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Integrable deformations of CFTs and the discrete Hirota equations

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Abstract

We solve the discrete Hirota equations (Kirillov-Reshetikhin Q-systems) for A_r , and their analogue for D_r , for the cases where the second variable ranges over either a finite set or over all integers. Until now only special solutions were known. We find all solutions for which no component vanishes, as required in the known applications. As an introduction we present the known solution where the second variable ranges over the natural numbers.

1 Introduction

Many conformally invariant quantum field theories in two dimensions (CFTs) can be described as limits of integrable massive quantum field theories. In such a situation, certain properties of the CFT can be understood from the perspective of the massive theory, in particular its spectrum. Suppose that the massive theory has K species of fermionic particles labeled by $a = 1, \dots, K$, and a diagonal scattering matrix $S_{ab} = \exp(i\Phi_{ab}(\theta^a - \theta^b))$ for particles of types a, b with rapidities θ^a, θ^b . Then the expected spectrum of the limiting CFT can be described as follows. Let $\Delta\Phi$ be the $K \times K$ matrix with entries $\Delta\Phi_{ab} = \Phi_{ab}(+\infty) - \Phi_{ab}(-\infty)$ and put $A = I - \Delta\Phi/2\pi$, where I denotes the identity matrix. Then for each conformal block of the CFT the partition function has the form

$$Z_L = \sum_{N \in \mathbb{N}^K} \frac{q^{Q(N)}}{(q)_{N_1} \cdots (q)_{N_K}}, \quad (1)$$

where we use the standard notation $(q)_n = (1-q) \cdots (1-q^n)$ for the q -factorial, and where

$$Q(N) = \frac{1}{2} \sum_{a,b=1}^K N_a A_{ab} N_b + L(N). \quad (2)$$

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L is an affine linear function which depends on the conformal block. One needs non-negative A for equation (1) to make sense. We shall assume that A is positive. Heuristically, the occurrence of such a partition function can be understood as follows. Consider a system with spatial periodicity R . For N_a free fermions of types $a = 1, \dots, K$, with antiperiodic boundary conditions, the contribution to the partition function is

$$\frac{q^{\sum N_a^2/2}}{(q)_{N_1} \cdots (q)_{N_K}}. \quad (3)$$

This yields equation (1) when $A = I$ and $L = 0$. To see how this contribution changes when Φ_{ab} does not vanish, consider the Bethe ansatz equations for N_a particles of type a with rapidities θ_i^a , where $i = 1, \dots, N_a$, and $a = 1, \dots, K$. They take the form

$$m_a R \sinh(\theta_i^a) + \sum_{b=1}^K \sum_{j=1}^{N_b'} \Phi_{ab}(\theta_i^a - \theta_j^b) = 2\pi n_i^a, \quad (4)$$

where m_a is the mass of a particle of type a , and the n_i^a are positive integers (or half integers in the antiperiodic case). The prime on the summation sign means that for $b = a$ the term with $j = i$ is to be omitted. For small R , particles moving in opposite directions should decouple, so that one can put $\Phi_{ab}(+\infty) = 0$ in the description of right movers. The quantisation of the total momentum of the system is not affected by the interaction. For a system of two particles of types a, b this means that $\Phi_{ab}(\theta) + \Phi_{ba}(-\theta)$ must be independent of θ , in agreement with standard constraints on the scattering matrix. This implies that $\Phi_{ab}(\theta) + \Phi_{ba}(-\theta) = -\Delta\Phi_{ab}$, since this is true for $\theta = \infty$.

Now let all θ_i^a be positive. Then equation (4) implies

$$\sum_{a=1}^K \sum_{i=1}^{N_a} m_a R \sinh(\theta_i^a) = -\pi \sum_{a,b=1}^K N_a \Delta\Phi_{ab} N_b + 2\pi \sum_{a=1}^K \sum_{i=1}^{N_a} n_i^a. \quad (5)$$

In the conformally invariant limit, rapidities are large and the difference between $\sinh(\theta)$ and $\cosh(\theta)$ can be neglected. Thus the previous equation yields the energy of the system. This differs from the free fermion case by an additive term, in agreement with equation (2). The linear part, $L(N)$, depends on the number of particles with zero rapidity and on the zero point energy, which needs a different treatment. The argument just given is heuristic, but in some cases exact formulae for the spectrum of the Hamiltonian have been obtained which agree with the preceding argument in the conformal limit.

Partition functions of conformal theories must be modular. This implies that they can be written as a sum over expressions of the form $\tilde{q}^{h-c/24}$, where $\log q \log \tilde{q} = 4\pi^2$, h is a conformal dimension of the theory and c its central charge. The smallest conformal dimension yields the exponent $-c_{eff}/24$. Evaluating Z_L close to $q = 1$ yields

$$\frac{\pi^2}{6} c_{eff} = \sum_{i=1}^K L(1 - x_i), \quad (6)$$

where L is Roger's dilogarithm and $x = (x_1, \dots, x_K)$ is the unique solution of the system of equations

$$\sum_{j=1}^K A_{ij} \log(x_j) = \log(1 - x_i), \quad (7)$$

with $i = 1, \dots, K$ and $(x_1, \dots, x_K) \in (0, 1)^K$.

According to an idea of Zagier the existence of a unique solution can be proved as follows. Consider the map $\phi : (0, 1)^K \rightarrow \mathbb{R}^K$ given by

$$\phi_i(x) = \sum_{j=1}^K A_{ij} \log x_j - \log(1 - x_i). \quad (8)$$

It is sufficient to show that ϕ is a homomorphism. Locally this is true since the Jacobian matrix with components $J_{ij} = \partial\phi_i/\partial x_j$ is nowhere degenerate. Indeed the product matrix $J\text{diag}(x_j)$ is everywhere positive. Moreover one easily sees that the inverse image under ϕ of every compact domain in \mathbb{R}^K is compact as follows. Let $C = K(\log 2)^2$ be the maximum of $\sum_{j=1}^K \log x_j \log(1 - x_j)$. Then $\phi_i(x) > a_i$ implies

$$\sum_{i,j=1}^K \log x_i A_{ij} \log x_j - \sum_{i=1}^K a_i \log x_i < -C, \quad (9)$$

such that $(\log x_1, \dots, \log x_K)$ lies in an ellipsoid. Replacing x_i by $1 - x_i$, and A by A^{-1} , one sees that the same is true for $(\log(1 - x_1), \dots, \log(1 - x_K))$. Let $x^{(1)}, x^{(2)}, \dots$ be any sequence in $(0, 1)^K$. If $\phi(x^{(k)})$ converges in \mathbb{R}^K , the $x^{(k)}$ lie in a compact set and must have an accumulation point. This implies that $\phi(x^{(k)})$ converges to an element in the image of ϕ . Since the image of ϕ is both open and closed, it must be equal to \mathbb{R}^K . It follows, by the monodromy theorem, that ϕ is a homomorphism.

To calculate non-leading terms in the modular transformation of Z_L one needs complex solutions of equation (7), where for each i one of the possible values of $\log x_i$ and $\log(1 - x_i)$ has to be chosen. The corresponding values of a suitably defined version of the Rogers dilogarithm yield the exponents $h - c/24$, multiplied by $\pi^2/6$. The main purpose of the present paper is the solution of equation (7) in some special cases. In this context it will be necessary to solve certain infinite systems of equations of independent interest.

Despite the occurrence of logarithms, equation (7) is essentially algebraic since exponentiation yields

$$\prod_{j=1}^K x_j^{A_{ij}} = 1 - x_i. \quad (10)$$

Omitting indices, these equations will be written as $x^A = 1 - x$. Since logarithms must be taken, one is only interested in solutions for which all x_i are different

from 0 and 1. These solutions yield torsion elements of the algebraic K-theory group [1, 2]. This property can be used to restrict the possible candidates for the matrix A . In particular, $K = 1$ allows only the three cases $A = 2, 1, 1/2$, see [3]. More generally, one may have

$$A = \mathcal{C}(X) \otimes \mathcal{C}(Y)^{-1}, \quad (11)$$

where X and Y are Dynkin diagrams of type ADET and \mathcal{C} denotes the corresponding Cartan matrices. Here K is the product of the ranks of X and Y , so for $K = 1$ one needs $\mathcal{C}(A_1) = 2$ and/or $\mathcal{C}(T_1) = 1$, in agreement with $A = 2, 1, 1/2$. We denote the set of vertices of X and Y by $V(X)$ and $V(Y)$ respectively, and write $V(X, Y) = V(X) \times V(Y)$, so that the components of x have the form x_{im} with $(i, m) \in V(X, Y)$. We will also consider more general auxiliary diagrams with an infinite number of vertices, but all will be simply laced, with a finite number of vertices adjacent to each given one. As usual, we put $V(A_r) = \{1, \dots, r\}$, with adjacencies between successive integers. We also introduce diagrams $Y = A_{\mathbb{N}}$ and $Y = A_{\mathbb{Z}}$ with $V(Y) = \mathbb{N}, \mathbb{Z}$, respectively, and adjacencies between successive integers.

With

$$\log(x_{im}) = \sum_{n \in V(Y)} \mathcal{C}(Y)_{mn} \log(z_{in}), \quad (12)$$

one obtains

$$z^{(2-\mathcal{C}(X)) \otimes I} + z^{I \otimes (2-\mathcal{C}(Y))} = z^2, \quad (13)$$

where indices and identity matrices have been suppressed in the notation. Given diagrams X and Y , we shall refer to the corresponding equations (13) as (X, Y) equations, with one component equation for each pair $(i, n) \in V(X, Y)$. Solutions are considered as elements $z \in \mathbb{C}^{V(X, Y)}$. Since Cartan matrices are invertible, $\log z_{in}$ must be finite for every $(i, n) \in V(X, Y)$, such that no component of z vanishes. The set of such solutions $z \in (\mathbb{C}^*)^{V(X, Y)}$ will be denoted $S^*(X, Y)$. A first result which shows that the (X, Y) equations simplify equation (7) is the following.

