DIAS-STP-02-5

Probabilistic derivation of a noncommutative version of Varadhan's theorem

T. C. Dorlas
Dublin Institute for Advanced Studies
School of Theoretical Physics
10 Burlington Road, Dublin 4, Ireland.

June 24, 2002

Abstract

We give a simple probabilistic derivation of a special case of a noncommutative version of Varadhan's theorem, first proved by Petz, Raggio and Verbeure. It is based on a Feynman-Kac representation combined with a standard large deviation argument. In the final section, this theorem is then extended to a more difficult situation with Bose-symmetry.

1 Introduction

Inspired by the work of Cegła, Lewis and Raggio [1] in which they use a combination of large deviation theory and group representation theory to derive a variational formula for the free energy of mean-field quantum spin systems, Petz, Raggio and Verbeure [2] derived a noncommutative version of Varadhan's theorem [3, 4] using C^*-algebraic methods. Their theorem was subsequently generalised to include inhomogeneous mean-field models by Raggio and Werner [5]. Here I present a probabilistic proof of a special case of the theorem of Petz, Raggio and Verbeure using a Feynman-Kac representation together with the standard Varadhan's theorem. This work is based on earlier work by the author on limit theorems and Feynman-Kac representations for bosons [6, 7]. Even though this proof applies only to a special case, it may nevertheless be of interest, especially as it may be easier to generalise to other situations. Indeed, one particular generalisation is considered here in Section 4. This concerns the case of two sets of creation and annihilation operators considered in [6], where there is Bose-symmetry and the unperturbed state is not a product state.

The setting for the theorem of Petz, Raggio and Verbeure is as follows. Let $\mathcal{M}$ be the algebra of all complex $m \times m$ matrices and let ρ be a state on $\mathcal{M}$ with density matrix D_ρ, i.e. $\rho(A) = \mathrm{Trace}(D_\rho A)$ and in particular, $\mathrm{Trace}\, D_\rho = 1$. (They consider a more general setting, where $\mathcal{M}$ is a more general unital C^*-algebra, but this simplest case is most relevant for quantum spin models.) Let $\mathcal{A} = \otimes_{k\in\mathbb{N}}\mathcal{M}$ be the infinite tensor product, and denote ω_ρ the infinite tensor product state $\otimes\rho$ on $\mathcal{A}$. Let $x \in \mathcal{M}$ be a fixed self-adjoint element, and suppose that f is a continuous, real-valued function on $[-||x||, ||x||]$. Denote by $x^{(n)}$ the element of $\mathcal{A}$ given by

$$x^{(n)} = \frac{1}{n}(x_1 + \ldots + x_n),$$

where x_k is a copy of x in the k-th factor of $\mathcal{A}$. Let $(\omega_\rho)^{nf(x^{(n)})}$ denote the (unnormalised) perturbed state on $\mathcal{A}$ with relative Hamiltonian $nf(x^{(n)})$. This state can be defined using the GNS representation of $\mathcal{A}$ associated with the product state ω_ρ: see [8].

In this setting we have:

Theorem 1.1 *The following holds:*

$$\lim_{n\to\infty} \frac{1}{n} \ln(\omega_\rho)^{nf(x^{(n)})}(1) = -\inf_{u\in[-||x||,||x||]} \{f(u) + I(u)\}, \qquad (1.1)$$

where $I(u)$ is the Legendre transform of the function

$$C(u) = \ln \mathrm{Trace} \exp(\ln D_\rho + ux). \qquad (1.2)$$

The special case we will be considering is the following. Let $e_1, \ldots, e_m$ be an orthonormal basis of $\mathbb{C}^m$ for which x is diagonal, and let $\lambda_1 \leq \ldots \leq \lambda_m$ be the corresponding eigenvalues. With respect to this basis, let $(h_{i,j})_{i,j=1,\ldots,m}$ be the matrix of h, where $D_\rho = e^{-h}$. We now assume that all off-diagonal matrix elements of h are negative or zero.

To prove this theorem using large deviation theory, we first derive a Feynman-Kac representation for the left-hand side expression.

We can define a Markov process with transition probabilities

$$\mathbb{P}(\xi(t+\delta t) = k' | \xi(t) = k) = \begin{cases} -h_{k',k}\delta t & \text{if } k' \neq k, \\ 1 + \sum_{k'' \neq k} h_{k'',k}\delta t & \text{if } k' = k. \end{cases} \tag{1.3}$$

Equivalently,

$$p_{t'-t}(k,k') = \mathbb{P}\left(\xi(t') = k' \,|\, \xi(t) = k\right) = \left(e^{-(t'-t)\tilde{h}}\right)_{k',k}, \tag{1.4}$$

where $\tilde{h}$ is defined by $\tilde{h}_{k',k} = h_{k',k}$ for $k' \neq k$ and $\tilde{h}_{k,k} = -\sum_{k'' \neq k} h_{k'',k}$. Using the Trotter product formula we then find

$$< k' \,|\, \exp[-h + f(x)] \,|\, k >=$$

$$\mathbb{E}\left\{\exp\left[\int_0^1 \left(f(\lambda(\xi(t))) + h_D(\xi(t))\right) dt\right] 1_{\{\xi(1)=k'\}} \,|\, \xi(0) = k\right\}. \tag{1.5}$$

Here $\lambda(u)$ is a continuous function such that $\lambda(k) = \lambda_k$ for $k = 1, \ldots, m$ and similarly, $h_D(u)$ is a continuous function such that $h_D(k) = \tilde{h}_{k,k} - h_{k,k}$.

More generally, we have for a continuous function F of n variables,

$$< k_1', \ldots, k_n' \,|\, \exp[-h^{(n)} + F(x_1, \ldots, x_n)] \,|\, k_1, \ldots, k_n >=$$

$$\mathbb{E}\left\{\exp\left[\int_0^1 \left(F(\lambda(\xi_1(t)), \ldots, \lambda(\xi_n(t))) + \sum_{i=1}^n h_D(\xi_i(t))\right) dt\right] \times \prod_{i=1}^n 1_{\{\xi_i(1)=k_i'\}} \,\middle|\, \xi_i(0) = k_i \;\forall i\right\}. \tag{1.6}$$

In particular we can apply this formula to the function $F(u_1, \ldots, u_n) = nf\left(\frac{1}{n}\sum_{i=1}^n u_i\right)$.

