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Fluctuations of the local magnetic field in a frustrated mean-field Ising model

T. C. Dorlas & W. M. B. Dukes
Dublin Institute for Advanced Studies
School of Theoretical Physics
10 Burlington Road, Dublin 4, Ireland.

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Abstract

We consider fluctuations of the local magnetic field in a mean-field Ising model with mixed ferromagnetic and anti-ferromagnetic interactions. We show that the distribution of the values of this local field does not converge to a stable distribution, but that the probability distribution of this distribution does converge. We compute the moments of this probability distribution on the space of measures and show in particular that it is not Gaussian.

1 The model and the local field distribution

We consider an Ising model with spins $s_x = \pm 1$ ($x = 1, 2, \ldots, N$) with $N = 4M + 1$ and coupling constants $J_{x,y}$ given by

$$J_{x,y} = \begin{cases} -1, & \text{if } 0 < |x-y| \leq M \text{ or } |x-y| \geq N - M; \\ 0, & \text{if } x = y; \\ +1, & \text{if } M < |x-y| < N - M = 3M + 1. \end{cases} \tag{1.1}$$

A similar model has been considered by Eisele and Ellis [1, 2, 3]. We define the fluctuation of the local magnetic field values by the measure

$$\mu_N = \frac{1}{N} \sum_{x=1}^{N} \delta_{\frac{1}{\sqrt{N}} \sum_{y=1}^{N} J_{x,y} s_y} \tag{1.2}$$

and consider its distribution in the space of probability measures. If the measure were to converge to a stable law γ then we would have

$$\mathbb{E}\left[e^{i\langle f, \mu_N \rangle}\right] \to \delta_\gamma \left[e^{i\langle f, \mu \rangle}\right] = e^{i\langle f, \gamma \rangle} \tag{1.3}$$

for continuous functions f. We will show that this is not the case. Indeed, we compute all moments $\lim_{N\to\infty} \mathbb{E}[\langle f, \mu_N \rangle^k]$ and also show that the series converges for bounded continuous functions f. Non-self averaging behaviour is quite common in frustrated spin systems, see for example [4], [5] and [6, 7]. However, we should remark, that (1.3) does hold (a.s.) for the Sherrington-Kirkpatrick model [8].

For the first moment, the convergence is easy: Let us introduce the notation

$$K_N(\{s_x\}) = N\langle f, \mu_N \rangle = \sum_{x=1}^{N} f\left(\frac{1}{\sqrt{N}} \sum_{y=1}^{N} J_{x,y} s_y\right). \tag{1.4}$$

Then

$$\frac{1}{N} \sum_{\{s_x\}} K_N(\{s_x\}) = 2 \sum_{k=-2M}^{2M} \binom{4M}{2M-k} f\left(\frac{2k}{\sqrt{N}}\right). \tag{1.5}$$

(To see this, write $\sum_{y=1}^{N} J_{x,y} s_y = -\sum_{y=x-M}^{x-1} s_y - \sum_{y=x+1}^{x+M} s_y + \sum_{y=x+M+1}^{x+3M} s_y$ and sum over the possible even values of this variable with possible number of occurrences at fixed x. The sum over s_x gives an additional factor 2.) Hence

$$\lim_{N\to\infty} \mathbb{E}\left[\langle \mu_N, f\rangle\right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-x^2/2} dx = \langle f, \gamma\rangle. \tag{1.6}$$

Therefore, if (1.3) were to hold the limiting measure γ would be a standard normal distribution. Computation of the higher moments is considerably more complicated. We first consider the case $k = 2$ is the next section.

2 The second moment

We need to compute the limit

$$\lim_{N\to\infty} \frac{1}{2^N N^2} \sum_{\{s_x\}} K_N(\{s_x\})^2. \tag{2.1}$$

Now

$$\sum_{\{s_x\}} K_N(\{s_x\})^2 = \sum_{x_1,x_2} \sum_{\{s_x\}} f\left(\frac{1}{\sqrt{N}} \sum_y J_{x_1,y} s_y\right) f\left(\frac{1}{\sqrt{N}} \sum_y J_{x_2,y} s_y\right). \tag{2.2}$$

To compute the limit of this expression we consider it as a quadratic form and insert $e^{it_1 z}$ and $e^{it_2 z}$ for f. We then need to compute

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{2^N N^2} \sum_{x_1,x_2} \sum_{\{s_x\}} \exp\left[\frac{i}{\sqrt{N}} \sum_y (t_1 J_{x_1,y} + t_2 J_{x_2,y}) s_y\right] \\ &= \lim_{N\to\infty} \frac{1}{2^N N^2} \sum_{x_1,x_2} \prod_y \sum_{s_y=\pm1} e^{iN^{-1/2}(t_1 J_{x_1,y} + t_2 J_{x_2,y}) s_y} \\ &= \lim_{N\to\infty} \frac{1}{N^2} \sum_{x_1,x_2} \prod_y \cos\left(\frac{1}{\sqrt{N}} (t_1 J_{x_1,y} + t_2 J_{x_2,y})\right). \end{aligned} \tag{2.3}$$

Now, if $|x_1 - x_2| \leq 2M$, the number of y for which $J_{x_1,y} = J_{x_2,y}$ is $2(2M - |x_1 - x_2| - 1)$ and the number of y for which $J_{x_1,y} = -J_{x_2,y}$ is $2|x_1 - x_2| + 1$. (Except of course if $x_1 = x_2$, but this case is negligible as we are dividing by N^2. Similarly, the cases $y = x_1, x_2$ are irrelevant.)

