Integrable deformations of CFTs and the discrete Hirota equations

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1 Introduction

One of the current major targets in physics is the understanding of integrable quantum field theories in two dimensions. In theories with diagonal scattering matrix there is a convincing conjecture for the spectrum of the Hamiltonian. Suppose that there are K species of particles with masses m_a, $a = 1, \ldots, K$, and a scattering matrix $S_{ab} = \exp(i\Phi_{ab}(\theta_a - \theta_b))$ for particles of types a, b with rapidities θ_a, θ_b. Let there be N_a particles of type a. Then the states are characterized by integers n_a^i, $i = 1, \ldots, N_a$, $a = 1, \ldots, K$. The corresponding energy $E(L)$ is given by

$$E(L) = -\frac{1}{2\pi}\int \cosh(\theta)\log(1+\exp(-\epsilon_a(\theta)))d\theta + \sum_{a=1}^{K}\sum_{i=1}^{N_a}\sinh(\theta_a^i).$$

Here the θ_a^i are complex numbers with imaginary part $i\pi/2$. They and the functions ϵ_a are determined by the equation

$$\epsilon(\theta_a^i) = 2\pi i(n_a^i - 1/2),$$

$$\begin{aligned}\epsilon_a(\theta) &= m_a L\cosh(\theta) - \frac{1}{2\pi}\sum_{b=1}^{K}(\Phi'_{ab} * \log(1+\exp(-\epsilon_b))\\ &- i\sum_{b=1}^{K}\sum_{i=1}^{N_b}(\Phi_{ab}(\theta-\theta_b^i)) - \Phi_{ab}(-\infty)).\end{aligned}$$

These equations can be solved numerically by iteration. When all θ_a^i are non-vanishing, the partition function factorizes for $L \to 0$ into terms for right and left movers. The limiting partition functions for the right movers is

$$\sum_{N \in \mathbb{N}^K} \frac{q^{Q(N)}}{(q)_{N_1} \cdots (q)_{N_K}} \ ,$$

where we use the standard notation $(q)_n = (1-q)\cdots(1-q^n)$ for the q-factorial, and where

$$Q(N) = \frac{1}{2} \sum_{a,b=1}^{K} N_a A_{ab} N_b - \frac{c}{24} \ ,$$

with a rational number c as effective central charge and

$$A_{ab} = \delta_{ab} - \frac{1}{2\pi}(\Phi_{ab}(+\infty) - \Phi_{ab}(-\infty)).$$

Partition functions of conformal theories must be modular. This implies that they can be written as a sum over expressions of the form $\tilde{q}^{h-c/24+n}$, where h is a conformal dimension of the theory, c its central charge, and $n \in \mathbb{N}$. Evaluating $Z(Q)$ close to $q = 1$ yields in particular

$$\frac{\pi^2}{6} c_{eff} = \sum_{i=1}^{r} L(1 - x_i),$$

where L is Roger's dilogarithm and $x = (x_1, \ldots, x_r)$ is the unique solution of the system of equations

$$\prod_{j=1}^{r} x_j^{A_{ij}} = 1 - x_i,$$

$i = 1, \ldots, r$, for which $(x_1, \ldots, x_r) \in (0,1)^r$. According to an idea of Zagier [6] the existence of a unique solution can be proved as follows. Consider the map $\phi : (0,1)^r \to \mathbb{R}^r$ given by

$$\phi_i(x) = \sum_{j=1}^{r} A_{ij} \log x_j - \log(1 - x_i).$$

It is sufficient to show that ϕ is a homomorphism. Locally this is true, since the Jacobian matrix with components $J_{ij} = \partial\phi_i/\partial x_j$ is nowhere degenerate. Indeed the product matrix $Jdiag(x_j)$ is everywhere positive. Moreover one easily sees that the inverse image under ϕ of every compact domain in $\mathbb{R}^r$ is compact. Let $C = r(\log 2)^2$ be the maximum of $\sum_{j=1}^{r} \log x_i \log(1 - x_i)$. Then $\phi_i(x) > a_i$ implies

$$\sum_{i,j=1}^{r} \log x_i A_{ij} \log x_j - \sum_{i=1}^{r} a_i \log x_i < -C,$$

such that $(\log x_1, \ldots, \log x_r)$ lies in an ellipsoid. Replacing x_i by $1 - x_i$ and A by its inverse, one sees that the same is true for $(\log(1 - x_1), \ldots, \log(1 - x_r))$. Taking limits, the inverse of each convergent sequence $\phi(x^{(k)})$ has an accumulation point, which implies that $\phi(x^{(k)})$ converges to an element in the image of ϕ. Since the image of ϕ is both open and closed, it must be equal to $\mathbb{R}^r$. Finally, due to the monodromy theorem, ϕ must be a homomorphism.

The Rogers dilogarithms of other solutions $x = (x_1, \ldots, x_r)$ with all x_i non-vanishing should yield conformal dimensions. This implies that they are given by torsion elements of the Bloch group [5]. Conjecturally, one obtains elements in the kernel of the Bloch group by putting

$$A = \mathcal{C}(X) \otimes \mathcal{C}(Y)^{-1},$$

where X, Y are Dynkin diagrams of type ADET and $\mathcal{C}$ yields the corresponding Cartan matrices. With $x = z^{\mathcal{C}(Y)}$, one obtains

$$z^{(2-\mathcal{C}(X))\otimes I} + z^{I\otimes(2-\mathcal{C}(Y))} = z^2.$$

With $X = A_\infty$ and $Y = A_n$, this is a discrete Hirota equation. More precisely, the graph of A_∞ may be infinite in one or in both directions. We refer to the corresponding systems as $\mathbb{N}$ equations and $\mathbb{Z}$ equations. Those solutions for which no component of z vanishes form part of an irreducible affine algebraic variety which we denote by $S(X, Y)$. We shall consider the cases $X = A_r, D_r$, $Y =$

$A_n, A_{\mathbb{N}}, A_{\mathbb{Z}}$. We recover the results of Kirillov and Reshetikhin for $Y = A_{\mathbb{N}}$. Since their proofs are partially unpublished, we provide complete proofs.

There also is a conjecture which relates the solutions of these equations to the representation theory of Yangians [4]. Let X have rank r. Consider the Yangian $Y(X)$, which contains the enveloping algebra $U(X)$ as sub-Hopf-algebra. The irreducible finite dimensional representations of $Y(X)$ have highest weights $\lambda = \sum_{i=1}^r n_i\lambda_i$, which are obtained by restricting them to representations of X. Here the λ_i are the fundamental weights of X and the n_i are non-negative integers. The representations with highest weight λ are characterized by r monic polynomials of degrees $n_1, \dots, n_r$. For $\lambda_0 \in \mathbb{R}$ there is an outer automorphism of $Y(X)$ which preserves $U(X)$ and acts on the polynomials by a shift $\lambda \mapsto \lambda + \lambda_0$. For $\lambda = n\lambda_i$ the irreducible representations are characterized by a single polynomial of order n. There is a basic representation given by the polynomial $\prod_{i=1}^n (\lambda - i)$ (up to shifts of λ). Let χ_{ni} be the corresponding character of X. These characters decompose into sums of irreducible characters which may be quite complicated, see e.g. [2]. Note that $\chi_{0i}(g) = 1$ for any $g \in Lie(X)$.

Kirillov and Reshetikhin conjectured that $z_{ni} = \chi^i_{\lambda_n}(g)$ solves the $\mathbb{N}$ equations for any $g \in Lie(X)$. This implies that any generic solution of the equations can be found in this way, since the z_{1i} can be chosen arbitrarily and generically determine all z_{ni} by recursion. The conjecture is true at least for $X = A_m$, and we will provide further support for $X = D_m$. Our proofs will use the generic solution of the $\mathbb{Z}$ equations for D_m.

Our original equations reduce to the $\mathbb{N}$ equations with $n = 1, \dots, m+1$ with $z_{m+1,i} = 1$ for all i and the the constraint that in the required range for n the z_{ni} are all different from 0.

