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Enumerating Low Rank Matroids and Their Asymptotic Probability of Occurrence

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Abstract

This paper shows the attractive enumerative relations between matroids of low rank. It differs from past work in that, rather than attempting to examine the numbers of non-isomorphic matroids as proposed by Crapo [4], it looks directly at the number of matroids and then extends to their non-isomorphic counterparts. We give the (heretofore unknown) numbers for matroids on at most eight elements. Furthermore, we consider a random collection of r-sets of an n-set and examine the probability that these satisfy the matroid basis exchange axioms. The asymptotic behavior of this probability shows interesting characteristics. The $r = 2$ case corresponds to a problem in random graphs.

1 Introduction

The matroid enumeration problem has long been forgotten. Research seemed to grind to a halt in the late '70s once sufficiently tight asymptotic bounds had been found [10, 8]. In this paper we revive the enumeration problem and see that by focusing on the number of matroids, rather than the number of non-isomorphic matroids (as proposed by Crapo [4]), more appealing expressions are obtained. We show how the numbers for rank-2 matroids are related to the Bell numbers and integer partitions, how numbers for the rank-3 matroids are related to 2-partitions and how Knuth's [8] lower bound for the number of combinatorial geometries may be used to improve Doyen's [5] lower bound on the number of 2-partitions. The rank-3 matroids are also seen to be *discretely self-similar* which partly answers a query made by Konvalina [7].

The probability that a random collection of k-sets forms the basis for a matroid is also examined. For 2-sets, the problem can be viewed as a random graph being t-partite and an exact recursion for the probability given. For $k = 3$ the same limiting behavior, as in the $k = 2$ case, is shown to hold but under a different scaling. We refer the reader unfamiliar with any concepts to the introductory chapter of Oxley [9].

1.1 Notation

Let S_n be a finite set of size n and S_n^d the collection of all d-element subsets of S_n. Let $\mathcal{M}_r^k(S_n)$ and $\mathcal{F}_r^k(S_n)$ be the classes of rank-r matroids and non-isomorphic rank-r matroids on S_n, respectively, both with all k-sets independent. We write $m_r^k(n) = \left|\mathcal{M}_r^k(S_n)\right|$ and $f_r^k(n) = \left|\mathcal{F}_r^k(S_n)\right|$. Define $\mathcal{M}_r(S_n) := \mathcal{M}_r^0(S_n)$ and similarly for $\mathcal{F}_r$, m_r and f_r. Let $\Pi_n(i)$ and $\Pi_n^\star(i)$ be the set of all partitions and non-isomorphic partitions, respectively, of the set S_n into i parts. Let $\Pi_n(i,j) := \Pi_n(i) \cup \Pi_n(i+1) \cup \cdots \cup \Pi_n(j)$

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and $\Pi_n := \Pi_n(1,n)$. Let $p_i(n)$ denote the number of partitions of the integer n into i parts and let $p(n) := p_1(n) + \ldots + p_n(n)$. The number of matroids and non-isomorphic matroids on S_n are given by

$$m(n) = \sum_{0 \leqslant r \leqslant n} m_r^0(n) \qquad\qquad f(n) = \sum_{0 \leqslant r \leqslant n} f_r^0(n)$$

Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a collection of distinct subsets of S_n. We say that $\mathcal{H}$ is a *d-partition* of S_n if,

1. $|H_i| \geqslant d$ for all $1 \leqslant i \leqslant k$,
2. $H_1 \cup \ldots H_k = S_n$,
3. Every $d-$element subset of S_n is contained in a unique $H_i \in \mathcal{H}$.

We see that the class of 1-partitions of S_n with k sets correspond to $\Pi_n(k)$. Let $h_d(n)$ be the number of d-partitions of the set S_n and $h_d^\star(n)$ the corresponding non-isomorphic number. It is well known that if $\mathcal{H}$ is such a d-partition with $k > 1$, then $\mathcal{H}$ satisfies the hyperplane axioms for a matroid M on S_n with rank $d+1$. Such a matroid is called a *paving matroid*.

2 Enumeration

The approach to counting matroids is through structural properties of the lattice of flats. The main results of this section are given in Theorems 3, 4 and an expression for the number of simple rank-r matroids given in equation 5. Enumerating rank-r matroids on S_n involves finding $m_r^0(n)$ and $f_r^0(n)$. The number of rank-0 and rank-1 matroids is trivial, $m_0^0(n) := 1$, $f_0^0(n) = 0$, $m_1^0(n) = 2^n - 1$ and $f_1^0(n) = n$ for all $n \geqslant 1$. Clearly $m_r^r(n) = f_r^r(n) = 1$ for all $1 \leqslant r \leqslant n$. The primary recursive relations between the first three classes of matroids are given in the Lemma 1. Note that the class $\mathcal{M}_r^1(S_n)$ is the class of rank-r matroids on S_n with no loops. Similarly, the class $\mathcal{M}_r^2(S_n)$ is the class of rank-r matroids with neither loops nor parallel elements (*simple matroids*). The class $\mathcal{M}_r^{r-1}(S_n)$ is the class of rank-r paving matroids on S_n.