Theorem 1.1 *When X and Y are Dynkin diagrams of type ADET, the components of any solution z of the (X, Y) equations are algebraic integers.*

Proof It suffices to show that the \mathbb{Z} -span of arbitrary monomials in the z_{in} has finite rank. Let the height of a monomial $\prod_{in} z_{in}^{N_{in}}$ be given by $\sum_{in} f_X^i f_Y^n N_{in}$, where f_X, f_Y are the Frobenius eigenvectors of $\mathcal{C}(X), \mathcal{C}(Y)$. When z_{in}^2 is replaced by the left hand side of the (i, n) component of the (X, Y) equations, the monomial is represented as the sum of two monomials of lower height. The height is lowered by at least the Frobenius eigenvalue of the Cartan matrices. After a finite number of such substitutions one is thus left with a sum of monomials for which all exponents are zero or one. There is only a finite number of such monomials. \square

In the following we will consider only diagrams Y of type A . To solve the (X, A_n) equations it is helpful to first consider $(X, A_{\mathbb{N}})$. The corresponding system of equations has been studied extensively, for example in [4, 5, 6], and is closely connected to the representation theory of Yangians [7]. Let X have rank r . Consider the Yangian $Y(X)$, which contains the enveloping algebra $U(X)$ as a sub-Hopf-algebra. The irreducible finite-dimensional representations of $Y(X)$ have highest weights $\lambda = \sum_{i=1}^r n_i \lambda_i$, which are obtained by restricting them to representations of X . Here the λ_i are the fundamental weights of X and the n_i are non-negative integers. The representations with highest weight λ are characterised by r monic polynomials of degrees n_1, \dots, n_r . For $\lambda_0 \in \mathbb{R}$ there is an outer automorphism of $Y(X)$ which preserves $U(X)$ and acts on the polynomials by a shift $\lambda \mapsto \lambda + \lambda_0$. For $\lambda = n\lambda_i$ the irreducible representations are characterised by a single polynomial of order n . There is a basic representation given by the polynomial $\prod_{i=1}^n (\lambda - i)$ (up to shifts of λ). Let χ_{ni} be the corresponding character of X . These characters decompose into sums of irreducible characters which may be quite complicated, see e.g. [8]. Note that $\chi_{0i}(g) = 1$ for any $g \in Lie(X)$. The $(A_r, A_{\mathbb{N}})$ equations were solved by Kirillov [9], and the $(D_r, A_{\mathbb{N}})$ equations by Kirillov and Reshetikhin [7], though in the latter case no complete proof was published. More precisely, they found the generic solution of these equations, which is all one needs in our context. For $X = A_r$ this solution can be written as $z_{ni} = \chi_{in}(g)$, with arbitrary g , and they conjectured that this is true for X of type D or E , too. For a proof and its history see [4], [10].

The $(X, A_{\mathbb{N}})$ and $(X, A_{\mathbb{Z}})$ equations can be considered as initial value problems. We formulate this property as follows. For $Y = A_n, A_{\mathbb{N}}$ let R_1 be the restriction map $R_1 : \mathbb{C}^{V(X,Y)} \mapsto \mathbb{C}^{V(X) \times \{1\}}$ and for $Y = A_{\mathbb{Z}}$ let R_{01} be the restriction map $R_{01} : \mathbb{C}^{V(X,Y)} \mapsto \mathbb{C}^{V(X) \times \{0,1\}}$.

Lemma 1.2 *For $m > 0$ let $z, z' \in V(X) \times \{1, \dots, m+1\}$ be solutions of the $V(X) \times \{1, \dots, m\}$ components of the $(X, A_{\mathbb{N}})$ equations. Let $R_1 z = R_1 z'$ and assume that no component of z vanishes. Then $z' = z$. For $m \leq 1 \leq m'$ let $z, z' \in V(X) \times \{m-1, m, \dots, m'+1\}$ be solutions of the $V(X) \times \{m, \dots, m'\}$ components of the $(X, A_{\mathbb{Z}})$ equations. Let $R_{01} z = R_{01} z'$ and assume that no component of z vanishes. Then $z' = z$.*

Proof First consider $Y = A_{\mathbb{Z}}$. Let $X(i)$ be the set of vertices of X which are adjacent to a vertex i . Then the (X, Y) equations imply

$$z_{i,m+1} = \frac{z_{im}^2 - \prod_{k \in X(i)} z_{km}}{z_{i,m-1}}, \quad (14)$$

such that $z_{i,m+1} = z'_{i,m+1}$ by induction on m , and analogously $z_{i,m'-1} = z'_{i,m'-1}$ by induction on $-m'$. For $Y = A_{\mathbb{N}}$ it is sufficient to specialise the argument to $m = 1$ and $z_{i0} = 1$ for all $i \in V(X)$. \square

In the following we will need the set of solutions of the (X, Y) equations which contains $S^*(X, Y)$ and all its limit points in $\mathbb{C}^{V(X,Y)}$. For convenience we

use the language of algebraic geometry. Let $\mathcal{I}(X, Y)$ be the ideal of polynomials on $\mathbb{C}^{V(X, Y)}$ which vanish for all $z \in S^*(X, Y)$. Let $S(X, Y)$ be the algebraic completion of $S^*(X, Y)$, i.e. the set of all z which are zeros of all elements of $\mathcal{I}(X, Y)$. In general, the set of all solutions of the (X, Y) equations has a countable infinity of other components, but these are irrelevant for our purpose.

Lemma 1.3 *Let X be any connected unordered graph, with finite $X(i)$ for any $i \in X$. Then all components of $z \in S(X, A_{\mathbb{N}})$ are polynomials in the components of $R_1 z$, such that $S(X, Y)$ is an algebraic variety isomorphic to $\mathbb{C}^{rk(X)}$ (or empty).*

Proof We use induction on m . Assume that the z_{in} are polynomials for $n < m$. Then equation (14) yields $z_{i, m+1}$ as the quotient of two polynomials. To show that the quotient is a polynomial, it suffices to show that $z_{im}^2 - \prod_{k \in X(i)} z_{k, m-1}$ vanishes whenever $z_{i, m-1}$ vanishes, and at least at the same order. Now $z_{i, m-1} = 0$ implies that $z_{i, m-2}^2 = \prod_{k \in X(i)} z_{k, m-2}$ and $z_{k, m-1}^2 = z_{k, m-2} z_{k, m}$ for $k \in X(i)$. Thus $z_{i, m-2}^2 (z_{im}^2 - \prod_{k \in X(i)} z_{k, m-1}) = 0$. By assumption $z_{i, m-2} \neq 0$ generically, and one checks easily that the order of vanishing works out correctly. \square

This polynomial property was proved in [11] using the Laurent phenomenon in cluster algebras, but our approach is much simpler. In the following we only consider X of type A or D . We will need more detailed information on the polynomial expressions for z , namely the explicit form of the polynomials given in [9] and [12].

2 The A Case

We first treat the case $X = A_r$. This yields a special case of the discrete Hirota equations, which has a well known explicit solutions.

Theorem 2.1 *Let $x \in \mathbb{C}^{\mathbb{Z}}$. For any positive integer m , and for $i \in \mathbb{Z}$, let $M_i^m(x)$ be the $m \times m$ Toeplitz matrix with entries*

$$(M_i^m(x))_{kl} = x_{l-k+i}.$$

Let $z_{im} = \det M_i^m(x)$. Then these z_{im} satisfy the $(A_{\mathbb{Z}}, A_{\mathbb{N}})$ equations. If $x_0 = 1$ and $x_i = 0$ for $i < 0$, the z_{im} with $(i, m) \in V(A_{\mathbb{N}}, A_{\mathbb{N}})$ satisfy the $(A_{\mathbb{N}}, A_{\mathbb{N}})$ equations. If in addition $x_{r+1} = 1$ and $x_i = 0$ for $i > r + 1$, the z_{im} with $(i, m) \in V(A_r, A_{\mathbb{N}})$ satisfy the $(A_r, A_{\mathbb{N}})$ equations.

Proof For $A_{\mathbb{Z}}$ this is an immediate consequence of Jacobi's determinant identity. When the determinants are written as wedge products, this identity states that

$$(v_1 \wedge U \wedge v_2) \otimes (w_1 \wedge U \wedge w_2) = (v_1 \wedge U \wedge w_2) \otimes (w_1 \wedge U \wedge v_2) - (v_1 \wedge U \wedge w_1) \otimes (w_2 \wedge U \wedge v_2), \quad (15)$$

where $U = u_1 \wedge \dots \wedge u_{n-2}$ and $v_1, v_2, w_1, w_2, u_1, \dots, u_{n-2}$ are vectors in \mathbb{C}^m .

For the convenience of the reader we give a proof of this identity. One may assume that $U \neq 0$. The equation is linear in the vectors v_i, w_i . Thus it is sufficient to prove it for generic w_1, w_2 , such that v_1, v_2 can be written as linear combinations of $w_1, w_2, u_1, \dots, u_{n-2}$. Linear combinations of the u_i drop out, and the equation reduces to an identity in the two-dimensional quotient space $\mathbb{C}^m / \langle u_1, \dots, u_{n-2} \rangle$, which is essentially the formula for the determinant of 2×2 matrices. To deduce the discrete Hirota equations one denotes the rows of M_i^m by $v_1, u_1, \dots, u_{n-2}, v_2$ and puts $w_1 = (1, 0, \dots, 0)$, $w_2 = (0, \dots, 0, 1)$.

