The ideal Bose gas in open harmonic trap systems and its condensation revisited.

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Abstract

We rigorously revisit a textbook model used to figure out the Bose-Einstein condensation (BEC) phenomenon created by dilute cold alkali atoms gases in a magnetic-optical trap. It consists of a d-dimensional ($d = 1, 2, 3$) ideal non-relativistic spin-0 Bose gas confined in a box and trapped in an isotropic harmonic potential. Throughout we review and clarify a series of methods involved in the derivation of the thermodynamics in the grand-canonical situation. To make the derivation consistent with the usual rules of the statistical mechanics, we assign through our *open-trap limit* approach the role of canonical parameter to a rescaled number of particles (instead of an effective density involving the pulsation of the trap). Within this approach, we formulate an Einstein-like and Penrose-Onsager-like criterion of BEC and show their equivalence. Afterwards, we focus on the spatial localization of the condensate/thermal gas. When dealing with the reduced density matrix, our method is similar to the loop path approach.

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Contents

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1 Introduction & A brief overview.

- *Bose-Einstein condensation (BEC) in dilute cold alkali atoms gases.*

BEC was for the first time observed in 1995 in a series of experiments on dilute cold alkali atoms gases, such as Rubidium ^{87}Rb [18], Sodium ^{23}Na [23] and Lithium ^{7}Li [21]. Although the first theoretical predictions go back to the 1920s [20, 25] and were made for the ideal Bose gas in isotropic cubic boxes [25], these recent experiments were realized in a magnetic-optical trap.

Let us give the two key principles of these experiments. The first step consists in pre-cooling the atoms by the *laser cooling method*. The dilute atoms gas is confined in a vacuum chamber and is cooled by two lasers facing each other in each direction at a frequency slightly lower than the resonance frequency of the atoms so that the moving atoms are slowed by Doppler effect. A temperature of the order of 10^{-4}K can be reached. The second step consists in lowering the temperature by the *magnetic evaporative cooling method*. An inhomogeneous magnetic field is introduced to trap the atoms. After switching off the laser beams, the magnetic evaporation allows to remove the high-energy atoms. The temperature is of the order of 10^{-6}K with about $10^4 - 10^6$ atoms in the magnetic trap. The temperature of the gas can be adjusted by moving the energy cutoff of the evaporating process, and then it can be below the predicted critical temperature $T_C \sim 10^{-6}K$ at the center of the trap. Note that this critical temperature is a good approximation for a dilute gas, see e.g. [22, 40, 41]. To observe the BEC, by an absorbing image technics, one can measure the spatial density profile of the atomic cloud. At high temperature (or at low density), one can observe a widely spread spatial distribution. At low temperature (or at high density), one can observe a *spatial condensation* through a peak of density.

Since 1995, a very large number of experiments have been realized to study more precisely the features of the BEC created by cold alkali atoms gases and, naturally there is a huge amount of literature on this topic. We refer the readers to those modern references [22, 40, 41, 52].

- *Figuring out BEC phenomenon - A textbook model.*

To figure out at first stage the BEC phenomenon created by cold alkali atom gases in magnetic-optical trap, the most simple widespread model in Physics literature consists in considering a d-dimensional ($d = 1, 2, 3$) ideal non-relativistic spin-0 Bose gas confined in a box Λ with Dirichlet boundary conditions and trapped in an external isotropic harmonic potential. The system is supposed to be at equilibrium with a thermal and particles bath. Within this textbook model, the one-particle Hamiltonian describing the dynamics is formally given by:

$$H_\Omega := \frac{1}{2m}(-i\hbar\nabla)^2 + \frac{m}{2}\Omega^2|\mathbf{x}|^2, \quad \Omega > 0. \tag{1.1}$$

At finite-volume, it has to be understood that H is defined with Dirichlet boundary conditions on the edges of Λ. Here Ω stands for the pulsation of the trap, it has the dimension of frequency.

We mention that the first theoretical investigations on existence of BEC with such a model go back long before the experimental works of Anderson *et al.* and Davis *et al.*, see e.g. [27, 19]. Although the 'ideal' assumption is obviously not realistic since the interactions between particles play an important role in BEC phenomenon, in most of experiments involving alkali atoms the Bose gas is diluted. Moreover, with such a model almost all predictions are analytical and quite simple. We also emphasize that taking into account interactions via a mean-field model bring only corrections to the critical temperature computed with this textbook model, see [41, Eq. (13.32)].