On the other hand, if $|x_1 - x_2| > 2M$ then the number of y with $J_{x_1,y} = J_{x_2,y}$ is $2(|x_1 - x_2| - 2M - 1)$ and the number of y with $J_{x_1,y} = -J_{x_2,y}$ is $2(N - |x_1 - x_2|) - 1$. Thus, the above limit equals

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N}\Bigg\{ \sum_{k=1}^{2M} \left(\cos\left(N^{-1/2}(t_1+t_2)\right)\right)^{2(2M-k-1)} \left(\cos\left(N^{-1/2}(t_1-t_2)\right)\right)^{2k+1} \\
&+ \sum_{k=2M+1}^{N-1} \left(\cos\left(N^{-1/2}(t_1+t_2)\right)\right)^{2(k-2M-1)} \left(\cos\left(N^{-1/2}(t_1-t_2)\right)\right)^{2(N-k)-1} \Bigg\} \\
&= \lim_{N\to\infty} \frac{1}{N}\Bigg\{ \sum_{k=0}^{2M} \left(1 - \frac{1}{2N}(t_1+t_2)^2\right)^{N-2k} \left(1 - \frac{1}{2N}(t_1-t_2)^2\right)^{2k} \\
&\quad + \sum_{k=2M+1}^{N} \left(1 - \frac{1}{2N}(t_1+t_2)^2\right)^{2k-N} \left(1 - \frac{1}{2N}(t_1-t_2)^2\right)^{2(N-k)} \Bigg\} \\
&= \frac{1}{2}\Bigg\{ \int_0^1 \exp\left[-\frac{1}{2}\{(1-s)(t_1+t_2)^2 + s(t_1-t_2)^2\}\right] ds \\
&\quad + \int_1^2 \exp\left[-\frac{1}{2}\{s(t_1+t_2)^2 + (2-s)(t_1-t_2)^2\}\right] ds \Bigg\}. \qquad (2.4)
\end{aligned}$$

This can be rewritten as

$$\frac{1}{2} e^{-(t_1^2+t_2^2)/2} \int_{-1}^{1} e^{st_1t_2} du. \qquad (2.5)$$

The limit (2.1) is therefore given by

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, \rho_2(x_1,x_2) f(x_1) f(x_2), \qquad (2.6)$$

where ρ_2 is the density of the measure with characteristic function given by (2.5), i.e.

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, \rho_2(x_1,x_2) e^{i(t_1x_1+t_2x_2)} = \frac{1}{2} e^{-(t_1^2+t_2^2)/2} \int_{-1}^{1} e^{st_1t_2} ds. \qquad (2.7)$$

Diagonalising $t_1^2 - 2st_1t_2 + t_2^2$ we find

$$\rho_2(x_1,x_2) = \frac{1}{4\pi} \int_{-1}^{1} \frac{e^{-\frac{(x_1+x_2)^2}{4(1-s)} - \frac{(x_1-x_2)^2}{4(1+s)}}}{\sqrt{1-s^2}} ds. \qquad (2.8)$$

Clearly, this is not equal to the density of $\gamma \otimes \gamma$, i.e. $\frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}$ which would be the result if (1.3) were to hold.

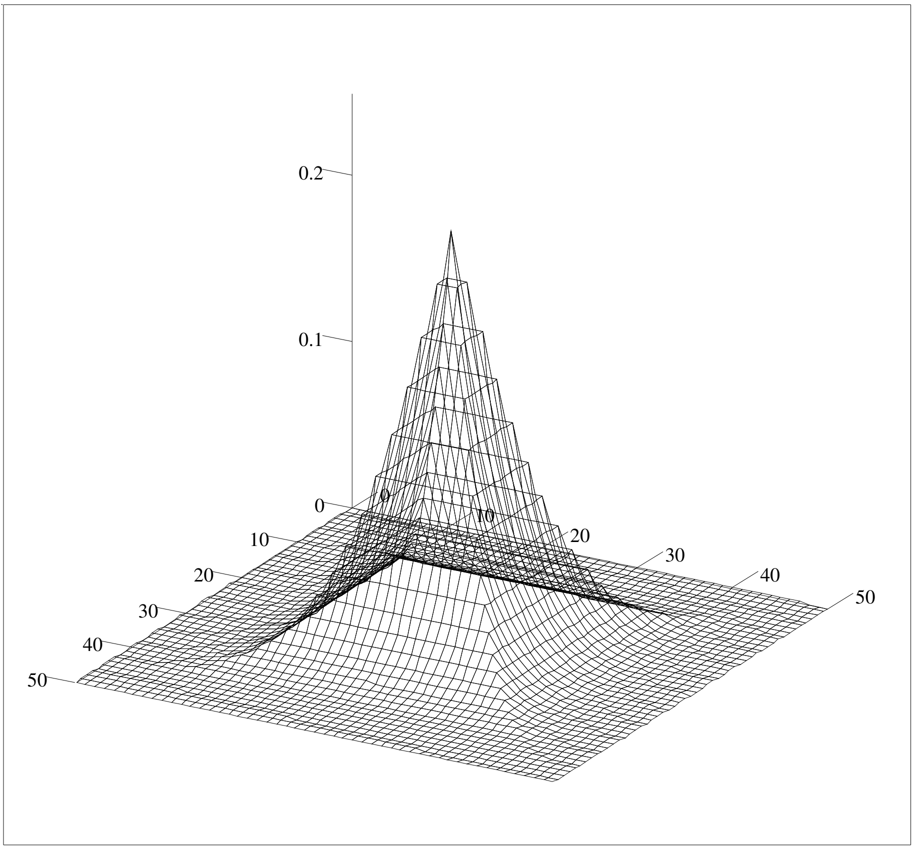

Figure 1: Plot of the function (2.8). It appears from this plot that ρ_2 is constant on squares, i.e. it only depends on $|x_1| \vee |x_2|$. This will be proved in the appendix.

3 Higher moments

The computation of the higher moments follows the same strategy but the result is more complicated and cannot be expressed in such a simple fashion.