If $x_0 = 1$ and $x_i = 0$ for $i < 0$, the matrices M_0^m are upper triangular with determinant $z_{0m} = 1$, such that the relevant components of the $(A_{\mathbb{Z}}, A_{\mathbb{N}})$ equations reduce to the $(A_{\mathbb{N}}, A_{\mathbb{N}})$ equations. If $x_{r+1} = 1$ and $x_i = 0$ for $i > r+1$, the matrices M_{r+1}^m are lower triangular with determinant $z_{r+1,m} = 1$, as required for the $(A_r, A_{\mathbb{N}})$ equations. \square

Theorem 2.2 *The map $R_1 : S(A_r, A_{\mathbb{N}}) \mapsto \mathbb{C}^r$ is an isomorphism. For any $z \in S(A_r, A_{\mathbb{N}})$ one has $z_{im} = \det M_i^m(x)$, where $x_i = z_{i1}$ for $i = 1, \dots, r$, $x_0 = x_{r+1} = 1$, and $x_i = 0$ otherwise.*

Proof Let $z \in S^*(A_r, A_{\mathbb{N}})$. Since $R_1 \circ M$ is the identity on \mathbb{C}^r , one has $R_1 M R_1 z = R_1 z$. By theorem 2.1 and equation (13) this implies the set of polynomial equations $M R_1 z = z$, which by definition are satisfied by all elements of $S(A_r, A_{\mathbb{N}})$. Thus R_1 yields an isomorphism from $S(A_r, A_{\mathbb{N}})$ to a subset of \mathbb{C}^r .

Moreover, $R_1(S^*(A_r, A_{\mathbb{N}}))$ consists of those vectors in \mathbb{C}^r for which none of the determinants $\det M_i^m(x)$ vanishes. This set is dense in \mathbb{C}^r , thus unconstrained by any polynomial equation. This implies that $S(A_r, A_{\mathbb{N}})$ is the solution set of the equations $M R_1 z = z$, which is isomorphic to \mathbb{C}^r . \square

Using equation (13) one obtains

Corollary 2.3 *There is a unique injective map $\phi_{rn} : S^*(A_r, A_n) \rightarrow S(A_r, A_{\mathbb{N}})$ such that $(\phi_{rn}(z))_{im} = z_{im}$ for $(i, m) \in V(A_r, A_n)$, namely $\phi_{rn} = M \circ R_1$. Moreover, $(\phi_{rn}(z))_{i,n+1} = 1$ for $i \in V(A_r)$.*

Theorem 2.4 *For $z \in S(A_r, A_{\mathbb{N}})$ one has $z_{im} = \det M_n^i(y)$, where $y_k = z_{1k}$ for $k > 0$, $y_0 = 1$ and $y_k = 0$ for $k < 0$. Moreover, $\det M_n^{r+1}(y) = 1$.*

Proof Let $z \in \mathbb{C}^{V(A_r, A_{\mathbb{N}})}$ and $x_i = z_{i1}$. By theorem 2.1, the component z_{im} is a polynomial of degree m in x_i , with leading term x_i^m , thus not identically zero. Thus one can apply equation (13) and theorem 2.1, as in the proof of the preceding theorem. \square

Theorem 2.5 *Let $x_i = z_{i1}$ for $i = 1, \dots, r$ and $x_0 = x_{r+1} = 1$. Put $z_{10} = 1$ and $z_{1,-n} = 0$ for $n = 1, \dots, r$. Then for any $n \in \mathbb{N}$ one has*

$$\sum_{k=0}^{r+1} (-)^k x_k z_{1,n-k} = 0.$$

Proof This is just the expansion of the matrix M_1^n in terms of minors with respect to the elements of the last column. \square

Theorem 2.6 *Let g_1, \dots, g_{r+1} be the roots of the polynomial*

$$p(\gamma) = \sum_{k=0}^{r+1} (-)^k x_{r+1-k} \gamma^k = 0,$$

in some order, and put $g = \text{diag}(g_1, \dots, g_{r+1})$. One has $g \in SL(r+1)$. Let $d(r+1)$ be the subgroup of $SL(r+1)$ consisting of the diagonal matrices and let $\lambda_1, \dots, \lambda_r$ be the fundamental highest weights of $SL(r+1)$ with respect to the maximal torus $d(r+1)$. Let $\chi_{n\lambda_i}(g)$ be the value at g of the character in the irreducible representation with highest weight $n\lambda_i$. Then

$$z_{1n} = \chi_{n\lambda_1}(g).$$

Proof Consider an element of $S(A_r, A_{\mathbb{N}})$ for which g is generic. The sequence z_{1n} is a linear combination of the sequences $(g_i)^n$, with $i = 1, \dots, r+1$. Since $g_1 \cdots g_{r+1} = 1$, one has $g \in SL(r+1)$. Note that the Weyl group of $SL(r+1)$ with respect to $d(r+1)$ is the permutation group of the diagonal entries, thus isomorphic to the permutation group \mathcal{S}^{r+1} on the set $\{1, \dots, r+1\}$. The map from (x_1, \dots, x_r) to g yields a bijection between $S(A_r, A_{\mathbb{N}})$ and the quotient of $d(r+1)$ by its Weyl group. Recall the following cases of Weyl's character formula for diagonal matrices. For $n \in \mathbb{Z}$ let

$$N_{ni}(g) = \sum_{\omega \in \mathcal{S}^{r+1}} \text{sgn}(\omega) \prod_{j=1}^i g_{\omega(j)}^{r+n+1-j} \prod_{j=i+1}^r g_{\omega(j)}^{r+1-j}. \quad (16)$$

The Weyl denominator is given by $D = N_{0i}$ independently of i . Then

$$\chi_{n\lambda_i}(g) = N_{ni}(g)/D(g). \quad (17)$$

In our case the Weyl denominator formula takes the Vandermonde form

$$D(g) = \prod_{i < j} (g_i - g_j). \quad (18)$$

For generic g one has $D(g) \neq 0$ such that the Weyl character formula implies that $z_{1n} = N_{n1}(g)/D(g)$ satisfies the recursion relation in Theorem 5. Moreover $z_{1n} = N_{n1}(g)/D(g)$ is true for $-n = 0, \dots, r$. Due to $x_{r+1} = 1$ the recursion relation has a unique solution with this property. Since we work on an irreducible variety, the restriction to $D \neq 0$ is irrelevant. \square

Theorem 2.7 (Kirillov and Reshetikin)

$$z_{im} = \chi_{m\lambda_i}(g).$$

Proof This is an immediate consequence of theorems 2.4 and 2.6, together with the character formula $\chi_{m\lambda_i}(g) = \det M_m^i(y)$, where $y_k = \chi_{k\lambda_1}(g)$. The latter was obtained by Schur in his PhD thesis [13]. Sometimes it is erroneously called Weyl's second character formula. \square

As a preparation for the more involved proof for $X = D_r$ we rederive Schur's result by an explicit calculation using Weyl's character formula.

Proof If one sums over all permutations ω with fixed $\omega(1), \dots, \omega(i)$ and uses the Weyl denominator formula for the Weyl group of $SL(r+1-i)$ one obtains

$$\chi_{n\lambda_i} = \sum_{\sigma \in I(i, r+1)} D_\sigma(g)^{-1} \prod_{j=1}^i g_{\sigma(j)}^{n+r+1-j}, \quad (19)$$

where $I(i, r+1)$ is the set of all injective maps from $\{1, \dots, i\}$ to $\{1, \dots, r+1\}$ and $D_\sigma = D_\sigma^1 D_\sigma^2$, with

$$D_\sigma^1 = \prod_{1 \leq j < k \leq r+1} (g_{\sigma(j)} - g_{\sigma(k)}), \quad (20)$$

$$D_\sigma^2 = \prod_{j=1}^i \prod_{k \in \sigma^c} (g_{\sigma(j)} - g_k), \quad (21)$$

where σ^c is the complement of the image of σ in $\{1, \dots, r+1\}$. In particular we have $z_{1n} = \sum_{j=1}^{r+1} \phi_{jn}$ with

$$\phi_{jn} = g_j^{n+r} / \prod_{k \neq j} (g_j - g_k). \quad (22)$$

Thus

$$z_{in} = \sum_{\mu \in I(i, r+1)} \det M_\mu^n, \quad (23)$$

where M_μ^n is the $i \times i$ matrix with elements

$$(M_\mu^n)_{kl} = \phi_{\mu k, l-k+n}. \quad (24)$$

Non-injective maps do not contribute since they lead to matrices with linearly dependent rows. The numerators of the M_μ^n matrix elements form a Vandermonde type matrix with determinant

$$\prod_{j=1}^i g_{\mu_j}^{n+r-j+1} \prod_{1 \leq k < l \leq i} (g_{\mu_l} - g_{\mu_k}). \quad (25)$$

This yields

$$z_{in} = \sum_{\mu \in I(i, r+1)} \prod_{1 \leq k < l \leq i} (g_{\mu_l} - g_{\mu_k})^{-1} \prod_{j=1}^i \left(g_{\mu_j}^{n+r-j+1} \prod_{s \in C(\mu)} (g_{\mu_j} - g_s)^{-1} \right), \quad (26)$$

where $C(\mu)$ is the complement of the image of μ in $\{1, \dots, r+1\}$. For given μ let $N(\mu)$ be the set of all injective maps from $\{i+1, \dots, r\}$ to $C(\mu)$. We have the Vandermonde determinant formula

$$\sum_{\nu \in N(\mu)} \prod_{j=i+1}^r g_{\nu j}^{r-j+1} \prod_{i \leq k < l \leq r} (g_{\nu l} - g_{\nu k})^{-1} = 1. \quad (27)$$