2 The A Case

We first treat the case $X = A_r$. This yields a special case of the discrete Hirota equations, which have well known explicit solutions.

Theorem 2.1. *There is an isomorphism from $\mathbb{C}^r$ to $S(A_r, A_{\mathbb{N}})$ defined as follows. Let $(x_1, \dots, x_r) \in \mathbb{C}^r$. Put $x_0 = x_{r+1} = 1$ and $x_i = 0$ for $i < 0$ or $i > r+1$. For any positive integer m and $i = 1, \dots, r$ let M^{mi} be the $m \times m$ Toeplitz matrix with entries*

$$(M^{mi})_{kl} = x_{l-k+i}.$$

Let

$$z_{im} = \det M^{mi}.$$

Then these z_{im} satisfy the $S(A_r, A_{\mathbb{N}})$-equations. The inverse map is given by $x_i = z_{i1}$ for $i = 1, \ldots, r$.

Proof. This is an immediate consequence of Jacobi's determinant identity. When the determinants are written as wedge products, this identity states that

$$(v_1 \wedge U \wedge v_2) \otimes (w_1 \wedge U \wedge w_2) =$$
$$(v_1 \wedge U \wedge w_2) \otimes (w_1 \wedge U \wedge v_2) - (v_1 \wedge U \wedge w_1) \otimes (w_2 \wedge U \wedge v_2),$$

where $U = u_1 \wedge \ldots \wedge u_{m-2}$ and $v_1, v_2, w_1, w_2, u_1, \ldots, u_{m-2}$ are vectors in $\mathbb{C}^m$. For the convenience of the reader we give a proof of this identity. One may assume that $U \neq 0$. The equation is linear in the v_i, w_i. Thus it is sufficient to prove it for generic w_1, w_2, so that v_1, v_2 can be written as linear combinations of $w_1, w_2, u_1, \ldots, u_{m-2}$. Linear combinations of the u_i drop out, and the equation reduces to an identity in the two-dimensional quotient space $\mathbb{C}^m / \langle u_1, \ldots, u_{m-2} \rangle$, which is essentially the formula for the determinant of 2×2 matrices. To deduce the discrete Hirota equations one denotes the rows of M^{mi} by $v_1, u_1, \ldots, u_{m-2}, v_2$ and puts $w_1 = (1, 0, \ldots, 0)$, $w_2 = (0, \ldots, 0, 1)$. That the map from $\mathbb{C}^r$ to $S(A_r, A_{\mathbb{N}})$ is an isomorphism follows by comparing dimensions. □

Corollary 2.2. *For any $n \in \mathbb{N}$ there is an embedding map $\Phi_{rn} : S(A_r, A_n) \to S(A_r, A_{\mathbb{N}})$ induced by the restriction to $(z_{11}, \ldots, z_{r1})$.*

Theorem 2.3. *For $z \in S(A_r, A_{\mathbb{N}})$, $m \in \mathbb{N}$ and $i = 1, \ldots, r+1$ let M_m^i be the $i \times i$ matrix with entries*

$$(M_m^i)_{kl} = z_{1,m-k+l},$$

with $z_{10} = 1$ and $z_{1,m-k+l} = 0$ for $m - k + l < 0$. Then

$$z_{im} = \det M_m^i. \tag{1}$$

Proof. As explained in the previous proof, but with exchange of i and m, Jacobi's identity implies that the determinants $\det M_m^i$ satisfy the $S(A_r, A_{\mathbb{N}})$ equations. Thus it is sufficient to show that for given generic z_{1m} there is at most one family z_{im} with this property. By the previous theorem, z_{im} is a polynomial whose highest order term in x_i is x_i^m. Thus one has generically $z_{im} \neq 0$. For such points the equation $z_{i,m} = (z_{i,m-1}^2 - z_{i+1,m-1} z_{i-1,m-1})/z_{i,m-2}$ and induction on m imply that z_{im} can be calculated in terms of z_{1k} with $m - i < k < m + i$ and $k > 0$. □

Theorem 2.4. *Let $x_i = z_{i1}$ for $i = 1, \ldots, r$ and $x_0 = x_{r+1} = 1$. Put $z_{10} = 1$ and $z_{1,-m} = 0$ for $m = 1, \ldots, r$. Then for any $m \in \mathbb{N}$ one has*

$$\sum_{k=0}^{r+1} (-)^k x_k z_{1,m-k} = 0.$$

Proof. This is just the expansion of the determinant of the matrix M_m^1 in terms of minors with respect to the elements of the last column. $\square$

In the following we need some facts about the representation theory of the A_r Lie algebra. Let $T(r+1)$ be the subgroup of $SL(r+1)$ consisting of the diagonal matrices, which we write as $g = diag(g_1, \ldots, g_{r+1})$. Let $\lambda_1, \ldots, \lambda_r$ be the corresponding fundamental highest weights of $SL(r+1)$, so that

$$\lambda_i(g) = g_1 \cdots g_i.$$

Let χ_λ be the character of the irreducible representation with highest weight λ. Note that the Weyl group of $SL(r+1)$ with respect to $T(r+1)$ is the permutation group of the diagonal entries, thus isomorphic to the permutation group $\mathcal{S}^{r+1}$ on the set $\{1, \ldots, r+1\}$.

Recall the following cases of Weyl's character formula. For $m \in \mathbb{Z}$ let

$$N_{mi}(g) = \sum_{\sigma \in \mathcal{S}^{r+1}} sgn(\sigma) \prod_{j=1}^{i} g_{\sigma(j)}^{r+m+1-j} \prod_{j=i+1}^{r} g_{\sigma(j)}^{r+1-j}.$$

Then for generic g

$$\chi_{m\lambda_i}(g) = N_{mi}(g)/D^{A_r}(g),$$

where the Weyl denominator is given by $D^{A_r} = N_{0i}$ independently of i. In our case the Weyl denominator formula takes the Vandermonde form

$$D^{A_r}(g) = \prod_{i<j} (g_i - g_j).$$

We define the singular locus of $T(r+1)$ by the equation $D^{A_r}(g) = 0$.

Remark 2.5. *If one sums over all permutations σ with fixed $\sigma(1), \ldots, \sigma(i)$ and uses the Weyl denominator formula for the Weyl group of $SL(r+1-i)$ one obtains for generic g*

$$\chi_{m\lambda_i} = \sum_{\mu \in I(i,r+1)} D_\mu(g)^{-1} \prod_{j=1}^{i} g_{\mu(j)}^{m+r+1-j},$$

where $I(i, r+1)$ is the set of all injective maps from $\{1, \ldots, i\}$ to $\{1, \ldots, r+1\}$ and $D_\mu = D_\mu^1 D_\mu^2$, with

$$\begin{aligned} D_\mu^1 &= \prod_{1 \le j < k \le i} \left(g_{\mu(j)} - g_{\mu(k)}\right), \\ D_\mu^2 &= \prod_{j=1}^{i} \prod_{k \in C(\mu)} \left(g_{\mu(j)} - g_k\right), \end{aligned}$$

where $C(\mu)$ is the complement of the image of μ in $\{1,\ldots,r+1\}$. For $i=1$ we have $D^1_\mu = 1$, so that for generic g

$$\chi_{m\lambda_1} = \sum_{j=1}^{r+1} \phi_{jm}, \tag{2}$$

with

$$\phi_{jm} = g_j^{m+r} / \prod_{k\neq j} (g_j - g_k).$$

For $j \neq i$ the term ϕ_{jm} is proportional to $(g_i - g_j)^{-1}$ and has no further dependence on g_i, which implies a simple generalization of equation (2) for arbitrary g. Let $I(g)$ be a subset of $\{1,\ldots,r+1\}$ such that for each $j \in \{1,\ldots,r+1\}$ there is exactly one $i \in I(g)$ with $g_i = g_j$. For given i let $E(i,g)$ be the set of such j. Then

$$\chi_{m\lambda_1}(g) = \sum_{i\in I(g)} F_i(m),$$

where $F_i(m)$ is the finite part of $h^{m+r}/\prod_{k\neq i}(h-g_k)$ in the Laurent expansion around $h = g_i$. In particular $F_i(m) = p_i(m) g_i^m$, where p_i is a polynomial of order $|E(i,g)|-1$.

