neutron respectively. In addition, by commuting two members of the set $\mathcal{M}_i^{\alpha}$, Lee obtained the relation $\mu_p^{\ 2} = \frac{1}{6} < r_p^{\ 2} >$ where $< r_p^{\ 2} >^{\frac{1}{2}}$ is the root mean square charge radius of the proton.

Both these results suffer from the difficulty that in view of the way the approximation used forces all states to have orbital angular momentum $L = 0$ (recall section IVa), the only consistent value for μ_p or μ_n is zero.[36)] This difficulty can be remedied either by putting the nucleon in a representation having non-zero angular momentum or by giving the quarks anomalous magnetic moments, but neither of these devices seems particularly attractive.

Subsequently a derivation of a magnetic moment sum rule free from these objections was given by Cabibbo and Radicati.[24)] Starting from the definition

$$D_t^{\,a} = \int d^3x\, J_o^{\,a}(x)\, x_i \tag{60}$$

where $J_o^{\,a}(x)$ is the time component of the isovector current, and the commutation relation

$$[D_1^{\,+}, D_1^{\,-}] = 2\int J_o^{\,3}(x)\, x_1^{\,2}\, d^3x \,, \tag{61}$$

these authors deduced, by means of the Fubini-Furlan-Adler-Weisberger technique, the sum rule

$$(\mu_p - \mu_n)^2 + \frac{1}{2\pi\alpha^2}\int \frac{dw}{w}\,[2\sigma_1^{\,V} - \sigma_3^{\,V}] = 1/3 < r^2 >, \tag{62}$$

where $\sigma_{1(3)}^{\,V}$ is the total cross section for photo-production on a proton of $I = \frac{1}{2}$ $(I = 3/2)$ states and $<r^2>^{\frac{1}{2}}$ is the root mean square isovector radius of the nucleon. An approximate evaluation of the integral, neglecting the $\sigma_1^{\,V}$ contribution and assuming that $\sigma_3^{\,V}$ is dominated by the (3,3) resonance, gives the result

$$|\mu_p - \mu_n| = 5.5 \text{ nuclear magnetons} ,$$

to be compared with the experimental value of 4.7.

A third approach is that due to Mathur and Pandit.[37] These authors noticed that on the basis of the quark model and the algebra of currents one has the relation

$$0 = \int d^4x\, \theta(x_o)\, [M^o(x), J^{el}_\mu(o)] \qquad (63)$$

where $J^{el}_\mu(o)$ is the electromagnetic current vector and $M^o(x)$ is the field operator of any of the neutral pseudo-scalar mesons $(\pi^o, K^o, \overline{K}^o, \eta)$. On the other hand one can define a causal amplitude

$$T(q, p_1, p_2) =$$

$$\sqrt{\frac{E_1}{M_1}\cdot\frac{E_2}{M_2}} \int d^4x\, e^{-iqx} \langle B_2(p_2)|[M^o(x), J_\mu(o)\varepsilon_\mu]\, \theta(x_o)|\, B_1(p_1)\rangle \qquad (64)$$

which is the invariant amplitude or reduced T-matrix element for the reaction

$$\gamma(k) + B_1(p_1) \rightarrow B_2(p_2) + M^o(q) ,$$

the photoproduction of a single neutral pseudoscalar meson on a baryon. (B_1 and B_2 represent baryon states.) Now eq.(63) shows that

$$\lim_{q\to 0} T(q, p_1, p_2) = 0 \qquad (65)$$

and on expanding $T(q, p_1, p_2)$ in CGLN invariant amplitudes[38) it is seen that eq.(65) represents a condition on only one of these. If one then assumes an unsubtracted relation for this amplitude and in the calculation of the imaginary part only the contributions of the baryons and of the first baryon resonances are retained, eq.(65) implies a relation between baryon magnetic moments. The relations so obtained appear to be in good agreement with experiment.

These results on baryon magnetic moments further encourage us to think that the algebra of currents represents a correct approach to the understanding of the properties of strongly interacting particles.

