On the thermodynamic limit of the 6-vertex model

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Abstract

We give a rigorous treatment to the thermodynamic limit of the 6-vertex model. We prove that the unique solution of the Bethe-Ansatz equation exists and the distribution of the roots converges to a continuum measure. We solve this problem for $0 < \Delta < 1$ using convexity arguments and for large negative Δ using the Fixed Point Theory of appropriately defined contracting operator.

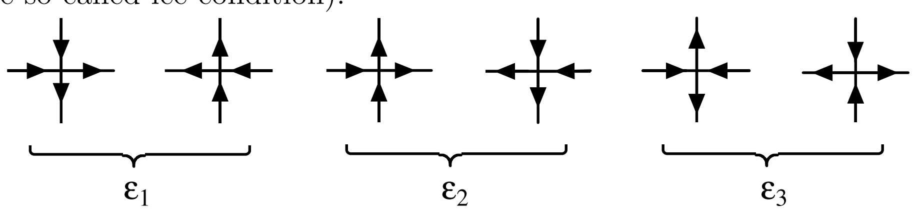
1 The 6-vertex model and formulation of the problem

The 6-vertex model is an exactly soluble model of classical statistical mechanics introduced and solved in various special cases by Lieb [1, 2, 3]. A solution of the most general case was obtained by Sutherland [4]. A clear description of this model and various other soluble models can be found in Baxter's book [5]. However, as Baxter remarks, an exact solution is not the same as a rigorous solution. In fact, already in his first article on the ice model [1], Lieb initiated the rigorous analysis of the model. A more extensive analysis was made by Lieb and Wu [6]. An important technical question was left unresolved, however. This concerns the convergence of the distribution of (quasi-) wavenumbers to a continuum measure in the thermodynamic limit. (Another technical problem, i.e. the independence of the free energy on the boundary conditions, was resolved by Brascamp et al. [7].) A similar problem was solved in the case of the nonlinear Schroedinger model in [8]. The 6-vertex model is more complicated because we cannot in all cases use the convexity argument of Yang and Yang [9] used there. However, their argument does extend to a certain domain of parameter space. Here we show how it can be used to prove the convergence of the Bethe Ansatz solutions in the thermodynamic limit in that case. In addition, we use another technique for proving the existence of a unique solution to the Bethe Ansatz equations in the thermodynamic limit in a different domain of parameter space. Uniqueness in other parts of parameter space is still an open problem, though numerical iteration does seem to converge to a unique solution.

1.1 Definition of the model and the free energy

We first recall the definition of the 6-vertex model and some general results concerning the existence of the thermodynamic limit. We then review the transfer matrix formulation of the model and the diagonalisation of the transfer matrix by means of the Bethe Ansatz.

The 6-vertex model is a model of classical statistical mechanics where the configurations are given by arrows on the bonds of a 2-dimensional square lattice. At each vertex only six different configurations of arrows are allowed (the so-called ice condition):



Each of these vertex configurations is assigned an energy and we assume spin-flip invariance, so that the first and the second, the third and the fourth and the fifth and the sixth configuration have the same energy. We denote these energies by ϵ_1, ϵ_2 and ϵ_3. If β is the inverse temperature, the corresponding Boltzmann weights are: $a = \exp[-\beta\epsilon_1]$, $b = \exp[-\beta\epsilon_2]$ and $c = \exp[-\beta\epsilon_3]$. The partition function is therefore

$$Z_{M,N}(a,b,c) = \sum_{\Gamma\in\mathcal{C}_{M,N}} e^{-\beta E(\Gamma)}, \tag{1.1}$$

where M is the number of rows and N is the number of columns in the lattice, $\mathcal{C}_{M,N}$ denotes the set of allowed configurations, and the total energy of a configuration $\Gamma \in \mathcal{C}_{M,N}$ is

$$E(\Gamma) = n_1(\Gamma)\,\epsilon_1 + n_2(\Gamma)\,\epsilon_2 + n_3(\Gamma)\,\epsilon_3, \tag{1.2}$$

if $n_i(\Gamma)$ is the number of vertices of type i in the configuration Γ. "Solving this model" now means: finding an explicit expression for the thermodynamic limit of the free energy density, i.e.

$$f(\epsilon_1, \epsilon_2, \epsilon_3; \beta) = -\frac{1}{\beta} \lim_{N,M\to\infty} \frac{1}{NM} \ln Z_{N,M}(a, b, c) \tag{1.3}$$

We shall assume periodic boundary conditions.

The first question that arises is whether the limit (1.3) exists. This was solved by Lieb and Wu [6]. In fact, we need the existence of the free energy at constant density ρ. In [6] this is highlighted as an open problem, but in fact, in the case of periodic boundary conditions, their method extends to this case. For convenience we repeat their argument here. (It was proved by Brascamp et.al. [7] that periodic boundary conditions are equivalent to free boundary conditions in the thermodynamic limit. It should be noted that not all boundary conditions are equivalent: see the recent solution of the model with domain-wall boundary conditions by Bleher et al. [10].)

The periodic boundary conditions imply that, in a given configuration, the number of up arrows in every row of vertical arrows is the same. We shall call this number divided by the maximum number N, the *density* ρ. The partition function with fixed density ρ is given by

$$Z_{M,N}(w_1, w_2, w_3; \rho) = \sum_{\alpha:\,\#\{\alpha_i=+1\}=N\rho} \sum_{\gamma} \sum_{\Gamma\in\mathcal{C}^p_{M,N}(\alpha,\gamma)} e^{-\beta E(\Gamma)}, \tag{1.4}$$

where $\mathcal{C}^p_{M,N}(\alpha, \gamma) = \mathcal{C}_{M,N}(\alpha, \alpha, \gamma, \gamma)$ and $\mathcal{C}_{M,N}(\alpha, \alpha', \gamma, \gamma')$ denotes the set of configurations with given boundary arrows: α and α' for the bottom and top rows of vertical arrows, and γ and γ' for the left- and right-hand columns of horizontal arrows.

Proposition 1.1 *Let $Z^p_{M,N}$ denote the partition function of the six-vertex model with periodic boundary conditions and let (M_l, N_l) be a sequence tending to infinity in the sense of Van Hove, and suppose that $(\rho_l)_{l=1}^\infty$ is a sequence of numbers $\rho_l \in [0, 1]$ tending to ρ such that $\rho_l N_l \in \mathbb{N}$. Then the corresponding free energy density $f^p(\epsilon_1, \epsilon_2, \epsilon_3; \beta, \rho)$ defined by*

$$f^p(\epsilon_1, \epsilon_2, \epsilon_3; \beta, \rho) = -\frac{1}{\beta} \lim_{l\to\infty} \frac{1}{M_l N_l} \ln\, Z^p_{M_l,N_l}(a, b, c; \rho_l) \tag{1.5}$$

exists and is independent of the sequences (M_l, N_l) and (ρ_l). Moreover, $f^p(\epsilon_1, \epsilon_2, \epsilon_3; \beta, \rho)$ is convex as a function of ρ and concave as a function of the variables $\epsilon_1, \epsilon_2, \epsilon_3$ and β.