Theorem 2.6. *Let $g_1,\ldots,g_{r+1}$ be the roots of the polynomial*

$$p(\gamma) = \sum_{k=0}^{r+1} (-)^k x_{r+1-k} \gamma^k$$

in some order, and put $g = diag(g_1,\ldots,g_{r+1})$. Then one has $g \in SL(r+1)$ and

$$z_{1m} = \chi_{m\lambda_1}(g).$$

Proof. Since $x_{r+1} = 1$ one has $g_1 \cdots g_{r+1} = 1$, thus $g \in SL(r+1)$. The map from $(x_1,\ldots,x_r)$ to g yields a bijection between $S(A_r, A_{\mathbb{N}})$ and the quotient of $T(r+1)$ by its Weyl group. Consider an element of $S(A_r, A_{\mathbb{N}})$ for which g is generic. The sequence z_{1m} is a linear combination of the sequences $(g_i)^m$, with $i = 1,\ldots,r+1$. For generic g one has $D^{A_r}(g) \neq 0$ such that the Weyl character formula implies that the sequence $N_{m1}(g)/D^{A_r}(g)$ satisfies the same recursion relation as z_{1m}. Moreover $z_{1m} = N_{m1}(g)/D^{A_r}(g)$ is true for $-m = 0,\ldots,r$. Due to $x_{r+1} = 1$ the recursion relation has a unique solution with this property. Since we work on an irreducible variety, the restriction to $D^{A_r} \neq 0$ is irrelevant. □

The following theorem is due to Kirillov [3] and follows immediately from the Littlewood-Richardson relations. The proof given here introduces some of the techniques which later will be used in the more complicated D-case.

Theorem 2.7. *For $i = 1, \ldots, r$*

$$z_{im} = \chi_{m\lambda_i}(g).$$

Proof. As mentioned in the proof of 2.3, z_{im} can be expressed in terms of z_{1k} with $m - i < k < m + i$. It follows that

$$z_{im} = \sum_{\mu \in I(i,r+1)} \det M_\mu^m,$$

where M_μ^m is the $i \times i$ matrix with elements

$$(M_\mu^m)_{kl} = \phi_{\mu k, m-k+l},$$

and $I(i, r+1)$ is the set of all injective maps from $\{1, \ldots, i\}$ to $\{1, \ldots, r+1\}$. Non-injective maps do not contribute since they lead to matrices with linearly dependent rows. The numerators of the M_μ^m matrix elements form a Vandermonde type matrix with determinant

$$\prod_{j=1}^{i} g_{\mu j}^{n+r-j+1} \prod_{1 \le k < l \le i} (g_{\mu l} - g_{\mu k}).$$

This yields

$$z_{im} = \sum_{\mu \in S(i,r+1)} \prod_{1 \le k < l \le 1} (g_{\mu l} - g_{\mu k})^{-1} \prod_{j=1}^{i} \left(g_{\mu j}^{m+r-j+1} \prod_{s \in C(\mu)} (g_{\mu j} - g_s)^{-1} \right),$$

where $C(\mu)$ is the complement of the image of μ in $\{1, \ldots, r+1\}$. For given μ let $N(\mu)$ be the set of all injective maps from $\{i+1, \ldots, r\}$ to $C(\mu)$. We have the Vandermonde determinant formula

$$\sum_{\nu \in N(\mu)} \prod_{j=i+1}^{r} g_{\nu j}^{r-j+1} \prod_{i \le k < l \le r} (g_{\nu l} - g_{\nu k})^{-1} = 1.$$

Inserting the left hand side in the previous formula and noting that the pairs $\sigma = (\mu, \nu)$ with $\nu \in N(\mu)$ form the permutation group $\mathcal{S}^{r+1}$, we obtain

$$z_{im} = \sum_{\sigma \in \mathcal{S}^{r+1}} \prod_{1 \le k < l \le r} (g_{\sigma k} - g_{\sigma l})^{-1} \prod_{j=1}^{i} g_{\sigma j}^{m+r-j+1} \prod_{k=i+1}^{r} g_{\sigma j}^{r-k+1}.$$

Note that z_{in} has only single poles at the locus where two g_i coincide. We shall refer to this subset of $T(r+1)$ as the singular locus. For all $\sigma \in \mathcal{S}^{r+1}$ we have

$$\prod_{1 \le k < l \le r} (g_{\sigma k} - g_{\sigma l}) = sgn(\sigma) D^{A_r},$$

where D^{A_r} is the Weyl denominator. This yields

$$D^{A_r}\, z_{im} = \sum_{\sigma \in \mathcal{S}^{r+1}} sgn\sigma \prod_{j=1}^{i} g_{\sigma j}^{m+r+1-j} \prod_{j=i+1}^{r} g_{\sigma j}^{r+1-j},$$

or according to the Weyl character formula

$$z_{im} = \chi_{m\lambda_i}(g).$$

$\square$

Remark 2.8. *The previous formula can be written somewhat more concisely as follows. We regard characters as polynomials in $g_1, \ldots, g_{r+1}$ and introduce the natural action of $\mathcal{S}^{r+1}$ on the ring of polynomials by $\sigma g_i = g_{\sigma i}$. With*

$$\mathcal{A}_r = \sum_{\sigma \in \mathcal{S}^{r+1}} (sgn\sigma)\sigma,$$

we have

$$D^{A_r}\, z_{im} = \mathcal{A}_r \prod_{j=1}^{i} g_j^{m+r+1-j} \prod_{j=i+1}^{r} g_j^{r+1-j}.$$

For completeness we also consider $S(A_r, A_{\mathbb{Z}})$.

Theorem 2.9. *$S(A_r, A_{\mathbb{Z}})$ is birationally isomorphic to the product of two maximal tori T_1, T_2 of $SL(r+1)$ modulo the simultaneous action of the group $\mathcal{S}^{r+1}$ on the two factors. Explicitly, let $g = diag(g_1, \ldots, g_{r+1})$ and $h = diag(h_1, \ldots, h_{r+1})$ be elements of T_1, T_2 and put*

$$N_{mi}(g,h) = \sum_{\sigma \in \mathcal{S}^{r+1}} sgn(\sigma) \prod_{j=1}^{i} g_{\sigma(j)}^{r+m+1-j} h_{\sigma(j)}^{r+1-j} \prod_{j=i+1}^{r} g_{\sigma(j)}^{r+1-j}. \tag{3}$$

For $D^{A_r}(g) \neq 0$ the corresponding element of $S(A_r, A_{\mathbb{Z}})$ is given by

$$z_{im} = N_{mi}(g,h)/D^{A_r}(g).$$

$S(A_r, A_{\mathbb{Z}})$ is a fibred space over $T_1/\mathcal{S}^{r+1}$, with fibres isomorphic to hypersurfaces in $\mathbb{C}^{r+1}$.

Proof. For generic $g \in SU(r+1)$, the group elements g^k, k a positive integer, are dense in $SU(r+1)$. Thus any algebraic relation between the matrix elements of g and g^k which is true for arbitrary positive integers k stays true when g^k is replaced by any diagonal matrix $h \in SU(r+1)$. By complexification the result also holds for $h \in SL(r+1)$. Now the $S(A_r, A_{\mathbb{Z}})$ relations between the z_{im} remain true when the z_{im} are replaced by $z_{i,m+k}$ for arbitrary positive integers k. According to equation (3) this implies that the equations remain true when g^m is replaced by $g^m h$. Since h can involve negative powers of g, the equations are valid for negative m. We conclude by comparing dimensions.