- *Figuring out BEC phenomenon - Thermodynamics within the weak-harmonic limit.*

A huge amount of literature deals with the thermodynamics of the harmonically trapped Bose gas for the model that we consider here, see e.g. [19, 28, 32, 34, 22] and references therein. We refer to [40, Chap. 2] and [41, Chap. X] in which the most basic results are collected. Based on the principle that the presence of the quadratic potential leads to some inhomogeneity, the framework in which the thermodynamics of the harmonically trapped Bose gas are derived differs from the one in which the thermodynamics of the perfect Bose gas confined in 'boxes' (named homogeneous systems) are derived. In the case of homogeneous systems, the underlying concept is the so-called *thermodynamic limit.* For a box Λ with volume $V(\Lambda)$, it consists in investigating the large-volume behavior (i.e. taking the limit $\Lambda \uparrow \mathbb{R}^d$) of the statistical quantities defined at finite-volume while the density of particles ρ is held fixed, see [30, Sec. 9.3]. This latter condition makes sense since ρ is given by the ratio $N(\Lambda)/V(\Lambda)$, where the number of particles $N(\Lambda) \uparrow \infty$ when $\Lambda \uparrow \mathbb{R}^d$. But for the harmonically trapped system, the previous method failed since $N(\Lambda, \Omega) \rightarrow N(\mathbb{R}^d, \Omega) < \infty$ when $\Lambda \uparrow \mathbb{R}^d$ because of the presence of the harmonic confinement. To solve this problem, the thermodynamic limit concept is replaced with the *weak-harmonic limit concept.* It consists in investigating the weak-pulsation behavior (i.e. taking the limit $\Omega \downarrow 0$) in the statistical quantities defined at *infinite-volume* while a certain quantity is kept fixed. The conserved quantity is chosen as $\tilde{\rho} := \Omega^d \times N(\mathbb{R}^d, \Omega)$ since $N(\mathbb{R}^d, \Omega) \uparrow \infty$ when $\Omega \downarrow 0$; $\tilde{\rho}$ is seen as an effective density since Ω has the dimension of a frequency. Due to an apparent similarity between these two methods, the weak-harmonic limit is often referred as to the 'thermodynamic limit' mistakenly, see e.g. [41]. The key idea behind is as follows: for a large-volume confining box and a weak-pulsation trap, the 'boundary' role played by the quadratic potential extending to the whole system overwhelms the boundary effects of the box. This leads to assign the role of an effective inverse volume to Ω^d.

It is a well-known result in Physics literature that BEC occurs in the harmonically trapped Bose gas for $d > 1$ (whereas for homogeneous systems, it occurs for $d > 2$). The usual rigorous way to derive the features of BEC within the framework of the weak-harmonic limit concept is as follows, see e.g. [54]. In the grand-canonical (G-C) situation, one assigns to the quantity:

$$\tilde{\rho}_\Omega(\beta, \mu) := \Omega^d \overline{N}_\Omega(\beta, \mu), \tag{1.2}$$

the role of an effective density of particles by seeing Ω^d as an effective inverse volume. Here $\overline{N}_\Omega$ is the *infinite-volume* G-C average number of particles at 'inverse temperature' $\beta := 1/(k_B T) > 0$ and chemical potential $\mu < d\hbar\Omega/2$. Within this approach, (1.2) is the counterpart of the G-C finite-volume density of particles for the confined ideal Bose gas (homogeneous systems):

$$\rho_\Lambda(\beta, \mu) := V(\Lambda)^{-1} \overline{N}_\Lambda(\beta, \mu). \tag{1.3}$$

Switching to the canonical parameters, they are (β, N, Ω^d) in (1.2) whereas they are $(\beta, N, V(\Lambda))$ in (1.3). By analogy, here we see the necessity to define beforehand the statistical quantities at infinite-volume within the weak-harmonic limit framework. The thermodynamic behavior of the effective density of particles $\tilde{\rho}_\Omega$ is obtained by performing the weak-harmonic limit (i.e. $\Omega \downarrow 0$) as the conventional thermodynamic limit (i.e. $\Lambda \uparrow \mathbb{R}^d$ for $\rho_\Lambda(\beta, \mu)$):

$$\tilde{\rho}(\beta, \mu) := \lim_{\Omega \downarrow 0} \tilde{\rho}_\Omega(\beta, \mu), \quad \mu < 0. \tag{1.4}$$

Defining the critical density as the limit $\tilde{\rho}_c(\beta) := \lim_{\mu\uparrow 0} \tilde{\rho}(\beta,\mu)$ within the Einstein's formulation, it is found to be finite if $d > 1$. The critical temperature is obtained by solving the equation:

$$\tilde{\rho} = \tilde{\rho}_c(\beta_c) = \zeta(d)(k_B T_c)^d/\hbar^d, \tag{1.5}$$

where ζ stands for the usual zeta-Riemann function.