Proof. We start by considering special sequences. Assume first that $\rho \in [0,1]\cap\mathbb{Q}$. Take $N_0 \in \mathbb{N}$ so large that $\rho N_0 \in \mathbb{N}$, and choose $M_0 \in \mathbb{N}$ arbitrary. Consider a *standard sequence* of rectangular boxes of height $M_l = 2^l M_0$ and width $N_l = 2^l N_0$. One then proves as in Lieb and Wu [6] that the limit (1.5) exists, using the inequalities

$$Z_{M_l,N_l}(\beta,\rho) \geq \Theta_{M_l,N_l}(\beta,\rho) \geq \left(\Theta_{M_{l-1},N_{l-1}}(\beta,\rho)\right)^4 \geq \left(Z_{M_{l-1},N_{l-1}}(\beta,\rho)\right) 2^{-4(M_0+N_0)}. \quad (1.6)$$

(We suppress the dependence on ϵ_i and on the periodic boundary conditions.) Here we define

$$\Theta_{M,N}(\beta,\rho) = \max_{\alpha:\,\#\{\alpha_i=+1\}=N\rho} \max_{\gamma} \sum_{\Gamma\in\mathcal{C}^p_{M,N}(\alpha,\gamma)} e^{-\beta E(\Gamma)}. \quad (1.7)$$

The inequalities (1.6) imply that the sequence

$$f_{M_l,N_l}(\beta,\rho) = -\frac{1}{\beta M_l N_l} \ln Z_{M_l,N_l}(\beta,\rho) \quad (1.8)$$

is essentially decreasing. As it is also bounded below, it converges. The concavity as a function of ϵ_i is standard. To prove the convexity as a function of the density, suppose that $\rho_1 < \rho_2$. Then if α_1 and α_2 are given sets of N vertical arrows with densities ρ_1 and ρ_2 respectively, and we write $\alpha = \alpha_1 \cup \alpha_2$ for the union,

$$\sum_{\gamma} \sum_{\Gamma\in\mathcal{C}^p_{M,2N}(\alpha,\gamma)} e^{-\beta E(\Gamma)} \geq \sum_{\gamma} \sum_{\Gamma_1\in\mathcal{C}^p_{M,N}(\alpha_1,\gamma)} \sum_{\Gamma_2\in\mathcal{C}^p_{M,N}(\alpha_2,\gamma)} e^{-\beta(E(\Gamma_1)+E(\Gamma_2))}. \quad (1.9)$$

Summing over α_1 and α_2 we have

$$Z^p_{M,2N}(\tfrac{1}{2}(\rho_1+\rho_2)) \geq Z^p_{M,N}(\rho_1) Z^p_{M,N}(\rho_2) \quad (1.10)$$

and hence

$$f(\beta, \tfrac{1}{2}(\rho_1+\rho_2)) \leq \frac{1}{2}\left(f(\beta,\rho_1) + f(\beta,\rho_2)\right). \quad (1.11)$$

Convexity implies continuity and we can thus extend the definition to all $\rho \in (0,1)$.

To show that the definition of $f(\beta,\rho)$ is independent of M_0 and N_0 we fill a general domain Λ with rectangles and use the condition $\epsilon_1 > 0$ and $\epsilon_2 > 0$ to decorate the remainder by vertices 1,2,3, and 4, as in [6]. □

In the exact solution of the six-vertex model one actually takes the limits $M \to \infty$ and $N \to \infty$ consecutively, but it was also shown by Lieb and Wu [6] that, for periodic boundary conditions, this yields the same limit as (1.5):

Proposition 1.2 *The double limit*

$$\tilde{f}(\epsilon_1, \epsilon_2, \epsilon_3; \beta, \rho) = -\frac{1}{\beta} \lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{NM} \ln Z^p_{M,N}(a, b, c) \tag{1.12}$$

exists and equals $f(\epsilon_1, \epsilon_2, \epsilon_3, \beta, \rho)$.

1.2 The transfer matrix and its diagonalisation

The transfer matrix method for solving models of classical statistical mechanics is common knowledge. Using periodic boundary conditions one writes the partition function as a trace

$$Z_{M,N} = \text{ Trace } (V_N^M) \tag{1.13}$$

where V_N is the transfer matrix with entries between two rows of vertical arrows α and α' given by

$$(V_N)_{\alpha,\alpha'} = \sum_\gamma \prod_{n=1}^N \exp\,[-\beta \epsilon_{\alpha_n}^{\alpha'_n}(\gamma_n, \gamma_{n+1})]. \tag{1.14}$$

The sum runs over a row of horizontal arrows $\gamma = (\gamma_1, \ldots, \gamma_N)$, where γ_n is the horizontal arrow between the $(n-1)$-th and the n-th vertex. It follows that we can take the limit $M \to \infty$ to obtain

$$f(\beta, \rho) = -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \ln \Lambda_{\max}(N), \tag{1.15}$$

where $\Lambda_{\max}(N)$ is the maximum eigenvalue of the transfer matrix V_N which exists because V_N satisfies the conditions of the Perron-Frobenius Theorem. The transfer matrix can be diagonalised by means of the Bethe Ansatz. If we write $|x_1, \ldots, x_n\rangle$ for the row configuration with n up-arrows then a general wave function in the subspace with n up-arrows can be expressed as

$$\psi = \sum_{1\le x_1 < \cdots < x_n \le N} \psi(x_1, \ldots, x_n)\, |x_1, \ldots, x_n\rangle. \tag{1.16}$$

The Bethe Ansatz for eigenfunctions of V_N then reads:

$$\psi(x_1, \ldots, x_n) = \sum_{\sigma \in \mathcal{S}_n} A_\sigma \exp\,[i \sum_{j=1}^n k_{\sigma(j)} x_j]. \tag{1.17}$$

Here, the sum runs over the set $\mathcal{S}_n$ of all permutations of$\{1, \ldots, n\}$ and the coefficients A_σ and the wave numbers $k_1, \ldots, k_n$ are to be determined by inserting into the eigenvalue equation. This yields the following conditions:

1. The wave numbers must satisfy the simultaneous nonlinear equations:

$$e^{iNk_j} = (-1)^{n-1} \prod_{l=1; l\neq j}^{n} e^{-i\theta(k_j,k_l)}, \tag{1.18}$$

where the function θ is defined by

$$\exp\left[-i\theta(k,k')\right] = \frac{1-2\Delta e^{ik}+e^{i(k+k')}}{1-2\Delta e^{ik'}+e^{i(k+k')}} \tag{1.19}$$

with

$$\Delta = \frac{a^2+b^2-c^2}{2ab}. \tag{1.20}$$

2. The corresponding eigenvalue is given by

$$\Lambda(k_1,\ldots,k_n) = a^N \prod_{j=1}^{n} L(e^{ik_j}) + b^N \prod_{j=1}^{n} M(e^{ik_j}), \tag{1.21}$$

where $L(z)$ and $M(z)$ are given by

$$L(z) = \frac{ab+(c^2-b^2)z}{a^2-abz}, \tag{1.22}$$

$$M(z) = \frac{a^2-c^2-abz}{ab-b^2z}. \tag{1.23}$$

Of course, (1.19) only defines the function θ up to a multiple of 2π. In taking the logarithm of (1.18), we shall assume that $-\pi < \theta(k,k') \leq \pi$. We obtain

$$Nk_j = 2\pi I_j - \sum_{l=1}^{n} \theta(k_j,k_l), \tag{1.24}$$

where $I_j \in \mathbb{Z}$ if n is odd, and $I_j \in \mathbb{Z}+\frac{1}{2}$ if n is even. These equations are identical to the BA equations found by Bethe [11] in his solution of the Heisenberg chain. They were analysed in detail by Yang and Yang [9], who showed that the ground state of the Heisenberg chain is obtained by choosing

$$I_j = j - \frac{1}{2}(n+1). \tag{1.25}$$

They also showed that, for this choice, the equations (1.24) have a real solution for $k_1,\ldots,k_n$. Lieb [1] then argued that, as the Heisenberg Hamiltonian also satisfies the conditions for the Perron-Frobenius Theorem, the corresponding eigenfunction must be positive, and hence it must also be the eigenfunction of the transfer matrix with maximum eigenvalue. We therefore have

$$\Lambda_{\max} = \Lambda(k_1, \ldots, k_n) \tag{1.26}$$

where $k_1, \ldots, k_n$ are the solutions of (1.24) in case the I_j are given by (1.25).