VI. COVARIANT FORMULATION AND CONNECTION WITH SOFT PION EMISSION.

As a final topic we deal with a very important step in current algebra theory, namely, the development of a method of deducing the consequences of current (or local operator) commutation relations in a formally covariant way. This method, which seems to have been discovered simultaneously by a number of authors,[39) is very closely related to that used by Nambu and his collaborators[40) in connection with soft pion emission. Basically what done is to use PCAC and current (or local operator) commutation relations to link up matrix elements for processes which differ only in the emission or absorption of one or more zero mass pions. Some of the results here are quite promising but there are also some difficulties which arise from the most common formulation of the method. For these reasons we shall now discuss this question in some detail.

The common formulation begins from the identity

$$0 = i\int d^4x\, \partial_\mu \left\{ e^{-iqx}\, \theta(x_o) < B(p')\left| \left[A^k_\mu(x),\, O(o) \right] \right| A(p)> \right\} , \tag{66}$$

where $A^k_\mu(x)$ is the k^{th} isospin component of the axial vector current, $O(o)$ is any local operator evaluated at $x_\mu = 0$, $\theta(x_o)$ is the usual step function in the time and $A(p)$ and $B(p')$ are arbitrary states having four momenta p and p' respectively, $A(p)$ being assumed not to contain a pion of denomination k. In order that this integral converge at $t = +\infty$ we impart to q_o a small positive imaginary part.

On performing the differentiation in the integrand here we obtain

$$\begin{aligned} 0 = &-i\, q_\mu M_\mu + \int d^4x\, e^{-iqx}\, \delta(x_o) < B(p')\left| \left[A^k_4(x),\, O(o) \right] \right| A(p)> \\ &+ i\int d^4x\, e^{-iqx}\, \theta(x_o) < B(p')\left| \left[\partial_\mu A^k_\mu(x),\, O(o) \right] \right| A(p)> , \end{aligned} \tag{67}$$

where

$$M_\mu = i\int d^4x\, e^{-iqx}\, \theta(x_o) <B(p)| \left[A^k_\mu(x), O(o) \right] |A(p)> . \tag{68}$$

Next we employ the PCAC relation

$$\partial_\mu A^k_\mu(x) = c_\pi \pi^k(x) \, , \tag{69}$$

where $\pi^k(x)$ is the renormalised pion field and then the last term in eq.(67) can be written as

$$\frac{-i c_\pi}{q^2 + m_\pi^2} \int d^4x\, e^{-iqx} (\Box - m_\pi^2)\, \theta(x_o) < B(p')| \left[\pi^k(x), O(o) \right] |A(p) > \tag{70}$$

and this is equal to

$$\frac{c_\pi}{q^2 + m_\pi^2} \sqrt{2q_o}\, (2\pi)^{3/2} < B(p'), \pi^k(q)|\, O(o)|\, A(p)> \tag{71}$$

i.e. $\frac{c_\pi}{q^2 + m_\pi^2}\sqrt{2q_o}\,(2\pi)^{3/2}$ times the matrix element of $O(o)$ between the state $A(p)$ and the state $B(p') + \pi^k(q)$. Hence rearranging eq.(67) we have

$$\frac{c_\pi \sqrt{2q_o}(2\pi)^{3/2}}{q^2 + m_\pi^2} <B(p')\, \pi^k(q)|\, O(o)|\, A(p)>$$

$$= -\int d^4x\, e^{-iqx}\, \delta(x_o) < B(p')| \left[A^k_4(x), O(o) \right] |A(p)> + i\, q_\mu M_\mu \, . \tag{72}$$

