Large Deviations and the Random Energy Model

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Abstract. We present a simple proof of the formula for the free energy of the random energy model using a large deviation property which holds almost surely with respect to the randomness. This proof is extended to the case with external magnetic field leading to the solution of a model with higher-order ferromagnetic term. It is shown that this model is useful for Sourlas' application to error-correcting codes as was already pointed out in a recent letter by the authors.

1. Large Deviations for the Random Energy Model.

The random energy model was introduced by Derrida [1] as a soluble approximation of the Sherrington-Kirkpatrick model of a spin glass. In his paper he presented an exact solution based on heuristic arguments. Later rigorous derivations of his expression for the free energy of this model have appeared [2,3]. Here we present another rigorous proof along the lines of Derrida's original argument. It is based on a large deviation theorem (for introductions to large deviation theory, see for example [4], [5] or [6]) and seems to be new. In the next section we shall extend our proof to the case with an external magnetic field as introduced in [7]. The results of the present section have appeared already in the second author's thesis [8].

The random energy model (REM) is a model of a spin glass given by the partition function

$$\mathcal{Z}_N(\beta) = \sum_{i=1}^{2^N} e^{-\beta E_i}, \tag{1.1}$$

where β is the inverse temperature and where the energies E_i $(i = 1, 2, \ldots, 2^N)$ are independent, identically distributed random variables with common distribution given by the Gaussian density

$$p_N(E) = \frac{1}{\sqrt{2\pi J^2 N}} e^{-E^2/(2NJ^2)}. \tag{1.2}$$

As usual, the free energy density is defined by the formula

$$f(\beta) = -\frac{1}{\beta} \lim_{N\to\infty} \ln \mathcal{Z}_N(\beta), \tag{1.3}$$

and we shall prove that the limit exists almost surely with respect to the distribution of energies $\{E_i\}$. We define, for any given (random) set $\{E_i\}$, the distribution functions F_N by

$$F_N(x) = \frac{1}{2^N} \#\{i : \, E_i \leq xN\}. \tag{1.4}$$

Then we can write

$$\mathcal{Z}_N(\beta) = 2^N \int_{-\infty}^{\infty} e^{-N\beta x} dF_N(x). \tag{1.5}$$

We now show that these distribution functions satisfy a large deviation property (LDP):

Theorem 1. *With probability 1, the random probability measures μ_N with distribution functions F_N defined by (1.4) satisfy a LDP with constants N and rate function I given by*

$$I(x) = \begin{cases} \frac{x^2}{2J^2} & \text{if } |x| \leq x_c, \\ +\infty & \text{if } |x| > x_c, \end{cases} \tag{1.6}$$

where

$$x_c = J\sqrt{2 \ln 2}. \tag{1.7}$$

Proof. Consider first the case $x > x_c$. (The case $x < -x_c$ is similar by symmetry.) It then suffices to prove that

$$\limsup_{N\to\infty} \frac{1}{N} \ln(1 - F_N(x)) = -\infty. \tag{1.8}$$

In fact, we show that $F_N(x) = 1$ with probability 1 if N is large enough. Now,

$$1 - F_N(x) = \frac{1}{2^N} \sum_{i=1}^{2^N} \mathbf{1}_{\{E_i > xN\}}. \tag{1.9}$$

Therefore,

$$\{\{E_i\} : F_N(x) = 1\} = \left\{ \{E_i\} : \sum_{i=1}^{2^N} \mathbf{1}_{\{E_i > xN\}} < 1 \right\}. \tag{1.10}$$

But, by Chebyshev's inequality,

$$\begin{aligned} \mathbf{P}\left\{ \sum_{i=1}^{2^N} \mathbf{1}_{\{E_i > xN\}} \geq 1 \right\} &\leq \mathbf{E}\left\{ \sum_{i=1}^{2^N} \mathbf{1}_{\{E_i > xN\}} \right\} \\ &= 2^N \mathbf{P}\left\{ E_i > xN \right\} \\ &= 2^N \int_{xN}^{\infty} p_N(E)\, dE \\ &\leq 2^N \frac{J}{x\sqrt{2\pi N}} e^{-Nx^2/(2J^2)}. \end{aligned} \tag{1.11}$$

Since $x > x_c$, $x^2/(2J^2) > \ln 2$ and the series

$$\sum_{N=1}^{\infty} \frac{J}{x\sqrt{2\pi N}} e^{N(\ln 2 - x^2/(2J^2))}$$

converges. Introducing the events

$$\mathcal{A}_N = \left\{ \{E_i\} : \sum_{i=1}^{2^N} \mathbf{1}_{\{E_i > xN\}} \geq 1 \right\}, \tag{1.12}$$

we have $\sum_{N=1}^{\infty} \mathbf{P}(\mathcal{A}_N) < +\infty$ and by the Borel-Cantelli lemma,

$$\mathbf{P}\left[\bigcap_{k=1}^{\infty} \bigcup_{N=k}^{\infty} \mathcal{A}_N \right] = 0, \tag{1.13}$$

i.e. with probability 1,

$$\{E_i\} \in \left(\bigcap_{k=1}^{\infty} \bigcup_{N=k}^{\infty} \mathcal{A}_N \right)^c = \bigcup_{k=1}^{\infty} \bigcap_{N=k}^{\infty} \mathcal{A}_N^c. \tag{1.14}$$

This means that for almost all $\{E_i\}$, there exists $k \in \mathbf{N}$ such that for $N \geq k$, $\{E_i\} \in \mathcal{A}_N^c$ and hence by (1.10), $F_N(x) = 1$.