By formula (1.15), the free energy is now given by

$$f(\beta, \rho) = \lim_{N\to\infty} \min \left\{ \epsilon_1 - \frac{1}{\beta N} \sum_{j=1}^{n} \ln L(e^{ik_j}), \epsilon_2 - \frac{1}{\beta N} \sum_{j=1}^{n} \ln M(e^{ik_j}) \right\}. \tag{1.27}$$

By the fact that the free energy is convex and symmetric in the density, the minimum is attained at $\rho = \frac{1}{2}$ and the solution for $k_1, \ldots, k_n$ corresponding to the integers (1.25).

In this paper we address the question of how to compute the thermodynamic limit (1.27). We want to take the limit $N \to \infty$, keeping $\rho = n/N$ fixed. One usually makes the reasonable assumption that, in this limit, the distribution of the wavenumbers $k_1, \ldots, k_n$ tends to a continuous distribution with density $\rho(k)$. In the following we shall investigate the validity of this assumption. Following Yang and Yang [9], we consider separately the cases $\Delta \in [0, 1)$ and $\Delta < 0$. (The case $\Delta \geq 1$ is trivial.) In the attractive case, $\Delta \in [0, 1)$, we can apply the same reasoning as in the case of the nonlinear Schroedinger model (see [8]) and use the convexity of a certain functional to prove the existence of a unique solution to (1.24). In the repulsive case, we can only treat the case $\Delta \ll -1$. The case of smaller negative Δ is more delicate. The difficulty is proving the uniqueness of the solution. We can show, however, that if one assumes that the solution is monotone, the limiting solution is unique. Numerical solution of the BA equations seems to suggest that it is unique even without this assumption, but we have so far been unable to prove that.

2 Thermodynamic limit in the case $\Delta \in [0, 1)$.

In taking the thermodynamic limit we distinguish the cases $\Delta > 1$, $\Delta \in [0, 1)$, $\Delta \in (-1, 0)$ and $\Delta < -1$. The case $\Delta > 1$ is trivial (Cf. Baxter [5]) so we start with the case $\Delta \in [0, 1)$. We first prove an analogue of the existence and uniqueness of a solution to the Bethe Ansatz equations in the thermodynamic limit. In the present case this is analogous to the nonlinear Schrödinger problem treated in [8].

Theorem 2.1 *Let* $m \in \mathcal{M}_+^b\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *with* $||m|| \leq 1/2$ *and* $\mathrm{supp}(m) \subset [-\pi||m||, \pi||m||]$. *In case* $||m|| = \frac{1}{2}$, *assume that there exists* $\delta_0 > 0$ *such that for* $0 < \delta \leq \delta_0$,

$$m\left(\left\{q \in [-\frac{\pi}{2}, \frac{\pi}{2}] : \frac{\pi}{2} - |q| \leq \delta\right\}\right) \leq \frac{1}{\pi}\delta. \tag{2.28}$$

(Notice that the uniform distribution satisfies this condition.) Let $\Delta = -\cos\mu$ *with* $\mu \in (\pi/2, \pi)$. *Then there exists a unique continuous function* $k : [-\pi/2, \pi/2] \to [-\pi + \mu, \pi - \mu]$ *such that*

$$k(q) = q - \int_{-\pi/2}^{\pi/2} \theta(k(q), k(q'))\, m(dq'). \tag{2.29}$$

Proof. Define the new function $g(q)$ by

$$e^{ik(q)} = \frac{e^{i\mu} - e^{g(q)}}{e^{i\mu + g(q)} - 1}. \tag{2.30}$$

Then $k(q) = K(g(q))$ where $K : \mathbb{R} \to (-\pi/2, \pi/2)$ is an increasing function given by

$$K(\alpha) = \int_0^\alpha \frac{\sin(\mu)}{\cosh(\beta) - \cos\mu} d\beta = 2\tan^{-1}\left(\frac{\tanh(\alpha/2)}{\tan(\mu/2)}\right). \tag{2.31}$$

It follows that $g(q)$ must satisfy:

$$K(g(q)) = q - \int_{-\pi/2}^{\pi/2} \omega(g(q) - g(q'))\, m(dq') \tag{2.32}$$

where

$$\omega(\alpha) = -2\tan^{-1}\left(\frac{\tanh(\alpha/2)}{\tan(\mu)}\right). \tag{2.33}$$

Notice that

$$\omega'(\alpha) = -\frac{\sin(2\mu)}{\cosh\alpha - \cos(2\mu)} > 0. \tag{2.34}$$

As in [8], we now define a functional $B[g]$ on the space $L^2(\mathbb{R}, m)$ by

$$\begin{aligned} B[g] \;=\; & \int S(g(q))\, m(dq) - \int q\, g(q) m(dq) \\ & + \frac{1}{2} \int\int \Omega(g(q) - g(q'))\, m(dq) m(dq'), \end{aligned} \tag{2.35}$$

where $S(\alpha) = \int_0^\alpha K(\beta) d\beta$ and $\Omega(\alpha) = \int_0^\alpha \omega(\beta) d\beta$.

The functional B is well-defined because $0 \leq S(\alpha) \leq \frac{1}{2}K'(0)\alpha^2$ and $0 \leq \Omega(\alpha) \leq \frac{1}{2}\omega'(0)\alpha^2$, where

$$K'(0) = \frac{\sin(\mu)}{1-\cos(\mu)} \text{ and } \omega'(0) = -\frac{\sin(2\mu)}{1-\cos(2\mu)}.$$

It is also easily seen to be continuous. The Gateaux derivative in the direction of a function f is given by

$$DB[g]f = \int \left\{ K(g(q)) - q + \int \omega(g(q)-g(q'))m(dq') \right\} f(q)m(dq). \quad (2.36)$$

It follows that the solution to (3.4) is a stationary point of B. Moreover, B is convex as

$$\begin{aligned} \frac{d^2}{dt^2}B[g+tf] &= \int K'(\alpha(q))f(q)^2 m(dq) \\ &+\frac{1}{2}\int\int \omega'(g(q)-g(q'))(f(q)-f(q'))^2 m(dq)\, m(dq') > 0 \end{aligned} \quad (2.37)$$

by (2.34) and the fact that $K'(\alpha) > 0$. This proves the uniqueness of the solution. To prove the existence, we need to find a compact set which contains the minimiser.