Analogous to (2.2) we have

$$\sum_{\{s_x\}} K_N(\{s_x\})^p \;=\; \sum_{x_1,\ldots,x_p} \sum_{\{s_x\}} \prod_{i=1}^{p} f\left(\frac{1}{\sqrt{N}} \sum_y J_{x_i,y} s_y\right). \tag{3.1}$$

Inserting exponentials $e^{it_i z}$ we have to compute

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{2^N N^p} \sum_{x_1,\ldots,x_p} \sum_{\{s_x\}} \exp\left[\frac{i}{\sqrt{N}} \sum_y s_y \sum_{i=1}^{p} t_i J_{x_i,y}\right] \\ &= \lim_{N\to\infty} \frac{1}{N^p} \sum_{x_1,\ldots,x_p} \prod_y \cos\left(\frac{1}{\sqrt{N}} \sum_{i=1}^{p} t_i J_{x_i,y}\right). \end{aligned} \tag{3.2}$$

We now show that a calculation as in the case of the second moment yields the following:

Theorem 3.1 *For bounded continuous functions f, the limit*

$$\lim_{N\to\infty} \mathbb{E}\left[\langle f, \mu_N\rangle^p\right] = \lim_{N\to\infty} \frac{1}{N^p 2^N} \sum_{\{s_x\}} K_N\left(\{s_x\}\right)^p$$

exists and equals

$$\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_p \, \rho_p(x_1,\ldots,x_p) f(x_1)\ldots f(x_p),$$

where the probability density ρ_p is given by

$$\rho_p(x_1,\ldots,x_p) \;=\; \frac{1}{(2\pi)^{p/2}} \int_0^1 \cdots \int_0^1 \frac{d\alpha_1 \ldots d\alpha_p}{\sqrt{\det\, S(\alpha_1,\ldots,\alpha_p)}} e^{-\frac{1}{2}\langle x,\, S(\alpha_1,\ldots,\alpha_p)^{-1} x\rangle} \tag{3.3}$$

and the matrix $S(\alpha_1,\ldots,\alpha_p)$ has matrix elements $s(\alpha_i - \alpha_j)$, $i,j = 1,\ldots,p$, where

$$s(\alpha) \;=\; \begin{cases} 1 - 4|\alpha| & \text{if } |\alpha| < \frac{1}{2}, \\ 4|\alpha| - 3 & \text{if } |\alpha| \geq \frac{1}{2}. \end{cases} \tag{3.4}$$

.

PROOF: First notice that for all pairs $i < j$,

$$\#\{y:\, J_{x_i,y} = J_{x_j,y}\} \;=\; \begin{cases} N - 2|x_j - x_i| & \text{if } |x_j - x_i| \le 2M, \\ 2|x_j - x_i| - N & \text{if } |x_j - x_i| > 2M. \end{cases} \tag{3.5}$$

(N.B. The right-hand side is correct up to an error of at most 2, which is irrelevant in the limit $N \to \infty$.) We now rewrite the limit (3.2) as follows:

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N^p} \sum_{x_1,\ldots,x_p} \prod_y \left\{ 1 - \frac{1}{2N} \left(\sum_{i=1}^p t_i J_{x_i,y} \right)^2 \right\} \\ &= \lim_{N\to\infty} \frac{1}{N^p} \sum_{x_1,\ldots,x_p} \prod_y \exp\left[-\frac{1}{2N} \sum_{i,j=1}^p t_i t_j J_{x_i,y} J_{x_j,y} \right] \\ &= e^{-\frac{1}{2}(t_1^2+\cdots+t_p^2)} \lim_{N\to\infty} \frac{1}{N^p} \sum_{x_1,\ldots,x_p} \exp\left[-\frac{1}{N} \sum_{i<j} t_i t_j \sum_y J_{x_i,y} J_{x_j,y} \right] \\ &= e^{-\frac{1}{2}(t_1^2+\cdots+t_p^2)} \lim_{N\to\infty} \frac{1}{N^p} \sum_{x_1,\ldots,x_p} \exp\left[-\sum_{i<j} t_i t_j N s((x_j - x_i)/N) \right] \end{aligned} \tag{3.6}$$

The last expression follows from the fact that

$$\begin{aligned} \sum_y J_{x_i,y} J_{x_j,y} &= \#\{y:\, J_{x_i,y} J_{x_j,y} = 1\} - \#\{y:\, J_{x_i,y} J_{x_j,y} = -1\} \\ &= \begin{cases} N - 2|x_j - x_i| - 2|x_j - x_i| & \text{if } |x_i - x_j| \le 2M, \\ 2|x_j - x_i| - N - (2N - 2|x_j - x_i|) & \text{if } |x_j - x_i| > 2M. \end{cases} \end{aligned} \tag{3.7}$$

Taking the limit $N \to \infty$ now yields

$$e^{-\frac{1}{2}(t_1^2+\cdots+t_p^2)} \int_0^1 d\alpha_1 \ldots \int_0^1 d\alpha_p \prod_{i<j} \exp\left[-t_i t_j s(\alpha_i - \alpha_j) \right]. \tag{3.8}$$

The result (3.3) then follows from the well-known Fourier transform formula for Gaussian functions. QED

The formula (3.3) can be simplified by a transformation of variables. We subdivide the domain of integration into subdomains as follows. First let $u_j := \alpha_{j+1} - \alpha_1$ and define $s_j := s(u_j)$ for $j = 1, \ldots, p-1$.

Lemma 3.2 *Let π be a permutation of $\{1,\ldots,p-1\}$ and let $\sigma_1,\ldots,\sigma_{p-1} \in \{\pm 1\}$. Define the region $\mathcal{R}(\pi,\sigma) \subset [0,1]^{p-1}$ by $(u_1,\ldots,u_{p-1}) \in \mathcal{R}(\pi,\sigma)$ iff*

$$0 \le u_{\pi(1)} - \frac{1}{4}(\sigma_{\pi(1)}+1) < \cdots < u_{\pi(p-1)} - \frac{1}{4}(\sigma_{\pi(p-1)}+1) \le \frac{1}{2}. \tag{3.9}$$