Next we consider the case $0 < x < x_c$. We shall prove that

$$\lim_{N\to\infty} \frac{1}{N} \ln(1 - F_N(x)) = -\frac{x^2}{2J^2}. \tag{1.15}$$

Let G_N be the scaled distribution function corresponding to the density p_N:

$$G_N(x) = \int_{-\infty}^{xN} p_N(E)\, dE. \tag{1.16}$$

Then we have, again by Chebyshev, for $\epsilon \in (0,1)$,

$$\mathbf{P}\left\{|G_N(x) - F_N(x)| \geq \epsilon(1 - G_N(x))\right\} \leq \frac{1}{\epsilon^2(1 - G_N(x))^2}\mathbf{E}\left\{|G_N(x) - F_N(x)|^2\right\}. \tag{1.17}$$

Now,

$$\begin{aligned}
\mathbf{E}\left\{|G_N(x) - F_N(x)|^2\right\} &= \mathbf{E}[F_N(x)^2] - 2\mathbf{E}[F_N(x)]G_N(x) + G_N(x)^2 \\
&= 2^{-2N} \sum_{i,j=1}^{2^N} \mathbf{E}\left\{\mathbf{1}_{\{E_i \leq xN\}} \cdot \mathbf{1}_{\{E_j \leq xN\}}\right\} - G_N(x)^2 \\
&= 2^{-2N} \sum_{i \neq j} G_N(x)^2 + 2^{-2N} \sum_{i=1}^{2^N} G_N(x) - G_N(x)^2 \\
&= 2^{-N} G_N(x)(1 - G_N(x)).
\end{aligned} \tag{1.18}$$

It follows that

$$\begin{aligned}
\mathbf{P}\left\{|G_N(x) - F_N(x)| \geq \epsilon(1 - G_N(x))\right\} &\leq \frac{G_N(x)}{\epsilon^2 2^N (1 - G_N(x))} \\
&\leq \frac{1}{\epsilon^2 2^N} \frac{\sqrt{2\pi}(J^2 + Nx^2)}{xJ\sqrt{N}} e^{Nx^2/(2J^2)},
\end{aligned} \tag{1.19}$$

where we have used the inequality (see [9])

$$\int_a^\infty e^{-u^2/2} du > \frac{1}{a + a^{-1}} e^{-a^2/2}.$$

If $x = 0$ we have $G_N(x) = \frac{1}{2}$ so that

$$\mathbf{P}\left\{|G_N(0) - F_N(0)| \geq \frac{1}{2}\epsilon\right\} \leq \frac{1}{\epsilon^2 2^N}. \tag{1.20}$$

In both cases the series with terms given by the right-hand side converges. As above we then have, by the Borel-Cantelli lemma, that with probability 1, for N large enough,

$$|G_N(x) - F_N(x)| < \epsilon(1 - G_N(x)), \tag{1.21}$$

that is, $(1-\epsilon)(1-G_N(x)) < 1 - F_N(x) < (1+\epsilon)(1-G_N(x))$, from which (1.15) follows.

Finally, we consider the case $x = x_c$. In that case we prove only that

$$\limsup_{N\to\infty} \frac{1}{N} \ln(1 - F_N(x_c)) \le -\ln 2 = -x_c^2/2J^2. \tag{1.22}$$

Replacing ϵ by N in the above argument we can write

$$\begin{aligned}
\mathbf{P}\left[1 - F_N(x_c) > N2^{-N}\right] &\le \frac{2^{2N}}{N^2}\mathbf{E}[(1 - F_N(x_c))^2] \\
&= \frac{1}{N^2}\sum_{i,j=1}^{2^N} \mathbf{E}\left[\mathbf{1}_{\{E_i > x_c N\}}\mathbf{1}_{\{E_j > x_c N\}}\right] \\
&= \frac{2^N}{N^2}(1 - G_N(x_c))\left(1 + (2^N - 1)(1 - G_N(x_c))\right) \\
&\le \frac{1}{N^2\sqrt{N}}.
\end{aligned} \tag{1.23}$$

Here we used the fact that

$$1 - G_N(x_c) \le \frac{1}{2\sqrt{\pi N \ln 2}} 2^{-N}$$

as in (1.11). Again, the series on the right-hand side converges and we have, with probability 1, that $1 - F_N(x_c) \le N2^{-N}$ for N large enough, which implies (1.22).

The derivation of the large deviation upper bound for general closed sets, and the lower bound for open sets is now a standard exercise: see for example [4,10]. Q.E.D.