We now give a short outline of the proof of the PRV theorem in our special case. As we have a product measure on the product of path spaces, it satisfies the large deviation property with rate function $I[\eta]$ given by the Legendre transform of the cumulant generating function $C[\phi]$ given by

$$C[\phi] = \ln \sum_{k=1}^{m} \mathbb{E}\left\{ \exp\left[\int_0^1 (\lambda(\xi(t))\phi(t) + h_D(\xi(t)))dt \right] 1_{\xi(1)=k} \,\middle|\, \xi(0) = k \right\}. \tag{1.7}$$

We conclude that the following variational formula holds for the left hand side in the PRV theorem:

$$\lim_{n\to\infty} \frac{1}{n} \ln(\omega_\rho)^{nf(x^{(n)})}(1) = - \inf_{\eta\in L^2[0,1]} \left\{ \int_0^1 f(\eta(t))\, dt + I[\eta] \right\}. \tag{1.8}$$

(Here η stands for $\lambda\circ\xi$.) We shall prove that the supremum is in fact attained for a constant function
$eta(t) = u$. Moreover, we show that in that case I is the Legendre transform of a function of a single real variable, namely $C[\phi]$ where ϕ is a constant function. This can be evaluated using (1.5) and yields simply

$$C(a) = \ln \mathrm{Trace}\, e^{-h+ax} = \ln \rho^{-ax}(\mathbf{1}). \tag{1.9}$$

The above is an outline of the main ideas of the proof. In the following section the Feynman-Kac formula and the large deviation property are proved and in section 3 the above statements about the maximiser will be proved.

2 The Feynman-Kac formula and the large deviation property

The Markov process with transition probabilities given by (1.4) is well-defined, and there exist measures $\mathbb{P}_k$ on the Skorohod space $D[0,1]$ with finite-dimensional marginals given by

$$\mathbb{P}_k\left(\xi(t_1) \in B_1, \ldots, \xi(t_p) \in B_p\right) = \sum_{k_1\in\{1,\ldots,m\}\cap B_1} \cdots \sum_{k_p\in\{1,\ldots,m\}\cap B_p} \prod_{i=1}^{p} p\left(\xi(t_i) = k_i \,|\, \xi(t_{i-1}) = k_{i-1}\right) \tag{2.1}$$

for $t_1 < \ldots < t_p$, writing $t_0 = 0$ and $k_0 = k$. (For a proof see [9] or [6].) The Feynman-Kac formula (1.5) then follows from the Trotter product formula (with $t_i = i/M$):

$$\begin{aligned} &< k' \,|\, e^{-h+f(x)} \,|\, k >= \\ &= \lim_{M\to\infty} \sum_{k_1,\dots,k_M=1}^{m} \prod_{i=1}^{M} < k_i \,|\, \exp[-\tilde{h}/M] \,|\, k_{i-1} > e^{(h_D(k_i)+f(\lambda_{k_i}))/M} \\ &= \lim_{M\to\infty} \sum_{k_1,\dots,k_M=1}^{m} \prod_{i=1}^{M} p\left(\xi(t_i) = k_i \,|\, \xi(t_{i-1}) = k_{i-1}\right) e^{(h_D(k_i)+f(\lambda_{k_i}))/M} \\ &= \lim_{M\to\infty} \mathbb{E}_k \left\{ \exp\left[\frac{1}{M} \sum_{i=1}^{M} h_D(\xi(t_i)) + f(\lambda(\xi(t_i))) \right] 1_{\{\xi:\, \xi(1)=k'\}} \right\}. \end{aligned} \tag{2.2}$$

The limit yields (1.5) by the continuity of $\int_0^1 f(\xi(t))dt$ as a function of $\xi \in D[0,1]$. In [6] it was shown that the inclusion $D[0,1] \to L^2[0,1]$ is continuous, so we also have a measure on $L^2[0,1]$. This derivation easily generalises to (1.6). We now a have a strong large deviation result by the Donsker-Varadhan theorem [10] (see also [11], Theorem (3.34)).

Theorem 2.1 *Consider the random variable X with values in $L^2[0,1]$ given by the measures ν defined by*

$$\int F[\eta]\, \nu[d\eta] = \sum_{k=1}^{m} \mathbb{E}_k \left(F(\lambda \circ \xi) \exp\left[\int_0^1 h_D(\xi(t))\, dt \right] 1_{\{\xi:\, \xi(1)=k\}} \right), \tag{2.3}$$

for any bounded continuous function F on $L^2[0,1]$. Define $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, where X_i is a copy of X. Then the sequence $(\overline{X}_n)_{n=1}^{\infty}$ satisfies the large deviation principle with rate function given by

$$I[\eta] = \sup_{\phi \in L^2[0,1]} \{ < \eta\,,\, \phi > - C[\phi] \}, \tag{2.4}$$

where

$$C[\phi] = \ln \sum_{k=1}^{m} \mathbb{E}_k \left\{ \exp\left[\int_0^1 \left(\lambda(\xi(t))\phi(t) + h_D(\xi(t)) \right) dt \right] 1_{\{\xi:\, \xi(1)=k\}} \right\}. \tag{2.5}$$

3 The variational formula

The perturbed state $\omega_\rho^{nf(x^{(n)})}$ is defined by

$$\omega_\rho^{nf(x^{(n)})}(A) = < \Omega_\rho^{nf(x^{(n)})} \,|A\, \Omega_\rho^{nf(x^{(n)})} >, \tag{3.1}$$

where $\Omega_\rho^{nf(x^{(n)})}$ is a vector in the Hilbert space of the GNS representation of the product state $\omega_\rho(A) =< \Omega \,|\, A\,\Omega >$ given by

$$\Omega_\rho^{nf(x^{(n)})} = e^{-(h^{(n)}+nf(x^{(n)}))/2} e^{h^{(n)}/2}\Omega. \tag{3.2}$$

Using the fact that $\omega_\rho(A) = \mathrm{Trace}\, A e^{-h^{(n)}}$, we get

$$\begin{aligned} \omega_\rho^{nf(x^{(n)})}(A) &= \langle e^{-(h^{(n)}+nf(x^{(n)}))/2} e^{h^{(n)}/2}\Omega \,|\, A e^{-(h^{(n)}+nf(x^{(n)}))/2} e^{h^{(n)}/2}\Omega\rangle \\ &= \omega_\rho\left(e^{h^{(n)}/2} e^{-(h^{(n)}+nf(x^{(n)}))/2} A e^{-(h^{(n)}+nf(x^{(n)}))/2} e^{h^{(n)}/2}\right) \\ &= \mathrm{Trace}\, A e^{-(h^{(n)}+nf(x^{(n)}))}. \end{aligned} \tag{3.3}$$