The usual way to derive the features of BEC in Physics literature leans on a discrete-continuous approximation (referred as to a 'semiclassical approximation'). It consists in describing the system by a continuum of states plus the discrete ground state, see e.g. [27, 19, 28, 22]. Such a procedure is considered to be good provided that the number of atoms is large and $\beta \gg \hbar\Omega$. Since the weak-harmonic limit concept justifies these assumptions, it establishes itself like a natural framework. Calculations performed with this semiclassical approximation leads to, see e.g. [41, Eq. (10.10)]:

$$k_B T_c := \hbar\Omega N^{\frac{1}{d}}(\zeta(d))^{-\frac{1}{d}}, \quad d = 2, 3.$$

Since the r.h.s. of (1.5) remains finite and constant when $N \uparrow \infty$ and $\Omega \downarrow 0$ while keeping fixed $N \times \Omega^d$, then it is considered as the 'right value' in 'thermodynamic limit'.

- *Towards a rigorous derivation consistent with the rules of the statistical mechanics.*

Some models dealing with the inhomogeneous Bose gas have come up in the 80s. One of them is the so-called *weak potential model.* In [45, 47, 53], they considered a scaled potential which is a positive power of one coordinate. The scaling is chosen as the inverse of the characteristic length of the confining box so that the effect of the potential is not so great as to destroy the thermodynamic behavior. The thermodynamics are then obtained in the framework of the conventional thermodynamic limit concept. However with such a model the features of the BEC are different from the ones derived from the model that we consider here. Later on, investigations taking into account some weak-field interaction models generated a lot of activity, see e.g. [54, 52].

Let us turn to our approach which is different than the ones mentioned above. The starting-point consists in introducing a rescaling of the finite-volume G-C quantities of interest. The rescaling in question takes into account the 'opening size' of the harmonic trap via a dimensionless parameter and depend on the local/global nature of the quantities. Performing successively the thermodynamic limit (to get the bulk behavior regardless of the boundary effects) and the so-called 'open-trap' limit (to get the leading term in the limit of large 'opening-size' of the trap), we recover the well-known features of BEC stated within the framework of the weak-harmonic limit concept. This approach restores the conventional thermodynamic limit concept and dodges the problem of having to consider an effective volume/density. Since the canonical parameters turns out to be $(\beta, \nu, V(\Lambda))$ where ν is a rescaled number of particles, our approach is thus consistent with the usual rules of the quantum statistical mechanics.

Let us go further into details. For the sake of simplicity, assume that the confining box is 'cubic', i.e. given by $\Lambda = \Lambda_L^d := (-L/2, L/2)^d$, with $L > 0$. Define the one-particle Hamiltonian:

$$H_{L,\kappa} := \frac{1}{2m}(-i\hbar\nabla_{\mathbf{x}})^2 + \frac{m}{2}\Omega_\kappa^2|\mathbf{x}|^2, \quad \Omega_\kappa := \kappa\Omega_0,\, \Omega_0 > 0,\, \kappa > 0. \tag{1.6}$$

Here Ω_κ is the pulsation of the trap, and we introduced the Ω_0-constant to make the *κ-parameter dimensionless.* At finite-volume (i.e. $L < \infty$), $H_{L,\kappa}$ is defined with Dirichlet boundary conditions on the edges of Λ_L^d. $H_{\infty,\kappa}$ denotes the Hamiltonian defined in the whole space $\mathbb{R}^d$. Under the G-C situation, we define the G-C κ-rescaled average number of particles at finite-volume as:

$$\nu_{L,\kappa}(\beta,\mu) := \kappa^d \overline{N}_{L,\kappa}(\beta,\mu),$$

which is nothing but the G-C average number of particles with a weight taking into account the 'opening-size' of the harmonic trap. Since κ is dimensionless, the same holds for the quantity $\nu_{L,\kappa}$. Turning to the bulk properties (i.e. regardless of the boundary effects), the thermodynamic limit (i.e. $L \uparrow \infty$ leading to $\Lambda_L^d \uparrow \mathbb{R}^d$) of the G-C rescaled average number of particles reads as:

$$\nu_{\infty,\kappa}(\beta,\mu) := \lim_{L\uparrow\infty} \nu_{L,\kappa}(\beta,\mu) = \kappa^d \overline{N}_{\infty,\kappa}(\beta,\mu), \quad \mu < d\hbar\Omega_\kappa/2.$$