Consider first the case that $||m|| < \frac{1}{2}$. Now, as $\alpha \to \pm\infty$, $K(\alpha) \to \pm(\pi-\mu)$ and $\omega(\alpha) \to \pm(2\mu-\pi)$. Let M be so large that $\pi - \mu - |K(\alpha)| < \epsilon$ and $(2\mu-\pi) - |\omega(\alpha)| < \epsilon$ for $|\alpha| > M$, where $\epsilon > 0$ is to be determined later. Consider the set

$$\Gamma_M = \{q \in [-\pi||m||, \pi||m||] : \, g(q) > M\}. \quad (2.38)$$

For M large enough, we can assume that $m(\Gamma_M) < \epsilon$. We now replace g on the set Γ_{2M} by $\pm 2M$, i.e. we set

$$\tilde{g}(q) = \mathrm{sgn}\,(g(q)) \min\{|g(q)|, 2M\}. \quad (2.39)$$

By convexity of the functions Ω and S we then have

$$
\begin{aligned}
&B[g]-B[\tilde g] = \\
&\quad = \int \left(S(g(q)) - S(\tilde g(q))\right) m(dq) - \int q\,(g(q)-\tilde g(q))\, m(dq) \\
&\qquad + \frac{1}{2}\int\int \left(\Omega(g(q)-g(q')) - \Omega(\tilde g(q)-\tilde g(q'))\right)\, m(dq)m(dq') \\
&\quad \geq \int_{\Gamma_{2M}} (|g(q)|-2M)(K(2M)-|q|)\, m(dq) \\
&\qquad + \int_{\Gamma_{2M}} m(dq) \int_{\Gamma_{2M}^c} m(dq')\left(\Omega(g(q)-g(q')) - \Omega(\tilde g(q)-g(q'))\right)
\end{aligned} \tag{2.40}
$$

where we used the convexity of the function S and the fact that if $q, q' \in \Gamma_{2M}$ then the second term in the double integral is zero whereas the first term is positive since $\Omega \geq 0$. Next using the convexity of Ω and the above bounds on the derivatives we get

$$
\begin{aligned}
&B[g]-B[\tilde g] = \\
&\quad \geq \int_{\Gamma_{2M}} (|g(q)|-2M)(K(2M)-|q|)\, m(dq) \\
&\qquad + \int_{\Gamma_{2M}} m(dq) \int_{\Gamma_{2M}^c} m(dq')\omega(2M-|g(q')|)(|g(q)|-2M) \\
&\quad \geq \int_{\Gamma_{2M}} (|g(q)|-2M)(\pi-\mu-|q|-\epsilon) m(dq) \\
&\qquad + \int_{\Gamma_{2M}} m(dq) \int_{\Gamma_{M}^c} m(dq')\left((2\mu-\pi)-\epsilon\right)(|g(q)|-2M) \\
&\quad \geq \int_{\Gamma_{2M}} m(dq)(|g(q)|-2M)\big(\pi-\mu-|q|-\epsilon+((2\mu-\pi)-\epsilon)\, m(\Gamma_M^c)\big) \\
&\quad \geq \int_{\Gamma_{2M}} m(dq)(|g(q)|-2M)\big(\pi-\mu-|q|-\epsilon+((2\mu-\pi)-\epsilon)\,(||m||-\epsilon)\big) \\
&\quad \geq \int_{\Gamma_{2M}} m(dq)(|g(q)|-2M)\left((\pi-\mu)(1-2||m||)-\epsilon(1+2\mu-\pi+||m||)\right) \\
&\quad > 0
\end{aligned} \tag{2.41}
$$

provided

$$
\epsilon < \frac{(\pi-\mu)(1-2||m||)}{1+2\mu-\pi+||m||}.
$$

We conclude that the minimiser must satisfy $||g||_\infty \leq 2M$ and is a fortiori contained in the ball $\{g \in L^2(m) : ||g||_2 \leq M\}$. This ball is bounded and therefore weakly compact. But the functional $B[g]$ is norm continuous and convex and therefore lower semicontinuous for the weak topology, see e.g. [12], Prop. 1.5 of Chap. 2. It follows that it attains its minimum on a compact set.

Next consider the case $||m|| = \frac{1}{2}$. In that case we cannot prove that the minimiser is bounded, so we need a more sophisticated bound. We use the function

$$f(q) = -2\ln\left(\frac{\pi}{2} - |q|\right).$$

Given $M > 0$ and $\delta > 0$, we define the sets

$$\Gamma_0^M = \{q \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi] : |g(q)| > M, |q| < \frac{\pi}{2} - \delta\} \tag{2.42}$$

and

$$\Gamma_k = \left\{q \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi] : |g(q)| > f(q), \frac{\pi}{2} - \gamma^{-k+1}\delta \leq |q| < \frac{\pi}{2} - \gamma^{-k}\delta\right\}, \tag{2.43}$$

where $\gamma > 1$ is a parameter to be determined later.

We now write

$$\Gamma^M = \Gamma_0^M \cup \bigcup_{k\geq 1} \Gamma_k$$

and consider the decomposition

$$\left\{(q, q') \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]^2 : q \in \Gamma^M \text{ or } q' \in \Gamma^M\right\} =$$
$$= \bigcup_{k\geq 0}\left(\Gamma_k \times \left(\bigcup_{l\geq k}\Gamma_l\right)^c \cup \left(\bigcup_{l\geq k}\Gamma_l\right)^c \times \Gamma_k \cup (\Gamma_k \times \Gamma_k)\right). \tag{2.44}$$

Note that this is a disjoint union. Replacing now $g(q)$ by

$$\tilde{g}(q) = \operatorname{sgn}(g(q)) \min\left\{|g(q)|, \left(f(q)\chi_{\Gamma^M\setminus\Gamma_0^M} + 2M\chi_{\Gamma_0^M}\right)\right\}$$

we have first of all

$$\begin{aligned}
&\int (S(g(q)) - S(\tilde{g}(q)))\, m(dq) - \int q\,(g(q) - \tilde{g}(q))\, m(dq) \\
&\quad \geq \int_{\Gamma_0^{2M}} m(dq)(|g(q)| - 2M)(K(2M) - |q|) \\
&\qquad + \sum_{k=1}^{\infty} \int_{\Gamma_k} m(dq)\,(|g(q)| - f(q))\,(K(f(q)) - |q|) \\
&\quad \geq \int_{\Gamma_0^{2M}} m(dq)(|g(q)| - 2M)(\pi - \mu - |q| - \eta) \\
&\qquad + \sum_{k=1}^{\infty} \int_{\Gamma_k} m(dq)\,(|g(q)| - f(q)) \\
&\qquad\quad \times \left(\pi - \mu - \frac{\pi}{\tan(\mu/2)}\left(\frac{\pi}{2} - |q|\right)^2 - |q|\right), \qquad (2.45)
\end{aligned}$$

where we used the bound

$$K(f(q)) > \pi - \mu - \frac{4}{\tan(\mu/2)}\left(\frac{\pi}{2} - |q|\right)^2 \qquad (2.46)$$

which follows from the inequalities

$$\tan^{-1}(x - \delta) \geq \tan^{-1}(x) - \delta$$

and

$$\tanh(x) > 1 - 2e^{-|x|}.$$

For the term

$$\frac{1}{2}\int\int \left(\Omega(g(q) - g(q')) - \Omega(\tilde{g}(q) - \tilde{g}(q'))\right)\, m(dq)m(dq')$$

we consider the contributions from the decomposition (2.44) separately:

$$\begin{aligned}
&\int_{\Gamma_0^M}\int_{(\Gamma^M)^c} \left(\Omega(g(q) - g(q')) - \Omega(\tilde{g}(q) - \tilde{g}(q'))\right)\, m(dq)m(dq') \\
&\quad \geq \int_{\Gamma_0^{2M}} m(dq)(2\mu - \pi - \eta)m((\Gamma^M)^c) \qquad (2.47)
\end{aligned}$$

as before. Combining this with the first term of (2.45) gives a positive contribution provided M is so large that $m(\Gamma^M) < \epsilon$ and $K(2M) > \pi - \mu - \eta$ and $\omega(M) > 2\mu - \pi - \eta$ where $\frac{3}{2}\eta + \pi\epsilon < \delta$.