Inverting the left hand side in the previous formula and noting that the pairs μ and ν with $\nu \in N(\mu)$ form the permutation group $W(r+1)$ of $\{1, \dots, r\}$, we obtain

$$z_{in} = \sum_{\sigma \in W(r+1)} \prod_{1 \leq k < l \leq r} (g_{\sigma k} - g_{\sigma l})^{-1} \prod_{j=1}^i g_{\sigma j}^{n+r-j+1} \prod_{k=i+1}^r g_{\sigma j}^{r-k+1}. \quad (28)$$

Note that z_{in} has only single poles at the locus where two g_i coincide. We shall refer to this subset of $U(r+1)$ as the singularity locus. For all $\sigma \in W(r+1)$ we have

$$\prod_{1 \leq k < l \leq r} (g_{\sigma k} - g_{\sigma l}) = \text{sgn}(\sigma) D(A_r), \quad (29)$$

where $D(A_r)$ is the Weyl denominator. Thus we have

$$D(A_r) z_{in} = \sum_{\sigma \in W(r+1)} \text{sgn}(\sigma) \prod_{j=1}^i g_{\sigma j}^{n+r+1-j} \prod_{j=i+1}^r g_{\sigma j}^{r+1-j}, \quad (30)$$

or according to the Weyl character formula

$$z_{in} = \chi_{n\lambda_i}(g). \quad (31)$$

□

Remark The previous formula can be written somewhat more concisely as follows. We regard characters as polynomials in g_1, \dots, g_{r+1} and introduce the natural action of \mathcal{S}^{r+1} on the ring of polynomials by $\sigma(g_i) = g_{\sigma(i)}$. With

$$\mathcal{A}_r = \sum_{\sigma \in \mathcal{S}^{r+1}} (\text{sgn}(\sigma)) \sigma, \quad (32)$$

we have

$$D(A_r) z_{in} = \mathcal{A}_r \prod_{j=1}^i g_j^{n+r+1-j} \prod_{j=i+1}^r g_j^{r+1-j}. \quad (33)$$

So far we have reproduced known results. We now consider $S(A_r, A_{\mathbb{Z}})$.

Theorem 2.8 $S(A_r, A_{\mathbb{Z}})$ is birationally isomorphic to the product of two maximal tori of $SL(r+1)$ modulo the simultaneous action of the Weyl group on the two factors. Explicitly, let $g = \text{diag}(g_1, \dots, g_{r+1})$ and $h = \text{diag}(h_1, \dots, h_{r+1})$ be elements in the torus and put

$$N_{ni}(g, h) = \sum_{w \in S^{r+1}} \text{sgn}(w) \prod_{j=1}^i g_{w(j)}^{r+n+1-j} h_{w(j)}^{r+1-j} \prod_{j=i+1}^r g_{w(j)}^{r+1-j}.$$

For $D(g) \neq 0$ the corresponding element of $S(A_r, A_{\mathbb{Z}})$ is given by

$$z_{in} = N_{ni}(g, h)/D(g).$$

Proof For generic $g \in SU(r+1)$, the group elements g^n , n a positive integer, are dense in $SU(r+1)$. Thus any algebraic relation between the diagonal matrix elements of g and g^n which is true for arbitrary positive integers n stays true when g^n is replaced by an arbitrary diagonal matrix $h \in SU(r+1)$. By complexification the result stays true for $SL(r+1)$. The algebraic relations between the z_{im} of $S(A_r, A_{\mathbb{N}})$ remain true when the z_{im} are replaced by $z_{i,m+n}$ for arbitrary positive integers n . Thus the formula for z_{in} given above yields elements of $S(A_r, A_{\mathbb{Z}})$. We conclude by comparing dimensions. \square

The case $h = 1$ yields

Corollary 2.9 There is a natural embedding $S(A_r, A_{\mathbb{N}}) \subset S(A_r, A_{\mathbb{Z}})$.

Theorem 2.10 $S(A_r, A_{\mathbb{Z}})$ is fibered over a maximal torus of $SL(r+1)$ modulo the action of the Weyl group.

Proof By theorems 2.1 and 2.4, any $z \in S(A_{\mathbb{N}}, A_{\mathbb{Z}})$ given by $z_{im} = \det M_m^i(y)$ satisfies the $(A_{\mathbb{N}}, A_{\mathbb{Z}})$ equations. In addition, $z_{r+1,m} = 1$ for $z \in S(A_r, A_{\mathbb{Z}})$, which implies $z_{r+2,m} = \det M_m^{r+2}(y) = 0$. This means that the vectors $v_m, v_{m+1}, \dots, v_{m+r+1}$ with $v_m = (y_m, y_{m+1}, \dots, y_{m+r+1})$ are linearly dependent for any m . On the other hand, $\det M_m^{r+1}(y) = 1$ implies that there is no linear dependency between $v_{m+1}, \dots, v_{m+r+1}$. This means that for each $z \in S(A_{\mathbb{N}}, A_{\mathbb{Z}})$ there is a unique relation $\sum_{k=0}^{r+1} (-)^k a_k v_{m+k} = 0$ with $a_0 = 1$. When z is given by g, h with $D(g) \neq 0$, the a_k are given by theorem 2.6 in terms of g , independently of m, h . Algebraic relations in $S(A_r, A_{\mathbb{Z}})$ which hold for generic elements hold for all of $S(A_r, A_{\mathbb{Z}})$. Thus the a_k are always independent of m and $a_{r+1} = 1$. As in theorem 2.6, this yields a unique element of the maximal torus of $SL(r+1)$ modulo the action of the Weyl group. \square

Now we determine the elements of $S(A_r, A_n)$.

Theorem 2.11 Let z be the image of an element of $S(A_r, A_n)$ in $(A_r, A_{\mathbb{N}})$. Then $z_{1m} = 0$ for $m = n+2, n+3, \dots, n+1+r$.

Proof By assumption we have $z_{i,n+1} = 1$ for all i . The $(A_r, A_{\mathbb{N}})$ equations immediately yield $z_{i,n+2} = 0$ for all i . Now we use induction on m . Assume that $z_{1k} = 0$ for $k = n+2, \dots, n+m$. By theorem 2.4 we have $z_{im} = \det M_m^i(y)$, where $y_k = z_{1k}$. The minors of $\det M_m^i(y)$ with respect to the first row have vanishing determinant, except for the last one, which yields 1. Thus

$$z_{m,n+2} = (-1)^{m+1} z_{1,n+m+1}. \quad (34)$$

□

Theorem 2.12

$$z_{1,m+n+2+r} = (-1)^r z_{1m}.$$

Proof For $m = -r, \dots, -1$ this is true by the previous theorem. For $m = 0$ it follows from that theorem by the recursion relation in Theorem 5. For larger m it follows from the recursion relation by induction. □

Theorem 2.13 *Let $d(r+1)$ be the diagonal subgroup of $SU(r+1)$. The elements of $S(A_r, A_n)$ are in one-to-one correspondence with the set of elements $g \in d(r+1)$ satisfying $g^{r+n+2} = (-1)^r$, where all eigenvalues of g are different, modulo permutations of the eigenvalues.*

Proof By l'Hopital's rule we can write for given g

$$\chi_{m\lambda_1} = \sum_{i=1}^{r+1} p_i(m)(g_i)^m, \quad (35)$$

where the degree of the polynomial p_i is bounded by the number of g_j with $g_j = g_i$, $i \neq j$. To have well-defined p_i we put $p_j = 0$ for $g_j = g_i$ with $j > i$. The natural continuation formula for $\chi_{m\lambda_1}(g)$ to negative m agrees with the continuation of z_{1m} by the recursion relation in Theorem 5. For large positive m the value of $\chi_{m\lambda_1}$ is dominated by the contribution of the g_i with largest absolute value and highest power of m among the corresponding p_i . Analogously, the behaviour for large negative m is dominated by the contribution of the g_i with smallest absolute values. From the periodicity of z_{1m} it follows that $\chi_{m\lambda_1}(g)$ remains bounded, such that $g \in d(r+1)$ and one can use constant p_i . Periodicity in m yields $g^{n+r+2} = 1$. The $r+1$ vectors $V_m = (z_{1m}, z_{1,m+1}, \dots, z_{1,m+r})$ for $m = n+1, n, \dots, n-r$ are linearly independent, since the matrix with row vectors $V_{n+1}, V_n, \dots, V_{n-r}$ is upper triangular with non-vanishing diagonal. Thus the $r+1$ vectors $1, g_i, \dots, (g_i)^r$ for $i = 0, \dots, r+1$, must be linearly independent too. This implies that the g_i are pairwise different. □

3 The D Case

Now let us consider $S(D_r, A_n)$. We choose a real form of D_r such that a maximal torus of the corresponding orthogonal group is parametrised as $g =$

$\text{diag}(g_1, \dots, g_{2r})$ with $g_{i+r} = g_i^{-1}$ for $i = 1, \dots, r$. We label the vertices of the D_r Dynkin diagram such that the nodes $r-1$ and r correspond to the half-spinor representations. The remaining nodes are numbered consecutively from 1 to $r-2$, starting at the outer vertex which corresponds to the vector representation.