When the rescaled number of particles is held fixed, one has:

$$\lim_{L\uparrow\infty}\nu_{L,\kappa}(\beta,\overline{\mu}_{L,\kappa}(\beta,\nu)) = \nu_{\infty,\kappa}(\beta,\overline{\mu}_{\infty,\kappa}(\beta,\nu)) = \nu, \tag{1.7}$$

where $\overline{\mu}_{L,\kappa}(\beta,\nu)$ is the unique solution of the equation $\nu_{L,\kappa}(\beta,\mu)=\nu$, $L\in(0,\infty]$.
When the opening-size of the harmonic potential is supposed to be large, the leading term of the bulk rescaled average number of particles is obtained by performing the 'open-trap' limit (i.e. the limit $\kappa\downarrow 0$). The so-called G-C open-trap rescaled average number of particles reads as:

$$\nu_{\infty,0}(\beta,\mu) := \lim_{\kappa\downarrow 0}\nu_{\infty,\kappa}(\beta,\mu),\quad \mu<0, \tag{1.8}$$

which is the counterpart of (1.4) in our 'open-trap' formulation. To investigate the features of the BEC, we define the critical rescaled average number of particles as the limit:

$$\nu_c(\beta) := \lim_{\mu\uparrow 0}\nu_{\infty,0}(\beta,\mu) = \zeta(d)/(\hbar\Omega_0\beta)^d, \tag{1.9}$$

which is finite if $d>1$. The critical temperature is derived by solving the equation:

$$\nu = \nu_c(\beta_c) = \zeta(d)(k_B T_c)^d/\hbar^d,$$

which is the result obtained in (1.5) within the weak-harmonic limit concept.

- *Investigating the reduced density matrix/local density function in the open-trap limit: A suitable sum decompositions method.*

The reduced density matrix (*rdm*) exhibits the fundamental properties of the Bose gas, and allows to formulate a more general concept of BEC since it can be defined for interacting Bose gases, see [39, 26, 42] for historical references and [52, 22, 40, 41] for recent developments.

From the notations introduced in (1.6)-(1.7), when the rescaled number of particles $\nu>0$ is kept fixed, we define the *rdm* as follows (kernel representation):

$$\rho_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu) = \frac{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{y})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa}(\beta,\nu))}-1} + \sum_{\mathbf{s}\in(\mathbb{N}^*)^d}\frac{\psi_{\infty,\kappa}^{(\mathbf{s})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{s})}(\mathbf{y})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{s})}-\overline{\mu}_{\infty,\kappa}(\beta,\nu))}-1},\quad (\mathbf{x},\mathbf{y})\in\mathbb{R}^{2d}. \tag{1.10}$$

Here $\{E_{\infty,\kappa}^{(\mathbf{s})}\}_{\mathbf{s}\in\mathbb{N}^d}$ and $\{\psi_{\infty,\kappa}^{(\mathbf{s})}\}_{\mathbf{s}\in\mathbb{N}^d}$ are the set of eigenvalues and associated eigenfunctions of $H_{\infty,\kappa}$ in (1.6), and $(\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{s})}-\overline{\mu}_{\infty,\kappa}(\beta,\nu))}-1)^{-1}$ is the expected number of particles (counting multiplicities) in an energy state $\mathbf{s}$ in Bose-Einstein statistics when the rescaled number of particles is fixed.

In Physics literature, the counterpart of the sum in the r.h.s. of (1.10) is usually replaced with an integral via the 'semiclassical approximation' (in which the number of atoms is supposed to be large and $\beta\gg\hbar\Omega$), see e.g. [22, 40, 41]. If $d=3$ and $T<T_c$ this term is found to be decreasing in $|\mathbf{x}-\mathbf{y}|$. As for the counterpart of the first term in the r.h.s. of (1.10), it exhibits the *off diagonal long range order* (ODLRO) since it tends to a finite positive value in the limit $|\mathbf{x}-\mathbf{y}|\uparrow\infty$.