Lemma 1 *For all* $1 \leqslant r \leqslant n$*,*

$$m_r^0(n) = \sum_{r \leqslant i \leqslant n} \binom{n}{i} m_r^1(i) \tag{1}$$

$$m_r^1(n) = \sum_{r \leqslant i \leqslant n} \left\{ {n \atop i} \right\} m_r^2(i). \tag{2}$$

Proof: Any matroid $M \in \mathcal{M}_r^0(S_n)$ can have at most $n-r$ loops. If M has loops $X \subseteq S_n$, $|X| = j$, then X may be chosen in $\binom{n}{j}$ ways. The resulting matroid is $M|_{S_n - X} \in \mathcal{M}_r^1(S_n - X)$ which has no loops since all 1-element subsets of $S_n - X$ are independent. Hence

$$\begin{aligned} m_r^0(n) &= \sum_{j=0}^{n-r} \binom{n}{j} m_r^1(n-j) \\ &= \sum_{i=r}^{n} \binom{n}{i} m_r^1(i), \end{aligned}$$

and equation 1 follows.

For equation 2 the argument is more involved. Let $M \in \mathcal{M}_r^1(S_n)$ have rank-1 flats $X_1, \ldots, X_i$ (note that $i \geqslant r$). There are no loops, so every element of S_n is contained in at least one rank-1 flat. If X_a and X_b are two distinct rank-1 flats, then $X_a \cap X_b := \emptyset$. Hence the collection $\{X_j\}_{1 \leqslant j \leqslant i}$ is simply a partition of S_n. Thus the natural bijection between the class of matroids in $\mathcal{M}_2^1(S_n)$ with i rank-1 flats and $\Pi_n(i)$. The collection $X_1, \ldots, X_i$ may be chosen in $\left\{ {n \atop i} \right\}$ ways where $\left\{ {n \atop i} \right\}$ are the Stirling numbers of the second kind.

Any flat of M is the union of some collection of the $\{X_j\}_{1 \leqslant j \leqslant i}$. Otherwise, there is some flat F and elements $a, b \in X_j$ such that $a \in F \not\ni b$. As F, X_j are both flats, $F \cap X_j$ is also a flat. But this forces $\emptyset \subset F \cap X_j \subset X_j$ (since $b \notin F$) which is a contradiction since there are no non-trivial flats which are properly contained in a rank-1 flat.

Choose any transversal $Y = \{x_1, \ldots, x_i\}$ of the family $\{X_j\}_{1 \leqslant j \leqslant i}$. Notice that $M|_Y \in \mathcal{M}_r^2(Y)$ since $r(\{x_j, x_k\}) = 2$ for all $1 \leqslant j \neq k \leqslant i$. Thus each matroid $M \in \mathcal{M}_r^1(S_n)$ is uniquely expressible by its collection of rank-1 flats and a simple rank-r matroid $M|_Y \in \mathcal{M}_r^2(Y)$. The number of such matroids with i rank-1 flats is given by $\left\{ {n \atop i} \right\} m_r^2(i)$ and the resulting equation 2 by summing from $i = r$ to n. $\square$

Lemma 2 *For all $n \geqslant 3$, $m_3^2(n) = h_2(n) - 1$.*

Proof: For any matroid $M \in \mathcal{M}_3^2(S_n)$, let $\mathcal{F}_2$ be the collection of rank-2 flats. Trivially we have $\mathcal{F}_1 = \{\, \{x\} \mid x \in S_n \,\}$ and so $r(\{x, y\}) = 2$ for all distinct $x, y \in S_n$. Thus for each pair of elements $x, y \in S_n$ there is a rank-2 flat $X \in \mathcal{F}_2$ containing both.

To show this flat to be unique, suppose there is another $Y \in \mathcal{F}_2$ such that $Y \supseteq \{x, y\}$. Now $2 = r(X) > r(X \cap Y) \geqslant r(\{x, y\}) = 2$. Thus there does not exist such a Y and X is unique. The only condition upon $\mathcal{F}_2$ in representing such a matroid is that $\mathcal{F}_2 \neq \{S_n\} =: \mathcal{F}_3$. Hence $|\mathcal{F}_2| \geqslant 2$. It follows that there is a natural bijection between the class of 2-partitions (excluding the trivial one $\{S_n\}$) of S_n and the class of simple rank-3 matroids on S_n. Hence $m_3^2(n) = h_2(n) - 1$. $\square$

For any rank-3 matroid $M \in \mathcal{M}_3^0(S_n)$, we see that by restricting it to any transversal Y of $\mathcal{F}_0 \cup \mathcal{F}_1$, the resulting matroid $M|_Y$ is self-similar in structure to M. This important fact allows us to enumerate rank-3 matroids. These two lemmas now suffice to prove the following recursions for the m numbers:

Theorem 3 *For all $n \geqslant 2, 3$, respectively,*

$$\begin{aligned} m_2(n) &= b(n+1) - 2^n \\ m_3(n) &= \sum_{3 \leqslant j \leqslant n} \left\{ {n+1 \atop j+1} \right\} (h_2(j) - 1)\,. \end{aligned}$$