Since the function $x \mapsto -\beta x$ is continuous and bounded on $[-x_c, x_c]$ an application of Varadhan's theorem (see [4,5,6]) yields immediately that

$$\begin{aligned}
-\beta f(\beta) &= \ln 2 + \sup_{x\in[-x_c,x_c]}\left\{-\beta x - \frac{x^2}{2J^2}\right\} \\
&= \begin{cases} \ln 2 + \frac{1}{2}\beta^2 J^2 & \text{if } \beta J^2 \le x_c, \\ \ln 2 + \beta x_c - \frac{x_c^2}{2J^2} = \beta J\sqrt{2\ln 2} & \text{if } \beta J^2 > x_c. \end{cases}
\end{aligned} \tag{1.24}$$

This is just Derrida's expression for the free energy. (Notice that we have adopted a different convention for the distribution p_N.) We finally remark that (1.24) can be written in the usual form $f(\beta) = \inf_{-x_c \le u \le x_c}\{u - Ts(u)\}$, where $T = 1/\beta$ ($k_B = 1$) is the temperature, $u = x$ is the energy (density) and the entropy $s(u)$ is given by

$$s(u) = \ln 2 - I(u) = \ln 2 - \frac{u^2}{2J^2}. \tag{1.25}$$

2. Including an external magnetic field.

The random energy model with external magnetic field is given by the partition function (see [1,7])

$$\mathcal{Z}_N(\beta, H) = \sum_{M=-N,-N+2,\ldots,N} \sum_{k=1}^{K(N,M)} \exp\{-\beta[E_{M,k} - H\,M]\}, \tag{2.1}$$

where

$$K(N,M) = \binom{N}{(N+M)/2} \tag{2.2}$$

is the number of possible states with total magnetisation M and the energy levels $E_{M,k}$ are again i.i.d. random variables with probability distribution given by the density (1.2).

In this case we want to consider a two-dimensional random variable corresponding to the observables $E_{M,k}/N$ and M/N. It has the distribution function given by

$$F_N(x,m) = \frac{1}{2^N}\#\{(M,k) : E_{M,k} \le xN \text{ and } M \le mN\}. \tag{2.3}$$

We prove the following analogue of Theorem 1:

Theorem 2. *With probability 1, the random probability measures ν_N on $\mathbf{R}^2$ with distribution functions F_N defined by (2.3) satisfy a LDP with constants N and rate function I given by*

$$I(x,m) = \begin{cases} I_0(m) + \frac{x^2}{2J^2} & \text{if } I_0(m) + \frac{x^2}{2J^2} \le \ln 2, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Here

$$I_0(m) = \begin{cases} \frac{1}{2}\{(1+m)\ln(1+m) + (1-m)\ln(1-m)\} & \text{if } -1 \le m \le 1, \\ +\infty & \text{otherwise} \end{cases} \tag{2.5}$$

is the usual expression for the distribution of independent Ising spins.

Proof. We first prove the large deviation upper bound. Let C be a closed set in $\mathbf{R}^2$. First suppose that C is contained in the complement of the essential domain D of I, which is the compact set depicted in Fig. 1, and given by

$$I_0(m) + \frac{x^2}{2J^2} \leq \ln 2. \tag{2.6}$$

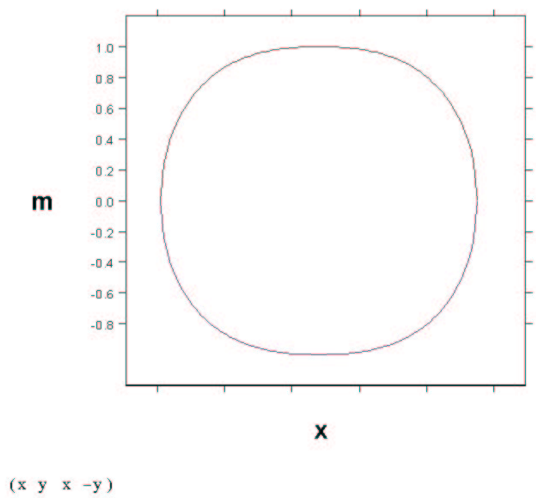

Figure 1. *The essential domain of $I(x,m)$.*

As in Theorem 1, we shall prove that in this case $\nu_N(C) = 0$ with probability 1. To this end first notice that C can be covered by a finite number of quadrants of the form $\{(x,m)|\, x \leq x_0, m \leq m_0\}$, $\{(x,m)|\, x \leq x_0, m \geq m_0\}$, $\{(x,m)|\, x \geq x_0, m \leq m_0\}$, or $\{(x,m)|\, x \geq x_0, m \geq m_0\}$, where $(x_0,m_0) \notin D$. (To see this let $\delta > 0$ be the distance from C to D and consider the annulus of points (x,m) with distance between δ and $1+\delta$ say from D. This annulus is compact and can be covered by a finite number of quadrants. But these quadrants must also cover C: take a ray from a point in C to 0 and intersect with the annulus.) It follows that it suffices to prove that $\nu_N(C) = 0$ with probability 1 if C is a quadrant. By symmetry this reduces to proving that $F_N(x_0,m_0) = 0$ with probability 1 if $(x_0,m_0) \notin D$ and $x_0, m_0 < 0$. This is done in essentially the same way as in Theorem 1 but using the large deviation property of the Bernoulli measure as well. The latter states that for any given $\epsilon > 0$ and for N large enough,