To obtain a description of $S(A_r, A_{\mathbb{Z}})$ as a fibred space consider the $r+2$ sequences $z_{1,m+k}$ with $k = 0, \ldots, r+1$. They are linearly dependent, since generically they are linear combinations of the sequences g_i^m with $i = 1, \ldots, r+1$. As in theorem 2.3 we have $z_{im} = \det M_m^i$, where M_m^i is the $i \times i$ matrix with entries $(M_m^i)_{kl} = z_{1,m-k+l}$,. Since $z_{r+1,m} = 1$, the $r+1$ sequences $z_{1,m+k}$ with $k = 0, \ldots, r$ are linearly independent. Thus for any $z \in S(A_r, A_{\mathbb{Z}})$ there is a unique linear recursion relation

$$\sum_{k=0}^{r+1} (-)^k x_k z_{1,m+k} = 0 \tag{4}$$

with $x_0 = 1$, $x_{r+1} \neq 0$. For generic g, thus everywhere we have $x_{r+1} = 1$. The x_k parametrise $T_1/\mathcal{S}^{r+1}$, as for $S(A_r, A_{\mathbb{N}})$, and constitute the coordinates of the basis of a fibration. The map to the vector $z_{10}, \ldots, z_{1r}$ imbeds each fibre in $\mathbb{C}^{r+1}$. Indeed by induction on m the recursion relation yields all other z_{1m}, and the z_{im} for higher i are given by the corresponding determinants. For $i = r+1$ the recursion relation yields $z_{r+1,m+1} = x_0 z_{r+1,m}$, thus a point of $\mathbb{C}^{r+1}$ yields an element of $S(A_r, A_{\mathbb{Z}})$, if it belongs to the determinantal hypersurface given by $z_{r+1,0} = 1$. □

The case $h = 1$ yields

Corollary 2.10. *There is a natural embedding $S(A_r, A_{\mathbb{N}}) \subset S(A_r, A_{\mathbb{Z}})$.*

Now we determine the elements of $S(A_r, A_n)$.

Theorem 2.11. *The image of $S(A_r, A_n)$ in $(A_r, A_{\mathbb{N}})$ is characterized by $z_{1,n+1} = 1$ and $z_{1m} = 0$ for $m = n+2, n+3, \ldots, n+1+r$.*

Proof. By assumption we have $z_{i,n+1} = 1$ for all i. By the $(A_r, A_{\mathbb{N}})$ equations and the condition $z_{in} \neq 0$ this is equivalent to $z_{1,n+1} = 1$ and $z_{i,n+2} = 0$ for $i = 1, \ldots, r$. Now we use induction on i. Assume that $1 < i < r$ and $z_{1k} = 0$ for $k = n+2, \ldots, n+i$. By equations (1, 4) one has

$$z_{i,n+2} = (-)^{i+1} z_{1,n+i+1}.$$

□

Theorem 2.12. *For the image of $S(A_r, A_n)$ in $S(A_r, A_{\mathbb{Z}})$ one has*

$$z_{1,m+n+2+r} = (-)^r z_{1m}.$$

Proof. For $m = -r, \ldots, -1$ this is true by the previous theorem. For $m \geq 0$ it follows from that theorem by equation (4) and induction on m. The Dynkin diagram of A_n is invariant under the involution given by $m \to n+1-m$. Under this involution equation (4) is transformed into

$$\sum_{k=0}^{r+1} (-)^k \tilde{x}_k z_{1,m-k} = 0, \tag{5}$$

where m is any integer with $m \leq n+r+1$. Since $z_{10} = 1$ and $\tilde{x}_0 = 1$, comparison of equation (4) and equation (5) for $m = 1, \ldots, r$ and induction on m yields $\tilde{x}_m = x_m$. $\square$

Theorem 2.13. *In the notation of theorem 2.7 the elements of $S(A_r, A_n)$ are given by elements $g \in T(r+1)$ which do not lie on the singular locus and satisfy $g^{r+n+2} = (-)^r$.*

Proof. By remark 2.5 we have

$$z_{1m} = \sum_{i \in I(g)} p_i(m)(g_i)^m,$$

where $g_i \neq g_j$ for any pair $i, j \in I(g)$ and the p_i are polynomials. A sequence of this form is bounded for positive and negative m, if and only if all g_i have modulus 1 and all p_i are constants. By theorem 2.12 the sequence z_{1m} is periodic, thus bounded. By remark 2.5 this implies that $|E(i,g)| = 1$ for all i, such that g cannot lie on the singular locus. Theorem 2.12 yields $g^{n+r+2} = (-)^r$. Conversely, by equation (3) the conditions $g^{r+n+2} = (-)^r$ and $D^{A_r}(g) \neq 0$ imply that $z_{1,n+1} = 1$ and $z_{1m} = 0$ for $m = n+2, n+3, \ldots, n+1+r$. One concludes by theorem 2.9. $\square$

3 The D Case

Now let us consider $S(D_r, A_n)$. Again we first consider $S(D_r, A_\mathbb{N})$. As for $S(A_r, A_\mathbb{N})$ the generic solutions of the discrete Hirota equations form an affine variety parametrized by z_{i1}. The formulas are slightly more complicated and were given in [1].

Theorem 3.1. *Given complex numbers $x_0, x_1, \ldots, x_r$ with $x_0 = 1$, and $j \in \mathbb{Z}$, let X_j, Y_j be defined by*

$$\begin{aligned} X_{2j+1} = x_j = -X_{4r-3-2j} \qquad & for \qquad 0 \leq j \leq r-2 \\ X_{2j} = -x_j \qquad & for \qquad j = r-1, r \\ X_j = 0 \qquad & otherwise \end{aligned}$$

$$Y_2 = -Y_6 = 1, \qquad Y_3 = x_r, \qquad Y_5 = x_{r-1}$$
$$Y_j = 0 \qquad otherwise.$$

We will consider square matrices T_m^i *of size* $2m-1$ *for* $i = 1, \ldots, r-2$, *and of size* $2m$ *for* $i = r-1, r$, *where* $m = 1, 2, \ldots$. *Their matrix elements have the property*

$$(T_m^i)_{jk} = (T_m^i)_{j+2,k+2}$$

if both sides of the equation are defined. Moreover,

$$\begin{aligned} (T_m^i)_{1j} &= X_{2i+j} \\ (T_m^i)_{2j} &= Y_j \end{aligned}$$

for $i = 1, \ldots, r-2$, *whereas*

$$\begin{aligned} (T_m^{r-1})_{1j} &= Y_{3+j} \\ (T_m^{r-1})_{2j} &= X_{2r-3+j} \\ (T_m^{r})_{1j} &= X_{2r-2+j} \\ (T_m^{r})_{2j} &= Y_{2+j} \ . \end{aligned}$$

Note that T_m^i *is antisymmetric for* $i = r-1, r$, *such that it has a Pfaffian. Put* $x_i = z_{i1}$ *for* $i = 1, \ldots, r$. *Then in* $S(D_r, A_n)$ *one has*

$$z_{im} = \det T_m^i,$$

for $i = 1, \ldots, r-2$ *and*

$$z_{im} = Pf\ T_m^i,$$

for $i = r-1, r$.

Proof. The published proof will be sketched in enough detail to allow the reader a straightforward reconstruction. For the equations

$$z_{i,m+1} z_{i,m-1} = z_{im}^2 - z_{i-1,m} z_{i+1,m},$$

$i = 1, \ldots, r-3$, it works exactly as in the proof of theorem 2.1, by using Jacobi's determinant identity for T_{m+1}^i. For the remaining equations one uses in addition some auxiliary cases of Jacobi's identity, where one of the three terms involves a determinant of an antisymmetric matrix of odd size and vanishes. The remaining two terms are explicit squares or allow the extraction of a square root due to the standard relation between determinants and Pfaffians. To use the equality between the two square roots one needs to fix signs by comparing the highest powers of

x_{r-1}, x_r. For $i = r-2$ one uses Jacobi's identities for T^i_{m+1} and for an antisymmetric auxiliary matrix which is defined like T^{r-1}_m but has size $2m+1$. For $i = r-1, r$ one first uses Jacobi's identity for T^{r-1}_{m+1} and T^r_{m+1} to write $Pf\ T^{r-1}_{m+1} Pf\ T^r_m$ and $Pf\ T^r_{m+1} Pf\ T^{r-1}_m$ as determinants of further auxiliary matrices. Then one applies Jacobi's identity to these auxiliary matrices, which yields

$$(z_{r-1,m+1} z_{rm})(z_{rm} z_{r-1,m-1}) = (z_{rm} z_{r-1,m})^2 - z^2_{rm} z_{r-2,m}$$

and

$$(z_{r,m+1} z_{r-1,m})(z_{r-1,m} z_{r,m-1}) = (z_{rm} z_{r-1,m})^2 - z^2_{r-1,m} z_{r-2,m}.$$

□

Remark 3.2. *Due to this theorem $S(D_r, A_{\mathbb{N}})$ has properties analogous to $S(A_r, A_{\mathbb{N}})$. In detail, $S(D_r, A_{\mathbb{N}})$ is isomorphic to $\mathbb{C}^r$, with elements parametrized by arbitrary vectors $(z_{11}, \ldots, z_{r1})$ and the map $S(D_r, A_n) \to S(D_r, A_{\mathbb{N}})$ induced by the restriction to $(z_{11}, \ldots, z_{r1})$ is an embedding.*

We now derive a recursion relation for the z_{1m}.