Proof: Applying $r = 2$ to equations 1 and 2 we have

$$\begin{aligned} m_2(n) &= m_2^0(n) \\ &= \sum_{2 \leqslant i \leqslant n} \binom{n}{i} m_2^1(i) \\ &= \sum_{2 \leqslant i \leqslant n} \binom{n}{i} \sum_{2 \leqslant j \leqslant i} \left\{ {i \atop j} \right\} m_2^2(j) \\ &= \sum_{2 \leqslant i \leqslant n} \binom{n}{i} \sum_{2 \leqslant j \leqslant i} \left\{ {i \atop j} \right\} 1 \\ &= \sum_{2 \leqslant i \leqslant n} \binom{n}{i} (b(i) - 1) \\ &= \sum_{2 \leqslant i \leqslant n} \binom{n}{i} b(i) - \sum_{2 \leqslant i \leqslant n} \binom{n}{i} \\ &= b(n+1) - nb(1) - b(0) - (2^n - n - 1) \\ &= b(n+1) - 2^n. \end{aligned}$$

Similarly, applying $r = 3$ to equations 1 and 2 and using lemma 2,

$$\begin{aligned} m_3(n) &= m_3^0(n) \\ &= \sum_{3\leqslant i\leqslant n} \binom{n}{i} m_3^1(i) \\ &= \sum_{3\leqslant i\leqslant n} \binom{n}{i} \sum_{3\leqslant j\leqslant i} \left\{ {i \atop j} \right\} m_3^2(j) \\ &= \sum_{3\leqslant i\leqslant n} \sum_{3\leqslant j\leqslant i} \binom{n}{i} \left\{ {i \atop j} \right\} m_3^2(j) \\ &= \sum_{3\leqslant j\leqslant n} \sum_{j\leqslant i\leqslant n} \binom{n}{i} \left\{ {i \atop j} \right\} m_3^2(j) \\ &= \sum_{3\leqslant j\leqslant n} m_3^2(j) \sum_{j\leqslant i\leqslant n} \binom{n}{i} \left\{ {i \atop j} \right\} \\ &= \sum_{3\leqslant j\leqslant n} m_3^2(j) \left\{ {n+1 \atop j+1} \right\}, \end{aligned}$$

from Knuth [11] equation 6.15. The result follows from Lemma 2. □

Turning our attention to the non-isomorphic numbers, we see the class of non-isomorphic rank-2 matroids can easily be singled out due to the structural properties revealed in Lemma 1. For the rank-3 case, isomorphisms prove more difficult to exclude but we give a lower bound.

Theorem 4 *For all $n \geqslant 2, 3$, respectively,*

$$f_2(n) = -n + \sum_{1\leqslant i\leqslant n} p(i) \tag{3}$$

$$f_3(n) \geqslant \sum_{i=3}^{n} (h_2^\star(i) - 1) \sum_{k=i}^{n} p_i(k). \tag{4}$$

Proof: Two matroids on ground sets of different cardinalities cannot be isomorphic, thus we may write the class $\mathcal{F}_r^0(S_n)$ as the disjoint union of the loopless classes

$$\mathcal{F}_r^0(S_n) = \bigcup_{r\leqslant i\leqslant n} \mathcal{F}_r^1(S_i),$$

and hence

$$f_r(n) = \sum_{r\leqslant i\leqslant n} f_r^1(i).$$

The class of matroids $\mathcal{M}_2^1(S_i)$ with j rank-1 flats corresponds precisely to the class of partitions of S_i into j sets, i.e. $\Pi_i\,(j)$. To rule out isomorphisms, we have the class of non-isomorphic partitions $\Pi_i^\star\,(j)$ through which we may view $\mathcal{F}_2^1(S_i)$. The number of these is simply the number of partitions of the integer i into j parts, $p_j(i)$. Thus

$$\begin{aligned} f_2^1(i) &= \sum_{j\geqslant 2} p_j(i) \\ &= p(i) - 1, \end{aligned}$$

and hence

$$\begin{aligned} f_2(n) &= \sum_{2\leqslant i\leqslant n} f_2^1(i) \\ &= \sum_{2\leqslant i\leqslant n} p(i) - 1 \\ &= -n + \sum_{1\leqslant i\leqslant n} p(i). \end{aligned}$$

For the inequality, we construct a sub-class of $\mathcal{F}_3^1(S_i)$. Let $\pi = \{X_1, \ldots, X_j\} \in \Pi_i^\star(j)$ and let $M \in \mathcal{F}_3^2(S_j)$. Let us now replace each element $x_k \in S_j$ by the set X_k in the partition π, for all $1 \leqslant k \leqslant j$. Two matroids in $\mathcal{M}_3^1(S_i)$ are isomorphic if and only if (1) the sequence of cardinalities of the rank-1 flats, when ordered, are the same, (2) both matroids, after restriction to a transversal of its rank-1 flats, are isomorphic (i.e. in $\mathcal{M}_3^2(\cdot)$) and (3) the assignment of rank-1 flats to the two restricted matroids just mentioned are in accordance. Essentially we are constructing matroids out of the non-isomorphic classes corresponding to (1) and (2) but which are never affected by condition (3). Thus

$$f_3^1(n) \quad \geqslant \quad \sum_{j=3}^{i} p_j(i) f_3^2(j)$$

and so

$$\begin{aligned} f_3(n) \quad &\geqslant \quad \sum_{i=3}^{n}\sum_{j=3}^{i} p_j(i) f_3^2(j) \\ &= \quad \sum_{i=3}^{n} f_3^2(i) \sum_{k=i}^{n} p_i(k) \\ &= \quad \sum_{i=3}^{n} (h_2^\star(i) - 1) \sum_{k=i}^{n} p_i(k). \end{aligned}$$