Next consider a term of the form

$$\int_{\Gamma_k} \int_{(\cup_{l\geq k}\Gamma_k)^c} \left(\Omega(g(q)-g(q')) - \Omega(\tilde{g}(q)-\tilde{g}(q'))\right)\, m(dq)m(dq').$$

Assuming $\delta < \delta_0$, this is bounded by

$$\begin{aligned}
&\int_{\Gamma_k} \int_{(\cup_{l\geq k}\Gamma_k)^c} \left(\Omega(g(q)-g(q')) - \Omega(\tilde{g}(q)-\tilde{g}(q'))\right)\, m(dq)m(dq') \\
&\quad \geq \int_{\Gamma_k} m(dq)\left(|g(q)| - f(q)\right) \\
&\qquad \times \left[2\mu - \pi - \frac{4}{\tan(\pi-\mu)}\left(\frac{\pi}{2} - |q|\right)^2\right] m\left[\left(\bigcup_{l\geq k}\Gamma_l\right)^c\right]. \qquad (2.48)
\end{aligned}$$

Since

$$\bigcup_{l\geq k}\Gamma_l \subset \left\{q \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] : \frac{\pi}{2} - \gamma^{-k+1}\delta \leq |q|\right\}$$

we have by the assumption about m,

$$m\left(\bigcup_{l\geq k}\Gamma_l\right) \leq \frac{1}{\pi}\gamma^{-k+1}\delta. \qquad (2.49)$$

Therefore

$$\begin{aligned}
&\int_{\Gamma_k} \int_{(\cup_{l\geq k}\Gamma_k)^c} \left(\Omega(g(q)-g(q')) - \Omega(\tilde{g}(q)-\tilde{g}(q'))\right)\, m(dq)m(dq') \\
&\quad \geq \int_{\Gamma_k} m(dq)\left(|g(q)| - f(q)\right) \\
&\qquad \times \left[2\mu - \pi - \frac{4}{\tan(\pi-\mu)}\left(\frac{\pi}{2} - |q|\right)^2\right]\left(\frac{1}{2} - \frac{1}{\pi}\gamma^{-k+1}\delta\right). \qquad (2.50)
\end{aligned}$$

Combining this with the corresponding term of (2.45) we have

$$\begin{aligned}
&\int \left(S(g(q)) - S(\tilde{g}(q))\right) m(dq) - \int q\left(g(q) - \tilde{g}(q)\right) m(dq) \\
&+ \int_{\Gamma_k} \int_{(\cup_{l\geq k}\Gamma_k)^c} \left(\Omega(g(q)-g(q')) - \Omega(\tilde{g}(q)-\tilde{g}(q'))\right)\, m(dq)m(dq') \\
&\quad \geq \int_{\Gamma_k} m(dq)\left(|g(q)| - f(q)\right) \\
&\qquad \times \left[\pi - \mu - \frac{\pi}{\tan(\mu/2)}\left(\frac{\pi}{2} - |q|\right)^2 - |q|\right. \\
&\qquad \left. + \left(2\mu - \pi - \frac{4}{\tan(\pi-\mu)}\left(\frac{\pi}{2} - |q|\right)^2\right)\left(\frac{1}{2} - \frac{1}{\pi}\gamma^{-k+1}\delta\right)\right]. \qquad (2.51)
\end{aligned}$$

Since $\frac{\pi}{2} - \gamma^{-k+1}\delta \leq |q| < \frac{\pi}{2} - \gamma^{-k}\delta$ for $q \in \Gamma_k$, we have

$$\begin{aligned}
&\pi - \mu - c_1 \left(\frac{\pi}{2} - |q|\right)^2 - |q| \\
&+ \left(2\mu - \pi - c_2 \left(\frac{\pi}{2} - |q|\right)^2\right) \left(\frac{1}{2} - \frac{1}{\pi}\gamma^{-k+1}\delta\right) \\
&\geq \frac{\pi}{2} - \mu - c_1 \left(\gamma^{-k+1}\delta\right)^2 + \gamma^{-k}\delta \\
&\quad +\mu - \frac{\pi}{2} - \frac{1}{2}c_2 \left(\gamma^{-k+1}\delta\right)^2 - \frac{2\mu - \pi}{\pi}\gamma^{-k+1}\delta \\
= &\left(1 - \frac{2\mu - \pi}{\pi}\gamma\right) \gamma^{-k}\delta - c\gamma^{-2k+2}\delta^2, \qquad (2.52)
\end{aligned}$$

where

$$c_1 = \frac{4}{\tan(\mu/2)}, \quad c_2 = \frac{4}{\tan(\pi - \mu)}, \text{ and } c = c_1 + \frac{1}{2}c_2.$$

Hence

$$\begin{aligned}
&\int (S(g(q)) - S(\tilde{g}(q)))\, m(dq) - \int q\, (g(q) - \tilde{g}(q))\, m(dq) \\
&+ \int_{\Gamma_k} \int_{(\cup_{l \geq k}\Gamma_k)^c} \left(\Omega(g(q) - g(q')) - \Omega(\tilde{g}(q) - \tilde{g}(q'))\right) m(dq) m(dq') \\
&\quad \geq \int_{\Gamma_k} m(dq)\, (|g(q)| - f(q)) \\
&\qquad \times \left[\left(1 - \frac{2\mu - \pi}{\pi}\gamma\right) \gamma^{-k}\delta - c\gamma^{-2k+2}\delta^2\right]. \qquad (2.53)
\end{aligned}$$

Finally consider the terms

$$\frac{1}{2}\int_{\Gamma_k} m(dq) \int_{\Gamma_k} m(dq')\, \left(\Omega(g(q) - g(q')) - \Omega(f(q) - f(q'))\right).$$

Since $0 \leq \Omega(\alpha) \leq (2\mu - \pi)|\alpha|$, these can be bounded by

$$\begin{aligned}
&\frac{1}{2}\int_{\Gamma_k} m(dq) \int_{\Gamma_k} m(dq')\, \left(\Omega(g(q) - g(q')) - \Omega(f(q) - f(q'))\right) \\
&\quad \geq -(\mu - \frac{\pi}{2}) \int_{\Gamma_k} m(dq) \int_{\Gamma_k} m(dq')\, |f(q) - f(q')| \\
&\quad \geq \pi \ln(\gamma^{-k}\delta) m(\Gamma_k)^2 \geq \pi(\gamma^{-k}\delta)^2 \ln(\gamma^{-k}\delta). \qquad (2.54)
\end{aligned}$$

In all, we get

$$\begin{aligned}
&\int (S(g(q)) - S(\tilde{g}(q)))\, m(dq) - \int q\, (g(q) - \tilde{g}(q))\, m(dq) \\
&+ \int_{\Gamma_k} \int_{(\cup_{l\geq k}\Gamma_k)^c} (\Omega(g(q) - g(q')) - \Omega(\tilde{g}(q) - \tilde{g}(q')))\; m(dq)m(dq') \\
&+ \frac{1}{2} \int_{\Gamma_k} m(dq) \int_{\Gamma_k} m(dq')\, (\Omega(g(q) - g(q')) - \Omega(f(q) - f(q'))) \\
&\quad \geq \int_{\Gamma_k} m(dq)\, (|g(q)| - f(q)) \\
&\qquad \times \left[\left(1 - \frac{2\mu - \pi}{\pi}\gamma\right) \gamma^{-k}\delta - c\gamma^{-2k+2}\delta^2 + \pi(\gamma^{-k}\delta)^2 \ln(\gamma^{-k}\delta) \right] \qquad (2.55)
\end{aligned}$$

Choosing $\gamma < \frac{2\mu-\pi}{\pi}$ (which is possible as $\mu < \pi$) and δ small enough, this is positive.