However, the first term in the r.h.s. of (1.10) does not converge in the open-trap limit. Indeed when $\nu>\nu_c(\beta)$ if $d=2,3$, it behaves like $\mathcal{O}(\kappa^{-\frac{d}{2}})$ when $\kappa\downarrow 0$ and then the ODLRO diverges. This comes from the fact that $\psi_{\infty,\kappa}^{(\mathbf{0})}\overline{\psi_{\infty,\kappa}^{(\mathbf{0})}} = \mathcal{O}(\kappa^{\frac{d}{2}})$, and when $\nu>\nu_c(\beta)$ if $d=2,3$, $E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa}(\beta,\nu) = \mathcal{O}(\kappa^d)$ when $\kappa\downarrow 0$, see Theorem 3.10 (iii). This means that the usual Penrose-Onsager criterion of BEC is not appropriate for the inhomogeneous case. To recover the well-known features of the *rdm* within our open-trap formulation, a rescaling with the κ-parameter is needed. Taking into account the local nature of the *rdm*, we define the rescaled *rdm* as follows:

$$r_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu) := \kappa^{\frac{d}{2}}\rho_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu).$$

When investigating the open-trap limit of the reduced *rdm*, we resort to a more suitable representation of the *rdm* than the one in (1.10) which is similar to the *loop gas approach* (or *long permutation* *cycles*) in [35, 36, 29] and involved in the Path-Integral Quantum Monte-Carlo numerical method [33]. The key-idea consists in rewriting the *rdm* for small values of $\kappa < 1$ as:

$$\rho_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu) = \sum_{l=1}^{N_{\kappa,\varepsilon}} e^{l\beta\overline{\mu}_{\infty,\kappa}(\beta,\nu)} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta) + \sum_{l=N_{\kappa,\varepsilon}+1}^{\infty} e^{l\beta\overline{\mu}_{\infty,\kappa}(\beta,\nu)} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta). \quad (1.11)$$

Here $N_{\kappa,\varepsilon} := \lfloor 1/\kappa^{\varepsilon} \rfloor$, $0 < \varepsilon < d$ and $G_{\infty,\kappa}^{(d)}(\cdot\,,\cdot\,;\beta)$ is the kernel of the Gibbs operator $\{\mathrm{e}^{-\beta H_{\infty,\kappa}}\}_{\beta>0}$ which is given by the so-called Melher formula reading as:

$$G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};\beta) = \prod_{j=1}^{d} \sqrt{\frac{m\Omega_\kappa}{2\pi\hbar\sinh(\beta\hbar\Omega_\kappa)}} \mathrm{e}^{-\frac{m\Omega_\kappa}{4\hbar}(x_j+y_j)^2\tanh(\frac{\beta\hbar\Omega_\kappa}{2})} \mathrm{e}^{-\frac{m\Omega_\kappa}{4\hbar}(x_j-y_j)^2\coth(\frac{\beta\hbar\Omega_\kappa}{2})},$$

where $\Omega_\kappa = \Omega_0\kappa,\ \Omega_0 > 0$. For κ small enough, one has the following approximations:

$$G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta) \sim \begin{cases} \lambda_\beta^{-d}\mathrm{e}^{-\frac{l\beta m\Omega_\kappa^2}{8}|\mathbf{x}+\mathbf{y}|^2}\mathrm{e}^{-\frac{\pi}{l\lambda_\beta^2}|\mathbf{x}-\mathbf{y}|^2}, & \text{if } l \ll \kappa^{-1} \\ \left(\frac{m\Omega_\kappa}{\hbar\pi}\right)^{\frac{d}{2}}\mathrm{e}^{-\frac{d}{2}\hbar\Omega_\kappa}\mathrm{e}^{-\frac{m\Omega_\kappa}{2\hbar}(|\mathbf{x}|^2+|\mathbf{y}|^2)} = \mathrm{e}^{-\frac{d}{2}\hbar\Omega_\kappa}\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{y})}, & \text{if } l \gg \kappa^{-1} \end{cases}, \quad (1.12)$$

where $\lambda_\beta := \sqrt{2\pi\hbar^2\beta/m}$ is the de Broglie thermal wavelength. By adjusting the value of the ε-parameter in (1.11), then for $\kappa < 1$ small enough, one can read off from (1.12) that when $\nu > \nu_c(\beta)$ if $d = 2, 3$, the ODLRO arises from the 'long loop' part of (1.11) (i.e. the second sum in the r.h.s.). In particular, when $\nu > \nu_c(\beta)$ if $d = 2, 3$ we prove the following:

$$\lim_{|\mathbf{x}-\mathbf{y}|\uparrow\infty} \lim_{\kappa\downarrow 0} r_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu) = \frac{\nu - \nu_c(\beta)}{\pi^{\frac{d}{2}}} > 0,$$

which is the analogous of the Penrose-Onsager criterion within our open-trap approach.