□

This is the point at which difficulties arise for the non-isomorphic matroid enumeration problem. However, the nice form of Theorem 3 gives future hope for the more general problem. It relies only upon knowledge of the number of 2-partitions. We may actually write down an expression for the number of rank-r matroids on S_n. For any collection of subsets λ of S_n, let us define $\Lambda(\lambda)$ as the family of collections of sets μ satisfying the following: If $Y \in \lambda$ and $A_1, \ldots, A_m$ are the sets in μ containing Y, then $\{A_1 - Y, A_2 - Y, \ldots, A_m - Y\}$ is a partition of the set $S_n - Y$. Then the number of simple rank-r matroids on S_n is given by the sum:

$$m_r^2(n) \quad = \quad \sum_{\lambda_1 \in \Lambda(S_n)} \sum_{\lambda_2 \in \Lambda(\lambda_1)} \cdots \sum_{\lambda_{r-1} \in \Lambda(\lambda_{r-2})} 1. \tag{5}$$

There is no known closed form expression for the number of 2-partitions of a finite set. Doyen [5] proved upper and lower bounds of $2^{\binom{n}{3}}$ and 2^n respectively. In the current setting, these bounds are very much trivial as the number of 2-partitions is less than the number of rank-3 matroids which in turn is less than $2^{\binom{n}{3}}$ (as can be seen by a simple argument involving the bases, i.e. $m_r^0(n) \leqslant 2^{\binom{n}{r}}$.) The lower bound is weak, it can be seen by choosing a single $X \subset S_n$ of cardinality $\geqslant 3$ (of which there are $\binom{n}{|X|}$) This X together with all those 2-element sets not contained in X form a 2-partition. We now form a better lower bound by slightly altering Knuth's [8] argument.

Lemma 5 *For all* $n \geqslant 3$,

$$h_2(n) \geqslant 2^{\frac{1}{12}(n-1)(n-2)} \quad \text{and} \quad h_2^\star(n) \geqslant \frac{1}{n!} 2^{\frac{1}{12}(n-1)(n-2)}.$$

Proof: Knuth's argument applies in more generality to prove the existence of $2^{\binom{n}{d}/2n}$ such $(d-1)$-partitions of S_n. Let H be the $n \times k$ matrix whose i^{th} row is the binary representation of i for all $1 \leqslant i \leqslant n$ and $k := \lfloor \log_2 n \rfloor + 1$. For any $X \in S_n^d$, let $\underline{X}$ be its binary representation. We define the partition $\mathcal{U}_j$ of S_n^d by

$$\mathcal{U}_j = \{\, X \in S_n^d \,|\, \underline{X}H = \text{binary representation of } j \,\}.$$

for all $1 \leqslant j \leqslant 2^k$. Now notice that if $X, Y \in \mathcal{U}_j$, then $|X \backslash Y| \geqslant 2$ for otherwise $(\underline{X} + \underline{Y})H \mod 2 = 0$ and this cannot happen as every row of H is distinct. Thus for any $X, Y \in \mathcal{U}_j$, $|X \cap Y| \leqslant 1$. Since the $\mathcal{U}_j$ partition S_n^d there exists some $\mathcal{U}_j$ with at least

$$|\mathcal{U}_j| \geqslant \binom{n}{d}/2^k > \binom{n}{d}/2n$$

sets. This particular $\mathcal{U}_j$ (or any collection of subsets of it), along with all $(d-1)$-sets not contained in any member of $\mathcal{U}_j$ defines a $(d-1)$-partition. Thus there are at least $2^{|\mathcal{U}_j|} \geqslant 2^{\binom{n}{d}/2n}$ $(d-1)$-partitions of S_n. We may divide this expression by $n!$ to rule out any isomorphisms. The lemma follows by choosing $d = 3$. □

Figure 1 shows the (previously unknown) values of $m_r^2(n)$ for all $2 \leqslant r \leqslant n \leqslant 8$. The numbers $m_r^0(n)$ and $m_r^1(n)$ may be calculated from this table by using Theorem 3. Figure 2 shows the number of non-isomorphic simple matroids, first given by Blackburn, Crapo and Higgs [6]. There is no direct way to calculate the numbers $f_r^1(n)$ from such a table, that was first done by Acketa [2].

r \ n	2	3	4	5	6	7	8
2	1	1	1	1	1	1	1
3		1	5	31	352	8389	433038
4			1	16	337	18700	7642631
5				1	42	2570	907647
6					1	99	16865
7						1	219
8							1
$m^2(n)$	1	2	7	49	733	29760	9000402

Figure 1: The value of $m_r^2(n)$ for $2 \leqslant r \leqslant n \leqslant 8$.

r \ n	2	3	4	5	6	7	8
2	1	1	1	1	1	1	1
3		1	2	4	9	23	68
4			1	3	11	49	617
5				1	4	22	217
6					1	5	40
7						1	6
8							1
$f^2(n)$	1	2	4	9	26	101	950

Figure 2: The value of $f_r^2(n)$ for $2 \leqslant r \leqslant n \leqslant 8$.