Inserting the representation (1.6) we get

$$\omega_\rho^{nf(x^{(n)})}(1) = \nu_n \left\{ \exp\left[-n\int_0^1 f(\eta(t))dt\right]\right\}, \tag{3.4}$$

where ν_n is the distribution of $\overline{X}_n$. Applying Varadhan's theorem then yields

$$\lim_{n\to\infty} \frac{1}{n} \ln \omega_\rho^{nf(x^{(n)})}(1) = \sup_{\eta\in L^2[0,1]} \left\{ -\int_0^1 f(\eta(t)))dt - I[\eta]\right\}. \tag{3.5}$$

We now want to show that the supremum is attained at a constant function. However, first of all we prove that if η is constant then the maximiser ϕ in (2.4) is also constant:

Theorem 3.1 *If η is a constant function then the supremum in*

$$I[\eta] = \sup_{\phi\in L^2} \{< \eta, \phi > -C[\phi]\}$$

is attained at a constant function ϕ.

Proof. We first remark that C is continuous as a function of $\phi \in L^2[0,1]$:

Lemma 3.1 *The generating functional $C[\phi]$ defined by (2.5) is convex and continuous as a function of $\phi \in L^2[0,1]$.*

Proof. The continuity follows immediately from the inequality

$$|C[\phi'] - C[\phi]| = \left| \ln \int e^{\langle \phi', \eta \rangle} \nu[d\eta] - \ln \int e^{\langle \phi, \eta \rangle} \nu[d\eta] \right| \leq ||x|| \, ||\phi' - \phi||_1$$

which follows from the fact that $|\lambda(\xi(t))| \leq ||x||$ with probability 1 (See e.g. [12]). The convexity follows from Hölder's inequality. QED

Let us introduce the Haar basis $\{h_p | \, p = 0, 1, \ldots\}$ for $L^2[0,1]$ consisting of the functions h_p defined by $h_0(t) = 1$ and if $2^m \leq p \leq 2^{m+1} - 1$,

$$h_p(t) = \begin{cases} 2^{m/2}, & \text{if } p2^{-m} - 1 \leq t < (p + \frac{1}{2})2^{-m} - 1; \\ -2^{m/2}, & \text{if } (p + \frac{1}{2})2^{-m} - 1 \leq t \leq (p+1)2^{-m} - 1; \\ 0, & \text{otherwise.} \end{cases} \quad (3.6)$$

It follows from the lemma that for every $\epsilon > 0$, there exists a ϕ in the space $\mathcal{H}_m$ spanned by $h_0, h_1, \ldots, h_{2^m-1}$ such that $I[\xi] < \langle \phi, \xi \rangle - C[\phi] + \epsilon$. Let $\xi(t) = u$ be constant and put $a_p = \langle \phi, h_p \rangle$ $(p = 0, 1, \ldots, 2^m - 1)$. The Euler-Lagrange equations for the variational problem (2.4) are then

$$u = \frac{\nu \left[\langle h_0, \eta \rangle e^{\langle \phi, \eta \rangle} \right]}{\nu \left[e^{\langle \phi, \eta \rangle} \right]} \quad \text{and} \quad 0 = \frac{\nu \left[\langle h_p, \eta \rangle e^{\langle \phi, \eta \rangle} \right]}{\nu \left[e^{\langle \phi, \eta \rangle} \right]}. \quad (3.7)$$

But the equations for $p > 0$ have the trivial solution $a_p = 0$ by the symmetry of the measure $\nu[\bullet \, e^{\langle \phi, \eta \rangle}]/\nu[e^{\langle \phi, \eta \rangle}]$ if $\phi = a_0$ is constant, and the first equation has a solution $a_0 \in \mathbb{R}$ provided $\lambda_- < u < \lambda_+$ where λ_+ and λ_- are the upper and lower bounds of the spectrum of x. If $u > \lambda_+$ or $u < \lambda_-$ then the maximum is zero as follows upon taking a_0 to $\pm\infty$. (See the lemma below.) Finally, if $u = \lambda_-$ or $u = \lambda_+$ then for any fixed a_0 the maximum is still obtained for $a_1 = \ldots = a_{2^m-1} = 0$ by convexity of C. This completes the proof of Theorem 3.1. QED

Theorem 3.2 *The supremum in (3.5) is attained at a constant function $\eta(t) = u$.*

Proof. We first prove the following lemma.

Lemma 3.2 *If* $\operatorname{ess\,sup}_{t \in [0,1]} \eta(t) > \lambda_+ = \sup_{\lambda \in \sigma(x)} \lambda$ *or* $\operatorname{ess\,inf}_{t \in [0,1]} \eta(t) < \lambda_- = \inf_{\lambda \in \sigma(x)} \lambda$ *then* $I(\eta) = +\infty$.

Proof. Suppose $\operatorname{ess\,sup}(\eta) > \lambda_+$. Then there exists $\epsilon > 0$ and a measurable subset $A \subset [0,1]$ with Lebesgue measure $|A| > 0$ such that $\eta \geq \lambda_+ + \epsilon$ for all $t \in A$. Then put $\phi = c\,1_A$ so that $\langle \phi, \eta \rangle \geq (\lambda_+ + \epsilon)c|A|$ and

$$
\begin{aligned}
C[\phi] &= \ln \int e^{\langle \phi, \eta \rangle} \nu[d\eta] \\
&= \ln \sum_{k=1}^{m} \mathbb{E}_k \left[e^{\langle \phi, \lambda \circ \xi \rangle} e^{\int_0^1 h_D(\xi(t))dt} 1_{\{\xi(1)=k\}} \right] \\
&\leq \lambda_+ c|A|.
\end{aligned} \tag{3.8}
$$

Hence $\langle \phi, \eta \rangle - C[\phi] \geq \epsilon c|A| \to +\infty$ as $c \to +\infty$. The case $\operatorname{ess\,inf}(\eta) < \lambda_-$ is similar. QED