It now follows that in this case the minimiser of $B[g]$ must satisfy

$$|g(q)| \leq f(q)\chi_{[-\frac{1}{2}\pi, -\frac{1}{2}\pi+\delta]\cup[\frac{1}{2}\pi-\delta, \frac{1}{2}\pi]} + 2M\chi_{[-\frac{1}{2}\pi+\delta, \frac{1}{2}\pi-\delta]}$$

and therefore

$$\begin{aligned}
||g||_2^2 &\leq 2M^2 + 2\int_{\frac{1}{2}\pi-\delta}^{\frac{1}{2}\pi} f(q)^2 m(dq) \\
&\leq 2M^2 + \frac{8}{\pi}\int_0^\delta (\ln x)^2 dx < +\infty. \qquad (2.56)
\end{aligned}$$

Again, it follows that $B[g]$ attains its minimum on this compact set.

We finally prove that the unique solution $g \in L^2(\mathbb{R}, m)$ of (2.32) in fact has a continuous version as a function $g : [-\pi||m||, \pi||m||] \to [-\infty, \infty]$. We have shown that there exists $\tilde{g} \in \mathcal{L}^2(\mathbb{R}, m)$ satisfying (2.32) for m-a.e. q. We define the image measure $\tilde{m} = \tilde{g}(m)$ and put

$$h(x) = K(x) + \int \omega(x - x')\tilde{m}(dx). \qquad (2.57)$$

Clearly, h is a $\mathcal{C}^\infty$-function on $\mathbb{R}$ and

$$h'(x) = K'(x) + \int \omega'(x - x')\tilde{m}(dx) > 0.$$

Therefore, the inverse function $g = h^{-1}$ is well-defined and $\mathcal{C}^\infty$ on the range of h. Since

$$h(x) \to \pm(\pi - \mu + (2\mu - \pi)||\tilde{m}||)$$

as $x \to \pm\infty$, the function g is defined on the interval

$$I_m = \big(-(\pi - \mu + (2\mu - \pi)||m||), \pi - \mu + (2\mu - \pi)||m||\big).$$

Notice that if $||m|| < \frac{1}{2}$, this interval contains $[-\pi||m||, \pi|m||]$, whereas if $||m|| = \frac{1}{2}$, $I_m = (-\frac{1}{2}\pi, \frac{1}{2}\pi)$. In the latter case, g extends continuously as a function $g : [-\frac{1}{2}\pi, \frac{1}{2}\pi] \to [-\infty, +\infty]$. Inserting $x = \tilde{g}(q)$ we have for $q \in \mathrm{supp}\,(m)$,

$$h(\tilde{g}(q)) = K(\tilde{g}(q)) + \int \omega(\tilde{g}(q) - \tilde{g}(q'))\, m(dq') = q$$

for m-a.e. q. Hence $g(q) = \tilde{g}(q)$ for m-a.e. q. Now inserting $x = g(q)$ we get

$$\begin{aligned} q = h(g(q)) &= K(g(q)) + \int \omega(g(q) - \tilde{g}(q'))m(dq') \\ &= K(g(q)) + \int \omega(g(q) - g(q'))m(dq'), \end{aligned}$$

so that g satisfies (2.32) for all q in its domain. It remains to show that the solution g is unique. It follows from the mean-value theorem that any continuous solution is differentiable and its derivative is given by

$$\begin{aligned} g'(q) &= \frac{1}{K'(g(q)) + \int \omega'(g(q) - g(q'))\, m(dq')} \\ &= \frac{1}{K'(g(q)) + \int \omega'(g(q) - \tilde{g}(q'))\, m(dq')}. \end{aligned} \tag{2.58}$$

Note that the function $\tilde{g}$ is uniquely defined modulo an m-null-set, so that the right-hand side only depends on the value of g at q. Since $g(q)$ is uniquely defined on $\mathrm{supp}\,(m)$ by continuity, its extension to $[-\pi||m||, \pi||m||]$ is also unique.

□

Theorem 2.2 *The mapping $m \mapsto k_m$ defined by (2.29) in Theorem 2.1 is continuous, that is, if $m_n \to m$ weakly then $k_{m_n} \to k_m$ in norm.*

Proof. Let $m_n^{(1)}$ be a subsequence. Notice that $||k_{m_n}|| \le \pi - \mu$ and k_{m_n} is also equicontinuous because

$$\frac{\partial}{\partial k}\theta(k, k') = \Delta \frac{\cos(k') + \cos(\mu)}{\Delta^2 \sin^2(k - k')/2 + [\cos(k + k')/2 - \Delta\cos(k - k')/2]^2} \ge 0 \tag{2.59}$$

for $-\pi+\mu \le k' \le \pi-\mu$. Hence

$$k'_{m_n}(q) = \left\{1 + \int \frac{\partial\theta}{\partial k}(k_{m_n}(q) - k_{m_n}(q'))m_n(dq')\right\}^{-1} \in (0,1) \qquad (2.60)$$

Therefore, $|k_{m_n}(q) - k_{m_n}(q')| \le |q-q'|$ uniformly in n. It follows that there exists a subsequence $m_n^{(2)}$ of $m_n^{(1)}$ such that $k_{m_n^{(2)}}$ converges to a continuous function k uniformly on $[-\pi/2, \pi/2]$. We must show that $k = k_m$. But θ is uniformly continuous on $[-\pi+\mu, \pi-\mu]^2$ so $\theta(k_{m_n^{(2)}}(q) - k_{m_n^{(2)}}(\cdot)) \to \theta(k(q) - k(\cdot))$ in norm, and hence

$$\int \theta(k_{m_n^{(2)}}(q) - k_{m_n^{(2)}}(q'))m_n^{(2)}(dq') \to \int \theta(k(q)-k(q'))m(dq').$$

It follows that $k(q) = k_m(q)$. □

Corollary 2.1 *If $m_n \to m$ weakly, and $\tilde{m}_n$ is the image measure of m_n under the mapping k_{m_n} then $\tilde{m}_n \to \tilde{m} = k_m(m)$.*

Proof. Let $F \in \mathcal{C}([-\pi+\mu, \pi-\mu])$. Then $\int F(k)\tilde{m}_n(dk) = \int F(k_{m_n}(q))m_n(dq)$ and

$$\begin{aligned}\left|\int F(k)\tilde{m}_n(dk) - \int F(k)\tilde{m}(dk)\right| &\le \\ \le \int |F(k_{m_n}(q)) - F(k_m(q))| m_n(dq) & \\ + \left|\int F(k_m(q))m_n(dq) - \int F(k_m(q))m(dq)\right|. & \qquad (2.61)\end{aligned}$$

The right-hand side tends to zero as $n \to \infty$ because $k_{m_n} \to k_m$ uniformly and k_m is continuous. □

Theorem 2.3 *Let*

$$m_N = \frac{1}{N}\sum_{j=1}^{n_N} \delta_{q_j}, \qquad (2.62)$$

where $q_j = \frac{2\pi}{N}(j - \frac{1}{2}(n_N+1))$ and $n_N \le N/2$. Assume that $n_N/N \to \rho$ as $N \to \infty$. Then $m_N \to \frac{1}{2\pi}dq$ on $[-\pi\rho, \pi\rho]$ and $\tilde{m}_n \to \tilde{m}$, where $\tilde{m}$ is absolutely continuous with respect to the Lebesgue measure and symmetric, and there exists $Q \in [0, \pi-\mu]$ such that $\mathrm{supp}(\tilde{m}) = [-Q, Q]$.