We also point out that a simple application of Theorem 4, Lemma 5 and a basic inductive argument reveals the inequality $f_2(n) < f_3(n)$. This is a first step in showing the validity of Welsh's conjecture that the sequence $\{f_r(n)\}_{0 \leqslant r \leqslant n}$ is unimodal.

3 Random Sets Representing Matroids

In this section we examine the probability that a random collection of subsets of S_n satisfy the basis exchange axioms for a matroid. The bases of a rank-r matroid on S_n is a non-empty collection $\mathcal{B} \subseteq S_n^r$ such that

$$X, Y \in \mathcal{B} \quad \Rightarrow \quad \forall x \in X \backslash Y, \exists y \in Y \backslash X \text{ with } X - \{x\} \cup \{y\} \in \mathcal{B}.$$

3.1 Asymptotic Behavior

Let $X_n^r(p)$ be a random subset of S_n^r generated in the following Bernoulli fashion:

$$\begin{aligned} \mathbb{P}\left(A \in X_n^r(p)\right) &= p \\ &= 1 - \mathbb{P}\left(A \notin X_n^r(p)\right), \end{aligned}$$

for all $A \in S_n^r$ and let $q := 1 - p$ throughout. Denote by $\varrho_n^r(p)$ the probability that the pair $(S_n, X_n^r(p))$ is a matroid on S_n (where $X_n^r(p)$ is the basis). An exact expression for $\varrho_n^r(p)$ would require in-depth knowledge about the exact structure of rank-r matroids. We shall see later that a nice recursion is possible for the $r = 2$ case. By definition

$$\varrho_n^r(p) \quad := \sum_{M(S_n,\mathcal{B})\in\mathcal{M}_r^0(S_n)} p^{|\mathcal{B}|} q^{\binom{n}{r}-|\mathcal{B}|}. \tag{6}$$

We may describe the general characteristics of $\varrho_n^r(p)$ through the use of inequalities. We see the same limiting behavior to hold in both the $r = 2, 3$ cases except under different scalings.

Theorem 6 *Let $c, r > 0$ be two fixed constants, r an integer; then*

$$\begin{aligned}\liminf_{n\to\infty} \varrho_n^r\left(\frac{c}{\binom{n}{r}}\right) &\geqslant ce^{-c}, \\ \liminf_{n\to\infty} \varrho_n^r\left(1-\frac{c}{\binom{n}{r}}\right) &\geqslant (1+c)e^{-c}.\end{aligned}$$

PROOF: From the class of rank-r matroids, let us focus upon $M_1(S_n, \mathcal{B}_1)$, $M_2(S_n, \mathcal{B}_2)$ and $M_3(S_n, \mathcal{B}_3) \in \mathcal{M}_r^0(S_n)$, where

$$\begin{aligned}\mathcal{B}_1 &= \{\{x_1, x_2, \ldots, x_r\}\}, \\ \mathcal{B}_2 &= S_n^r \backslash \{\{x_1, x_2, \ldots, x_r\}\}, \\ \mathcal{B}_3 &= S_n^r,\end{aligned}$$

are the bases for the matroids. The number of such matroids M_1 in $\mathcal{M}_r^0(S_n)$ is $\binom{n}{r}$ and the probability of any one of them arising is $pq^{\binom{n}{r}-1}$. Similarly, for M_2, the number is $\binom{n}{r}$ each with probability $p^{\binom{n}{r}-1}q$ and for M_3, the number is 1 with probability $p^{\binom{n}{r}}$. Thus we may lower bound $\varrho_n^r(p)$ by

$$\varrho_n^r(p) \quad \geqslant \quad \binom{n}{r} pq^{\binom{n}{r}-1} + \binom{n}{r} p^{\binom{n}{r}-1} q + p^{\binom{n}{r}}. \tag{7}$$

Fixing $c > 0$ we have

$$\varrho_n^r\left(\frac{c}{\binom{n}{r}}\right) \geqslant \binom{n}{r}\frac{c}{\binom{n}{r}}\left(1-\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1} + \binom{n}{r}\left(\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1}\left(1-\frac{c}{\binom{n}{r}}\right) + \left(\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}}.$$