We now continue with the proof of the theorem. Let $\eta \in L^2[0,1]$. We want to prove that

$$
I[\eta] \geq \int_0^1 I(\eta(t))\, dt, \tag{3.9}
$$

where $I(u)$ denotes $I[\xi]$ for the constant function $\xi(t) = u$. The theorem then follows immediately since this would imply that

$$
\begin{aligned}
\int_0^1 f(\eta(t))dt + I[\eta] &\geq \int_0^1 \left(f(\eta(t)) + I(\eta(t)) \right) dt \\
&\geq \inf_{u \in \mathbb{R}} \{ f(u) + I(u) \}.
\end{aligned} \tag{3.10}
$$

By the lemma, we may clearly assume that $\eta(t) \in [\lambda_-, \lambda_+]$ for a.e. t. By Theorem 3.1, $I(u)$ is given by

$$
I(u) = \sup_{a \in \mathbb{R}} \{ au - C(a) \}, \tag{3.11}
$$

where $C(a)$ is given by

$$
C(a) = \ln \int e^{a \int_0^1 \eta(t)dt} \nu[d\eta] = \ln \operatorname{Trace} e^{-h+ax}. \tag{3.12}
$$

This is formula (1.9). The function $C(a)$ is clearly convex and infinitely differentiable. Its derivative increases from λ_- to λ_+ as a runs from $-\infty$ to $+\infty$. Therefore, if $\eta(t) \in (\lambda_-, \lambda_+)$ there is a $\phi(t)$ such that $I(\eta(t)) = \phi(t)\,\eta(t) - C(\phi(t))$ On the other hand, if $\eta(t) = \lambda_\pm$ then the supremum is attained as $\phi(t) \to \pm\infty$. In that case we truncate $\phi(t)$ to $\pm n$. An important technical point here is that since $C'(a)$ is increasing, it maps intervals to intervals, which implies that the function $\phi(t)$ (and also the truncated function $\phi_n(t)$) is measurable as it is determined by $C'(\phi(t)) = \eta(t)$. We conclude that there exists, for any given $\epsilon > 0$, a function $\phi_n \in L^\infty[0,1]$ such that $I(\eta(t)) < \phi_n(t)\,\eta(t) - C(\phi_n(t)) + \epsilon$ for a.e. t. Clearly, $I[\eta] \geq \langle \phi_n, \eta \rangle - C[\phi_n]$ so it remains to show that $C[\phi_n] \leq \int_0^1 C(\phi_n(t))\,dt$. By continuity, we may assume that $\phi_n \in \mathcal{H}_m$ for some $m \in \mathbb{N}$. Then ϕ_n can be written in the form $\phi_n = \sum_{k=1}^{2^m} a_k\, 1_{[(k-1)2^{-m}, k2^{-m}]}$ and we need to prove that $C[\phi_n] \leq 2^{-m} \sum_{k=1}^{2^m} C(a_k)$. This is a consequence of the Hölder inequality for trace norms: see [13]. Indeed, we can write $C[\phi_n]$ as follows:

$$C\left[\sum_{k=1}^{2^m} a_k 1_{[(k-1)2^{-m}, k2^{-m}]}\right] = \ln \text{Trace} \prod_{k=1}^{2^m} e^{-2^{-m}(h - a_k x)}. \tag{3.13}$$

Writing

$$A_k = e^{-2^{-m}(h - a_k x)} \tag{3.14}$$

we therefore need to prove that

$$\text{Trace}\left(\prod_{k=1}^{2^m} A_k\right) \leq \prod_{k=1}^{2^m} ||A_k||_{2^m}. \tag{3.15}$$

This follows by repeated application of the Hölder inequality [13], Prop. 5 of Appendix IX.4. In fact, a more elementary proof is obtained by repeated application of the Cauchy-Schwarz inequality as in the proof of the Golden-Thompson inequality [14], [15].

We repeat the Golden-Thompson proof here for convenience:

Lemma 3.3 *For positive definite matrices A and B, the following holds:*

$$\text{Trace}(AB)^{2^m} \leq \text{Trace}(A^2B^2)^{2^{m-1}} \tag{3.16}$$

for all integers $m \geq 1$.

Proof. For $m = 1$ the result follows directly from the Cauchy-Schwarz inequality:

$$\text{Trace}(AB)^2 \leq \text{Trace}(AB)^*(AB) = \text{Trace}\, A^2B^2. \tag{3.17}$$

Now suppose the inequality is proved for all positive definite matrices A and B and all integers $\leq m-1$. We introduce the matrices X_n and Y_n as follows:

$$X_n = (AB)^{2^n}(BA)^{2^n} \text{ and } Y_n = (BA)^{2^n}(AB)^{2^n}. \tag{3.18}$$

Clearly, X_n and Y_n are also positive definite. Moreover, the following identities hold:

$$\text{Trace}(X_n)^p = \text{Trace}(Y_n)^p = \text{Trace}(X_{n-1}Y_{n-1})^p \tag{3.19}$$

for all integers $p \geq 1$. Now, $\text{Trace}(AB)^{2^m} \leq \text{Trace}\, X_{m-1}$ by the Cauchy-Schwarz inequality as in (3.17). We now prove that

$$\text{Trace}(X_{m-k})^{2^{k-1}} \leq \text{Trace}(X_{m-k-1})^{2^k} \tag{3.20}$$

for $k = 1, \ldots, m-1$. By the above identities,

$$\text{Trace}(X_{m-k})^{2^{k-1}} = \text{Trace}(X_{m-k-1}Y_{m-k-1})^{2^{k-1}} \tag{3.21}$$

and applying the induction hypothesis repeatedly with A replaced by X_{m-k-1} and B by Y_{m-k-1} and m by $k-1, k-2, \ldots, 1$, we have

$$\text{Trace}(X_{m-k-1}Y_{m-k-1})^{2^{k-1}} \leq \text{Trace}\left(X_{m-k-1}^{2^{k-1}} Y_{m-k-1}^{2^{k-1}}\right). \tag{3.22}$$

Another application of the Cauchy-Schwarz inequality shows that this is bounded by

$$\left(\text{Trace}\, X_{m-k-1}^{2^k}\right)^{1/2} \left(\text{Trace}\, Y_{m-k-1}^{2^k}\right)^{1/2} = \text{Trace}\, X_{m-k-1}^{2^k}. \tag{3.23}$$