Theorem 3.3. *Let $x_i = z_{i1}$ for $i = 1, \ldots, r$ and $x_0 = x_{r+1} = 1$. Put $z_{10} = 1$ and $z_{1,-n} = 0$ for $n = 1, \ldots, 2r-1$. Then for any $n \in \mathbb{N}$ one has*

$$\sum_{k=0}^{2r} (-)^k \chi_k z_{1,n-k} = 0,$$

where $\chi_0 = 1$, $\chi_1 = x_1$, $\chi_i = x_i - x_{i-2}$ for $i = 2, \ldots, r-2$, $\chi_{r-1} = x_{r-1}x_r - x_{r-3}$, $\chi_r = x^2_{r-1} + x^2_r - 2x_{r-2}$ and $\chi_{2r-k} = \chi_k$ for $k = 0, \ldots, r$.

Proof. Expanding $\det(T^1_k)$ minors with respect to the first row one finds

$$\sum_{i=0}^{r-2} (-)^i x_i (\det(T^1_{k-i}) - \det(T^1_{k-2r+2+i}))$$
$$+ x_{r-1} \det(\tilde{T}_{2k-2r+3}) - x_r \det(\tilde{T}_{2k-2r+1}) = 0\ ,$$

and

$$\det(\tilde{T}_{2k+3}) - x_r \det(T^1_{k+1}) - x_{r-1} \det(T^1_k) - \det(\tilde{T}_{2k-1}) = 0\ .$$

Here $\tilde{T}_{2k-2r+3}$ is the matrix obtained by suppressing rows $1, \ldots, 2r-5$ and $2r-3$ and columns $1, \ldots, 2r-4$ in T^1_k. For small values of k the recursion relations are to be interpreted so that $\det T^1_{-k} = 0$ for $k = 1, \ldots, 2r-3$, $\det T^1_{-2r+2} = 1$ and $\det \tilde{T}_{-2k+1} = 0$ for $k = 1, \ldots, r$. Eliminating the $\tilde{T}$-determinants by subtracting the recursion relations for $\det\left(T^1_k\right)$ and $\det\left(T^1_{k-2}\right)$ yields

$$\sum_{i=0}^{r-1} (-)^i \chi_i \left(\det\left(T^1_{k-i}\right) + \det\left(T^1_{k-2r+i}\right)\right) + (-)^r \chi_r \det\left(T^1_{k-r}\right) = 0\ .$$

□

For generic z_{i1} the solutions of the recursion relations have the form

$$z_{1k} = \sum_{i=1}^{2r} \alpha_i g_i^k,$$

where the g_i are the roots of the polynomial

$$p(\gamma) = \sum_{i=0}^{2r} (-)^i \chi_i \gamma^i.$$

One may put $g_{i+r} = g_i^{-1}$ since the polynomial coefficients are invariant under reversal of their order. The diagonal matrices $g = diag(g_1, \ldots, g_{2r})$ with $g_{i+r} = g_i^{-1}$ for $i = 1, \ldots, r$ form a maximal torus $T(\eta^r)$ of the Lie group $SO(\eta^r)$, where

$$\eta^r(y_1, \ldots, y_{2r}) = \sum_{i=1}^{r} y_i y_{i+r}.$$

A maximal torus of the simply connected double cover $Spin(\eta^r)$ of $SO(\eta^r)$ is given by a double cover of $T(\eta^r)$ on which a variable Γ with $\Gamma^2 = g_1 \cdots g_r$ is defined. The Lie algebra of $SO(\eta^r)$ is isomorphic to D_r, and its fundamental highest weights λ_i, $i = 1, \ldots, r$ are given by

$$\lambda_i(g) = \prod_{j=1}^{i} g_j$$

for $j = 1, \ldots, r-2$. By abuse of notation we write

$$\begin{aligned} \lambda_{r-1}(g) &= \Gamma g_r^{-1} \\ \lambda_r(g) &= \Gamma, \end{aligned}$$

where a choice of the sign of Γ is implied. The automorphism group of $T(\eta^r)$ is generated by the permutations of the g_i, $i = 1, \ldots, r$ and the involution ξ for which $\xi(g_r) = g_r^{-1}$ and $\xi(g_i) = g_i^{-1}$ for $i = 1, \ldots, r-1$. Permutations lift immediately to the double cover, for ξ a lift is given by $\xi(\Gamma) = g_r^{-1}\Gamma$. The Weyl group is of index two in the automorphism group. Since ξ interchanges λ_{r-1} and λ_r, it is not contained in the Weyl group. We define the singular locus of $D(r)$ as the set of points on which the Weyl group does not act freely. This is equivalent to $g_i = g_j$ or $g_i = g_j^{-1}$ for some pair i, j.

Theorem 3.4. *Let $g_1, \ldots, g_{2r}$ with $g_{i+r} = g_i^{-1}$ for $i = 1, \ldots, r$ be the roots of the polynomial*

$$p(\gamma) = \sum_{i=0}^{2r} (-)^i \chi_i \gamma^i.$$

Then

$$z_{1k} = \chi_{k\lambda_1}(g).$$

Proof. The proof is analogous to the A_r case. For generic g let

$$N_k(g) = \sum_{\epsilon\in\{-1,1\}^{r-1}} \sum_{\sigma\in\mathcal{S}^r} sgn(\sigma) g_{\sigma(1)}^{\epsilon_1(r-1+k)} g_{\sigma(2)}^{\epsilon_2(r-2)} g_{\sigma(3)}^{\epsilon_3(r-3)} \cdots g_{\sigma(r-1)}^{\epsilon_{r-1}} ,$$

so that the Weyl character formula can be written as $\chi_{k\lambda_1}(g) = N_k(g)/N_0(g)$. Clearly the N_k satisfies the recursion formula and one has $N_{-k}(g) = 0$ for $k = 1, \ldots, 2r-3$ and $N_{-2r+2}(g) = N_0(g)$. Thus $N_k(g)/N_0(g) = z_{1k}$ for $k = -2r+2, \ldots, 1$, which implies the same relation for all k due to the recursion. ☐