Then the region $\mathcal{R}(\pi,\sigma)$ is equivalent to

$$-1 < \sigma_{\pi(1)} s_{\pi(1)} < \cdots < \sigma_{\pi(p-1)} s_{\pi(p-1)} < 1. \tag{3.10}$$

and the elements of the matrix S are given by

$$\begin{array}{lllll} & & S_{ii} & = & 1 & \text{for all } 1 \le i \le p \\ S_{1i} & = & S_{i1} & = & s_{i-1} & \text{for all } 1 < i \le p \\ S_{ij} & = & S_{ji} & = & \sigma_{j-1}s_{i-1} - \sigma_{i-1}s_{j-1} + \sigma_{i-1}\sigma_{j-1} & \text{if } \sigma_{i-1}s_{i-1} < \sigma_{j-1}s_{j-1}. \end{array}$$

Moreover, if we denote $b_i := \sigma_{\pi(i)} s_{\pi(i)}$, then

$$\det S(\alpha_1,\ldots,\alpha_p) = 2^{p-2}(1+b_1)(1-b_{p-1}) \prod_{i=1}^{p-2}(b_{i+1}-b_i). \tag{3.11}$$

Proof: The equivalence of regions follows immediately from (3.4) which implies

$$s_k = \begin{cases} 1-4u_k, & \text{if } \sigma_k = -1 \\ 4u_k - 3, & \text{if } \sigma_k = +1 \end{cases}$$

which can be written as

$$s_k = 4\sigma_k u_k - 1 - 2\sigma_k \tag{3.12}$$

from which

$$u_k - \frac{1}{4}(\sigma_k + 1) = \frac{1}{4}(\sigma_k s_k + 1). \tag{3.13}$$

To determine the matrix elements of S, notice first that $S_{i,i} = s(0) = 1$ and $S_{i,1} = S_{1,i} = s(|u_{i-1}|) = s_{i-1}$ for $i > 1$. Moreover, if $1 < i < j$, $S_{i,j} = S_{j,i} = s(|u_{j-1} - u_{i-1}|)$, so we may assume $\sigma_{i-1}s_{i-1} < \sigma_{j-1}s_{j-1}$. Now, by (3.12) we have

$$s(u_{i-1} - u_{j-1}) = 4\sigma|u_{i-1} - u_{j-1}| - 1 - 2\sigma, \tag{3.14}$$

where $\sigma = -1$ if $|u_{i-1} - u_{j-1}| \leq \frac{1}{2}$ and $\sigma = +1$ otherwise. By the above equivalence, $0 \leq u_{i-1} - \frac{1}{4}(\sigma_{i-1}+1) < u_{j-1} - \frac{1}{4}(\sigma_{j-1}+1) \leq \frac{1}{2}$ and hence

$$\frac{1}{4}(\sigma_{j-1} - \sigma_{i-1}) \leq u_{j-1} - u_{i-1} \leq \frac{1}{4}(\sigma_{j-1} - \sigma_{i-1}) + \frac{1}{2}. \tag{3.15}$$

¿From this it is easy to see that

$$|u_{i-1} - u_{j-1}| = \begin{cases} u_{j-1} - u_{i-1}, & \text{if } \sigma_{i-1} \leq \sigma_{j-1} \\ u_{i-1} - u_{j-1}, & \text{if } \sigma_{i-1} > \sigma_{j-1} \end{cases}$$

and

$$\sigma = \begin{cases} -1, & \text{if } \sigma_{i-1} \geq \sigma_{j-1}, \\ +1, & \text{if } \sigma_{i-1} < \sigma_{j-1}. \end{cases}$$

This implies

$$\sigma = -1 + \frac{1}{2}(1 - \sigma_{i-1})(1 + \sigma_{j-1}) \tag{3.16}$$

and

$$\frac{|u_{i-1} - u_{j-1}|}{u_{i-1} - u_{j-1}} = \sigma_{i-1}\sigma_{j-1}\sigma. \tag{3.17}$$

Inserting these identities into (3.14) we obtain

$$s(u_{i-1} - u_{j-1}) = 4\sigma_{i-1}\sigma_{j-1}(u_{i-1} - u_{j-1}) + \sigma_{i-1}\sigma_{j-1} - \sigma_{j-1} + \sigma_{i-1},$$

which is the stated result.

To evaluate $\det S$ we perform several elementary row and column operations and show the resulting matrix to be the matrix B, given in the appendix. For all $2 \leq i \leq p$ multiply each entry in row i by σ_i and each entry in column i by σ_i. Notice that if $1 < i < j \leq p$ and $\sigma_{i-1}s_{i-1} < \sigma_{j-1}s_{j-1}$, then the resulting matrix $\tilde{S}$ satisfies

$$\begin{aligned} \tilde{S}_{ij} &= \sigma_{i-1}\sigma_{j-1}(\sigma_{j-1}s_{i-1} - \sigma_{i-1}s_{j-1} + \sigma_{i-1}\sigma_{j-1}) \\ &= \sigma_{i-1}s_{i-1} - \sigma_{j-1}s_{j-1} + 1 \\ &= b_{\pi^{-1}(i)} - b_{\pi^{-1}(j)} + 1. \end{aligned}$$

The entries in the row 1 now read $(1 \ b_{\pi^{-1}(1)} \ \ldots \ b_{\pi^{-1}(p-1)})$. Reorder the rows and columns, according to the permutation π, so that the b indices are increasing in the first row and column. The resulting matrix is B. The total number of row and column operations is even, due to the symmetry of the matrix S, so the sign of the determinant is preserved. Thus $\det S = \det B$, which is evaluated in the Lemma A.2 in the appendix. QED

The inverse of the matrix B can also be worked out: see Lemma A.2 in the Appendix. This leads to the following representation of the density ρ_p:

Corollary 3.3 *The density ρ_p of (3.3) can be written as*

$$\rho_p(x_0, x_1, \ldots, x_{p-1}) = \frac{2^{-p+1}}{(2\pi)^{p/2}} \sum_{\vec{\sigma},\pi} g_p(x_0, x_1, \ldots, x_{p-1}; \vec{\sigma}, \pi), \tag{3.18}$$

where

$$g(x_0, x_1, \ldots, x_{p-1}; \sigma, \pi) \;=\; \int_{\substack{v_1^2+\cdots+v_{p-2}^2 \le 4 \\ v_i \ge 0,\,\forall i}} dv_1 \ldots dv_{p-2} \int_{-\pi/2}^{\pi/2} d\alpha \tag{3.19}$$

$$\exp\left\{ -\frac{1}{2} \left(\frac{(x_0 + \sigma_{\pi(1)} x_{\pi(1)})^2}{v_1^2} + \frac{(x_0 - \sigma_{\pi(p-1)} x_{\pi(p-1)})^2}{\frac{1}{2}(4 - \sum_{i=1}^{p-2} v_i^2)(1 - \sin\alpha)} \right.\right. \tag{3.20}$$

$$+ \sum_{i=1}^{p-2} \frac{\left(\sigma_{\pi(i)} x_{\pi(i)} - \sigma_{\pi(i+1)} x_{\pi(i+1)}\right)^2}{v_{i+1}^2} \tag{3.21}$$

$$\left.\left. + \frac{\left(\sigma_{\pi(p-2)} x_{\pi(p-2)} - \sigma_{\pi(p-1)} x_{\pi(p-1)}\right)^2}{\frac{1}{2}(4 - \sum_1^{p-2} v_i^2)(1 + \sin\alpha)} \right) \right\}. \tag{3.22}$$

In particular, $g_3(x_0, x_1, x_2; \vec{\sigma}, \pi)$ is given by

$$\begin{aligned} g_3(x_0, x_1, x_2; \vec{\sigma}, \pi) \;=\; & \int_0^2 dv_1 \int_{-\pi/2}^{\pi/2} d\alpha \\ & \exp\left\{ -\frac{1}{2} \left[\frac{(x_0 + \sigma_{\pi(1)} x_{\pi(1)})^2}{v_1^2} + \frac{(x_0 - \sigma_{\pi(2)} x_{\pi(2)})^2}{(2 - v_1^2/2)(1 - \sin\alpha)} \right.\right. \\ & \left.\left. + \frac{(\sigma_{\pi(1)} x_{\pi(1)} - \sigma_{\pi(2)} x_{\pi(2)})^2}{(2 - v_1^2/2)(1 + \sin\alpha)} \right] \right\}. \end{aligned} \tag{3.23}$$

Figure 3 shows a contour plot of the density ρ_3 at fixed x_2 from which it is apparent that the simple property of ρ_2 mentioned in the caption of Figure 2 does not generalise to higher p.

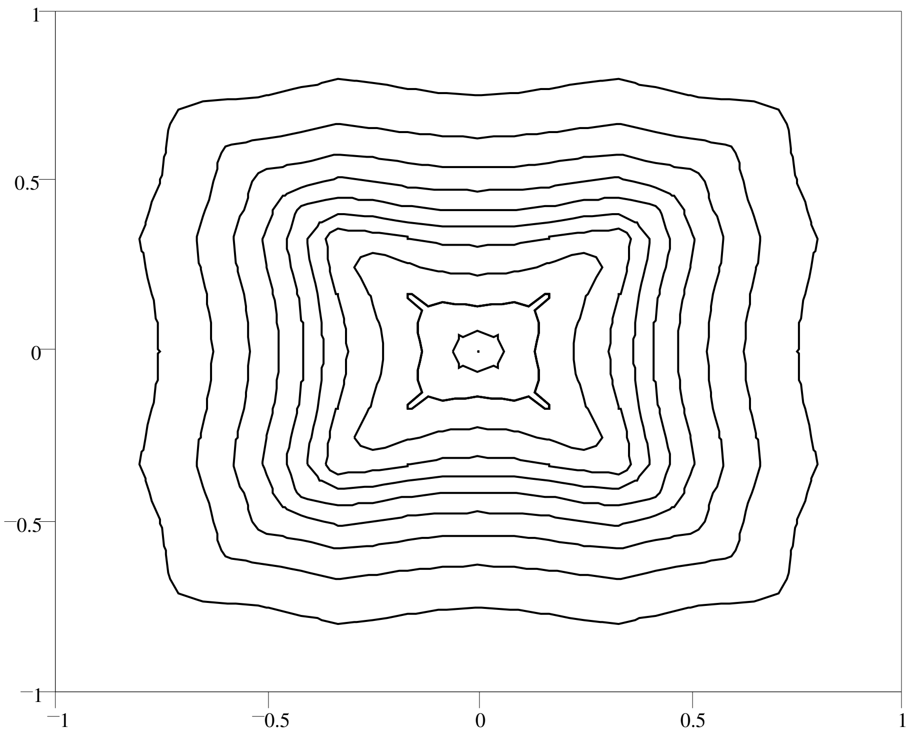

Figure 2: Contour plot of the function $\rho_3(x_0, x_1, 0.6)$ generated using the integrals in (3.23).

4 Convergence of the probability distribution on the space of measures

The convergence of the characteristic functions $\mathbb{E}\left[e^{i\langle f,\mu_N\rangle}\right]$ follows immediately from the theorem of Section 3, but this does not yet imply that the corresponding distributions converge (this is only true for probability distributions on finite-dimensional spaces). This therefore requires a proof:

Theorem 4.1 *The sequence of probability distributions of the measures (1.2) converges weakly to the probability distribution on $\mathcal{M}^1(\mathbb{R})$ with characteristic function given by*

$$\mathbb{E}\left[e^{i\langle f,\cdot\rangle}\right] = \sum_{p=0}^{\infty} \frac{i^p}{p!} \int dx_1 \ldots \int dx_p \rho_p(x_1, \ldots, x_p) f(x_1) \ldots f(x_p). \quad (4.1)$$

This theorem follows from

Lemma 4.2 *For all $\epsilon > 0$, there is a compact $K_\epsilon \subset \mathcal{M}^1(\mathbb{R})$ and $N_0 \in \mathbb{N}$ such that*

$$\mathbb{P}(\mu_N \notin K_\epsilon) < \epsilon$$

for all $N = 4M + 1$ with $M \in \mathbb{N}$.