The proof of the lemma is now complete since we can iterate (3.20), and $\text{Trace}\, X_0^{2^{m-1}} = \text{Trace}(A^2B^2)^{2^{m-1}}$. QED

We then have:

Lemma 3.4 *For all matrices A and B and all $k \geq 1$,*

$$||AB||_{2^k} \leq ||A||_{2^{k+1}} ||B||_{2^{k+1}}. \tag{3.24}$$

Proof. We have, using the iterated version of the above lemma,

$$\begin{aligned} \text{Trace}((AB)^*(AB))^{2^{k-1}} &= \text{Trace}(B^*A^*AB)^{2^{k-1}} \\ &= \text{Trace}(BB^*A^*A)^{2^{k-1}} \\ &\leq \text{Trace}\left((BB^*)^{2^{k-1}}(A^*A)^{2^{k-1}}\right) \\ &\leq \left[\text{Trace}(BB^*)^{2^k}\ \text{Trace}(A^*A)^{2^k}\right]^{1/2} \end{aligned} \tag{3.25}$$

The lemma follows by taking the 2^k-th root. QED

Iterating this lemma we obtain

$$
\begin{aligned}
\text{Trace}(A_1 \ldots A_{2^m}) &\le ||A_1 \ldots A_{2^{m-1}}||_2 ||A_{2^{m-1}+1} \ldots A_{2^m}||_2 \\
&\le \left(\prod_{k=1}^{2^{m-1}} ||A_k||_{2^m} \right) \left(\prod_{k=2^{m-1}+1}^{2^m} ||A_k||_{2^m} \right), \quad (3.26)
\end{aligned}
$$

which proves (3.19). QED

Corollary *Theorem 1.1 holds in the special case that $D_\rho = e^{-h}$ where h has a matrix $(h_{i,j})_{i,j=1}^m$ with respect to an orthonormal basis $\{e_1, \ldots, e_m\}$ of eigenvectors of x, satisfying $h_{i,j} \le 0$ for $i \ne j$.*

4 Bose-Einstein Statistics

As an example of a more complicated case, we now prove a large deviation result for the measure considered in [6]. A physically more interesting case will be analysed in [17].

Let $a_\pm^*$ and $a_\pm$ be creation and annihilation operators satisfying the commutation relations $[a_\pm, a_\pm^*] = 1$ and $[a_\pm, a_\mp^*] = 0$. Define

$$c_\pm = \frac{1}{\sqrt{2}}(a_+ + a_-) \text{ and } \Delta = a_+^* a_+ - a_-^* a_-. \quad (4.1)$$

It was shown in [6] that the following Feynman-Kac formula holds:

$$\frac{\text{Trace}_n\, e^{-\beta c_-^* c_- + f(\Delta/n)}}{\text{Trace}_n\, e^{-\beta c_-^* c_-}} = \mathbb{K}_\beta^{(n)} \left[\exp\left(\int_0^1 f(\xi(t)) dt \right) \right], \quad (4.2)$$

where $\mathbb{K}_\beta^{(n)}$ is the measure on $L^2[0,1]$ given by

$$\mathbb{K}_\beta^{(n)}[F] = \frac{1}{Z_n(\beta)} \sum_{k=0}^n \sum_{\substack{x'_1, \ldots, x'_n = \pm 1 \\ \#\{i:\, x'_i = 1\} = k}} \mathbb{E}_{(x_1, \ldots, x_n)}^{(n)} \left[F\left(\frac{1}{n} \sum_{i=1}^n \xi_i \right) \prod_{i=1}^n 1_{\{\xi:\, \xi_i(1) = x'_i\}} \right] \quad (4.3)$$

for any continuous function $F : L^2[0,1] \to \mathbb{R}$, where $\mathbb{E}_{(x_1, \ldots, x_n)}^{(n)}$ is the expectation w.r.t. the n-fold product measure over paths $\xi_i : [0,1] \to \mathbb{R}$ with values ± 1 and with hopping probabilities given by

$$\mathbb{P}_\beta[\xi(t + \delta t) = x' \,|\, \xi(t) = x] = \begin{cases} 1 - \frac{\beta}{2}\delta t, & \text{if } x' = x; \\ \frac{\beta}{2}\delta t, & \text{if } x' \ne x. \end{cases} \quad (4.4)$$

In (4.3), $x_i = +1$ for $i \le k$ and $x_i = -1$ for $k < i \le n$. We now want to show that the measure (4.3) satisfies a large deviation principle. A trivial modification of Theorem 3.2 yields a formula for the generating function:

$$\mathbb{K}_\beta^{(n)}\left[\exp\left(n\int_0^1 u(t)\xi(t)dt\right)\right] = \frac{1}{Z_n(\beta)}\frac{(\lambda_+[u])^{n+1} - (\lambda_-[u])^{n+1}}{\lambda_+[u] - \lambda_-[u]}, \tag{4.5}$$

where $\lambda_\pm[u]$ are the maximal and minimal eigenvalues of the 2×2-matrix $A[u]$ given by

$$\langle x' \,|\, A[u] \,|\, x\rangle = \mathbb{E}_x\left[\exp\left(\int_0^1 u(t)\xi(t)dt\right) 1_{\{\xi:\, \xi(1)=x'\}}\right]. \tag{4.6}$$

Taking limits we obtain

$$C[u] = \ln \lambda_+[u]. \tag{4.7}$$

Theorem 4.1 *The measures $\mathbb{K}_\beta^{(n)}$ on $L^2[0,1]$ satisfy a large deviation principle with rate function given by the Legendre transform of the function $C[u]$ given by (4.7).*

Proof. We follow the procedure as in [16]. We first prove the upper bound for compact sets $K \subset L^2[0,1]$. This is straightforward: using Gärtner's lemma there exists, given $\gamma < \inf_{\phi\in K} I[\phi]$, a finite set $u_1, \ldots, u_r \in L^2[0,1]$ such that $K \subset \cup_{j=1}^r \{\phi \in L^2[0,1] :\, \langle\phi, u_j\rangle - C[u_j] \ge \gamma\}$. But then,

$$\begin{aligned} \mathbb{K}_\beta^{(n)}(K) &\le \sum_{j=1}^r \mathbb{K}_\beta^{(n)}\left(\{\phi \in L^2[0,1] :\, \langle\phi, u_j\rangle - C[u_j] \ge \gamma\}\right) \\ &\le \sum_{j=1}^r e^{-n(C[u_j]+\gamma)} \int_{L^2[0,1]} e^{n\langle\phi,u_j\rangle}\mathbb{K}_\beta^{(n)}[d\phi] \\ &= e^{-n\gamma}\sum_{j=1}^r e^{n\{C_n[u_j]-C[u_j]\}}. \end{aligned} \tag{4.8}$$