Theorem 3.5. *For $i = 1, \ldots, r-2$*

$$z_{im} = \sum_{K(i,m)} \chi_{\sum_{j=1}^{i} k_j\lambda_j}(g),$$

where for odd i the set $K(i,m)$ consists of the tuples $k_1, \ldots, k_i$ such that $k_j = 0$ for even j and

$$\sum_{j=1}^{i} k_j = m,$$

whereas for even i the set $K(i,m)$ consists of the tuples $k_1, \ldots, k_i$ such that $k_j = 0$ for odd j and

$$\sum_{j=2}^{i} k_j \leq m.$$

Proof. We put

$$Z_{im} = det M_m^i,$$

where $i = 1, \ldots, r$ and M_m^i is the $i \times i$ matrix with elements

$$\left(M_m^i\right)_{jk} = z_{1,m-j+k}.$$

Up to $i = r-2$ the Dynkin diagrams of A_r and D_r are equal, so that $Z_{im} = z_{im}$ for $i = 1, \ldots, r-2$. On the other hand, the relation $z_{r-2,m}^2 = z_{r-2,m-1} z_{r-2,m+1} + z_{r-3,m}\left(z_{r-1,m} z_{r,m}\right)$ yields

$$Z_{r-1,m} = z_{r-1,m} z_{rm},$$

and a short calculation shows that the relations for $i = r-1, r$ yield

$$Z_{rm} = z_{rm}^2 + z_{r-1,m}^2 - z_{r-2,m}.$$

By theorem 3.4 the Z_{in} are $\mathbb{Z}$-linear combinations of $SO(\eta^r)$ characters and in particular $\mathbb{Z}$-linear combinations of the $SO(\eta^r)$ weights λ. Instead of $\lambda(g)$ we use the notation

$$g^\lambda = \prod_{i=1}^{r} g_i^{l_i},$$

or equivalently $\lambda = (l_1, \ldots, l_r)$, where the l_i are either all integral or all half integral. For later use we need a notation which suppresses the dependence on $g_1, \ldots, g_k$ with $k \in \{0, \ldots, r\}$. We write

$$g^\lambda[k] = \prod_{i=k+1}^{r} g_i^{n_i}.$$

Dominant weights are those for which $n_1 \geq n_2 \geq \ldots \geq n_{r-1} \geq |n_r|$.

We will use the weights

$$\Lambda_i = (1, \ldots, 1, 0, \ldots, 0)\,,$$

for $i = 1, \ldots, r$. In terms of the fundamental weight λ_i one has $\Lambda_i = \lambda_i$ for $i \leq r-2$, $\Lambda_{r-1} = \lambda_{r-1} + \lambda_r$ and $\Lambda_r = 2\lambda_r$.

Recall that $\mathbb{Z}[x]$ is the ring of polynomials in x with integer coefficients and $\mathbb{Z}(x)$ is the corresponding ring of power series. By abuse of notation we denote by $\mathbb{Z}[x_I]$ the ring of polynomials in x_i with $i \in I$ and analogously for $\mathbb{Z}(x_I)$. Instead of $\mathbb{Z}[x](y)/\langle xy-1\rangle$ we use the abbreviated notation $\mathcal{R}[x](x^{-1})$.

Let $P \in End\,\mathbb{Z}[g_1, \ldots, g_r](g_1^{-1}, \ldots, g_r^{-1})$ be the projection to the span of the dominant weights and $P_\geq$, P_0, $P_\leq$ the subprojections to dominant weights with $n_r \geq 0$, $n_r = 0$, and $n_r \leq 0$ respectively. In particular, $P_\leq = \xi P_\geq \xi$. We use $P[k]$ to denote the projection to the span of $g^\lambda[k]$, with λ any dominant weight, and analogously for $P_\geq[k]$.

When χ_λ is the character of an irreducible representation with highest weight λ and Δ_r is the Weyl denominator, we have

$$g^\lambda = P g^{-\rho} \Delta_r \chi.$$

Here $\rho = \sum_{i=1}^{r} \lambda_i$, such that

$$g^\rho = \prod_{i=1}^{r} g_i^{r-i}.$$

Since $\xi Z_{im} = Z_{im}$, one has

$$P g^{-\rho} \Delta_r Z_{im} = (1 + \xi - P_0) P_\geq g^{-\rho} \Delta_r Z_{im},$$

so it is sufficient to calculate $P_\geq g^{-\rho} \Delta_r Z_{im}$ to obtain the decomposition of Z_{im} into characters of irreducible representations.

We write $\Delta_r = \Delta_r^+ \Delta_r^-$, where

$$\Delta_r^+ = \prod_{1 \leq i < j \leq r} (g_i - g_j),$$

and

$$\Delta_r^- = \prod_{1\le i<j\le r}(1-g_i^{-1}g_j^{-1}).$$

Using $(1-x)^{-1} = 1+x+x^2+\dots$ we can regard the inverse of Δ_r^- as an element of $\mathbb{Z}(g_1^{-1},\dots,g_r^{-1})$.

By the Weyl character formula for D_r and the Weyl denominator formula for D_{r-1} we have

$$z_{1m} = \sum_{j=1}^r \phi_j^m + \phi_-^m,$$

where

$$\phi_{jm} = g_j^{m+r-1}/\prod_{k\neq j}(g_j-g_k)(1-g_j^{-1}g_k^{-1}),$$

and

$$\phi_-^m = \sum_{j=1}^r g_j^{-(m+r-1)}/\prod_{k\neq j}\left((g_j-g_k)(1-g_j^{-1}g_k^{-1})\right).$$

In ϕ_-^m the poles at the singularity locus are removable, so that

$$\phi_-^m \in \prod_{j=1}^r g_j^{-1}\mathbb{Z}(g_1^{-1},\dots,g_r^{-1}).$$

One has

$$Z_{im} = \sum_{\mu\in\nu(i,r)} \det M_m^\mu,$$

where $\nu(i,r)$ is the set of maps from $\{1,\dots,i\}$ to $\{1,\dots,r,-\}$ and the $i\times i$ matrices M_m^μ have matrix elements

$$(M_m^\mu)_{jk} = \phi_{\mu j}^{m-j+k}.$$

For a given map μ let $n(\mu)$ be the subset of $\{1,\dots,i\}$ which is mapped to $-$, and let $p(\mu)$ be its complement. If μ is not injective on $p(\mu)$ then $\det M_m^\mu = 0$. Otherwise $\det M_m^\mu$ has only single poles at the singular locus. The development into minors with respect to $p(\mu)$ yields

$$\Delta_r^+ \det M_n^\mu \in \Delta_\mu^c \prod_{i=1}^r g_i^{-|n(\mu)|}\mathbb{Z}[g_{\mu p(\mu)}]\left(g_1^{-1},\dots,g_r^{-1}\right),$$

where

$$\Delta_\mu^c = \prod_{i,j\in c(\mu)_{i<j}}(g_i-g_j),$$

and $c(\mu)$ is the complement of $\mu p(\mu)$ in $\{1,\ldots,r\}$. Using the Weyl denominator formula for Δ^c_μ, we see that every monomial in $\Delta^c_\mu \prod_{i=1}^r g_i^{-|n(\mu)|}$ has at least $|n(\mu)|$ strictly negative exponents for g_i with $i \in c(\mu)$. Thus

$$P_{\geq} g^{-\rho} \Delta_r \det M^\sigma_m = 0,$$

unless $n(\sigma) = \emptyset$. This implies

$$P_{\geq} g^{-\rho} \Delta_r Z_{im} = P_{\geq} g^{-\rho} \Delta_r \sum_{\sigma \in S(i,r)} \det M^\sigma_m,$$

where $S(i,r)$ is the set of injective maps from $\{1,\ldots,i\}$ to $\{1,\ldots,r\}$. For $\sigma \in S(i,r)$ the calculation of $\det M^\sigma_m$ proceeds as for $S(A_r, A_{\mathbb{N}})$ and yields

$$P_{\geq} g^{-\rho} \Delta_r Z_{im} = P_{\geq} g^{-\rho} \mathcal{A}(0,r) (N^0_i)^{-1} N^i_{r-i} g^{m\Lambda_i + \rho},$$

where

$$N^t_{i-t} = \prod_{t<j<k\leq i} \left(1 - g_j^{-1} g_k^{-1}\right),$$

and

$$\mathcal{A}(t,s) = \sum_{\sigma \in \Pi(t,s)} \sigma(sgn\sigma),$$

and $\Pi(t,s)$ is the permutation group of $I(t,s) = \{t+1,\ldots,s\}$. Since all terms of N^i_{r-i} except the identity yield terms which project to zero, the preceeding formula can be simplified to

$$P_{\geq} g^{-\rho} \Delta_r Z_{in} = P_{\geq} g^{-\rho} \mathcal{A}(0,r) (N^0_i)^{-1} g^{n\Lambda_i + \rho}.$$