Again we first consider $S(D_r, A_{\mathbb{N}})$. As for $S(A_r, A_{\mathbb{N}})$ the generic solutions of the discrete Hirota equations form an affine variety parametrised by z_{i1} . The formulas are slightly more complicated and were given in [12].

Theorem 3.1 (Kuniba, Nakamura and Hirota) *Given complex numbers x_0, x_1, \dots, x_r with $x_0 = 1$, and $j \in \mathbb{Z}$, let X_j and Y_j be defined by*

$$\begin{aligned} X_{2j+1} = x_j = -X_{4r-3-2j} & \quad \text{for} & \quad 0 \leq j \leq r-2 \\ X_{2j} = -x_j & \quad \text{for} & \quad j = r-1, r \\ X_j = 0 & \quad \text{otherwise} \end{aligned}$$

$$\begin{aligned} Y_2 = -Y_6 = 1, \quad Y_3 = x_r, \quad Y_5 = x_{r-1} \\ Y_j = 0 \quad \text{otherwise.} \end{aligned}$$

We will consider square matrices T_k^a of size $2k-1$ for $a = 1, \dots, r-2$, and of size $2k$ for $a = r-1, r$, where $k = 1, 2, \dots$. Their matrix elements have the property

$$(T_k^a)_{ij} = (T_k^a)_{i+2, j+2}$$

if both sides of the equation are defined. Moreover,

$$\begin{aligned} (T_k^a)_{1j} &= X_{2a+j} \\ (T_k^a)_{2j} &= Y_j \end{aligned}$$

for $a = 1, \dots, r-2$, whereas

$$\begin{aligned} (T_k^{r-1})_{1j} &= Y_{3+j} \\ (T_k^{r-1})_{2j} &= X_{2r-3+j} \\ (T_k^r)_{1j} &= X_{2r-2+j} \\ (T_k^r)_{2j} &= Y_{2+j}. \end{aligned}$$

Note that T_k^a is antisymmetric for $a = r-1, r$, such that it has a Pfaffian. Put $x_i = z_{i1}$ for $i = 1, \dots, r$. Then in $S(D_r, A_n)$ one has

$$z_{ak} = \det(T_k^a),$$

for $a = 1, \dots, r-2$ and

$$z_{ak} = \text{Pf}(T_k^a),$$

for $a = r-1, r$.

Proof The proof is again an immediate consequence of Jacobi's identity for suitably chosen matrices, together with the standard relation between determinants and Pfaffians. \square

Remark Consequently $S(D_r, A_{\mathbb{N}})$ has properties analogous to $S(A_r, A_{\mathbb{N}})$. More specifically, $S(D_r, A_{\mathbb{N}})$ is isomorphic to \mathbb{C}^r , with elements parametrised by arbitrary vectors (z_{11}, \dots, z_{r1}) , and the map $\phi_{rn} : S(D_r, A_n) \rightarrow S(D_r, A_{\mathbb{N}})$, induced by the restriction to (z_{11}, \dots, z_{r1}) , is an embedding.

We now derive a recursion relation for the z_{1m} .

Theorem 3.2 *Let $x_i = z_{i1}$ for $i = 1, \dots, r$ and $x_0 = x_{r+1} = 1$. Put $z_{10} = 1$ and $z_{1,-n} = 0$ for $n = 1, \dots, 2r - 1$. Then for any $n \in \mathbb{N}$ one has*

$$\sum_{k=0}^{2r} (-)^k \chi_k z_{1,n-k} = 0,$$

where $\chi_{2r-k} = \chi_k$ for $k = 0, \dots, r$ and $\chi_0 = 1$, $\chi_1 = x_1$, $\chi_i = x_i - x_{i-2}$ for $i = 2, \dots, r-2$, $\chi_{r-1} = x_{r-1}x_r - x_{r-3}$ and $\chi_r = x_{r-1}^2 + x_r^2 - 2x_{r-2}$.

Proof Expanding the minors of $\det(T_k^1)$ with respect to the first row one finds

$$\begin{aligned} & \sum_{i=0}^{r-2} (-)^i x_i \left(\det(T_{k-i}^1) - \det(T_{k-2r+2+i}^1) \right) \\ & + x_{r-1} \det(\tilde{T}_{2k-2r+3}) - x_r \det(\tilde{T}_{2k-2r+1}) = 0, \end{aligned} \quad (36)$$

and

$$\det(\tilde{T}_{2k+3}) - x_r \det(T_{k+1}^1) - x_{r-1} \det(T_k^1) - \det(\tilde{T}_{2k-1}) = 0. \quad (37)$$

Here $\tilde{T}_{2k-2r+3}$ is the matrix obtained by suppressing rows $1, \dots, 2r-5$ and $2r-3$ and columns $1, \dots, 2r-4$ in T_k^1 . For small values of k the recursion relations are to be interpreted so that $\det(T_{-k}^1) = 0$ for $k = 1, \dots, 2r-3$, $\det(T_{-2r+2}^1) = 1$ and $\det(\tilde{T}_{-2k+1}) = 0$ for $k = 1, \dots, r$. Eliminating the \tilde{T} -determinants by subtracting the recursion relations for $\det(T_k^1)$ and $\det(T_{k-2}^1)$ yields

$$\sum_{i=0}^{r-1} (-)^i \chi_i \left(\det(T_{k-i}^1) + \det(T_{k-2r+i}^1) \right) + (-)^r \chi_r \det(T_{k-r}^1) = 0. \quad (38)$$

□

Remark The solutions of the recursion relations have the form

$$z_{1k} = \sum_{i=1}^{2r} \alpha_i g_i^k, \quad (39)$$

where the g_i are the solutions of the polynomial equation in γ

$$\sum_{i=0}^{r-1} (-)^i \chi_i (\gamma^i + \gamma^{2r-i}) + (-)^r \chi_r \gamma^r = 0, \quad (40)$$

and one may put $g_{i+r} = g_i^{-1}$ since the polynomial coefficients are invariant under reversal of their order. Expressing χ_i in terms of g_1, \dots, g_r one notes that it can be interpreted as the character of the i -fold antisymmetric product of the fundamental representation of D_r , evaluated at the group element $g = \text{diag}(g_1, \dots, g_{2r})$. This implies that x_{r-1} and x_r can be interpreted as characters of the half-spin representations and that all x_i can be understood as Yangian characters.

Theorem 3.3 *Let $\{g_1, \dots, g_{2r}\}$ with $g_{i+r} = g_i^{-1}$ for $i = 1, \dots, r$ be the solution set of the polynomial equation in γ*

$$\sum_{i=0}^{2r} (-)^i \chi_i \gamma^i = 0.$$

Then

$$z_{1k} = \chi_{k\lambda_1}(g).$$

Proof The proof is analogous to the A_r case. Let

$$N_k(g) = \sum_{\epsilon \in \{-1,1\}^{r-1}} \sum_{\sigma \in \mathcal{S}^r} \text{sgn}(\sigma) g_{\sigma(1)}^{\epsilon_1(r-1+k)} g_{\sigma(2)}^{\epsilon_2(r-2)} g_{\sigma(3)}^{\epsilon_3(r-3)} \cdots g_{\sigma(r-1)}^{\epsilon_{r-1}}. \quad (41)$$

Then the Weyl character formula can be written as $\chi_{k\lambda_1}(g) = N_k(g)/N_0(g)$. Clearly the N_k satisfy the recursion formula, and one also has $N_{-k}(g) = 0$ for $k = 1, \dots, 2r-3$, and $N_{-2r+2}(g) = 1$. Thus $N_k(g)/N_0(g) = z_{1k}$ for $k = -2r+2, \dots, 1$, which implies that the same relation holds for all k due to the recursion relation. \square

For completeness we use the preceding result to give a direct derivation of the Kirillov-Reshetikhin formula for D_r .

Theorem 3.4 *For $i = 1, \dots, r-2$*

$$z_{in} = \sum K(i, n) \chi_{\sum_{j=1}^i k_j \lambda_j}(g),$$

where for odd i the set $K(i, n)$ consists of the tuples k_1, \dots, k_i such that $k_j = 0$ for even j and

$$\sum_{j=1}^i k_j = n,$$

whereas for even i the set $K(i, n)$ consists of the tuples k_1, \dots, k_i such that $k_j = 0$ for odd j and

$$\sum_{j=2}^i k_j \leq n.$$

Proof We put

$$Z_{in} = \det M_n^i, \quad (42)$$

where $i = 1, \dots, r$ and M_n^i is the $i \times i$ matrix with elements

$$\left(M_n^i\right)_{jk} = z_{1, n-j+k}. \quad (43)$$

For $i = 1, \dots, r-2$ the Dynkin diagrams of A_r and D_r are identical. Therefore we must have $Z_{in} = z_{in}$ for $i = 1, \dots, r-2$. On the other hand, the relation $z_{r-2, n}^2 = z_{r-2, n-1} z_{r-2, n+1} + z_{r-3, n} z_{r-1, n} z_{r, n}$ yields

$$Z_{r-1, n} = z_{r-1, n} z_{r, n}, \quad (44)$$

and a short calculation shows that the relations for $i = r-1, r$ yield

$$Z_{rn} = z_{rn}^2 + z_{r-1, n}^2 - z_{r-2, n}. \quad (45)$$

We have seen that $z_{1n} = \chi_{n\lambda_1}(g)$, where g is a diagonal $SO(r, r)$ matrix with entries $g_j, g_j^{-1}, j = 1, \dots, r$. We again refer to the set where the two g_i coincide as the singularity locus. Since the z_{in} are invariant under permutation of the g_i and under the replacement of any g_i by its inverse, the same is true for the Z_{in} . Permutations and replacements of an even number of the g_i by their inverses generates the Weyl group of $SO(r, r)$. On the other hand the single flip ξ defined by $\xi g_r = g_r^{-1}, \xi g_i = g_i$ for $i < r$ yields an outer automorphism which fixes the fundamental weights λ_i for $i = 1, \dots, r-2$, but interchanges λ_r and λ_{r-1} .