It follows that

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{K}_\beta^{(n)}[K] \le -\gamma, \tag{4.9}$$

which implies the large deviation upper bound since $\gamma < \inf_{\phi\in K} I(\phi)$ is arbitrary. To extend this result to arbitrary closed sets, we use the Donsker-Varadhan theorem. Introducing the auxiliary measure

$$\mu[A] = \frac{1}{4}\sum_{x,x'} \mathbb{P}_{\beta,x}\left[A \,|\, \xi(1) = x'\right], \tag{4.10}$$

we have that if X_i are independent random variables with distribution given by μ, $\frac{1}{n}\sum_{i=1}^n X_i$ satisfies the large deviation principle. In particular, there exists, for all $L > 0$, a compact set $K \subset L^2[0,1]$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu^n \left[\frac{1}{n} \sum_{i=1}^n X_i \in K^c \right] < -L. \tag{4.11}$$

It follows that

$$\mathbb{P}^{(n)}_{\beta,(x_1,\ldots,x_n)} \left[\frac{1}{n} \sum_{i=1}^n \xi_i \in K^c \,|\, \xi_i(1) = x_i' \, \forall i = 1, \ldots, n \right] < e^{-nL} 4^n = e^{-nL'}. \tag{4.12}$$

The same bound therefore also holds for $\mathbb{K}^{(n)}_\beta$. Now, if F is a general closed set, we can write

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{K}^{(n)}_\beta[F] &= \limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{K}^{(n)}_\beta[F \cap K] \\ &\qquad \vee \limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{K}^{(n)}_\beta[F \cap K^c] \\ &\le - \inf_{\xi \in F \cap K} I[\xi] \vee (-L') \\ &\le - \inf_{\xi \in F} I[\xi] \end{aligned} \tag{4.13}$$

if L is large enough.

To prove the lower bound, we again emulate the strategy of [16] and define a shifted measure. Let O be a given open subset of $L^2[0,1]$ and $\epsilon > 0$. There exists $\xi_0 \in O$ such that $I[\xi] \ge I[\xi_0] - \epsilon$ for all $\xi \in O$. Let $\delta > 0$ be so small that $B(\xi_0, 2\delta) \subset O$. By (4.12) there exists a compact set K such that $\mathbb{K}^{(n)}_\beta[K^c] < e^{-nL'}$ given $L' > I[\xi_0] + \epsilon$. By compactness there exists $m \in \mathbb{N}$ so that $\sum_{p \ge 2^m} |\xi^{(p)}|^2 < \delta^2$ for all $\xi \in K$ and also for $\xi = \xi_0$. We now claim that there exists $u_0 \in \mathcal{H}_m$ such that

$$\left. \frac{\partial C[u]}{\partial u^{(p)}} \right|_{u=u_0} = \xi_0^{(p)} \tag{4.14}$$

for $p = 0, 1, \ldots, 2^m - 1$. To see this, we compute the derivatives of C_n. Since

$$C_n[u] = \frac{1}{n} \ln \sum_{k=0}^n \binom{n}{k}^{-1} \sum^{(k)}_{\substack{x_1,\ldots,x_n \\ x_1',\ldots,x_n'}} \prod_{i=1}^n \mathbb{E}_{x_i} \left[e^{\langle \xi_i, t \rangle} \chi_{x_i'}(\xi_i) \right], \tag{4.15}$$

where the symbol (k) above the sum indicates that $\#\{i: x_i = +1\} = \#\{i : x'_i = +1\} = k$, and $\chi_{x'}(\xi) = 1_{\{\xi:\, \xi(1)=x'\}}$, we have

$$\frac{\partial C_n[u]}{\partial u^{(p)}} = e^{-nC_n[u]} \sum_{k=0}^{n} \binom{n}{k}^{-1} \sum_{\substack{x_1,\dots,x_n \\ x'_1,\dots,x'_n}}^{(k)} \mathbb{E}_{(x_1,\dots,x_n)} \left[\left(\frac{1}{n} \sum_{j=1}^{n} \xi_j^{(p)} \right) \prod_{i=1}^{n} e^{\langle \xi_i, u\rangle} \chi_{x'_i}(\xi_i) \right]. \tag{4.16}$$

In this formula we can interpret $\xi_i^{(p)}$ and $u^{(p)}$ as the components w.r.t. the basis $\{h_p\}$ or, alternatively, as the constant values on the intervals $[p2^{-m}, (p+1)2^{-m}]$. With the latter interpretation, we now have that these derivatives converge to ± 1 as $u^{(p)} \to \pm\infty$, *uniformly w.r.t. the other components.* (This does not hold with the former interpretation!) To prove this, it suffices to show that

$$\frac{\mathbb{E}_x \left[\xi^{(p)} e^{\langle \xi, u\rangle} \chi_{x'}(\xi) \right]}{\mathbb{E}_x \left[e^{\langle \xi, u\rangle} \chi_{x'}(\xi) \right]} \to 1 \text{ as } u^{(p)} \to \infty \tag{4.17}$$

uniformly in the other components of u. This can be written as

$$\frac{\mathbb{E}_x \left[\xi^{(p)} e^{\langle \xi, u\rangle} \chi_{x'}(\xi) \right]}{\mathbb{E}_x \left[e^{\langle \xi, u\rangle} \chi_{x'}(\xi) \right]} = \frac{\sum_{x_1,\dots,x_{2^m-1}} \prod_{i=0}^{2^m-1} \bar\rho_i(x_i, x_{i+1}) \Gamma_p(x_p, x_{p+1})}{\sum_{x_1,\dots,x_{2^m-1}} \prod_{i=0}^{2^m-1} \bar\rho_i(x_{i-1}, x_i)}, \tag{4.18}$$

where $\bar\rho_i(x_i, x_{i+1}) = \mathbb{E}_{x_i} \left[e^{u^{(i)} \int_{i2^{-m}}^{(i+1)2^{-m}} \xi(t)dt} \chi_{x_{i+1}}(\xi) \right]$ and

$$\Gamma_p(x, x') = \frac{\mathbb{E}_x \left[2^m \left(\int_{p2^{-m}}^{(p+1)2^{-m}} \xi(t)dt \right) e^{u^{(p)} \int_{p2^{-m}}^{(p+1)2^{-m}} \xi(t)dt} \chi_{x'}(\xi) \right]}{\bar\rho_p(x, x')}. \tag{4.19}$$