Proof: Let

$$K_\epsilon \;:=\; \left\{ \mu \in \mathcal{M}^1(\mathbb{R}) \;\middle|\; \mu(a, +\infty) < \frac{e^{-a^2/4}}{\epsilon} \text{ for all } a \geq \sqrt{2} \right\}$$

Clearly this K_ϵ is compact, since for all $\delta > 0$ there exists $a > \sqrt{2}$ such that: $\mu(a, +\infty) < \delta$ for all $\mu \in K_\epsilon$. Using Chebychev's inequality we have

$$\mathbb{P}\left(\mu_N(a, +\infty) > \frac{e^{-a^2/2}}{\epsilon}\right) \;<\; \frac{\epsilon \mathbb{E}(\langle \mu_N, 1_{(a,+\infty)} \rangle)}{e^{-a^2/2}} \tag{4.2}$$

where, as in (1.5),

$$\begin{aligned}
\mathbb{E}(\langle \mu_N, 1_{(a,+\infty)} \rangle) &= \frac{1}{2^N} \sum_{\{s_x\}} K_N(\{s_x\}) \\
&= \frac{1}{2^{4M+1}} \sum_{k=-2M}^{2M} \binom{4M}{2M-k} 1_{(a,+\infty)}\left(\frac{2k}{\sqrt{4M+1}}\right) \\
&= \frac{1}{2^{4M+1}} \sum_{k=[a\sqrt{M}]+1}^{2M} \binom{4M}{2M+k}.
\end{aligned}$$

We now use the bounds $\binom{4M}{2M} < \frac{2^{4M}}{\sqrt{2M}}$ for all $M \geq 1$ and $\frac{2M+1-i}{2M+i} < \exp\left(\frac{1-2i}{4M}\right)$ for all $1 \leq i \leq 2M$ (which follows from $e^{-x} > 1 - x$) to bound the coefficients

$$\begin{aligned}
\binom{4M}{2M+k} &= \binom{4M}{2M} \prod_{i=1}^{k} \left(\frac{2M+1-i}{2M+i}\right) \\
&< \binom{4M}{2M} \prod_{i=1}^{k} \exp\left(\frac{1-2i}{4M}\right) \\
&= \binom{4M}{2M} \exp\left(\frac{-k^2}{4M}\right) \\
&< \frac{2^{4M}}{\sqrt{2M}} \exp\left(\frac{-k^2}{4M}\right).
\end{aligned}$$

This gives the following bound

$$\begin{aligned}
\mathbb{E}(\langle \mu_N, 1_{(a,+\infty)}\rangle) &< \frac{1}{2^{4M+1}} \sum_{k=[a\sqrt{M}]+1}^{2M} \frac{2^{4M}}{\sqrt{2M}} \exp\left(\frac{-k^2}{4M}\right) \\
&= \frac{1}{2\sqrt{2M}} \int_{a\sqrt{M}}^{+\infty} e^{(-x^2/4M)} dx \\
&< \frac{1}{2} \int_{a/\sqrt{2}}^{+\infty} e^{-u^2/2} du \\
&< \frac{e^{-a^2/4}}{a\sqrt{2}} \\
&< \frac{1}{2} e^{-a^2/4}.
\end{aligned}$$

Applying to (4.2) gives the required result:

$$\mathbb{P}\left(\mu_N(a,\infty) \geq \frac{e^{-a^2/2}}{\epsilon}\right) \quad < \quad \epsilon.$$

QED

Proof: (of Theorem 4.1.) By Prokhorov's theorem (see [9]) the lemma implies that the set of probability measures $\{\mu_N\}$ is relatively compact. This means that every subsequence has a convergent subsequence which must have the characteristic function given by (4.1) and is therefore uniquely determined. It follows by the usual subsequence argument that the sequence μ_N itself must converge to this measure. QED

Appendix

Lemma A.1 *The value $\rho_2(x,y)$ given in (2.8) depends on $|x| \vee |y|$ only.*

Proof: Notice that it is clearly symmetric under interchange of x and y and also under sign change of x or y. We can therefore assume that $0 \leq x < y$. Differentiating w.r.t. x then yields

$$\frac{d}{dx}\rho_2(x,y) = \frac{1}{4\pi} \int_{-1}^{1} (x+sy) \frac{e^{-\frac{x^2+2sxy+y^2}{2(1-s^2)}}}{(1-s^2)^{3/2}} ds$$

Notice that the exponent can be rewritten as

$$\frac{(x+sy)^2}{2(1-s^2)} + \frac{1}{2}y^2.$$

Since $x < y$, $x + sy$ has a zero inside the integration interval $[-1,1]$. We therefore divide this range into the intervals $[-1,-x/y]$ and $(-x/y,1]$. On the second interval we change variables to s' in such a way that

$$s = 1 \implies s' = -1 \text{ and } s = -x/y \implies s' = -x/y$$

whereas

$$\frac{x+sy}{\sqrt{1-s^2}} = -\frac{x+s'y}{\sqrt{1-s'^2}}$$

and

$$\frac{ds}{1-s^2} = -\frac{ds'}{1-s'^2}$$

Solving the latter yields

$$s' = \frac{c(1-s)-(1+s)}{c(1-s)+(1+s)}$$

and inserting the boundary conditions then gives

$$c = \left(\frac{y-x}{y+x}\right)^2.$$

A simple calculation shows that the other identity also holds. The integral over the interval $(-x/y,1]$ now transforms into minus the integral over $[-1,-x/y)$ so that the two contributions cancel and the derivative is zero. QED

Lemma A.2 *Let B be the symmetric matrix with entries:*

$$\begin{aligned} B_{ii} &:= 1 && \text{for } 1 \le i \le p \\ B_{1,i} = B_{i,1} &:= b_{i-1} && \text{for } 2 \le i \le p \\ B_{i,j} = B_{j,i} &:= b_{i-1} - b_{j-1} + 1, && \text{for } 1 \le i < j \le p. \end{aligned}$$