The trick now is to set $q_\mu = 0$ in this relation. When this is done the first term on the right hand side becomes

$$- \langle B(p')| \left[I_5^k(o), O(o) \right] | A(p) \rangle$$

where $I_5^k(t)$ is the k^{th} component of the axial isospin defined by

$$I_5^k(t) = \int d^3x \, A_4^k(x, t) \; .$$

The commutator here $\left[I_5^k(o), O(o) \right]$, will in all cases of interest be equal to some other local operator - call it $O_5^k(o)$ - as a result of some assumptions about the operator $O(o)$ (such as: that it is a certain function of quark field operators). This term is therefore equal to

$$- \langle B(p')| O_5^k(o) | A(p) \rangle \; .$$

With regard to the second term in (72), namely $i \, q_\mu M_\mu$, we see that this vanishes as $q_\mu \to 0$ except when M_μ has a pole at that value. This situation occurs when, on the insertion of a complete set of states between the two operators in the commutator in the expression for M_μ, eq.(68), there is a contribution from a state degenerate in mass with either of the states $A(p)$ or $B(p')$. The contribution here will also be generally calculable in terms of other known parameters.

Thus we arrive at a relation of the form

$$\frac{c_\pi (2\pi)^{3/2}}{m_\pi^2} \lim_{q_\mu \to 0} \sqrt{2q_o} \langle B(p') \pi^k(q)| O(o) | A(p) \rangle$$

$$= - \langle B(p')| O_5^k(o) | A(p) \rangle + \lim_{q_\mu \to 0} i \, q_\mu M_\mu \, , \qquad (73)$$

where the first term on the right hand side represents the current (or local operator) commutation relation contribution.

As an example we consider the case when $O(o) = V^1_\mu(o)$, the $I_3 = -\frac{1}{2}$ component of the strangeness changing vector current of weak interactions, $A(p) = K^+(p)$ a single K^+ state of momentum p and $B(p') = 0$, the vacuum state. In this instance there is no contribution from the term $\lim_{q_\mu \to 0} i\, q_\mu M_\mu$ and the operator $O^k_5(o)$ is $\frac{1}{\sqrt{2}} A^1_\mu(o)$, the $I_3 = -\frac{1}{2}$ component of the strangeness changing axial current. We therefore get the relation

$$\frac{c_{\pi^o}}{m^2_{\pi^o}}(2\pi)^{3/2} \lim_{q_\mu \to 0} \sqrt{2q_o} <\pi^o(q)|V^1_\mu(o)|K^+(p)> = \tfrac{1}{2} <o|A^1_\mu(o)|K^+(p)> . \tag{74}$$

Then with the definitions

$$<\pi^o(q)|V^1_\mu(o)|K^+(p)>$$
$$= -\frac{1}{\sqrt{2}}\frac{1}{(2\pi)^3}\frac{-i}{\sqrt{4p_o q_o}}\Big\{ f_+[-p^2, -q^2, -(p-q)^2]\,(p+q)_\mu + f_-[-p^2, -q^2, -(p-q)^2]\,(p-q)_\mu \Big\} \tag{75}$$

and

$$<o|A^1_\mu(o)|K^+(p)> = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2p_o}}\, i\, p_\mu f_K \tag{76}$$

and the relations

$$\frac{c_{\pi^o}}{m^2_{\pi^o}} = \frac{1}{\sqrt{2}} \frac{c_{\pi^\pm}}{m^2_\pi} = \frac{1}{\sqrt{2}} f_\pi \quad \text{and} \quad f_K = \frac{c_K}{m_K^2} \quad \text{[see eq.(51)]}$$

we find

$$f_+(m_k^2, o, m_k^2) + f_-(m_k^2, o, m_k^2) = \frac{f_\pi}{f_K} \quad . \quad (77)$$

This is the Callan-Treiman relation[41] linking the sum of the form factors f_+ and f_- for K_{l_3} decay extrapolated to zero pion mass with the ratio of the K_{l_2} and π_{l_2} decay constants.