Thus it suffices if $\Gamma_p(x, x') \to \pm 1$. But this expression can be written as

$$\frac{\langle x' |\, \sigma_z e^{2^{-m}(u^{(p)}\sigma_z - \beta(1-\sigma_x)/2)} \,| x\rangle}{\langle x' |\, e^{2^{-m}(u^{(p)}\sigma_z - \beta(1-\sigma_x)/2)} \,| x\rangle}, \tag{4.20}$$

which converges to ± 1 as $u^{(p)} \to \pm\infty$ by explicit computation.

We now shift $\mathbb{K}_n$ over u_0 and define

$$\tilde{\mathbb{K}}_\beta^{(n)}[d\xi] = e^{n(\langle \xi, u_0\rangle - C_n[u_0])} \mathbb{K}_\beta^{(n)}[d\xi]. \tag{4.21}$$

Denote $u_0^{(p)} = \langle u_0, h_p \rangle$. It follows from (4.16) and the fact that $||\xi||_2 \le ||\xi||_\infty \le 1$ for a.e. ξ w.r.t. the measure $\mathbb{P}_x$, that

$$\sum_{p=0}^{\infty} \left| \frac{\partial C_n[u]}{\partial u^{(p)}} \right|^2 < +\infty. \tag{4.22}$$

This means that there is a maximiser ξ_1 for the expression $\langle \xi, u_0 \rangle - C[u_0]$ and it satisfies

$$\xi_1^{(p)} = \frac{\partial C_n[u_0]}{\partial u^{(p)}} \tag{4.23}$$

for all $p = 0, 1, \ldots$. We now show that $\tilde{\mathbb{K}}_\beta^{(n)}$ converges to δ_{ξ_1} in $L^2[0,1]$. To this end we first compute the Laplace transform:

$$\begin{aligned} \int e^{\langle \xi, u \rangle} \tilde{\mathbb{K}}_\beta^{(n)}[d\xi] &= \int e^{\langle \xi, u \rangle + n\langle \xi, u_0 \rangle - nC_n[u_0]} \mathbb{K}_\beta^{(n)}[d\xi] \\ &= e^{n(C_n[u_0 + \frac{1}{n}u] - C_n[u_0])}. \end{aligned} \tag{4.24}$$

By Griffiths' lemma we have

$$n(C_n[u_0 + \frac{1}{n}u] - C_n[u_0]) \to \nabla_u C_n[u_0] = \sum_{p=0}^\infty u^{(p)} \frac{\partial C_n[u_0]}{\partial u^{(p)}} = \langle \xi_1, u \rangle. \tag{4.25}$$

This proves that $\tilde{\mathbb{K}}_\beta^{(n)} \to \delta_{\xi_1}$ provided there exists, for any $\epsilon > 0$, a compact set $\tilde{K} \subset L^2[0,1]$ such that $\tilde{\mathbb{K}}_\beta^{(n)}[\tilde{K}^c] < \epsilon$ for all n. (Cf. Lemma 4.2 of [6].) For this we can use again the Donsker-Varadhan theorem, this time with the shifted measure

$$\tilde{\mu}[A] = \frac{\sum_{x,x'} \mathbb{E}_x \left[e^{\langle \xi, u_0 \rangle} 1_{A \cap \{\xi:\, \xi(1) = x'\}} \right]}{\sum_{x,x'} \mathbb{E}_x \left[e^{\langle \xi, u_0 \rangle} 1_{\{\xi:\, \xi(1) = x'\}} \right]}. \tag{4.26}$$

We in fact deduce more than we need, namely that there exists a compact set $\tilde{K}$ such that (4.11) holds with μ replaced by $\tilde{\mu}$, i.e.

$$\tilde{\mu}^n \left[\frac{1}{n} \sum_{i=1}^n \xi_i \in \tilde{K}^c \right] < e^{-nL}. \tag{4.27}$$

This implies as above that $\tilde{\mathbb{K}}_\beta^{(n)}[\tilde{K}^c] < e^{-nL'}$.

We can now complete the proof of the lower bound. Suppose $\xi_1 \in K^c$. Define $N_\epsilon = \{\xi : |\langle \xi - \xi_1, u_0 \rangle| < \epsilon\}$. Then $\tilde{\mathbb{K}}_n[K^c \cap N_\epsilon] \to 1$, and it follows that

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \ln \mathbb{K}_\beta^{(n)}[K^c \cap N_\epsilon] &> -\langle \xi_1, u_0 \rangle + C[u_0] - \epsilon \\ &= -\langle \xi_0, u_0 \rangle + C[u_0] - \epsilon \\ &\geq -I[\xi_0] - \epsilon, \end{aligned} \tag{4.28}$$

where we used the fact that $u \in \mathcal{H}_m$. On the other hand,

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{K}_\beta^{(n)}[K^c \cap N_\epsilon] < -L' < -I[\xi_0] - \epsilon. \tag{4.29}$$

This is obviously a contradiction so that $\xi_1 \in K$. This implies that $||\xi_1 - \xi_0||_2 \leq 2\delta$ and therefore $\xi_1 \in O$. It then follows that

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbb{K}_\beta^{(n)}[O \cap N_\epsilon] > -I[\xi_0] - \epsilon, \tag{4.30}$$

which proves the lower bound since ϵ is arbitrary. QED

Applying Varadhan's theorem now yields immediately

Corollary. *For any real-valued continuous function f on $[-1, 1]$ the following holds:*

$$\lim_{n\to\infty} \frac{1}{n} \ln \frac{\mathrm{Trace}_n\, e^{-\beta c_-^* c_- + n f(\Delta/n)}}{\mathrm{Trace}_n\, e^{-\beta c_-^* c_-}} = \sup_{\xi \in L^2[0,1]} \{\int_0^1 f(\xi(t))dt - I_\beta[\xi]\}, \tag{4.31}$$

where the rate function I_β is given by

$$I_\beta[\xi] = \sup_{u \in L^2[0,1]} \{\langle \xi, u\rangle - C[u]\} \tag{4.32}$$

and $C[u]$ is given by (4.7).