Then $\det B = 2^{p-2}(1+b_1)(b_2-b_1)\dots(b_{p-1}-b_{p-2})(1-b_{p-1})$. *Define* $b_0 := -1$, $b_p := 1$ *and* $w_i := (2(b_{i+1}-b_i))^{-1}$ *for all* $i = 0,\dots,p$. *Then the inverse* B^{-1}

is given by:

$$
\begin{array}{rcl}
& B^{-1}_{1,1} & = w_{p-1} + w_0 \\
& B^{-1}_{i,i} & = w_{i-2} + w_{i-1} \quad \textit{for } 2 \le i \le p \\
B^{-1}_{1,2} = & B^{-1}_{2,1} & = w_0 \\
B^{-1}_{i,i+1} = & B^{-1}_{i+1,i} & = -w_{i-1} \quad \textit{for } 2 \le i \le p-1 \\
B^{-1}_{1,p} = & B^{-1}_{p,1} & = -w_{p-1}.
\end{array}
$$

PROOF: The determinant is obtained by elementary row and column operations: subtract row 1 from all other rows; then add column 2 to column 1; and then successively subtract column $i+1$ from column i for $i = 2, \ldots, p-1$. The resulting matrix is upper triangular and the product of diagonal elements is the said value.

To prove the statement about the inverse of B^{-1}, we multiply row x of B and column y of B^{-1} and consider various cases. If $x = y = 1$ we have $(BB^{-1})_{11} = b_{p-1}(-w_{p-1} + (w_0 + w_{p-1}) + b_1 w_0 = 1$. If $x = y = 2$ we have $(BB^{-1})_{22} = b_1 w_0 + (w_0 + w_1) + (b_1 - b_2 + 1)(-w_1) = 1$, and if $x = y > 1$, $(BB^{-1})_{yy} = (b_{y-2} - b_{y-1} + 1)(-w_{y-2}) + (w_{y-2} + w_{y-1}) + (b_{y-1} - b_y + 1)(-w_{y-1}) = 1$.

The (1,2)- and (2,1) elements are: $(BB^{-1})_{12} = w_0 + b_1(w_0 + w_1) + b_2(-w_1) = 0$ and $(BB^{-1})_{21} = (b_1 - b_{p-1} + 1)(-w_{p-1}) + b_1(w_0 + w_{p-1}) + w_0 = 0$. For $y > 2$ we get $(BB^{-1})_{1y} = b_{y-2}(-w_{y-2}) + b_{y-1}(w_{y-2} + w_{y-1}) + b_y(-w_{y-1}) = 0$ and $(BB^{-1})_{y1} = (b_{y-1} - b_{p-1} + 1)(-w_{p-1}) + b_{y-1}(w_{p-1} + w_0) + (b_1 - b_{y-1} + 1)w_0 = 0$.

For the cases $|x - y| \ge 2$, first assume, $1 < x < y < p$. Then $(BB^{-1})_{xy} = (b_{x-1} - b_{y-2} + 1)(-w_{y-2}) + (b_{x-1} - b_{y-1} + 1)(w_{y-2} + w_{y-1}) + (b_{x-1} - b_y + 1)(-w_{y-1}) = 0$. If $y = p$ we have $(BB^{-1})_{xp} = (b_{x-1} - b_{p-2} + 1)(-w_{p-2}) + (b_{x-1} - b_{p-1} + 1)(w_{p-2} + w_{p-1}) + b_{x-1}(-w_{p-1}) = 0$. If $1 < y < x < p$, $(BB^{-1})_{xy} = (b_{y-2} - b_{x-1} + 1)(-w_{y-2}) + (b_{y-1} - b_{x-1} + 1)(w_{y-2} + w_{y-1}) + (b_y - b_{x-1} + 1)(-w_{y-1}) = 0$. Finally, if $2 < y < p-1$, $(BB^{-1})_{py} = (b_{y-2} - b_{p-1} + 1)(-w_{y-2}) + (b_{y-1} - b_{p-1} + 1)(w_{y-2} + w_{y-2}) + (b_y - b_{p-1} + 1)(-w_{y-1}) = 0$ and $(BB^{-1})_{p2} = b_{p-1} w_0 + (b_1 - b_{p-1} + 1)(w_0 + w_1) + (b_2 - b_{p-1} + 1)(-w_1) = 0$. QED

Lemma A.3 *The density $\rho_p(x_1,\ldots,x_p)$ may be expressed as*

$$\rho_p(x_0,x_1,\ldots,x_{p-1}) = \frac{1}{2^{p-1}(2\pi)^{p/2}} \sum_{\substack{\sigma_1,\ldots,\sigma_{p-1}\in\pm1 \\ \pi\in Perm[1,p-1]}} g(x_0,x_1,\ldots,x_{p-1};\vec{\sigma},\pi)$$

where the sum is over all permutations π of $\{1,\ldots,p-1\}$ and

$$g(x_0,x_1,\ldots,x_{p-1};\sigma,\pi) = \int_{\substack{v_1^2+\cdots+v_{p-2}^2\le 4 \\ v_i\ge 0,\,\forall i}} dv_1\ldots dv_{p-2} \int_{-\pi/2}^{\pi/2} d\alpha$$

$$\exp\Bigg\{-\frac{1}{2}\Bigg(\frac{(x_0+\sigma_{\pi(1)}x_{\pi(1)})^2}{v_1^2} + \frac{(x_0-\sigma_{\pi(p-1)}x_{\pi(p-1)})^2}{\frac{1}{2}(4-\sum_{i=1}^{p-2}v_i^2)(1-\sin\alpha)}$$

$$+\sum_{i=1}^{p-2}\frac{\big(\sigma_{\pi(i)}x_{\pi(i)}-\sigma_{\pi(i+1)}x_{\pi(i+1)}\big)^2}{v_{i+1}^2}$$

$$+\frac{\big(\sigma_{\pi(p-2)}x_{\pi(p-2)}-\sigma_{\pi(p-1)}x_{\pi(p-1)}\big)^2}{\frac{1}{2}(4-\sum_1^{p-2}v_i^2)(1+\sin\alpha)}\Bigg)\Bigg\}.$$