Hence,

$$\begin{aligned}\liminf_{n\to\infty} \varrho_n^r\left(\frac{c}{\binom{n}{r}}\right) &\geqslant \liminf_{n\to\infty} \binom{n}{r}\frac{c}{\binom{n}{r}}\left(1-\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1} + \liminf_{n\to\infty}\binom{n}{r}\left(\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1}\left(1-\frac{c}{\binom{n}{r}}\right) + \liminf_{n\to\infty}\left(\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}} \\ &= \liminf_{n\to\infty} c\left(1-\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1} + \liminf_{n\to\infty}\binom{n}{r}\left(\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1}\left(1-\frac{c}{\binom{n}{r}}\right) + \liminf_{n\to\infty}\left(\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}} \\ &= \liminf_{n\to\infty} c\left(1-\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1} + 0 \\ &= ce^{-c}.\end{aligned}$$

Similarly, for $p = 1 - \frac{c}{\binom{n}{r}}$ we have

$$\varrho_n^r\left(1-\frac{c}{\binom{n}{r}}\right) \geqslant \binom{n}{r}\left(1-\frac{c}{\binom{n}{r}}\right)\left(\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1} + \binom{n}{r}\left(1-\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1}\left(\frac{c}{\binom{n}{r}}\right) + \left(1-\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}}.$$

Hence,

$$\begin{aligned}
\liminf_{n\to\infty} \varrho_n^r \left(1-\frac{c}{\binom{n}{r}}\right) &\geqslant \liminf_{n\to\infty} \binom{n}{r}\left(1-\frac{c}{\binom{n}{r}}\right)\left(\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1} + \liminf_{n\to\infty} \binom{n}{r}\left(1-\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1}\left(\frac{c}{\binom{n}{r}}\right) \\
&\quad + \liminf_{n\to\infty} \left(1-\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}} \\
&= 0 + \liminf_{n\to\infty} c\left(1-\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}-1} + \liminf_{n\to\infty}\left(1-\frac{c}{\binom{n}{r}}\right)^{\binom{n}{r}} \\
&= ce^{-c} + e^{-c} \\
&= (1+c)e^{-c}.
\end{aligned}$$

$\square$

Lemma 7 *For* $0 < p < 1$,

$$\varrho_n^r(p) \quad \leqslant \quad m_r(n)\max\{p,q\}^{\binom{n}{r}}.$$

Proof: For $p \leqslant q$ we have $\frac{p}{q} \leqslant 1$. From Expression 6,

$$\begin{aligned}
\varrho_n^r(p) &:= \sum_{M(S_n,\mathcal{B})\in\mathcal{M}_r^0(S_n)} p^{|\mathcal{B}|}q^{\binom{n}{r}-|\mathcal{B}|} \\
&\leqslant |\mathcal{M}_r^0(S_n)| \max_{M(S_n,\mathcal{B})\in\mathcal{M}_r^0(S_n)} \left\{p^{|\mathcal{B}|}q^{\binom{n}{r}-|\mathcal{B}|}\right\} \\
&= m_r(n)q^{\binom{n}{r}} \max_{M(S_n,\mathcal{B})\in\mathcal{M}_r^0(S_n)} \left\{\left(\frac{p}{q}\right)^{|\mathcal{B}|}\right\} \\
&\leqslant m_r(n)q^{\binom{n}{r}} \max_{M(S_n,\mathcal{B})\in\mathcal{M}_r^0(S_n)} \left\{1^{|\mathcal{B}|}\right\} \\
&= m_r(n)q^{\binom{n}{r}}.
\end{aligned}$$

For $q \leqslant p$, $\frac{q}{p} \leqslant 1$ and hence

$$\begin{aligned}
\varrho_n^r(p) &= \sum_{M(S_n,\mathcal{B})\in\mathcal{M}_r^0(S_n)} p^{|\mathcal{B}|}q^{\binom{n}{r}-|\mathcal{B}|} \\
&\leqslant |\mathcal{M}_r^0(S_n)| \max_{M(S_n,\mathcal{B})\in\mathcal{M}_r^0(S_n)} \left\{p^{|\mathcal{B}|}q^{\binom{n}{r}-|\mathcal{B}|}\right\} \\
&= m_r(n)p^{\binom{n}{r}} \max_{M(S_n,\mathcal{B})\in\mathcal{M}_r^0(S_n)} \left\{\left(\frac{q}{p}\right)^{\binom{n}{r}-|\mathcal{B}|}\right\} \\
&\leqslant m_r(n)p^{\binom{n}{r}} \max_{M(S_n,\mathcal{B})\in\mathcal{M}_r^0(S_n)} \left\{1^{\binom{n}{r}-|\mathcal{B}|}\right\} \\
&= m_r(n)p^{\binom{n}{r}}.
\end{aligned}$$

$\square$

The following lemma gives a rather coarse upper bound on the numbers $m_r(n)$ but is essential in showing the limit approaches 0 for p fixed.