We emphasize that from (1.11), we can get some information about the non-condensate part of the *rdm* (i.e. the sum in the r.h.s. of (1.10)) and prove that when $\nu > \nu_c(\beta)$ it converges only for $d = 3$. Additionally, the diagonal part of (1.10) (obtained by setting $\mathbf{x} = \mathbf{y}$) is of a great interest since it can be interpreted as the local density of particles at the point $\mathbf{x} \in \mathbb{R}^d$. Actually, it turns out to be the correct notion of density of particles per unit volume since in our open-trap approach, the fixed canonical parameter is the rescaled number of particles ν. From a scaling of the spatial variables in the diagonal part of (1.11), we state some rigorous results about the spatial localization of the condensate gas in the open-trap limit. When $\nu > \nu_c(\beta)$ if $d = 3$, we show in particular that the 'long loop' part of the local density function exhibits a *spatial condensation* in the center of the trap (the number of particles per unit of volume is of the order of $\mathcal{O}(\kappa^{-\frac{d}{2}})$) with a width of the order of $\sqrt{\hbar/(2m\Omega_\kappa)}$, whereas the 'short loop' part is spread out (the number of particles per unit of volume is of the order of $\mathcal{O}(1)$) on a large scale of the order of $\sqrt{k_B T/(m\Omega_\kappa^2)}$.

- *The content of the paper.*

Sec. 2 is devoted to the rigorous derivation of the thermodynamic limits for the finite-volume G-C quantities of interest. The method that we use is borrowed from [9, 13, 10, 11] and allows to prove the thermodynamic limit of the G-C potential and all its derivatives w.r.t. the fugacity uniformly on compacts w.r.t. β, z, κ, see Theorem 2.3 and Corollary 2.7. It requires a quite sharp estimate estimate between the difference of the traces for semigroups (finite-infinite volume), see Theorem 2.4 and Remark 2.5. The proof of this key estimate lies in Annex B. It is based on the so-called geometric perturbation theory for semigroups, see e.g. [14, 15, 16] for further applications.

Sec. 3 is devoted to the derivation of the thermodynamics within the open-trap formulation. In Sec. 3.1, we introduce the open-trap rescaled average number of particles and its critical value in Definition 3.1 and 3.3 respectively. In Sec. 3.2, we focus on the concept of BEC in the open-trap limit. To do that, we introduce an Einstein-like criterion named 'open-trap BEC criterion' in Definition 3.9. The main result is Theorem 3.10. In Sec. 3.3, we are interested in the reduced density matrix and its rescaling, see Definitions 3.12 and 3.16. We successively investigate their open-trap limit, see Propositions 3.14 and 3.18. This allows to formulate a Penrose-Onsager-like criterion in the open-trap limit criterion named 'open-trap ODLRO', see Definition 3.19. The main result is as follows: there is equivalence between the open-trap BEC criterion and open-trap ODLRO criterion, see Theorem 3.20. We interpret this result in Corollary 3.21: the long range order is due to the condensate on the ground-state. In Sec. 3.4, we focus on the spatial localization of the condensate and the thermal gas in the open-trap limit. To do so, we introduce a scaling of the spatial variable in the local density function and its rescaling. In the open-trap limit, we state a barometric formula-like in Theorem 3.23. Corollary 3.25 interprets this result.

In Sec. 4 is collected some concluding remarks. In particular, we discuss the methods that we used and give some perspectives related to the generalized BEC concept in the open-trap limit.

2 Some statistical quantities related to the Bose gas.

2.1 The setting.

Consider a confined d-dimensional ($d = 1, 2, 3$) ideal non-relativistic spin-0 Bose gas confined in a box and trapped in an external isotropic harmonic potential. The box in which the gas is confined is given by $\Lambda_L^d := (-L/2, L/2)^d$ with $L \in (0,\infty)$; hereafter $|\Lambda_L^d|$ denotes the Lebesgue-measure of Λ_L^d. Furthermore, the Bose gas is at equilibrium with a thermal and particles bath.