PROOF: Because of periodicity of the function $s(\alpha)$ the p-fold integral in (3.3) can be transformed into the following $p-1$-fold integral over $u_i=\alpha_{i+1}-\alpha_1$. Since s is an even function of u, we have that 2^{p-1} regions are similar and we obtain

$$\rho_p(x_1,\ldots,x_p) = \frac{2^{p-1}}{(2\pi)^{p/2}}\int_{[0,1]^{p-1}}\frac{du_1\ldots du_{p-1}}{\sqrt{\det\ S(u_1,\ldots,u_{p-1})}} \times\exp\left\{-\frac{1}{2}\langle x,S(\vec{u})^{-1}x\rangle\right\}$$

Next, we perform a change of variables to $\vec{s}$. By (3.12) we have $ds_i/du_i=4\sigma_i$, and the above integral may be written as

$$\rho_p(x_1,\ldots,x_p) = \frac{2^{-p+1}}{(2\pi)^{p/2}}\sum_{\vec{\sigma},\pi}\int_{-1\le\sigma_{\pi(1)}s_{\pi(1)}\le\cdots\le\sigma_{\pi(p-1)}s_{\pi(p-1)}\le 1}\frac{ds_1\ldots ds_{p-1}}{\sqrt{\det\ S(\vec{s},\vec{\sigma},\pi)}} \times\exp\left\{-\frac{1}{2}\langle\vec{x},S(\vec{s},\vec{\sigma},\pi)^{-1}\vec{x}\rangle\right\}.$$

Now change variables to $\vec{b}$. This yields

$$\rho_p(x_1,\ldots,x_p) = \frac{1}{2^{p-1}(2\pi)^{p/2}}\sum_{\vec{\sigma},\pi}\int_{-1\le b_1\le\cdots\le b_{p-1}\le 1}\frac{db_1\ldots db_{p-1}}{\sqrt{\det\ S(\vec{b},\vec{\sigma},\pi)}} \exp\left\{-\frac{1}{2}\langle\vec{x},S(\vec{b},\vec{\sigma},\pi)^{-1}\vec{x}\rangle\right\}$$

We now recall that $\det S(\vec{b},\vec{\sigma},\pi) = 2^{p-2}(1+b_1)(b_2-b_1)\dots(b_{p-1}-b_{p-2})(1-b_{p-1})$. Moreover, the scalar product $\langle \vec{x}, S(\vec{b},\vec{\sigma},\pi)^{-1}\vec{x}\rangle$ simplifies by reordering the vector $\vec{x}$ according to the permutation π, multiplying each entry by it's corresponding σ value and then using the matrix B rather than S. Thus we have

$$\begin{aligned}
&\langle (x_0, x_1, \dots, x_{p-1}), S(\vec{b},\vec{\sigma},\pi)^{-1}\ (x_0, x_1, \dots, x_{p-1})\rangle \\
&\quad = \ \langle (x_0, \sigma_{\pi(1)}x_{\pi(1)}, \dots, \sigma_{\pi(p-1)}x_{\sigma(p-1)}),\ B\,(x_0, \sigma_{\pi(1)}x_{\pi(1)}, \dots, \sigma_{\pi(p-1)}x_{\sigma(p-1)})\rangle
\end{aligned}$$

A brief calculation shows that this value is indeed

$$\frac{(x_0+\sigma_{\pi(1)}x_{\pi(1)})^2}{2(1+b_1)} + \frac{(x_0-\sigma_{\pi(p-1)}x_{\pi(p-1)})^2}{2(1-b_{p-1})} + \sum_{i=1}^{p-2}\frac{(\sigma_{\pi(i)}x_{\pi(i)}-\sigma_{\pi(i+1)}x_{\pi(i+1)})^2}{2(b_{i+1}-b_i)}.$$

and so

$$g(x_0, x_1, \dots, x_{p-1}; \sigma, \pi) \ = \ \int_{-1\le b_1\le\cdots\le b_{p-1}\le 1} db_1\dots db_{p-1}$$
$$\frac{\exp\left\{-\frac{1}{2}\left(\begin{array}{c}(x_0+\sigma_{\pi(1)}x_{\pi(1)})^2/(2(1+b_1))\\ +(x_0-\sigma_{\pi(p-1)}x_{\pi(p-1)})^2/(2(1-b_{p-1}))\\ +\sum_{i=1}^{p-2}(\sigma_{\pi(i)}x_{\pi(i)}-\sigma_{\pi(i+1)}x_{\pi(i+1)})^2/(2(b_{i+1}-b_i))\end{array}\right)\right\}}{\sqrt{2^{p-2}(1+b_1)(b_2-b_1)\dots(b_{p-1}-b_{p-2})(1-b_{p-1})}}.$$

Finally, we perform the following change of variables. For shorthand in this proof, let $V := v_1^2+\dots+v_{p-2}^2$. For $1\le i\le p-2$, let $b_i = (v_1^2+\cdots+v_i^2)/2-1$ and $b_{p-1} = \frac{1}{4}(V+(4-V)\sin\alpha)$. Then the Jacobian is $\left|\frac{\partial(b_1,\dots,b_{p-1})}{\partial(v_1,\dots,v_{p-2},\alpha)}\right| = \frac{1}{4}v_1v_2\dots v_{p-2}(4-V)\cos\alpha$. This is also the value of the denominator in the integral when expressed in the new variables. Thus the two cancel to leave only an exponential term. Under this change of variables, the region $\{\vec{b}\in\mathbb{R}^{p-1} \mid -1\le b_1\le\cdots\le b_{p-1}\le 1\}$ is equivalent to the region $\{\vec{v}\in\mathbb{R}_+^{p-2}, \alpha \mid \|\vec{v}\|\le 2, -\pi/2\le\alpha\le\pi/2\}$. The resulting integral is just the one stated in the lemma, which is also (3.19).

QED

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