Proof. Let $F \in \mathcal{C}([-\pi/2, \pi/2])$. Then

$$\int F(q)\, m_N(dq) = \frac{1}{N} \sum_{j=1}^{n_N} F\left(\frac{2\pi}{N}(i - \frac{1}{2}(n_N + 1))\right) \to \int_{-\pi\rho}^{\pi\rho} F(q) \frac{dq}{2\pi}. \quad (2.63)$$

It follows that $\tilde{m}_N \to \tilde{m}$, and we must show that $\tilde{m}$ is absolutely continuous and even. The latter follows from the fact that k_m is even, which is a consequence of the uniqueness. To prove the absolute continuity, let $\epsilon > 0$. We must show that there exists $\delta > 0$ such that $\tilde{m}(k_0 - \delta, k_0 + \delta) < \epsilon$ for all k_0. Now, $\tilde{m}(k_0 + \delta, k_0 + \delta) = \int_{k_m^{-1}(k_0 - \delta, k_0 + \delta)} \frac{dq}{2\pi}$ and we have seen that k_m is continuous and increasing: $[-\pi/2, \pi/2] \to [-\pi + \mu, \pi - \mu]$. Therefore k_m^{-1} is continuous and $\forall \epsilon > 0\, \exists \delta > 0 :\, k_m^{-1}(k_0 - \delta, k_0 + \delta) \subset (q_0 - \pi\epsilon, q_0 + \pi\epsilon)$ where $k_m(k_0) = q_0$. Hence $\int_{k_m^{-1}(k_0 - \delta, k_0 + \delta)} \frac{dq}{2\pi} < \epsilon$. □

Writing the free energy (1.27) in the form

$$\begin{aligned} f(\beta, \rho) &= \lim_{N \to \infty} \min \Big\{ \epsilon_1 - \frac{1}{\beta} \int \ln L(e^{ik(q)})\, m_N(dq), \\ &\qquad \epsilon_2 - \frac{1}{\beta} \int \ln M(e^{ik(q)})\, m_N(dq) \Big\} \\ &= \lim_{N \to \infty} \min \Big\{ \epsilon_1 - \frac{1}{\beta} \int \ln L(e^{ik})\, \tilde{m}_N(dk), \\ &\qquad \epsilon_2 - \frac{1}{\beta} \int \ln M(e^{ik})\, \tilde{m}_N(dk) \Big\} \end{aligned} \quad (2.64)$$

we obtain

$$\begin{aligned} f(\beta, \rho) &= \min \Big\{ \epsilon_1 - \frac{1}{2\pi\beta} \int \ln L(e^{ik(q)})\, dq, \epsilon_2 - \frac{1}{2\pi\beta} \int \ln M(e^{ik(q)})\, dq \Big\} \\ &= \min \Big\{ \epsilon_1 - \frac{1}{\beta} \int \ln L(e^{ik})\, \tilde{m}(dk), \epsilon_2 - \frac{1}{\beta} \int \ln M(e^{ik})\, \tilde{m}(dk) \Big\}. \end{aligned} \quad (2.65)$$

By transformation to the variable $y = g(q)$ this becomes

$$\begin{aligned} f(\epsilon_1, \epsilon_2, \epsilon_3; \beta) &= \min \Big\{ \epsilon_1 - \frac{1}{2\pi\beta} \int_{-y_1}^{y_1} \ln L(e^{iK(y)}) R(y) dy, \\ &\qquad \epsilon_2 - \frac{1}{2\pi\beta} \int_{-y_1}^{y_1} \ln M(e^{iK(y)}) R(y) dy \Big\}. \end{aligned} \quad (2.66)$$

Here $y_1 = g(\pi/2)$ and $R(y) = g'(q)^{-1}$ is given by (2.58):

$$
\begin{aligned}
R(y) &= K'(y) + \int_{-\pi/2}^{\pi/2} \omega'(y - g(q)) \frac{dq}{2\pi} \\
&= \frac{\sin(\mu)}{\cosh(y) - \cos(\mu)} - \frac{1}{2\pi} \int_{-y_1}^{y_1} \frac{\sin(2\mu)}{\cosh(y-\alpha) - \cos(2\mu)} R(\alpha) d\alpha.
\end{aligned}
\tag{2.67}
$$

If we assume that $y_1 = +\infty$ then this can be evaluated by Fourier transformation as in [5]. With

$$
\hat{R}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\alpha) e^{i\alpha x} d\alpha \tag{2.68}
$$

we have $\hat{R}(x) = e^{-\mu x} - e^{-2\mu x} \hat{R}(x)$ and hence

$$
\hat{R}(x) = \frac{1}{2\cosh(\mu x)}. \tag{2.69}
$$

This is consistent with the fact that the minimum in (2.65) is attained at $\rho = \frac{1}{2}$ since f is convex in ρ. Indeed,

$$
\hat{R}(0) = \frac{1}{2\pi} \int_{-y_1}^{y_1} R(\alpha) d\alpha = ||m|| = \rho.
$$

Now, $R(y) = (g^{-1})'(y)$ determines g^{-1} given that $g^{-1}(0) = 0$. By uniqueness of $g(q)$, it must be the solution. Eventually, one finds

Corollary 2.2 *Assume $\Delta \in [0, 1)$. Then, the free energy is given by*

$$
\begin{aligned}
f(\beta) &= \epsilon_1 - \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{\sinh[(\mu + w)x] \sinh[(\pi - \mu)x]}{2x \sinh[\pi x] \cosh[\mu x]} dx \\
&= \epsilon_2 - \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{\sinh[(\mu - w)x] \sinh[(\pi - \mu)x]}{2x \sinh[\pi x] \cosh[\mu x]} dx,
\end{aligned}
\tag{2.70}
$$

where the parameter w is defined by

$$
a \,:\, b \,:\, c = \sin \tfrac{1}{2}(\mu - w) \,:\, \sin \tfrac{1}{2}(\mu + w) \,:\, \sin(\mu). \tag{2.71}
$$

3 The case $\Delta \ll -1$.

For $\Delta < -1$ we write $\Delta = -\cosh\lambda$, assuming $\lambda > 0$. Define the new function $\alpha(q)$ by

$$e^{ik(q)} = \frac{e^{\lambda} - e^{-i\alpha(q)}}{e^{\lambda - i\alpha(q)} - 1}. \tag{3.1}$$

Then $k(q) = \omega_\lambda(\alpha(q))$ where $\omega_\lambda : [-\pi, \pi] \to [-\pi, \pi]$ is an increasing function given by

$$\omega_\lambda(x) = \int_0^x \frac{\sinh(\lambda)}{\cosh(\lambda) - \cos(u)} du = 2\tan^{-1}\left(\frac{\tan(x/2)}{\tanh(\lambda/2)}\right). \tag{3.2}$$

In terms of $\alpha(q)$ the Bethe Ansatz equations read

$$\omega_\lambda(\alpha(q)) = q + \int_{-\pi/2}^{\pi/2} \omega_{2\lambda}(\alpha(q) - \alpha(q'))\, m(dq'). \tag{3.3}$$

Here the function $\omega_{2\lambda}$ is defined as in (3.2) with the understanding that for $|x| > \pi$ the integral expression is assumed so that $\omega_{2\lambda}$ is continuous. The measure $m \in \mathcal{M}_+^b\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ satisfies $||m|| \leq \frac{1}{2}$. For the finite lattice, it is given by

$$m = \frac{1}{N}\sum_{j=1}^{n} \delta_{q_j}, \qquad q_j = -\frac{n+1-2j}{2n}\pi. \tag{3.4}$$

Theorem 3.1 *Assume $\lambda > \lambda_0$, where $\lambda_0 = \ln(3 + 2\sqrt{5})$, when $||m|| \leq 1/2$. Then, for any measure $m \in \mathcal{M}_+^b[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $||m|| \leq \frac{1}{2}$, there exists a unique function $\alpha \in L^\infty(m)$ such that (3.3) holds for all $q \in \operatorname{supp}(m)$. Moreover, if m is symmetric then α extends uniquely to a continuous function on $[-\pi/2, \pi/2]$ with values in $[-\pi, \pi]$ which satisfies (3.3) for all $q \in [-\pi/2, \pi/2]$.*