Introduce the one-particle Hamiltonian. On $\mathcal{C}_0^\infty(\Lambda_L^d)$, define $\forall\kappa > 0$ the family of operators:

$$H_{L,\kappa} := \frac{1}{2}(-i\nabla_{\mathbf{x}})^2 + \frac{1}{2}\kappa^2|\mathbf{x}|^2, \tag{2.1}$$

where we set $\hbar, m, \Omega_0 = 1$ in (1.6). It is well-know that $\forall\kappa > 0$, (2.1) extends to a family of self-adjoint and bounded from below operators for any $L \in (0,\infty)$, denoted again by $H_{L,\kappa}$, with domain $D(H_{L,\kappa}) = \mathcal{H}_0^1(\Lambda_L^d) \cap \mathcal{H}^2(\Lambda_L^d)$ (here $\mathcal{H}^k(\Lambda_L^d)$ denotes the usual Sobolev space of order k). This definition corresponds to choose Dirichlet boundary conditions on the boundary $\partial\Lambda_L^d$. Moreover $\forall\kappa > 0$ $H_{L,\kappa}$ has a purely discrete spectrum with an accumulation point at infinity. Due to the property of separation of variables, the eigenvalues and eigenfunctions of the multidimensional case (i.e. $d = 2, 3$) are respectively related to those of the one-dimensional case by:

$$E_{L,\kappa}^{(\mathbf{s})} := \sum_{j=1}^{d} \epsilon_{L,\kappa}^{(s_j)}, \quad \psi_{L,\kappa}^{(\mathbf{s})}(\mathbf{x}) := \prod_{j=1}^{d} \phi_{L,\kappa}^{(s_j)}(x_j), \quad \mathbf{s} = \{s_j\}_{j=1}^d \in \mathbb{N}^d,\ \ \mathbf{x} = \{x_j\}_{j=1}^d \in \Lambda_L^d.$$

Here $\{\epsilon_{L,\kappa}^{(s)}\}_{s\in\mathbb{N}}$ denotes the set of eigenvalues counting multiplicities and in increasing order of the one-dimensional problem, and $\{\phi_{L,\kappa}^{(s)}\}_{s\in\mathbb{N}}$ the set of corresponding normalized eigenfunctions.

When Λ_L^d fills the whole space (when $L \uparrow \infty$), define $\forall\kappa > 0$ on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ the family of operators:

$$H_{\infty,\kappa} := \frac{1}{2}(-i\nabla_{\mathbf{x}})^2 + \frac{1}{2}\kappa^2|\mathbf{x}|^2. \tag{2.2}$$

Due to [6, Thm. X.28], $\forall\kappa > 0$ (2.2) is essentially self-adjoint and its self-adjoint extension, denoted again by $H_{\infty,\kappa}$, is semi-bounded. Moreover from [7, Thm. XIII.16], the spectrum of $H_{\infty,\kappa}$ is purely discrete with eigenvalues increasing to infinity. From the one-dimensional problem, the eigenvalues and eigenfunctions of the multidimensional case can be written down explicitly. The eigenvalues of the one-dimensional problem are all non-degenerate and given by, see e.g. [2, Sec. 1.8]:

$$\epsilon_{\infty,\kappa}^{(s)} := \kappa(s + \frac{1}{2}), \quad s \in \mathbb{N}. \tag{2.3}$$

The corresponding eigenfunctions, which form an orthonormal basis in $L^2(\mathbb{R})$ read as:

$$\forall x \in \mathbb{R}, \quad \phi_{\infty,\kappa}^{(s)}(x) := \frac{1}{\sqrt{2^s s!}}\left(\frac{\kappa}{\pi}\right)^{\frac{1}{4}} \exp(-\frac{\kappa}{2}x^2)\mathcal{H}_s(\sqrt{\kappa}x), \quad s \in \mathbb{N}. \tag{2.4}$$

Here the functions $\mathscr{H}_s$, $s \in \mathbb{N}$ are the Hermite polynomials defined by:

$$\forall x \in \mathbb{R}, \quad \mathscr{H}_s(x) := (-1)^s \exp(x^2) \frac{\mathrm{d}^s}{\mathrm{d}x^s}(\exp(-x^2)).$$