$$2^{-N} \sum_{M \leq mN} K(N,M) \leq e^{-N(I_0(m)-\epsilon)}. \tag{2.7}$$

Hence we have analogous to (1.11),

$$\begin{aligned}\mathbf{P}\{F_N(x_0,m_0)>0\} &= \mathbf{P}\left\{\sum_{M\le m_0N}\sum_{k=1}^{K(N,M)}\mathbf{1}_{\{E_{M,k}\le x_0N\}}\ge 1\right\}\\ &\le \mathbf{E}\left\{\sum_{M\le m_0N}\sum_{k=1}^{K(N,M)}\mathbf{1}_{\{E_{M,k}\le x_0N\}}\right\}\\ &= \sum_{M\le m_0N}K(N,M)\mathbf{P}\{E_{M,k}\le x_0N\}\\ &\le 2^Ne^{-N(I_0(m_0)-\epsilon)}\frac{J}{\sqrt{2\pi N}|x_0|}e^{-Nx_0^2/(2J^2)}.\end{aligned}\tag{2.8}$$

Taking $\epsilon < I_0(m_0)+x_0^2/(2J^2)-\ln 2$ the corresponding series converges and the result follows by the Borel-Cantelli lemma as before.

Next we consider the case that $C\cap D\neq\emptyset$. In that case $b=\inf_{(x,m)\in C}I(x,m)$ is finite. Given $\delta>0$ we can cover C again by quadrants with corner points (x_0,m_0) satisfying $I(x_0,m_0)\ge b-\delta$. As before we can consider a single quadrant and assume $x_0,m_0<0$. We put

$$G_N(x_0,m_0)=2^{-N}\sum_{M\le m_0N}K(N,M)G_N(x_0),\tag{2.9}$$

where $G_N(x_0)=\int_{-\infty}^{x_0N}p_N(E)dE$ is the same as (1.16). Next we compute, as in (1.17) and (1.18),

$$\begin{aligned}&\mathbf{P}\{|F_N(x_0,m_0)-G_N(x_0,m_0)|\ge\epsilon G_N(x_0,m_0)\}\\ &\quad\le\frac{1}{\epsilon^2G_N(x_0,m_0)^2}\mathbf{E}\left\{|F_N(x_0,m_0)-G_N(x_0,m_0)|^2\right\}\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}&\mathbf{E}\left\{|F_N(x_0,m_0)-G_N(x_0,m_0)|^2\right\}\\ &=\mathbf{E}\{F_N(x_0,m_0)^2\}-G_N(x_0,m_0)^2\\ &=2^{-2N}\sum_{M\le m_0N}K(N,M)G_N(x_0)(1-G_N(x_0))\\ &\le 2^{-N}G_N(x_0,m_0).\end{aligned}\tag{2.11}$$

Again using the LDP for the Bernoulli distribution,

$$\begin{aligned}&\mathbf{P}\{|F_N(x_0,m_0)-G_N(x_0,m_0)|\ge\epsilon G_N(x_0,m_0)\}\\ &\le 2^{-N}\epsilon^{-2}(G_N(x_0,m_0))^{-1}\\ &\le 2^{-N}\epsilon^{-2}\frac{\sqrt{2\pi}(J^2+Nx_0^2)}{|x_0|J\sqrt{N}}e^{N(I_0(m_0)+\delta+x_0^2/(2J^2))}.\end{aligned}\tag{2.12}$$

(If $x_0 = 0$ then $G_N(x_0) = \frac{1}{2}$ and we can write simply

$$\mathbf{P}\left\{|F_N(0, m_0) - G_N(0, m_0)| \geq \epsilon G_N(0, m_0)\right\} \leq 2^{-N}\epsilon^{-2}2e^{N(I_0(m_0)+\delta)}.$$

This yields the same result as for $x_0 \neq 0$.) Taking $\epsilon = \frac{1}{2}$ and $\delta < \ln 2 - b$ we conclude that $\frac{1}{2}G_N(x_0, m_0) \leq F_N(x_0, m_0) \leq \frac{3}{2}G_N(x_0, m_0)$ with probability 1. Using the Borel-Cantelli lemma once again we have, with probability 1,

$$\begin{aligned} \limsup_{N\to\infty} \frac{1}{N} \ln F_N(x_0, m_0) &= \limsup_{N\to\infty} \frac{1}{N} \ln G_N(x_0, m_0) \\ &= I_0(m_0) + \frac{x_0^2}{2J^2} = I(x_0, m_0). \end{aligned} \tag{2.13}$$

This completes the proof of the large deviation upper bound.