By construction, the Z_{in} are \mathbb{Z} -linear combinations of $SO(r, r)$ characters and in particular \mathbb{Z} -linear combinations of the $SO(r, r)$ weight λ . One writes

$$g^\lambda = \prod_{i=1}^r g_i^{l_i}, \quad (46)$$

or equivalently $\lambda = (l_1, \dots, l_r)$, where the l_i are either all integral or all half integral. For later use we need a notation which suppresses the dependence on g_1, \dots, g_k with $k \in \{0, \dots, r\}$. We write

$$g^\lambda[k] = \prod_{i=k+1}^r g_i^{n_i}. \quad (47)$$

Dominant weights are those for which $n_1 \geq n_2 \geq \dots \geq n_{r-1} \geq |n_r|$. We will use the weights

$$\Lambda_i = (\underbrace{1, \dots, 1}_i, 0, \dots, 0), \quad (48)$$

for $i = 1, \dots, r$. In terms of the fundamental weight λ_i one has $\Lambda_i = \lambda_i$ for $i \leq r-2$, $\Lambda_{r-1} = \lambda_{r-1} + \lambda_r$ and $\Lambda_r = 2\lambda_r$. Recall that $\mathbb{Z}[x]$ is the ring of polynomials in x with integer coefficients and $\mathbb{Z}(x)$ is the corresponding ring of power series. By abuse of notation we denote by $\mathbb{Z}[x_I]$ the ring of polynomials

in x_i with $i \in I$ and analogously for $\mathbb{Z}(x_I)$. Instead of $\mathbb{Z}[x](y)/\langle xy - 1 \rangle$ we use the abbreviated notation $\mathcal{R}[x](x^{-1})$.

Let $P \in \text{End}\mathbb{Z}[g_1, \dots, g_r](g_1^{-1}, \dots, g_r^{-1})$ be the projection to the dominant weight and P_{\geq} , P_0 , P_{\leq} the subprojections to dominant weights with $n_r \geq 0$, $n_r = 0$, and $n_r \leq 0$ respectively. In particular, $P_{\leq} = \xi P_{\geq} \xi$. We use $P[k]$ to denote the projection to the span of $g^\lambda[k]$, with λ any dominant weight, and analogously for $P_{\geq}[k]$.

When χ_λ is the character of an irreducible representation with highest weight λ and Δ_r is the Weyl denominator, we have

$$g^\lambda = P g^{-\rho} \Delta_r \chi. \quad (49)$$

Here $\rho = \sum_{i=1}^r \lambda_i$, such that

$$g^\rho = \prod_{i=1}^r g_i^{r-i}. \quad (50)$$

Since $\xi Z_{in} = Z_{in}$, one has

$$P g^{-\rho} \Delta_r Z_{in} = (1 + \xi - P_0) P_{\geq} g^{-\rho} \Delta_r Z_{in}, \quad (51)$$

so it is sufficient to calculate $P_{\geq} g^{-\rho} \Delta_r Z_{in}$ to obtain the decomposition of Z_{in} into characters of irreducible representations.

We write $\Delta_r = \Delta_r^+ \Delta_r^-$, where

$$\Delta_r^+ = \prod_{1 \leq i < j \leq r} (g_i - g_j), \quad (52)$$

and

$$\Delta_r^- = \prod_{1 \leq i < j \leq r} (1 - g_i^{-1} g_j^{-1}). \quad (53)$$

Using $(1-x)^{-1} = 1+x+x^2+\dots$ we can regard the inverse of Δ_r^- as an element of $\mathbb{Z}(g_1^{-1}, \dots, g_r^{-1})$.

By the Weyl character formula for D_r and the Weyl denominator formula for D_{r-1} we have

$$z_{1n} = \sum_{j=1}^r \phi_j^n + \phi_-^n, \quad (54)$$

where

$$\phi_{jn} = g_j^{n+r-1} / \prod_{k \neq j} (g_j - g_k) (1 - g_j^{-1} g_k^{-1}), \quad (55)$$

and

$$\phi_-^n = \sum_{j=1}^r g_j^{-(n+r-1)} / \prod_{k \neq j} ((g_j - g_k) (1 - g_j^{-1} g_k^{-1})). \quad (56)$$

In ϕ_-^n the poles at the singularity locus are removable, so that

$$\phi_-^n \in \prod_{j=1}^r g_j^{-1} \mathbb{Z}(g_1^{-1}, \dots, g_r^{-1}). \quad (57)$$

One has

$$Z_{in} = \sum_{\mu \in \nu(i,r)} \det M_n^\mu, \quad (58)$$

where $\nu(i,r)$ is the set of maps from $\{1, \dots, i\}$ to $\{1, \dots, r, -\}$ and the $i \times i$ matrices M_n^μ have matrix elements

$$(M_n^\mu)_{jk} = \phi_{\mu j}^{n-j+k}. \quad (59)$$

For a given map μ let $n(\mu)$ be the subset of $\{1, \dots, i\}$ which is mapped to $-$, and let $p(\mu)$ be its complement. If μ is not injective on $p(\mu)$ then $\det M_n^\mu = 0$. Otherwise $\det M_n^\mu$ has only single poles at the singularity locus. The development into minors with respect to $p(\mu)$ yields

$$\Delta_r^+ \det M_n^\mu \in \Delta_\mu^c \prod_{i=1}^r g_i^{-|n(\mu)|} \mathbb{Z}[g_{\mu p(\mu)}] (g_1^{-1}, \dots, g_r^{-1}), \quad (60)$$

where

$$\Delta_\mu^c = \prod_{i,j \in c(\mu), i < j} (g_i - g_j), \quad (61)$$

and $c(\mu)$ is the complement of $\mu p(\mu)$ in $\{1, \dots, r\}$. Using the Weyl denominator formula for Δ_μ^c , we see that every monomial in $\Delta_\mu^c \prod_{i=1}^r g_i^{-|n(\mu)|}$ has at least $|n(\mu)|$ strictly negative exponents for g_i with $i \in c(\mu)$. Thus

$$P_{\geq} g^{-\rho} \Delta_r \det M_n^\sigma = 0, \quad (62)$$

unless $n(\sigma) = \phi$. This implies

$$P_{\geq} g^{-\rho} \Delta_r Z_{in} = P_{\geq} g^{-\rho} \Delta_r \sum_{\sigma \in S(i,r)} \det M_n^\sigma, \quad (63)$$

where $S(i,r)$ is the set of injective maps from $\{1, \dots, i\}$ to $\{1, \dots, r\}$. For $\sigma \in S(i,r)$ the calculation of $\det M_n^\sigma$ proceeds as for $S(A_r, A_{\mathbb{N}})$ and yields

$$P_{\geq} g^{-\rho} \Delta_r Z_{in} = P_{\geq} g^{-\rho} \mathcal{A}(0,r) (N_i^0)^{-1} N_{r-i}^i g^{n\Lambda_i + \rho}, \quad (64)$$

where

$$N_{i-t}^t = \prod_{t < j < k \leq i} (1 - g_j^{-1} g_k^{-1}), \quad (65)$$

and

$$\mathcal{A}(t,s) = \sum_{\sigma \in \Pi(t,s)} \sigma(\text{sgn}(\sigma)), \quad (66)$$

and $\Pi(t, s)$ is the permutation group of $I(t, s) = \{t + 1, \dots, s\}$. Since all terms of N_{r-i}^i except the identity yield terms which project to zero, the preceding formula can be simplified to

$$P_{\geq} g^{-\rho} \Delta_r Z_{in} = P_{\geq} g^{-\rho} \mathcal{A}(0, r) (N_i^0)^{-1} g^{n\Lambda_i + \rho}. \quad (67)$$

Now

$$N_s^0 = N_{s-1}^1 \sum_{M \subset I(1, s)} (-)^{|M|} g_1^{-|M|} \prod_{a \in M} g_a^{-1}, \quad (68)$$

yields

$$(N_s^0)^{-1} = (N_{s-1}^1)^{-1} - (N_s^0)^{-1} \sum_{\substack{M \subset I(1, s) \\ M \neq \emptyset}} (-)^{|M|} g_1^{-|M|} \prod_{a \in M} g_a^{-1}. \quad (69)$$