Proof. We expand $\tan^{-1}(\tan(x/2)/\tanh(\lambda/2))$ into a Fourier series. For this, we compute first

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sinh(\lambda)}{\cosh(\lambda) - \cos(x)} e^{inx} dx = e^{-\lambda|n|}.$$

Hence

$$\frac{\sinh(\lambda)}{\cosh(\lambda) - \cos(x)} = \sum_{n \in \mathbb{Z}} e^{-\lambda|n|} e^{-inx} = 1 + 2\sum_{n=1}^{\infty} e^{-\lambda n}\cos(nx).$$

Integrating, we obtain the following Fourier expansion:

$$\omega_\lambda(x) = x + 2\sum_{n=1}^{\infty} \frac{e^{-\lambda n}}{n}\sin(nx). \tag{3.5}$$

Inserting this into the Bethe Ansatz equation (3.3), we have

$$\alpha(q) = \frac{q}{1-||m||} - \frac{1}{1-||m||}\int \alpha(q')m(dq') - \frac{2}{1-||m||}\sum_{n=1}^{\infty} e^{-n\lambda}\frac{\sin(n\alpha(q))}{n} + \\ + \frac{2}{1-||m||}\sum_{n=1}^{\infty} e^{-2\lambda n}\int \frac{\sin(n(\alpha(q)-\alpha(q')))}{n} m(dq'). \tag{3.6}$$

Notice next that it follows from (3.6):

$$\int \alpha(q')m(dq') = \int q'm(dq') - 2\sum_{n=1}^{\infty}\frac{1}{n}\int e^{-n\lambda}\sin(n\alpha(q'))m(dq').$$

Let us introduce an operator:

$$T[\alpha] = \\ -\frac{1}{1-||m||}\int q'm(dq') + \frac{2}{1-||m||}\sum_{n=1}^{\infty}\frac{1}{n}\int e^{-n\lambda}\sin(n\alpha(q'))m(dq') + \frac{q}{1-||m||} - \\ -\frac{2}{1-||m||}\sum_{n=1}^{\infty} e^{-n\lambda}\frac{\sin(n\alpha(q))}{n} + \frac{2}{1-||m||}\sum_{n=1}^{\infty} e^{-2\lambda n}\int \frac{\sin(n(\alpha(q)-\alpha(q')))}{n} m(dq').$$

We consider (3.6) as a fixed point problem $\alpha = T[\alpha]$ and show that the map $\alpha \mapsto T[\alpha]$ is contraction w.r.t. the L^∞ norm for sufficiently large λ. This is straightforward:

$$||T(\alpha_1) - T(\alpha_2)|| \le \left(\frac{1+||m||}{1-||m||}\frac{2}{e^\lambda - 1} + \frac{4||m||}{1-||m||}\frac{1}{e^{2\lambda}-1}\right)||\alpha_1 - \alpha_2||.$$

Clearly, the factor in front of $||\alpha_1 - \alpha_2||$ is less than 1 if $\lambda > \ln(3 + 2\sqrt{5})$.

The same inequalitie holds for functions $\alpha \in \mathcal{C}([-\frac{\pi}{2}, \frac{\pi}{2}])$, so there also exists a unique continuous function satisfying (3.3) for all $q \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Clearly, restricting this function to supp(m) yields the solution $\alpha \in L^\infty(m)$. Finally, note that if m is symmetric, uniqueness implies that the function α must be odd. This in turn implies that $\alpha(q) \in [-\pi, \pi]$ for if $\alpha_1\, \alpha_2 \in [0, \pi]$ then

$$\omega_{2\lambda}(\alpha_1 - \alpha_2) + \omega_{2\lambda}(\alpha_1 + \alpha_2) \le 2\omega_{2\lambda}(\alpha_1)$$

as follows easily by differentiation. □

We now also have an analogue of Theorem 2.3:

Lemma 3.1 *The map $m \mapsto \alpha_m$ defined by Theorem 3.1 is continuous.*

Proof. This follows from (3.6) using

$$
\begin{aligned}
& |\int \sin(n\alpha_{m_1}(q'))m_1(dq') - \int \sin(n\alpha_{m_2}(q'))m_2(dq')|, \\
& |\int \cos(n\alpha_{m_1}(q'))m_1(dq') - \int \cos(n\alpha_{m_2}(q'))m_2(dq')| \leq \frac{1}{2}n||m_1 - m_2||.
\end{aligned}
$$

□

Corollary 3.1 *If $m_n \to m$ weakly, and $\tilde{m}_n$ is the image measure of m_n under the mapping k_{m_n} then $\tilde{m}_n \to \tilde{m} = k_m(m)$.*

As in the case $\Delta \in [0, 1)$ we can now conclude that the free energy is given by

$$
\begin{aligned}
f(\beta, \rho) &= \min \left\{ \epsilon_1 - \frac{1}{2\pi\beta} \int \ln L(e^{ik(q)})\, dq, \right. \\
&\qquad \left. \epsilon_2 - \frac{1}{2\pi\beta} \int \ln M(e^{ik})\, dq \right\}. \qquad (3.7)
\end{aligned}
$$

Transforming to the variable α we have

$$
\begin{aligned}
f(\epsilon_1, \epsilon_2, \epsilon_3; \beta) &= \min \left\{ \epsilon_1 - \frac{1}{2\pi\beta} \int_{-\pi}^{\pi} \ln L(e^{i\omega_\lambda(\alpha)}) R(\alpha) d\alpha, \right. \\
&\qquad \left. \epsilon_2 - \frac{1}{2\pi\beta} \int_{-\pi}^{\pi} \ln M(e^{i\omega_\lambda(\alpha)}) R(\alpha) d\alpha \right\}, \qquad (3.8)
\end{aligned}
$$

where $R(\alpha) = \alpha'(q)^{-1}$. Again, it can be evaluated by Fourier transformation, but now on $[-\pi, \pi]$:

$$
\hat{R}_p = \frac{1}{2\cosh(\lambda p)}. \qquad (3.9)
$$

The resulting free energy is

Corollary 3.2 *Assume $\Delta = -\cosh(\lambda)$ with $\lambda > \lambda_0$. Then, the free energy of the 6-vertex model with periodic boundary conditions is given by*

$$
\begin{aligned}
f(\beta) &= \epsilon_1 - \frac{1}{\beta} \left\{ \frac{\lambda + v}{2} + \sum_{p=1}^{\infty} \frac{\sinh[(\lambda + v)p]\, e^{-p\lambda}}{p \cosh[p\lambda]} \right\} \\
&= \epsilon_2 - \frac{1}{\beta} \left\{ \frac{\lambda - v}{2} + \sum_{p=1}^{\infty} \frac{\sinh[(\lambda - v)p]\, e^{-p\lambda}}{p \cosh[p\lambda]} \right\}, \qquad (3.10)
\end{aligned}
$$

where the parameter v is given by

$$
a : b : c = \sinh \tfrac{1}{2}(\lambda - v) : \sinh \tfrac{1}{2}(\lambda + v) : \sinh(\lambda). \qquad (3.11)
$$

4 Concluding remarks.

For values of $\Delta < -1$ which are not large negative, numerical iteration of the equations (3.3) with $m = m_N$ seems to indicate that there is in fact a unique solution. We have so far not been able to prove this, although it is possible to show that the solution is unique and increasing for small $|q|$.

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