For the lower bound we need to prove that for every open set $O \subset \mathbf{R}^2$,

$$\liminf_{N\to\infty} \frac{1}{N} \ln \nu_N(O) \geq - \inf_{(x,m)\in O} I(x, m) \tag{2.14}$$

with probability 1. First notice that if $O \cap D = \emptyset$ then there is nothing to prove. On the other hand, if $O \cap D \neq \emptyset$ then the infimum is attained at a boundary point of O in the interior of D. By the continuity of I on D, for every $\eta > 0$, there exists a point $(x_0, m_0) \in I \cap \mathrm{int}(D)$ such that $I(x_0, m_0) < \inf_{(x,m)\in O} I(x, m) + \eta$. It follows that for $\delta > 0$ small enough, the square $S = \{(x, m) | \, |x - x_0| \leq \delta, |m - m_0| \leq \delta\}$ is contained in $I \cap \mathrm{int}(D)$ and for every $(x', m') \in S$, $I(x', m') < \inf_{(x,m)\in O} I(x, m) + \eta$. Now,

$$\nu_N(O) \geq \nu_N(S) = 2^{-N} \sum_{(m_0-\delta)N \leq M \leq (m_0+\delta)N} \sum_{k=1}^{K(N,M)} \mathbf{1}_{\{E_{M,k}/N \in [x_0-\delta, x_0+\delta]\}}. \tag{2.15}$$

For the latter expression we can argue as above ((2.10) and (2.11)):

$$\begin{aligned} &\mathbf{P}\left\{ \left| 2^{-N} \sum_{(m_0-\delta)N \leq M \leq (m_0+\delta)N} \sum_{k=1}^{K(N,M)} \left(\mathbf{1}_{\{E_{M,k}/N \in [x_0-\delta, x_0+\delta]\}} - \int_{x_0-\delta}^{x_0+\delta} p_N(E)dE \right) \right| \right. \\ &\quad \left. \geq \epsilon 2^{-N} \sum_{(m_0-\delta)N \leq M \leq (m_0+\delta)N} K(N, M) \int_{x_0-\delta}^{x_0+\delta} p_N(E)dE \right\} \\ &\qquad \leq \epsilon^{-2} 2^{-N} \left(2^{-N} \sum_{(m_0-\delta)N \leq M \leq (m_0+\delta)N} K(N, M) \int_{x_0-\delta}^{x_0+\delta} p_N(E)dE \right)^{-1}. \end{aligned} \tag{2.16}$$

The integral on the right-hand side behaves at worst like $e^{-N(|x_0|+\delta)^2/(2J^2)}$, and the sum over M behaves like $e^{-NI_0(m_0\pm\delta)}$. But $S \subset \mathrm{int}(D)$ so $I(|x_0|+\delta, m_0\pm\delta) < \ln 2$. It follows that the series with terms given by the right-hand side converges and by the Borel-Cantelli lemma (taking $\epsilon = \frac{1}{2}$),

$$
\begin{aligned}
&\frac{1}{2}2^{-N} \sum_{(m_0-\delta)N\le M\le(m_0+\delta)N} K(N,M)\int_{x_0-\delta}^{x_0+\delta} p_N(E)dE \\
&\qquad \le 2^{-N} \sum_{(m_0-\delta)N\le M\le(m_0+\delta)N} \sum_{k=1}^{K(N,M)} \mathbf{1}_{\{E_{M,k}/N\in[x_0-\delta,x_0+\delta]\}} \qquad (2.17)\\
&\qquad\quad \le \frac{3}{2}2^{-N} \sum_{(m_0-\delta)N\le M\le(m_0+\delta)N} K(N,M)\int_{x_0-\delta}^{x_0+\delta} p_N(E)dE
\end{aligned}
$$

with probability 1. The large deviation lower bound now follows from

$$
\begin{aligned}
&\liminf_{N\to\infty}\frac{1}{N}\ln\left[2^{-N} \sum_{(m_0-\delta)N\le M\le(m_0+\delta)N} K(N,M)\int_{x_0-\delta}^{x_0+\delta} p_N(E)dE\right] \\
&\quad = \lim_{N\to\infty}\frac{1}{N}\ln\left[2^{-N} \sum_{(m_0-\delta)N\le M\le(m_0+\delta)N} K(N,M)\right] \\
&\qquad + \lim_{N\to\infty}\frac{1}{N}\ln\int_{x_0-\delta}^{x_0+\delta} p_N(E)dE \qquad (2.18)\\
&\quad = -I_0(|m_0|-\delta) - \frac{(|x_0|-\delta)^2}{2J^2} \\
&\quad = -I(|x_0|-\delta, |m_0|-\delta) \\
&\quad \ge -\inf_{(x,m)\in O} I(x,m) - \eta.
\end{aligned}
$$

This completes the proof of the theorem. Q.E.D.