The analogues of Theorems 3.1 and 3.2 are

Theorem 4.2 *The supremum in the right-hand side of (4.31) is attained at a constant function ξ, and if ξ is a constant function then the supremum in (4.32) is attained at a constant function u.*

Proof. The proof of the second part is a carbon copy of the proof of Theorems 3.1. The proof of the first part is along the same lines as Theorem 3.2 but the formula (3.12) has to be replaced with

$$C(a) = \lim_{n\to\infty} \frac{1}{n} \ln \int e^{na \int_0^1 \xi(t)dt} \mathbb{K}_\beta^{(n)}[d\xi] = \ln \lambda_+(a), \tag{4.33}$$

where $\lambda_+(a)$ is the maximal eigenvalue of the matrix A given by

$$\begin{aligned} A_{x,x'} &= \mathbb{E}_x\left[e^{a\int_0^1 \xi(t)dt} 1_{\{\xi(1)=x'\}}\right] \\ &= \langle x'|\, e^{a\sigma_z - \frac{1}{2}\beta(1-\sigma_x)}\,|x\rangle. \end{aligned} \tag{4.34}$$

Obviously $\lambda_\pm(a) = \exp\left[\frac{1}{2}(\pm\sqrt{\beta^2+4a^2}-\beta)\right]$ so that

$$C(a) = \frac{1}{2}(\sqrt{\beta^2+4a^2}-\beta). \tag{4.35}$$

Similarly, (3.13) has to be replaced by

$$C\left[\sum_{k=1}^{2^m} a_k 1_{[(k-1)2^{-m},k2^{-m}]}\right] = \ln\lambda_+(A[u]), \tag{4.36}$$

where $A[u]$ for $u = \sum_{k=1}^{2^m} a_k 1_{[(k-1)2^{-m},k2^{-m}]}$ is given by

$$\begin{aligned} A[u] &= \mathbb{E}_x\left[\exp\left(2^{-m}\sum_{k=1}^{2^m} a_k \int_{(k-1)2^{-m}}^{k2^{-m}} \xi(t)dt\right) 1_{\{\xi(1)=x'\}}\right] \\ &= \langle x'\,|\prod_{k=1}^{2^m} e^{2^{-m}(a_k\sigma_z - \frac{1}{2}\beta(1-\sigma_x))}|\,x\rangle, \end{aligned} \tag{4.37}$$

that is, $A[u] = \prod_{k=1}^{2^m} A_k$ where

$$A_k = e^{2^{-m}(a_k\sigma_z - \frac{1}{2}\beta(1-\sigma_x))}. \tag{4.38}$$

The inequality (3.15) now has to be replaced by

$$\left\|\prod_{k=1}^{2^m} A_k\right\| \le \prod_{k=1}^{2^m} ||A_k^{2^m}||^{2^{-m}} = \prod_{k=1}^{2^m} ||A_k|| \tag{4.39}$$

for positive definite matrices. QED

Corollary *For any real-valued continuous function f on $[-1,1]$ the following holds:*

$$\lim_{n\to\infty}\frac{1}{n}\ln\frac{\mathrm{Trace}_n\, e^{-\beta c_-^* c_- + f(\Delta/n)}}{\mathrm{Trace}_n\, e^{-\beta c_-^* c_-}} = \sup_{x\in[-1,1]}\{f(x)-\tilde{I}_\beta(x)\}, \tag{4.40}$$

where the rate function $\tilde{I}_\beta$ is given by

$$\tilde{I}_\beta(x) = \frac{1}{2}\beta(1-\sqrt{1-x^2}). \tag{4.41}$$

Proof. We simply compute the Legendre transform of the function $C(a)$ given by (4.35). QED

References

[1] W. Cegła, J. T. Lewis & G. A. Raggio, *The free energy of quantum spin systems and large deviations.* Commun. Math. Phys. **118**, 337-354 (1988).

[2] D. Petz, G. A. Raggio & A. Verbeure, *Asymptotics of Varadhan-type and the Gibbs variational principle.* Commun. Math. Phys. **121**, 271-282 (1989).

[3] S. R. S. Varadhan, *Asymptotic probabilities and differential equations.* Commun. Pure & Appl. Math. **19**, 261-286 (1966).

[4] S. R. S. Varadhan, *Large Deviations and Applications.* S.I.A.M. 1984.

[5] G. A. Raggio & R. F. Werner, *Quantum statistical mechanics of general mean-field syatems.* Helv. Phys. Acta **62**, 980-1003 (1989).

[6] T. C. Dorlas *A non-commutative central limit theorem.* J. Math. Phys. **37**, 4662-4682 (1996).

[7] T. C. Dorlas, *Proof of a general non-commutative central limit theorem using a Feynman-Kac representation.* Markov Proc. & Rel. Fields **7**, 225-250 (2001).

[8] H. Araki, *Relative hamiltonian for faithful normal states of a von Neumann algebra.* Publ. RIMS **9**, 165-209 (1973).

[9] I. I. Gikhman & A. V. Skorokhod, *Introduction to the Theory of Random Processes.* W. B. Saunders, Philadelphia, 1969.

[10] M. Donsker & S. R. S. Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time III.* Commun. Pure & Appl. Math. **29**, 389-461 (1976).

[11] D. W. Stroock, *An Introduction to the Theory of Large Deviations.* Springer Verlag, Heidelberg etc. 1984.

[12] R. B. Israel, *Convexity in the Theory of Lattice Gases.* Princeton University Press, Princeton NJ, 1979.

[13] M. Reed & B. Simon, *Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness.* Acad. Press, New York etc. 1975.

[14] S. Golden, *Lower bounds for the Helmholtz function.* Phys. Rev. **137**, B1127-B1128 (1965).

[15] C. Thompson, *Inequality with applications in statistical mechanics.* J. Math. Phys. **6**, 1812-1813 (1965).

[16] M. van den Berg, T. C. Dorlas, J. T. Lewis & J. V. Pulé, *A perturbed mean-field model of an interacting boson gas and the large deviation principle.* Commun. Math. Phys. **127**, 41-69 (1990).

[17] T. C. Dorlas, *Variational formula for the pressure of a mean-field Bose-Hubbard model.* In preparation.