Moreover $\mathcal{A}(0, r)$ commutes with N_s^0 and factorises through $\mathcal{A}(1, r)$. When we put

$$G(t, s) = \prod_{t < a \leq s} g_a^{-a}, \quad (70)$$

we have

$$\mathcal{A}(1, s) G(1, s) \prod_{a \in M} g_a^{-1} = 0, \quad (71)$$

unless M is of the form $I(t, s)$ with $1 \leq t \leq s$, since for $a \in M$, $a + 1 \in M$, $a + 1 \leq s$ one gets equal exponents of g_a and g_{a+1} , thus zero when one averages over a group which contains the transposition of a , $a + 1$. For $M = I(t, s)$ one obtains

$$\mathcal{A}(0, r) g_1^{t-s} \prod_{a \in I(t, s)} g_a^{-1} = 0 \quad \text{unless } t = s/2, \quad (72)$$

since for $t < s/2$ the exponents of g_1 and g_{s-t} , for $t > s/2$ those of g_1 and g_{s-t+1} , coincide. Thus the sum over M yields no contribution for odd i , whereas for even i only the term $M = I(1, i/2)$ contributes. For odd i this yields

$$\begin{aligned} & P g^{-\rho} \mathcal{A}(0, r) (N_i^0)^{-1} g^{n\Lambda_i + \rho} \\ &= g_1^n P[1] g^{-\rho}[1] \mathcal{A}(1, r) (N_{i-1}^1)^{-1} g^{n\Lambda_i + \rho}[1], \end{aligned} \quad (73)$$

and for even i

$$\begin{aligned} & P g^{-\rho} \mathcal{A}(0, r) (N_i^0)^{-1} g^{n\Lambda_i + \rho} \\ &= g_1^n P[1] g^{-\rho}[1] \mathcal{A}(1, r) (N_{i-1}^1)^{-1} g^{n\Lambda_i + \rho}[1] \\ &+ P g^{-\rho} \mathcal{A}(0, r) (N_i^0)^{-1} g^{(n-1)\Lambda_i + \rho}. \end{aligned} \quad (74)$$

By induction on n and k one obtains

$$\begin{aligned} & P_{\geq}[k] g^{-\rho}[k] \mathcal{A}(k, r) (N_{i-k}^k)^{-1} \prod_{j=k+1}^i g_j^{n+r-j} \prod_{j=i+1}^r g_j^{r-j} \\ &= \sum_{\lambda \in L(i, n, k)} g^\lambda, \end{aligned} \quad (75)$$

where $L(i, n, k)$ is the set of weakly decreasing integral sequences such that $l_{k+1} \leq r$, $l_m = 0$ for $m > i$ and $l_{i-2j-1} = l_{i-2j}$ for $j = 0, \dots, [i/2] - 1$. For $k = 0$ this yields the desired decomposition of Z_{in} in terms of characters of $SO(r, r)$. Equivalently one can write for $i = 1, \dots, r - 2$

$$z_{in} = \sum K(i, n) \chi_{\sum_{j=1}^i k_j \lambda_j}(g), \quad (76)$$

where for odd i the set $K(i, n)$ consists of the tuples k_1, \dots, k_i such that $k_j = 0$ for even j and

$$\sum_{j=1}^i k_j = n, \quad (77)$$

whereas for even i the set $K(i, n)$ consists of the tuples k_1, \dots, k_i such that $k_j = 0$ for odd j and

$$\sum_{j=2}^i k_j \leq n. \quad (78)$$

□

For $i = r - 1, r$ they stated the following result

Theorem 3.5 For $i = r - 1, r$

$$z_{in} = \chi_{n\lambda_i}(g).$$

Proof We verify this result by proving that

$$Z_{r-1,n} = \chi_{n\lambda_{r-1}}(g) \chi_{n\lambda_r}(g), \quad (79)$$

and

$$Z_{r,n} = (\chi_{n\lambda_{r-1}}(g))^2 + (\chi_{n\lambda_r}(g))^2 - z_{r-2,n}. \quad (80)$$

The latter equations show that the pair $z_{r-1,n}, z_{r,n}$ agrees with $\chi_{n\lambda_{r-1}}(g), \chi_{n\lambda_r}(g)$ in some order. Once the order is fixed for $n = 1$, the Kirillov-Reshetikhin result follows from $z_{r-1,n} z_{r,n-1} = z_{rn}^2 - z_{r-2,n}$ by induction on n and comparison of the highest powers of g_n .

To prove the statements note that

$$Pg^{-\rho} \Delta_r \chi_\lambda \chi_{n\lambda_r} = Pg^{-\rho} \chi_\lambda \mathcal{A}(0, r) \sum_{\epsilon \in E^+} \prod_{i=1}^r g_i^{\epsilon_i(n/2+r-i)}, \quad (81)$$

where

$$E^\pm = \left\{ (\epsilon_1, \dots, \epsilon_r) \in \{+, -\}^r : \prod_{i=1}^r \epsilon_i = \pm 1 \right\}. \quad (82)$$

For $\lambda = n\lambda_r$ or $\lambda = n\lambda_{r-1}$ all g_i exponents in χ_λ are less than or equal to $n/2$, such that only $\epsilon = (1, \dots, 1)$ contributes when the projection P is taken. Evaluating the Vandermonde determinant we get

$$Pg^{-\rho}\Delta_r\chi_\lambda\chi_{n\lambda_r} = Pg^{-\rho}\chi_\lambda\Delta_r^+ \prod_{i=1}^r g_i^{n/2}. \quad (83)$$

For $\lambda = n\lambda_{r-1}$ this yields

$$Pg^{-\rho}\Delta_r\chi_{n\lambda_{r-1}}\chi_{n\lambda_r} = Pg^{-\rho}(\Delta_r^-)^{-1}\mathcal{A}(0, r) \sum_{\epsilon \in E^-} \prod_{i=1}^r g_i^{n/2+\epsilon_i(n/2+r-i)}. \quad (84)$$

Only the term with $\epsilon = (1, \dots, 1, -1)$ survives the projection. Thus

$$Pg^{-\rho}\Delta_r\chi_{n\lambda_{r-1}}\chi_{n\lambda_r} = Pg^{-\rho}(\Delta_r^-)^{-1}\mathcal{A}(0, r) \prod_{i=1}^{r-1} g_i^{(n+r-i)}. \quad (85)$$

Since $\Delta_r^- = N_r^0$, the right hand side is equal to $Z_{r-1, n}$ as claimed. For $\lambda = n\lambda_r$ one obtains

$$Pg^{-\rho}\Delta_r\chi_{n\lambda_r}\chi_{n\lambda_r} = Pg^{-\rho}(\Delta_r^-)^{-1}\mathcal{A}(0, r) \sum_{\epsilon \in E^+} \prod_{i=1}^r g_i^{n/2+\epsilon_i(n/2+r-i)}. \quad (86)$$

Only the term with $\epsilon = (1, \dots, 1)$ survives the projection, so that

$$Pg^{-\rho}\Delta_r(\chi_{n\lambda_r})^2 = Pg^{-\rho}(\Delta_r^-)^{-1}\mathcal{A}(0, r) \prod_{i=1}^r g_i^{(n+r-i)} = P_{\geq}Z_{r, n}. \quad (87)$$

Since $P_0Z_{r, n} = PZ_{r-2, n}$, we also have

$$P_0g^{-\rho}\Delta_r\chi_{n\lambda_r}^2 = PZ_{r-2, n}. \quad (88)$$

Since

$$Pg^{-\rho}\Delta_rZ_{in} = (1 + \xi - P_0)P_{\geq}g^{-\rho}\Delta_rZ_{in}, \quad (89)$$

this confirms the Kirillov-Reshetikhin result for $z_{r-1, n}$ and z_{rn} . \square

Remark As for A_r , the elements of $S(D_r, A_{\mathbb{Z}})$ are parametrized by two group elements g, h . In particular one has $z_{1k} = N_k(g, h)/N_0(g)$, where

$$N_k(g, h) = \sum_{\epsilon \in \{-1, 1\}^{r-1}} \sum_{\sigma \in S^r} \text{sgn}(\sigma) h_{\sigma(1)}^{\epsilon_1} g_{\sigma(1)}^{\epsilon_1(r-1+k)} g_{\sigma(2)}^{\epsilon_2(r-2)} g_{\sigma(3)}^{\epsilon_3(r-3)} g_{\sigma(r-1)}^{\epsilon_{r-1}}. \quad (90)$$

Theorem 3.6 *Let z be the image of an element of $S(D_r, A_n)$ in $(D_r, A_{\mathbb{N}})$. Then $z_{1m} = 0$ for $m = n+2, n+3, \dots, n+r$. Moreover $dz_{1, n+r} = 0$ in $S(D_r, A_{\mathbb{Z}})$ at the image of any element of $S(D_r, A_n)$ in this variety.*

Proof By assumption we have $z_{i,n+1} = 1$ for all i . The $(D_r, A_{\mathbb{N}})$ equations immediately yield $z_{i,n+2} = 0$ for all i . Now we put $u_{im} = z_{im}$ for $i = 1, \dots, r-2$ and $u_{r-1,m} = z_{r-1,m}z_{rm}$, but leave u_{im} undefined for $i = r$. Note that $du_{r-1,n+2} = 0$ at the image of any element of $S(D_r, A_n)$ in $S(D_r, A_{\mathbb{Z}})$. To the extent that quantities are defined, the (D_r, A_n) equations for the u_{im} are invariant under an interchange of r, n . The discrete Hirota equations can be solved iteratively with initial values z_{1m} . Accordingly, u_{im} is the determinant of an $m \times m$ matrix with matrix elements $N_{jk} = x_{i+j-k}$, where $x_j = z_{1j}$. We use induction on m . For $m = n+2$ we already have seen that the statement is true. Assume that $z_{1k} = 0$ for $k = n+2, \dots, n+m$. Then consider the determinant which yields $z_{m,n+2}$. The minors with respect to the first row have vanishing determinant, except for the last one, which is 1. Thus

$$u_{m,n+2} = (-)^{m+1} z_{1,n+m+1}. \quad (91)$$

□

Theorem 3.7

$$z_{1,m+2n+4r-2} = z_{1m}.$$

Proof Evaluating the derivatives of $z_{1,n+r}$ with respect to the h_i and imposing $dz_{1,n+r} = 0$ yields

$$g_i^{n+2r-1} = g_i^{-(n+2r-1)}, \quad (92)$$

for all i . This implies

$$z_{1,m+2n+4r-2} = z_{1m}. \quad (93)$$

□

Remark As in the (A_r, A_n) case one can show that this periodicity condition cannot be satisfied when the Weyl denominator vanishes, in other words when there is a pair i, j such that $g_i = g_j$ or $g_i = g_j^{-1}$. Indeed a calculation identical to the one for the (A_r, A_n) case yields a contribution to z_{1n} proportional to $n^{k-1}g_i^n$, where k is the number of vanishing factors in $\sum_{j \neq i} (g_i - g_j)(g_i - g_j^{-1})$.