Now

$$N^0_s = N^1_{s-1} \sum_{M \subset I(1,s)} (-)^{|M|} g_1^{-|M|} \prod_{a \in M} g_a^{-1}$$

yields

$$(N^0_s)^{-1} = (N^1_{s-1})^{-1} - (N^0_s)^{-1} \sum_{M \subset I(1,s) M \neq 0} (-)^{|M|} g_1^{-|M|} \prod_{a \in M} g_a^{-1}.$$

Moreover $\mathcal{A}(0,r)$ commutes with N^0_s and factorizes through $\mathcal{A}(1,r)$. When we put

$$G(t,s) = \prod_{t<s\leq s} g_a^{-a}$$

we have

$$\mathcal{A}(1,s) G(1,s) \prod_{a \in M} g_a^{-1} = 0,$$

unless M is of the form $I(t,s)$ with $1 \leq t \leq s$, since for $a \in M$, $a+1 \in M$, $a+1 \leq s$ one gets equal exponents of g_a and g_{a+1}, thus zero when one averages over a group which contains the transposition of $a,\ a+1$. For $M = I(t,s)$ one obtains

$$\mathcal{A}(0,r) g_1^{t-s} \prod_{a \in I(t,s)} g_a^{-1} = 0 \quad \text{unless} \quad t = s/2,$$

since for $t < s/2$ the exponents of g_1 and g_{s-t}, for $t > s/2$ those of g_1 and g_{s-t+1} coincide. Thus the sum over M yields no contribution for odd i, whereas for even i only the term $M = I(1, i/2)$ contributes. For odd i this yields

$$\begin{aligned} & P g^{-\rho} \mathcal{A}(0,r)(N_i^0)^{-1} g^{n\Lambda_i+\rho} \\ = \ & g_1^n P[1] g^{-\rho}[1] \mathcal{A}(1,r)(N_{i-1}^1)^{-1} g^{n\Lambda_i+\rho}[1] \end{aligned}$$

and for even i

$$\begin{aligned} & P g^{-\rho} \mathcal{A}(0,r)(N_i^0)^{-1} g^{n\Lambda_i+\rho} \\ = \ & g_1^n P[1] g^{-\rho}[1] \mathcal{A}(1,r)(N_{i-1}^1)^{-1} g^{n\Lambda_i+\rho}[1] \\ + \ & P g^{-\rho} \mathcal{A}(0,r)(N_i^0)^{-1} g^{(n-1)\Lambda_i+\rho} \ . \end{aligned}$$

By induction on m and k one obtains

$$\begin{aligned} & P_{\geq}[k] g^{-\rho}[k] \mathcal{A}(k,r)(N_{i-k}^k)^{-1} \prod_{j=k+1}^{i} g_j^{m+r-j} \prod_{j=i+1}^{r} g_j^{r-j} \\ = \ & \sum_{\lambda \in L(i,m,k)} g^{\lambda}, \end{aligned}$$

where $L(i,m,k)$ is the set of weakly decreasing integral sequences such that $l_{k+1} \leq r$, $l_a = 0$ for $a > i$ and $l_{i-2j-1} = l_{i-2j}$ for $j = 0, \ldots, [i/2]-1$. For $k = 0$ this yields the wanted decomposition of Z_{im} in terms of characters of $SO(\eta^r)$. Equivalently one can write for $i = 1, \ldots, r-2$

$$z_{im} = \sum K(i,m) \chi_{\sum_{j=1}^{i} k_j \lambda_j}(g),$$

where for odd i the set $K(i,m)$ consists of the tuples $k_1, \ldots, k_i$ such that $k_j = 0$ for even j and

$$\sum_{j=1}^{i} k_j = m,$$

whereas for even i the set $K(i,m)$ consists of the tuples $k_1, \ldots, k_i$ such that $k_j = 0$ for odd j and

$$\sum_{j=2}^{i} k_j \leq m.$$

This result was known to Kirillov and Reshetikhin, but their proof is unpublished. $\square$

For $i = r-1, r$ they stated the following result

Theorem 3.6. *For $i = r-1, r$*

$$z_{im} = \chi_{m\lambda_i}(g).$$

Proof. We verify this result by proving that

$$Z_{r-1,m} = \chi_{n\lambda_{r-1}}(g)\chi_{m\lambda_r}(g)$$

and

$$Z_{r,m} = \left(\chi_{n\lambda_{r-1}}(g)\right)^2 + \left(\chi_{m\lambda_r}(g)\right)^2 - z_{r-2,m}.$$

The latter equations show that the pair $z_{r-1,m}$, $z_{r-2,m}$ agrees with $\chi_{m\lambda_{r-1}}(g)$, $\chi_{m\lambda_r}(g)$ in some order. Once the order is fixed for $m = 1$, the Kirillov-Reshetikhin result follows from $z_{r-1,m}z_{r,m-1} = z_{rm}^2 - z_{r-2,m}$ by induction on m and comparison of the highest powers of g_n.

To prove the statements note that

$$Pg^{-\rho}\Delta_r\chi_\lambda\chi_{m\lambda_r} = Pg^{-\rho}\chi_\lambda\mathcal{A}(0,r)\sum_{\epsilon\in E^+}\prod_{i=1}^{r} g_i^{\epsilon_i(m/2+r-i)},$$

where

$$E^{\pm} = \left\{(\epsilon_1,\ldots,\epsilon_r)\in\{+,-\}^r \mid \prod_{i=1}^{r}\epsilon_i = \pm 1\right\}.$$

For $\lambda = m\lambda_r$ or $\lambda = m\lambda_{r-1}$ all g_i exponents in χ_λ are less than or equal to $m/2$, such that only $\epsilon = (1,\ldots,1)$ contributes when the projection P is taken. Evaluating the Vandermonde determinant we get

$$Pg^{-\rho}\Delta_r\chi_\lambda\chi_{m\lambda_r} = Pg^{-\rho}\chi_\lambda\Delta_r^+\prod_{i=1}^{r} g_i^{m/2}.$$

For $\lambda = m\lambda_{r-1}$ this yields

$$Pg^{-\rho}\Delta_r\chi_{m\lambda_{r-1}}\chi_{m\lambda_r} = Pg^{-\rho}\left(\Delta_r^-\right)^{-1}\mathcal{A}(0,r)\sum_{\epsilon\in E^-}\prod_{i=1}^{r} g_i^{m/2+\epsilon_i(m/2+r-i)}.$$

Only the term with $\epsilon = (1,\ldots,1,-1)$ survives the projection. Thus

$$Pg^{-\rho}\Delta_r\chi_{m\lambda_{r-1}}\chi_{m\lambda_r} = Pg^{-\rho}\left(\Delta_r^-\right)^{-1}\mathcal{A}(0,r)\prod_{i=1}^{r-1} g_i^{(m+r-i)}.$$

Since $\Delta_r^- = N_r^0$, the right hand side is equal to $Z_{r-1,m}$ as claimed. For $\lambda = m\lambda_r$ one obtains

$$Pg^{-\rho}\Delta_r\chi_{m\lambda_r}\chi_{m\lambda_r} = Pg^{-\rho}\left(\Delta_r^-\right)^{-1}\mathcal{A}(0,r)\sum_{\epsilon\in E^+}\prod_{i=1}^{r} g_i^{m/2+\epsilon_i(m/2+r-i)}.$$

Only the term with $\epsilon = (1,\ldots,1)$ survives the projection, so that

$$Pg^{-\rho}\Delta_r\left(\chi_{m\lambda_r}\right)^2 = Pg^{-\rho}\left(\Delta_r^-\right)^{-1}\mathcal{A}(0,r)\prod_{i=1}^{r} g_i^{(m+r-i)} = P_{\geq}Z_{r,m}.$$