A precision check of this relation requires a knowledge of (1) the functional dependence of $f_+ + f_-$ on the pion mass and the momentum transfer squared and (2) the value of the ratio f_π/f_K - this depends on the ratio $G_A^{\,o}/G_A^{\,1}$ of the bare coupling constants of strangeness conserving and strangeness changing axial vector currents and therefore on the model of weak interactions. At present such detailed information is not available but as a rough check let us assume that the form factors f_+ and f_- are constant, that $f_+ = 1$ in accordance with the Ademollo-Gatto theorem[42] and that $f_K/f_\pi = 1.28$, which is the value given by Cabibbo's theory[43]. In that event we obtain for $\xi = f_-/f_+$ the value 0.28 which is very different from the presently indicated one of about -1[43]. This result should not be taken as evidence against the relation (77) but, as is more likely, that the assumption of constant form factors is incorrect.

The above is a simple application of formula (73). A more complicated one arises in the case of weak non-leptonic decays. In that event $O(o)$ is $H^{NL}(o)$, the non-leptonic weak interaction Hamiltonian, and $A(p)$ and $B(p')$ are single baryon states differing in strangeness by one unit. This problem was first studied by Sugawara[44] and Suzuki[45] and later by other authors and a number of sum rules for the various amplitudes were obtained. We shall not reproduce these relations but shall rather focus our attention on the term $\lim_{q_\mu \to 0} i\, q_\mu M_\mu$, which here gives rise to one of the difficulties alluded to at the beginning of this section.

We recall that this term is non-vanishing only if, on inserting a complete set of states in the expression for M_μ, eq.(68), there is a contribution from a state degenerate in mass with either $A(p)$ or $B(p')$. In the case of $\pi^\pm$ emission there are no such contributions because there are no two baryons degenerate in mass and strangeness but differing by the unit of charge (here we do not neglect mass differences between the members of an isomultiplet); on the other hand for π^o emission there are non-vanishing contributions because now the states $A(p)$ and $B(p')$ themselves contribute to the sum over intermediate states. Thus we have the difficulty that the term $\lim_{q_\mu \to 0} i\, q_\mu M_\mu$ gives no contribution to charged pion decay but does contribute to neutral pion decay.

One resolution of this difficulty is simply to neglect mass differences within multiplets. With this assumption the term $i\, q_\mu M_\mu$ has a well defined $q_\mu \to 0$ limit for both charged and neutral pion decay. One then gets sum rules for the p-wave amplitudes which are experimentally indistinguishable from the $\Delta I = \frac{1}{2}$ predictions, without postulating that the Hamiltonian is a pure $I = \frac{1}{2}$ operator (it can have an $I = 3/2$ part also).

Another way out comes from an examination of the Born term in the amplitude. In this term one finds, in the $q_\mu \to 0$ limit, an asymmetry between charged and neutral pion decay analogous to that in the term $i\, q_\mu M_\mu$; in fact this time the neutral decay term vanishes while the charged one does not. So if we denote the term on the left hand side of eq.(73) by $R(q)$, the Born contribution to this quantity by $R_{Born}(q)$, and the difference $R(q) - R_{Born}(q)$ by $\tilde{R}(q)$ we can then write eq.(73) as

$$\tilde{R}(o) = \lim_{q_\mu \to 0} [R(q) - R_{Born}(q)]$$

$$= - \langle B(p')| [Q_5^k(o), H_{pc}^{NL}(o)]| A(p)\rangle + \lim_{q_\mu \to 0} [i\, q_\mu M_\mu - R_{Born}(q)]. \tag{78}$$

In this form we have a quantity, $\tilde{R}(q)$, which tends to a well defined value in the $q_\mu \to 0$ limit and if we assume that $\tilde{R}(q)$ does not vary very much with q, so that we can replace $\tilde{R}(q)$ everywhere by $\tilde{R}(o)$, then we can write for the whole amplitude

$$R(q) = R_{Born}(q) + \tilde{R}(o) . \qquad (79)$$

From this formula we can obtain the same $\Delta I = \frac{1}{2}$ - like sum rules as above. A full discussion of this matter will be given elsewhere.