3. The phase diagram.

Applying Varadhan's theorem [4,5,6] we can now conclude that for any continuous function $g : \mathbf{R} \to \mathbf{R}$,

$$
\begin{aligned}
f(\beta) &= -\lim_{N\to\infty}\frac{1}{\beta N}\ln \sum_{M=-N,-N+2,\dots,N}\sum_{k=1}^{K(N,M)} \exp\{-\beta[E_{M,k} - Ng(M/N)]\} \\
&= -\beta^{-1}\ln 2 - \sup_{(x,m)\in D}[g(m) - x - \beta^{-1}I(x,m)].
\end{aligned}
\qquad (3.1)
$$

We consider in particular the case $g(m) = \lambda|m|^p$ for some power $p \geq 2$ and constant $\lambda > 0$. In the case $p = 2$ this amounts to the usual ferromagnetic interaction term; for $p > 2$ this is a generalised ferromagnetic interaction term. The variational problem (3.1) is identical to that for the tree model analysed in [11]. The supremum over x can be performed explicitly and yields

$$f(\beta) = -\beta^{-1}\ln 2 - \sup_{m\in[-1,1]} [g(m) - \beta^{-1}\tilde{I}(\beta, m)], \tag{3.2}$$

where

$$\tilde{I}(\beta, m) = \begin{cases} I_0(m) - \frac{1}{2}\beta^2 J^2 & \text{if } \frac{1}{2}\beta^2 J^2 \leq \ln 2 - I_0(m), \\ \ln 2 - \beta J\sqrt{2(\ln 2 - I_0(m))} & \text{if } \frac{1}{2}\beta^2 J^2 \geq \ln 2 - I_0(m). \end{cases} \tag{3.3}$$

This is more simply written as

$$f(\beta) = -\sup_{m\in[-1,1]} [g(m) + \beta^{-1}s(\beta, m)], \tag{3.4}$$

where $s(\beta, m) = \ln 2 - I(\beta, m)$, i.e.

$$s(\beta, m) = \begin{cases} s_0(m) + \frac{1}{2}\beta^2 J^2 & \text{if } \frac{1}{2}\beta^2 J^2 \leq s_0(m), \\ \beta J\sqrt{2s_0(m)} & \text{if } \frac{1}{2}\beta^2 J^2 \geq s_0(m), \end{cases} \tag{3.5}$$

where $s_0(m) = \ln 2 - I_0(m)$ is the free spin entropy.

We now recap briefly the derivation of the phase diagram in the case $g(m) = |m|^p$ with $p > 2$. It is depicted in Figure 2 below.

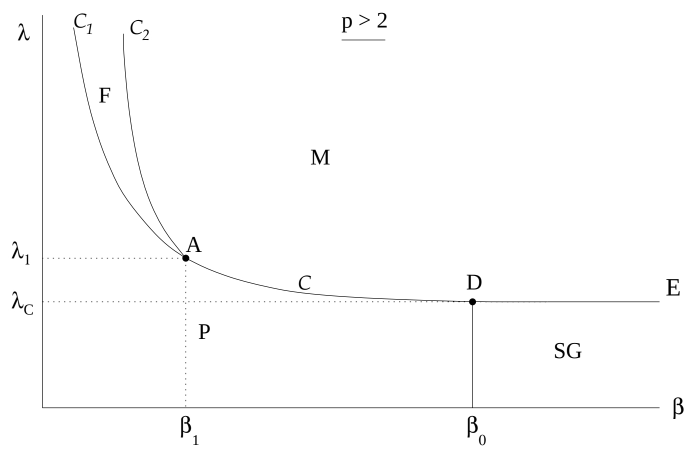

Fig. 2. *The phase diagram.*

If $\beta > \beta_c$, where β_c is the critical inverse temperature of the REM,

$$\beta_c = J^{-1}\sqrt{2\ln 2}, \tag{3.6}$$

then the second case of (3.3) applies and we can write

$$f(\lambda) = -\sup_m [\lambda m^{p-1} + J\sqrt{2s_0(m)}], \tag{3.7}$$

It follows that in this case there is a transition at $\lambda = \lambda_c(p)$ independent of β. The value of λ_c is found by combining the maximisation condition for (3.4) given by

$$g'(m) = \frac{J \tanh^{-1}(m)}{\sqrt{2s_0(m)}}, \tag{3.8}$$

with the condition

$$g(m) + J\sqrt{2s_0(m)} = J\sqrt{2s_0(0)} = \beta_c J^2. \tag{3.9}$$

The maximisation condition can be written as

$$m = \tanh\left[p\lambda\bar{\beta}(m)m^{p-1}\right], \tag{3.10}$$

where we define

$$\bar{\beta}(m) = J^{-1}\sqrt{2s_0(m)}. \tag{3.11}$$

Eliminating λ we see that across the transition $\lambda = \lambda_c$, the magnetisation jumps from zero to the value m_c given by the solution of

$$pJ^2\bar{\beta}(m)[\beta_c - \bar{\beta}(m)] - m\tanh^{-1}(m) = 0. \tag{3.12}$$