Lemma 8 *For all* $n \geqslant 2, 3$, *respectively,*

$$\begin{aligned}
m_2(n) &\leqslant (n+1)^{n+1} \\
m_3(n) &\leqslant \prod_{i=3}^{n} i^i.
\end{aligned}$$

Proof: From Theorem 3, we have that $m_2(n) = b(n+1) - 2^n$ for all $n \geqslant 2$. Notice that the Bell numbers satisfy the inequality $b(n) \leqslant n^n$ for all $n \geqslant 1$ (proof by induction). Thus we have $m_2(n) \leqslant (n+1)^{n+1}$. We may represent any $M \in \mathcal{M}_3(S_n)$ as $n-2$ rank-2 matroids. Let $\mathcal{B}$ be the basis for M and define

$$\mathcal{B}_i(M) \quad = \quad \{\,\{x_j, x_k\}\,|\,\{x_j, x_k, x_i\} \in \mathcal{B} \text{ and } 1 \leqslant j < k < i\}$$

for all $3 \leqslant i \leqslant n$. Each matroid $M_i'\,(S_{i-1}, \mathcal{B}_i(M)) \in \mathcal{M}_2^0(S_{i-1})$ and so we may upper bound $|\mathcal{M}_3^0(S_n)|$ by

$$m_3(n) \quad < \quad \prod_{i=3}^{n} m_2^0(i-1).$$

The result now follows from direct application of the first inequality. □

We now show for fixed $p \neq 0, 1$, the values $\varrho_n^2\,(p)$ and $\varrho_n^3\,(p)$ converge to 0 for large n.

Theorem 9 *For fixed p, $0 < p < 1$, and $r = 2, 3$,*

$$\lim_{n\to\infty} \varrho_n^r\,(p) \quad = \quad 0.$$

Proof: For $r = 2$, $\varrho_n^2\,(p) \leqslant m_2^0(n) \max\{p, q\}^{\binom{n}{2}} < (n+1)^{n+1} \max\{p, q\}^{\binom{n}{2}}$ which tends to 0 for n large. From Lemma 7, let us assume that $0 < p \leqslant \frac{1}{2}$. Then,

$$\begin{aligned} \varrho_n^3\,(p) &\leqslant m_3(n) q^{\binom{n}{3}} \\ &\leqslant q^{\binom{n}{3}} \prod_{i=3}^{n} i^i, &&\text{from Lemma 8.} \end{aligned}$$

Now, as $\binom{n}{3} = \binom{n-1}{2} + \binom{n-2}{2} + \ldots + \binom{2}{2}$, we have

$$= \prod_{i=3}^{n} i^i q^{\binom{i-1}{2}} =: A(n).$$

Since $A(n)$ is a sequence of positive real numbers, then if we can show that $\lim\limits_{n\to\infty} \dfrac{A(n+1)}{A(n)}$ exists and is less than 1, then $A(n)$ converges and $\lim\limits_{n\to\infty} A(n) = 0$ (see Bartle & Sherbert [3] Theorem 3.2.11):

$$\begin{aligned} \lim_{n\to\infty} \frac{A(n+1)}{A(n)} &= \lim_{n\to\infty} (n+1)^{n+1} q^{\binom{n}{2}} \\ &= 0. \end{aligned}$$

Since the sequence $A(n)$ dominates $\varrho_n^3\,(p)$, we have

$$\limsup_{n\to\infty} \varrho_n^3\,(p) \quad \leqslant \quad \limsup_{n\to\infty}\ A(n) \quad \leqslant \quad \lim_{n\to\infty} A(n) \quad = \quad 0,$$

Because of non-negativity, the limit exists and is zero. For the case $\dfrac{1}{2} \leqslant q < 1$ the same result clearly holds. □

3.2 The Rank-2 Case and Random Graphs

A rank-2 matroid may be represented by a simple graph, with the vertices representing the elements of the ground set and the edges representing the sets in the bases. This is what Acketa [1] termed a "matroidic graph". The condition on the graph for it to be matroidic is that it have at least one edge and the collection of non-isolated vertices constitutes a complete k-partite graph for some $k \geqslant 2$. The set of isolated vertices are the loops of the matroid. We give a recursion for the probability that the standard random graph $G(n, p)$ (with edge probability p) represents such a matroidic graph, i.e. a rank-2 matroid.

For any $\pi \in \Pi_n\,(i)$ where $\pi = X_1, \ldots, X_i$, let the weight of π be

$$w(\pi) := \sum_{j=1}^{i} \binom{|X_j|}{2}.$$

We now have the precise expression:

$$\varrho_n^2(p) = \sum_{i=2}^{n}\binom{n}{n-i}\sum_{\pi\in\Pi_i(2,i)} p^{\binom{i}{2}-w(\pi)}q^{\binom{n}{2}-\binom{i}{2}+w(\pi)} \tag{8}$$

$$= q^{\binom{n}{2}}\sum_{i=2}^{n}\binom{n}{i}\left(\frac{1}{z}\right)^{\binom{i}{2}}\left\{-z^{\binom{i}{2}}+\sum_{\pi\in\Pi_i} z^{w(\pi)}\right\},$$

where $z := q/p$.