A second and more serious difficulty with the above procedure is the following. We recall that we have used in eqs.(70) and (71) the reduction formula

$$\sqrt{2q_o}\,(2\pi)^{3/2} < B(p'), \pi^k(q) | O(o) | A(p) >$$
$$= i \int d^4x\, e^{-iqx} (\Box - m_\pi^2) < B(p') | \theta(x_o) [\pi^k(x), O(o)] | A(p) >$$
$$= i(q^2 + m_\pi^2) \int d^4x\, e^{-iqx} < B(p') | \theta(x_o) [\pi^k(x), O(o)] | A(p) > . \qquad (80)$$

Now it is to be emphasised that this formula is derived precisely on the assumption that the pion is on its mass shell and consequently the term on the right hand side is to be understood in the sense of the limit of this quantity as $q^2 \to -m_\pi^2$. Thus if we write

$$\int d^4x\, e^{-iqx} < B(p') | \theta(x_o) [\pi^k(x), O(o)] | A(p) >$$
$$= \frac{A(q^2)}{q^2 + m_\pi^2} + B(q^2) \qquad (81)$$

for arbitrary values of q^2, where $B(q^2)$ has no pole at $q^2 = -m_\pi^2$, eq.(80) is simply

$$\sqrt{2q_o}\,(2\pi)^{3/2} < B(p'), \pi^k(q) | O(o) | A(p) > = i A(-m_\pi^2) \qquad (82)$$

The matrix element for the pion at its physical mass is then $i A(-m_\pi^2)$.

Consider now what happens when we set $q_\mu = 0$ in eq.(80). From eqs.(80) and (81) we find

$$\lim_{q_\mu \to 0} \sqrt{2q_o}(2\pi)^{3/2} \langle B(p'), \pi^k(q) | O(o) | A(p) \rangle = i\,A(o) + i\,m_\pi^2 B(o). \qquad (83)$$

Comparing this with eq.(82) we see that what we have got is not simply the physical matrix element with the pion mass continued to zero but we have an additional term besides. Thus the limit $q_\mu \to 0$ is not equivalent to continuing the external pion mass to zero. In order to identify the limit $q_\mu \to 0$ with the zero pion mass limit it is necessary to assume that $B(o)$ is zero or small and it is not obvious that this is in general the case. This difficulty could be quite a serious one for the present formulation.

Conclusion.

This concludes our presentation of current algebra theory and its applications. The present treatment is far from exhaustive[48] but it does trace the development of the subject along a fairly central path through the subject. It is seen that the ideas contained in this development have proved fairly fruitful and have led to some real progress in understanding elementary particle interactions. The situation is still not completely satisfactory, however, and in particular methods of computing the effects of mass continuation are badly needed. Also the need to take the $q_\mu \to 0$ limit ought to be overcome in some way. One topic of those omitted deserves special mention, namely that of Okubo and his collaborators[49] on models and distinguishing between them by means of current algebra techniques. While the first heady enthusiasm for the subject has now waned somewhat, it is felt that current algebra ideas will continue to prove useful in elementary particle physics for a long time to come.

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47. The phase convention was not correctly applied in this paper.

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ERRATA

P.40, line 8: The references in question are 4, 5, and 48.

p. 43, last line: The factor 10^{-12} should read 10^{-13}

p.44, line 3: do.

p.45, eq. (54): The factor 10^{-10} should read 10^{-11} .

Eq. (56) should read:

$$(G_A^1)^2 M^4 \leq \left[\left\{ \begin{matrix} 0.49 \\ 0.48 \\ 0.73 \end{matrix} \right\} + 0.081 \right] \times 10^{-11}. \qquad (56)$$

p.58, Eq. (78): Delete subscript PC from $H_{PC}^{NL}(6)$, to read $H^{NL}(6)$.

p.59, line 3: after "above" insert the phrase "if the Born terms can be neglected".

p.66. Insert reference 49: S. Okubo, Suppl. Prog. Theor. Phys. 37-38, 114 (1966).