Since $P_0Z_{r,m} = PZ_{r-2,m}$, we also have

$$P_0g^{-\rho}\Delta_r\chi_{m\lambda_r}^2 = PZ_{r-2,m}.$$

Since

$$Pg^{-\rho}\Delta_rZ_{im} = (1+\xi-P_0)\,P_{\geq}g^{-\rho}\Delta_rZ_{im},$$

this confirms the Kirillov-Reshetikhin result for $z_{r-1,m}$ and z_{rm}. □

Remark 3.7. *As for A_r, the elements of $S(D_r, A_{\mathbb{Z}})$ are parametrized by two group elements g, h. In particular one has $z_{1k} = N_k(g,h)/N_0(g)$, where*

$$N_k(g,h) = \sum_{\epsilon\in\{-1,1\}^{r-1}}\sum_{\sigma\in\mathcal{S}^r} sgn(\sigma)h_{\sigma(1)}^{\epsilon_1}g_{\sigma(1)}^{\epsilon_1(r-1+k)}g_{\sigma(2)}^{\epsilon_2(r-2)}g_{\sigma(3)}^{\epsilon_3(r-3)}g_{\sigma(r-1)}^{\epsilon_{r-1}}.$$

Theorem 3.8. *Let z be the image of an element of $S(D_r, A_n)$ in $(D_r, A_{\mathbb{N}})$. Then $z_{1m} = 0$ for $m = n+2, n+3, \ldots, n+r$. Moreover $dz_{1,n+r} = 0$ in $S(D_r, A_{\mathbb{Z}})$ at the image of any element of $S(D_r, A_n)$ in this variety.*

Proof. By assumption we have $z_{i,n+1} = 1$ for all i. The $(D_r, A_{\mathbb{N}})$ equations immediately yield $z_{i,n+2} = 0$ for all i. Now we put $u_{im} = z_{im}$ for $i = 1,\ldots,r-2$ and $u_{r-1,m} = z_{r-1,m}z_{rm}$, but leave u_{im} undefined for $i = r$. Note that $du_{r-1,n+2} = 0$ at the image of any element of $S(D_r, A_n)$ in $S(D_r, A_{\mathbb{Z}})$. To the extent that quantities are defined, the (D_r, A_n) equations for the u_{im} are invariant under an interchange of r, n. The discrete Hirota equations can be solved iteratively with initial values z_{1m}. Accordingly, u_{im} is the determinant of an $m\times m$ matrix with matrix elements $N_{jk} = x_{i+j-k}$, where $x_j = z_{1j}$. We use induction on m. For $m = 1$ we already have seen that the statement is true. Assume that $z_{1k} = 0$ for $k = n+2,\ldots,n+m$. Then consider the determinant which yields $z_{m,n+2}$. The minors with respect to the first row have vanishing determinant, except for the last one, which is 1. Thus

$$u_{m,n+2} = (-)^{m+1}z_{1,n+m+1}.$$

□

Theorem 3.9. *For the image of $S(D_r, A_n)$ in $(D_r, A_{\mathbb{Z}})$ one has*

$$z_{1,m+2n+4r-2} = z_{1m}.$$

Proof. Evaluating the derivatives of $z_{1,n+r}$ with respect to the h_i and imposing $dz_{1,n+r} = 0$ yields

$$g_i^{n+2r-1} = g_i^{-(n+2r-1)}$$

for all i. This implies

$$z_{1,m+2n+4r-2} = z_{1m}.$$

□

Remark 3.10. *As in the (A_r, A_n) case this periodicity condition cannot be satisfied at the singular locus. The derivation is analogous to the one in remark 2.8. Let $J(g)$ be a subset of $\{1, \dots, r\}$ such that for each $j \in \{1, \dots, 2r\}$ there is exactly one $i \in J(g)$ with $g_j \in \{g_i, g_i^{-1}\}$. Then*

$$z_{1m}(g) = \sum_{i \in J(g)} F_i^D(m),$$

where $F_i^D(m)$ is the finite part of the Laurent expansion of

$$(h^{m+r-1} + h^{-m-r+1}) / \prod_{j \neq i} (h - g_j)(1 - h^{-1} g_j^{-1})$$

at $h = g_i$. Let $E^D(i)$ be the set of $j \in \{1, \dots, r\}$ for which $g_j \in \{g_i, g_i^{-1}\}$. One obtains $F_i^D(m) = p_i^+(m) g_i^m + p_i^-(m) g_i^{-m}$, where for $g_i^2 \neq 1$ the degrees of p_i^+ and p_i^- are equal to $|E^D(i)| - 1$, whereas for $g_i^2 = 1$ their degree as well as the degree of their sum is equal to $2|E^D(i)| - 2$. When the sequence z_{1m} is periodic, it is bounded, such that $|E^D(i)| = 1$ for all i and g cannot lie on the singular locus of $T(\eta^r)$.

We now sharpen the result $g_i^{2(n+2r-1)} = 1$ obtained above to $g_i^{n+2r-1} = 1$.

Theorem 3.11.

$$g_i^{n+2r-1} = 1.$$

Proof. By the Weyl character formula we have $z_{1n} = N_n(g)/N_0(g)$, where

$$N_n(g) = \sum_{j=1}^{r} \left(g_j^{r-1+n} + g_j^{-(r-1+n)} \right) A_j,$$

and

$$A_j = \sum_{\substack{\sigma\in\mathcal{S}^r \\ \sigma(1)=j}} \sum_{\epsilon_2,\ldots,\epsilon_{r-1}\in\{1,-1\}} sgn(\sigma) g_{\sigma(2)}^{(r-2)\epsilon_2} \cdots g_{\sigma(r-1)}^{\epsilon_{r-1}}.$$

Note that the Weyl denominator N_0 can be written as

$$N_0(g) = \sum_{\sigma\in\mathcal{S}^r} \sum_{\epsilon_1,\ldots,\epsilon_{r-1}\in\{1,-1\}} sgn(\sigma) g_{\sigma(1)}^{(r-1)\epsilon_1} \cdots g_{\sigma(r-1)}^{\epsilon_{r-1}}$$

or

$$N_0(g) = \prod_{i=1}^{r} g_i^{-(r-i)} \prod_{i>j} (g_i - g_j)(g_i - g_j^{-1}).$$

By the Weyl character formula, $g_i^{n+2r-1} = g_i^{-(n+2r-1)}$ for all i implies

$$z_{1,n+r+k} = z_{1,n+r-k}.$$

In particular, $z_{1,n+r+k} + z_{1,n+r-k} = 0$ for $k = 0, 1, \ldots, r-2$ and
$z_{1,n+r+k} + z_{1,n+r-k} = 2$ for $k = r-1$.

When one writes

$$A_k = \sum_{w\in W(D_{r-1})[k]} sgn(1k).sgn(w).w\left((1k)(a_2^{r-2} a_3^{r-3} \ldots a_{r-2}^2 a_{r-1})\right),$$

this implies

$$\begin{aligned} & \sum_{j=1}^{r} \left(g_j^{n+2r-1+k} + g_j^{-n-2r+1-k} + g_j^{n+2r-1-k} + g_j^{-n-2r+1+k}\right) A_j \\ = \quad & 2\sum_{j=1}^{r} \left(g_j^k + g_j^{-k}\right) A_j \end{aligned}$$

for $k = 0, \ldots, r-1$, since

$$\sum_{k=1}^{r} \left(a_k^{r-j} + a_k^{j-r}\right) A_k = 0,$$

for $j = 2, \ldots, r$.

By the Weyl character formula for D_{r-1} all A_k are different from zero if all g_i are different. Moreover the matrix with entries $g_k^{r-j} + g_k^{j-r}$, $j, k = 1, \ldots, r$ has the same determinant as the Vandermonde matrix with entries $(g_k + g_k^{-1})^{r-j}$, which also has non-vanishing determinant. Thus $g_i^{n+2r-1} - 2 + g_i^{-(n+2r-1)} = 0$ for all i, which is equivalent to $g_i^{n+2r-1} = 1$.

Conversely, the latter identities imply $z_{1,n+1} = 1$ and $z_{1,n+k} = 0$ for $k = 0, \ldots, r$. This implies $z_{i,n+2} = 0$ for $i = 2, \ldots, r-2$ and $z_{i,n+1} = 1$ for $i = 2, \ldots, r-2$ by the (D_r, A_n) equations. To fix the values of $z_{r-1,n+1}$ and $z_{r,n+1}$ one needs to choose a preimage of g in the connected cover of $SO(r,r)$, or equivalently a choice of squareroot of $g_1 \cdots g_r$. $\square$

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