Theorem 10 *Let $\gamma_0(x) = 1$, $\gamma_1(x) = 1$ and for all $n > 0$ define*

$$\gamma_{n+1}(x) := \sum_{0\leqslant k\leqslant n}\binom{n}{k}x^{-k(n+1-k)}\gamma_k(x).$$

Then for all $n \geqslant 2$,

$$\varrho_n^2(p) = q^{\binom{n}{2}}\sum_{0\leqslant i\leqslant n}\binom{n}{i}\{\gamma_i(z)-1\}.$$

PROOF: Let $\gamma_0(x) = 1$ and $\gamma_1(x) = 1$. For all $n \geqslant 2$ define

$$\gamma_n(x) := \frac{1}{x^{\binom{n}{2}}}\sum_{\pi\in\Pi_n} x^{w(\pi)}.$$

Then we see that

$$\begin{aligned}
\gamma_{n+1}(x) &= \frac{1}{x^{\binom{n+1}{2}}}\sum_{\pi\in\Pi_{n+1}} x^{w(\pi)}\\
&= \frac{1}{x^{\binom{n+1}{2}}}\sum_{k=0}^{n}\binom{n}{n-k}\sum_{\pi'\in\Pi_k} x^{w(\pi')+\binom{1+n-k}{2}}\\
&= \frac{1}{x^{\binom{n+1}{2}}}\sum_{k=0}^{n}\binom{n}{k}x^{\binom{1+n-k}{2}}\sum_{\pi'\in\Pi_k} x^{w(\pi')}\\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{x^{\binom{1+n-k}{2}}}{x^{\binom{n+1}{2}}}\frac{x^{\binom{k}{2}}}{x^{\binom{k}{2}}}\sum_{\pi'\in\Pi_k} x^{w(\pi')}\\
&= \sum_{k=0}^{n}\binom{n}{k}x^{\binom{1+n-k}{2}+\binom{k}{2}-\binom{n+1}{2}}\gamma_k(x).
\end{aligned}$$

Now $\binom{1+n-k}{2}+\binom{k}{2}-\binom{n+1}{2} = -k(n-k+1)$ so the above expression becomes

$$\gamma_{n+1}(x) = \sum_{k=0}^{n}\binom{n}{k}x^{-k(n-k+1)}\gamma_k(x).$$

From equation 8,

$$\begin{aligned}
\varrho_n^2(p) &= q^{\binom{n}{2}}\sum_{i=2}^{n}\binom{n}{i}\left(\frac{1}{z}\right)^{\binom{i}{2}}\left\{-z^{\binom{i}{2}}+\sum_{\pi\in\Pi_i} z^{w(\pi)}\right\}\\
&= q^{\binom{n}{2}}\sum_{i=2}^{n}\binom{n}{i}\left\{-1+\left(\frac{1}{z}\right)^{\binom{i}{2}}\sum_{\pi\in\Pi_i} z^{w(\pi)}\right\}\\
&= q^{\binom{n}{2}}\sum_{i=2}^{n}\binom{n}{i}\{-1+\gamma_i(z)\},
\end{aligned}$$

and since $\gamma_0(x) = \gamma_1(x) = 1$,

$$\varrho_n^2(p) = q^{\binom{n}{2}}\sum_{i=0}^{n}\binom{n}{i}\{\gamma_i(z)-1\}.$$

$\square$

By definition, $\varrho_n(0) = 0$ and $\varrho_n(1) = 1$. Figure 3 shows $\varrho_n^2(p)$ for small values of n and we see its evolving nature with regard to Theorems 6 and 9.

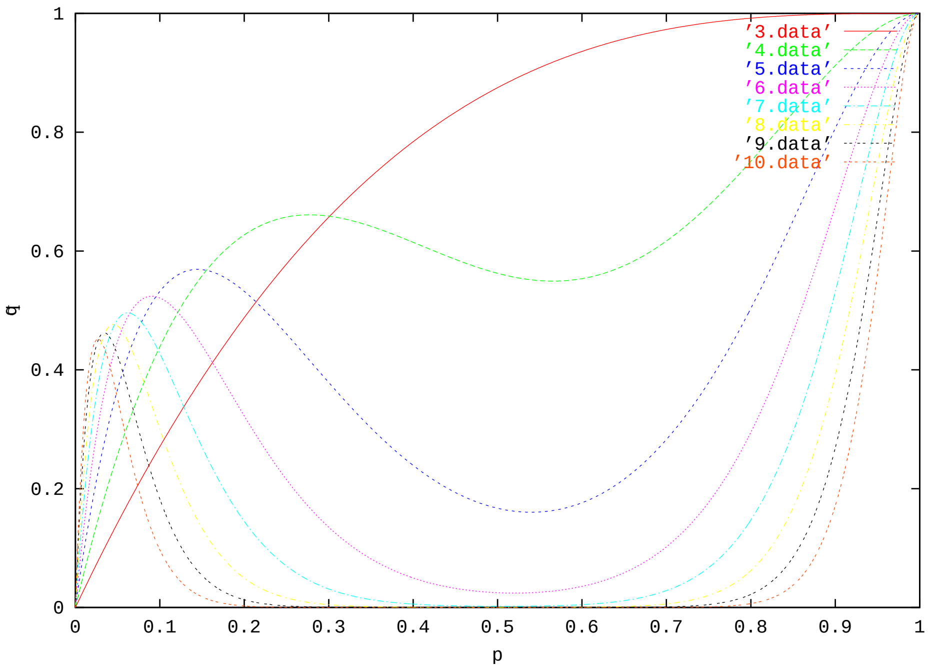

Figure 3: The graph of $\varrho_n^2(p)$ for small values of n.

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