We now sharpen the result $g_i^{2(n+2r-1)} = 1$ obtained above to $g_i^{n+2r-1} = 1$.

Theorem 3.8

$$g_i^{n+2r-1} = 1.$$

Proof By the Weyl character formula we have $z_{1n} = N_n(g)/N_0(g)$, where

$$N_n(g) = \sum_{j=1}^r \left(g_j^{r-1+n} + g_j^{-(r-1+n)} \right) A_j, \quad (94)$$

and

$$A_j = \sum_{\substack{\sigma \in S^r \\ \sigma(1)=j}} \sum_{\epsilon_2, \dots, \epsilon_{r-1} \in \{1, -1\}} \text{sgn}(\sigma) g_{\sigma(2)}^{(r-2)\epsilon_2} \cdots g_{\sigma(r-1)}^{\epsilon_{r-1}}. \quad (95)$$

Note that the Weyl denominator N_0 can be written as

$$N_0(g) = \sum_{\sigma \in \mathcal{S}^r} \sum_{\epsilon_1, \dots, \epsilon_{r-1} \in \{1, -1\}} \text{sgn}(\sigma) g_{\sigma(1)}^{(r-1)\epsilon_1} \cdots g_{\sigma(r-1)}^{\epsilon_{r-1}}, \quad (96)$$

or

$$N_0(g) = \prod_{i=1}^r g_i^{-(r-i)} \prod_{i>j} (g_i - g_j)(g_i - g_j^{-1}). \quad (97)$$

By the Weyl character formula, $g_i^{n+2r-1} = g_i^{-(n+2r-1)}$ for all i implies

$$z_{1,n+r+k} = z_{1,n+r-k}. \quad (98)$$

In particular, $z_{1,n+r+k} + z_{1,n+r-k} = 0$ for $k = 0, 1, \dots, r-2$ and $z_{1,n+r+k} + z_{1,n+r-k} = 2$ for $k = r-1$. This implies

$$\sum_{j=1}^r \left(g_j^{n+2r-1+k} + g_j^{-n-2r+1-k} + g_j^{n+2r-1-k} + g_j^{-n-2r+1+k} \right) A_j = 0, \quad (99)$$

for $k = 0, \dots, r-2$ and

$$\begin{aligned} & \sum_{j=1}^r \left(g_j^{n+2r-1+k} + g_j^{-n-2r+1-k} + g_j^{n+2r-1-k} + g_j^{-n-2r+1+k} \right) A_j \\ &= 2 \sum_{j=1}^r \left(g_j^{r+1} + g_j^{-r-1} \right) A_j. \end{aligned} \quad (100)$$

Since

$$\sum_{k=1}^r \left(a_k^{r-j} + a_k^{j-r} \right) A_k = 0, \quad (101)$$

for $j = 2, \dots, r$, we can write

$$\begin{aligned} & \sum_{i=1}^r \left(g_i^{n+2r-1} + g_i^{-(n+2r-1)} \right) \left(g_i^k + g_i^{-k} \right) A_i \\ &= 2 \sum_{i=1}^r \left(g_i^k + g_i^{-k} \right) A_i, \end{aligned} \quad (102)$$

for $k = 0, 1, \dots, r-1$.

By the Weyl character formula for D_{r-1} , all A_k are different from zero if all g_i are different. Moreover the matrix with entries $a_k^{r-j} + a_k^{j-r}$, $j, k = 1, \dots, r$ has the same determinant as the Vandermonde matrix with entries $(g_k + g_k^{-1})^{r-j}$, which also has non-vanishing determinant. Thus $g_i^{n+2r-1} - 2 + g_i^{-(n+2r-1)} = 0$ for all i , which is equivalent to $g_i^{n+2r-1} = 1$.

Conversely, the latter identities imply $z_{1,n+1} = 1$ and $z_{1,n+k} = 0$ for $k = 2, \dots, r$. This implies $z_{i,n+2} = 0$ for $i = 2, \dots, r-2$ and $z_{i,n+1} = 1$ for $i = 2, \dots, r-2$ by the (D_r, A_n) equations. To fix the values of $z_{r-1,n+1}$ and $z_{r,n+1}$ one needs to choose a preimage of g in the connected cover of $SO(r, r)$, or equivalently a choice of squareroot of $g_1 \cdots g_r$. \square

Conjecture 3.9 $S(D_r, A_{\mathbb{Z}})$ is fibered over a maximal torus of $SO(r, r)$ modulo the action of the Weyl group.

This statement seems to be much more difficult to prove than the analogous one for $S(A_r, A_{\mathbb{Z}})$. Our current understanding is the following. We have given a map from a dense subset of maximal torus pairs (g, h) to $S(D_r, A_{\mathbb{Z}})$. For any point z in the latter space we can find a sequence of pairs g, h such that the image converges to z . We first show that the sequence can be restricted to a compact domain.

Lemma 3.10 For a sequence with limiting image in $S(D_r, A_{\mathbb{Z}})$ the values of $|g_j|$, $|g_j|^{-1}$, $|h_j|$, $|h_j|^{-1}$ remain bounded for all $j = 1, \dots, r$.

Proof Consider a subsequence such that for a maximal subset I of $\{1, \dots, r\}$ the values of $|g_i|$, $|h_i|$ and their inverses remain bounded. Let J be the complement of I . For $j \in J$ one can choose a subsequence that either $|h_j g_j^n| \rightarrow \infty$ or $|h_j g_j^n| \rightarrow 0$ for sufficiently large n . In the first case put $\epsilon_j = 1$, in the second $\epsilon_j = -1$. The character z_{1n} can be written as

$$z_{1n} = \frac{\sum_{j=1}^r (h_j g_j^{n+r-1} + h_j^{-1} g_j^{-n-r+1})}{\prod_{k \neq j} (g_j - g_k) (1 - g_j^{-1} g_k^{-1})}. \quad (103)$$

We relabel the indices so that $J = \{1, \dots, k\}$. The character z_{1n} can be written as

$$z_{1n} = \frac{\sum_{j=1}^r (h_j g_j^{n+r-1} + h_j^{-1} g_j^{-n-r+1})}{\prod_{k \neq j} (g_j - g_k) (1 - g_j^{-1} g_k^{-1})}. \quad (104)$$

For $k > 0$ consider the character z_{kn} , expressed as a determinant, as described in the proof of Theorem 18. Looking only at the dominating term gives

$$\begin{aligned} z_{kn} &= \det \left(\frac{\sum_{j=1}^{|J|} h_j g_j^{n+r-1}}{\prod_{k \neq j} (g_j - g_k) (1 - g_j^{-1} g_k^{-1})} \right), \\ &\sim h_1^{\epsilon_1} \dots h_k^{\epsilon_k} (g_1^{\epsilon_1} \dots g_k^{\epsilon_k})^n \times \mathcal{F}, \end{aligned} \quad (105)$$

where \mathcal{F} denotes some non-zero factor independent of n . In the limit one finds $|z_{kn}| \rightarrow \infty$, which is clearly impossible. \square

The previous lemma implies that any element $z \in S(D_r, A_{\mathbb{Z}})$ can be obtained as an image of pairs g, h where g is fixed. The remaining question is whether different choices of g can lead to the same z . As in the A -case one calculates for a generic element of $S(D_r, A_{\mathbb{Z}})$,

$$Z_{2r,k} = \prod_{i=1}^r (g_i - g_i^{-1})^2, \quad (106)$$

and $Z_{2r+1,k} = 0$ for all k . If none of the g_i is equal to $+1$ or -1 this implies that the sequence z_{1k} and its translates span a $2r$ -dimensional vector space. In particular, the dimension of this vector space is bounded above by $2r$ for all elements in $S(D_r, A_{\mathbb{Z}})$. Let $p(T)$ be the monic polynomial of minimal order which annihilates the sequence z_{1k} . Whenever the dimension is exactly $2r$, the map to the roots of p yields a fibration over a maximal torus of $SO(r, r)$ modulo the action of the Weyl group. More generally, consider the factorisation $p(T) = (T-1)^a(T+1)^bq(T)$, where q is prime to T^2-1 . Let s be the order of q . When $a+b+2+s = 2r$, the solution belongs to the fiber corresponding to

$$g = \left(\underbrace{1, \dots, 1}_{a+1}, \underbrace{-1, \dots, -1}_{ab+1}, g_{a+b+3}, \dots, g_{2r} \right), \quad (107)$$

where g_{a+b+3}, \dots, g_{2r} are the roots of q . It is possible, however, that $a+b+2+s < 2r$. In this case different choices of g, h may lead to the same sequence z_{1k} . The simplest example occurs for $a = b = 0$, $s = 2r - 2$. Let $h_1 = i, g_1 = 1$ and $g_k = i$ for $k = 2, \dots, r$, with h_2, \dots, h_r chosen such that the sequence z_{1k} remains finite. Then one obtains exactly the same sequence for $g'_1 = -1$ and $g'_k = g_k$ for $k \neq 1$ and $h'_k = -h_k$. The sequences $z_{2r-1,k}$ and $z_{2r,k}$ turn out to be different, however. We expect that this will be true in all such situations.

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