The value m_c is independent of β and behaves for large p asymptotically as

$$m_c \sim 1 - \frac{1}{8p^2 \ln 2} \ln(8p^2 \ln 2). \tag{3.13}$$

Next we consider the case $\beta < \beta_c$. For small β and λ the first expression for s holds and we obtain by differentiation the maximisation condition

$$m = \tanh(\beta\lambda p m^{p-1}). \tag{3.14}$$

For small λ this equation has only the zero solution, but for larger λ there are also two positive solutions, $\tilde{m}$ and $\bar{m} > \tilde{m}$. The former is a local minimum and the latter a local maximum, but it is only a global maximum if λ exceeds the critical value $\lambda_1(\beta)$ for which $\beta^{-1}s(\beta,0) = \lambda_1 m^p + \beta^{-1}s(\beta,m)$, i.e. $\beta\lambda m^p = I_0(m)$. Eliminating λ between this and equation (3.14) yields the equation

$$\left[1+\left(1-\frac{1}{p}\right)\right]\ln(1+m)+\left[1-\left(1-\frac{1}{p}\right)\right]\ln(1-m)=0 \tag{3.15}$$

for the magnetisation m_p just above the transition. This transition between the paramagnetic and the ferromagnetic phase occurs for $\lambda = \lambda_1(\beta)$ given by

$$\lambda_1(\beta) = \frac{\tanh^{-1}(m_p)}{\beta p m_p^{p-1}}. \tag{3.16}$$

If λ is increased further, the magnetisation increases as can be deduced from (3.14). Eventually it reaches the value at which $\frac{1}{2}\beta^2 J^2 = s_0(m)$. The second formula for $s(\beta, m)$ then takes over and there is another phase transition to the mixed phase. We indicate the corresponding value of the magnetisation by $m(\beta)$ so that

$$s_0(m(\beta)) = \frac{1}{2}\beta^2 J^2. \tag{3.17}$$

The transition line between the ferromagnetic phase and the mixed phase is then, according to (3.14), given by

$$\lambda_2(\beta) = \frac{\tanh^{-1}(m(\beta))}{p\beta[m(\beta)]^{p-1}}. \tag{3.18}$$

As β is increased, the value of $m(\beta)$ decreases until it reaches m_p. At this point, indicated by A in the phase diagram, the transition lines C_1 and C_2 meet and the ferromagnetic phase disappears. This happens at $\beta = \beta_1$ where β_1 satisfies $m(\beta_1) = m_p$, i.e. $\frac{1}{2}\beta_1^2 J^2 = s_0(m_p)$. For $\beta_1 \leq \beta < \beta_c$ there is only one transition, namely from the paramagnetic phase to the mixed phase. The transition line is determined by the equation

$$\lambda \bar{m}(\lambda)^p + \beta^{-1}s(\beta, \bar{m}(\lambda)) = \beta^{-1}\ln 2 + \frac{1}{2}\beta^2 J^2, \tag{3.19}$$

where $\bar{m}(\lambda)$ is the maximal solution of (3.10). Notice that for large p, m_p tends to 1 asymptotically as $m_p \sim 1 - 2^{-(2p-1)}$ so that β_1 according to (3.17) must tend to zero. This shows that for large p, the point A moves to small values of β, that is, high temperatures and there is definitely a gap between the points A and D in the phase diagram. Indeed, in the limit $p \to \infty$ the ferromagnetic phase disappears altogether. This was shown in [12] to be relevant for Sourlas' error correcting coding scheme [13.14]. In the following section we give a short review of this coding method.

4. Sourlas' error correcting code.

Consider a Gaussian communication channel and suppose we want to transmit N bits of information, written as Ising spins $\epsilon_1, \ldots, \epsilon_N$. We encode this information as follows. For fixed, large p, we form all products $b_{i_1,\ldots,i_p} = \epsilon_{i_1} \cdots \epsilon_{i_p}$ with $1 \leq i_1 < \ldots < i_p \leq N$. These are transmitted as voltage signals $J^0_{i_1,\ldots,i_p} = v b_{i_1,\ldots,i_p}$, where v is a constant amplitude. Due to noise in the channel, a slightly perturbed version of the signal is received as output:

$$J_{i_1,\ldots,i_p} = J^0_{i_1,\ldots,i_p} + \Delta J_{i_1,\ldots,i_p}, \tag{4.1}$$

where, in the Gaussian case, the $\Delta J_{i_1,\ldots,i_p}$ are i.i.d. Gaussian random variables with mean zero and variance ω^2. To recover the original message, use the received signals as interaction parameters in a p-spin Hamiltonian as follows:

$$\mathcal{H}_p = - \sum_{1 \leq i_1 < \ldots < i_p \leq N} J_{i_1,\ldots,i_p} \sigma_{i_1} \ldots \sigma_{i_p}. \tag{4.2}$$

In the absence of noise the ground state of $\mathcal{H}_p$ would be given by the original message $\sigma_i = \epsilon_i$. In the presence of noise we can perform a gauge transformation to new variables $\sigma'_i = \epsilon_i \sigma_i$ and write $\mathcal{H}_p$ in the form

$$\mathcal{H}_p = -N\lambda m_N^p - \sum_{1 \leq i_1 < \ldots < i_p \leq N} \tilde{J}_{i_1,\ldots,i_p} \sigma'_{i_1} \ldots \sigma'_{I_p}, \tag{4.3}$$

where $m_N = N^{-1} \sum_{i=1}^N \sigma'_i$ and $\lambda = vN^{p-1}/p!$, and where $\tilde{J}_{i_1,\ldots,i_p} = (\Delta J)_{i_1,\ldots,i_p} \times \epsilon_{i_1} \ldots \epsilon_{i_p}$. Here we have omitted terms which are irrelevant in the limit of large p and N. Notice that if the magnetization m_N is close to 1 then the bit error probability, i.e. the probability that in the ground state of $\mathcal{H}_p$, $\sigma_i \neq \epsilon_i$, is low because

$$p_e = \frac{1}{N} \#\{i : \sigma_i \neq \epsilon_i \ = \frac{1}{2}(1 - m_N). \tag{4.4}$$

The Hamiltonian $\mathcal{H}_p$ is a p-spin generalization of the Sherrington-Kirkpatrick model of a spin glass [15,16]. Even the 2-spin case has not been solved exactly (or at least rigorously). However, Derrida has argued that in the limit $p \to \infty$ the second term of (4.3) becomes a sum of i.i.d. random energy variables E_i

($i = 1, \ldots, 2^N$) provided the variance ω^2 is scaled according to $\omega^2 = J^2 p!/N^{p-1}$. This yields the random energy model. If we do the same in (4.3) but keeping p fixed in the first term, the resulting model is given by (3.1) with $g(m) = \lambda m^p$. It is reasonable to suspect that the phase diagram of this model in the limit $p \to \infty$ is the same as that of (4.3) in the limit $p \to \infty$, i.e. if we take both parameters $p \to \infty$ at the same time.

The channel capacity C is defined as the maximum of the mutual information content of the messages sent down the channel and received at the end with respect to all possible input distributions. For the Gaussian channel it has been computed [17] to be

$$C = \frac{1}{2} \log_2 \left(1 + \frac{v^2}{\omega^2} \right). \tag{4.5}$$

Inserting the rescaled coupling parameter λ and replacing ω^2 by $J^2 p!/N^{p-1}$ we get

$$C = \frac{1}{2} \log_2 \left(1 + \frac{\lambda^2 p!}{J^2 N^{p-1}} \right) \approx \frac{\lambda^2 p!}{2 \ln 2\, J^2 N^{p-1}} \tag{4.6}$$

assuming that $p \ll N$. It was proved by Shannon that the rate of transmission, i.e. the number of bits N in the original message divided by the number of bits M in the encoded message cannot exceed C. This is one half of his celebrated channel coding theorem; the other half states that it is possible, in principle to approximate C arbitrarily closely with negligible error probability. However, his proof of this fact is not constructive. In our case we have $M = \binom{N}{p} \approx N^p/p!$ so that the rate of the Sourlas code is given by $R = p!/N^{p-1}$. Shannon's bound thus becomes

$$\lambda > \beta_c J^2 = \lambda_c(\infty), \tag{4.7}$$

where β_c is the critical inverse temperature of the random energy model, and $\lambda_c(\infty)$ is the value of $\lambda(p)$ in the limit $p \to \infty$ which can be deduced from (3.9). Taking λ slightly above this value we can therefore approximate Shannon's bound and if we lower the temperature in an annealing process we cross the transition line from the paramagnetic to the mixed phase near point D in Figure 2. The magnetisation then jumps to the value (3.13) after which it remains constant. This value is very close to 1 for large values of p, so the error probability is low. This is therefore a new, more constructive 'proof' of the second part of Shannon's coding theorem. (It is not strictly a proof because of the assumptions about the limits $p \to \infty$.) It must be said however, that it is still not very practical as the rate of this code is very low for large p.

We finish with a remark about the scaling involved in the limit $p \to \infty$. We have in effect scaled the noise by the rate of the code. In order to compare the performance of this coding scheme with other schemes we should do the same scaling. For example, consider the simple repetition code, repeating every bit p times. In that case, after scaling we find that

$$C = \frac{1}{2} \log_2(1 + \frac{\lambda^2}{pJ^2}) \approx \frac{\lambda^2}{2pJ^2 \ln 2},$$

whereas the rate is $1/p$. To recover the message we take the sign of the sum of the received variables $\sigma_1, \ldots, \sigma_p$. The effect of the scaling is here that the total voltage is $\pm pv = \pm\lambda$ whereas the added random noise has a variance $p\omega^2 = J^2$. Thus, in order that the bit error probability becomes small we need to take $\lambda \gg J$ in which case $R \ll C$. Notice that without scaling we get essentially the same result: in that case we need to take p large in order that the bit error probability is small and that means that the rate tends to zero, whereas now the capacity is independent of p so again $R \ll C$. This coding scheme can in fact also be modelled with a Hamiltonian, namely $H = (\sigma_1 + \ldots + \sigma_p)\sigma'$ and the rescaling ensures again that the energy per spin σ' remains finite in the limit $p \to \infty$.

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