The eigenvalues and eigenfunctions of the multidimensional case (i.e. $d = 2, 3$) are respectively related to those of the one-dimensional case by:

$$E_{\infty,\kappa}^{(\mathbf{s})} := \sum_{j=1}^{d} \epsilon_{\infty,\kappa}^{(s_j)} = \kappa \sum_{j=1}^{d} (s_j + \frac{1}{2}), \quad \mathbf{s} = \{s_j\}_{j=1}^{d} \in \mathbb{N}^d, \tag{2.5}$$

$$\psi_{\infty,\kappa}^{(\mathbf{s})}(\mathbf{x}) := \prod_{j=1}^{d} \phi_{\infty,\kappa}^{(s_j)}(x_j), \quad \mathbf{x} = \{x_j\}_{j=1}^{d} \in \mathbb{R}^d. \tag{2.6}$$

From (2.3)-(2.5) and by the use of the min-max principle, one has for any $L \in (0, \infty)$:

$$\forall \kappa > 0, \quad \inf \sigma(H_{L,\kappa}) \geq \inf \sigma(H_{\infty,\kappa}) = E_{\infty,\kappa}^{(\mathbf{0})} = d\epsilon_{\infty,\kappa}^{(0)} > 0, \quad \epsilon_{\infty,\kappa}^{(0)} := \frac{\kappa}{2}.$$

For the need of the following section, let us introduce the one-parameter semigroup generated by $H_{L,\kappa}$, $L \in (0, \infty]$ and $\kappa > 0$. It is defined by $\{G_{L,\kappa}(t) := \mathrm{e}^{-tH_{L,\kappa}} : L^2(\Lambda_L^d) \to L^2(\Lambda_L^d)\}_{t \geq 0}$ with the convention $\Lambda_\infty := \mathbb{R}$. $\{G_{L,\kappa}(t)\}_{t\geq 0}$ is a self-adjoint and positive operator on $L^2(\Lambda_L^d)$. For reader's convenience, we collect some properties on it and its integral kernel in Appendix 1, see Sec. 2.3. As well, we need to introduce a particular function of the semigroup generated by $H_{L,\kappa}$. Define $\forall d \in \{1, 2, 3\}$, $\forall L \in (0, \infty]$, $\forall \kappa > 0$, $\forall \beta > 0$ and $\forall \mu < E_{L,\kappa}^{(\mathbf{0})}$ the operator on $L^2(\Lambda_L^d)$:

$$\mathfrak{f}_{BE}(\beta, \mu; H_{L,\kappa}) := (\mathrm{e}^{\beta(H_{L,\kappa} - \mu)} - 1)^{-1}, \tag{2.7}$$

where $\mathfrak{f}_{BE}(\beta, \mu; \cdot\,)$ is nothing but the Bose-Einstein distribution function. The operator in (2.7) is defined via the Dunford functional calculus as bounded operator on $L^2(\Lambda_L^d)$, see e.g. [5, Sec. VII.9]. In Lemma 2.13 we focus on some useful properties of this operator, see Sec. 2.3.

2.2 G-C potential and G-C (rescaled) average number of particles.

Here we introduce some of the relevant quantities associated with the harmonically trapped Bose gas formulated within the grand-canonical (G-C) ensemble. Each quantity is beforehand defined at finite-volume, subsequently we prove the thermodynamic limit. When dealing with the finite-volume system, $(\beta, z, |\Lambda_L^d|)$ stand for the fixed external parameters in the G-C ensemble. Here $\beta := (k_B T)^{-1} > 0$ is the 'inverse temperature' (k_B is the Boltzmann constant) and $z := \mathrm{e}^{\beta\mu} \in (0, \mathrm{e}^{\beta E_{L,\kappa}^{(\mathbf{0})}})$ is the fugacity (μ is the chemical potential). Remind that $E_{L,\kappa}^{(\mathbf{0})} := \inf \sigma(H_{L,\kappa}) > 0$.

For reader's convenience, the proves of intermediary results lie in Appendix 2, see Sec. 2.4.

2.2.1 The G-C potential.

The G-C partition function reads $\forall d \in \{1, 2, 3\}$, $\forall L > 0$, $\forall \kappa > 0$, $\forall \beta > 0$, $\forall z \in (0, \mathrm{e}^{\beta E_{L,\kappa}^{(\mathbf{0})}})$ as:

$$\Xi_{L,\kappa}(\beta, z) := \prod_{\mathbf{s} \in \mathbb{N}^d} (1 - z\mathrm{e}^{-\beta E_{L,\kappa}^{(\mathbf{s})}})^{-1}. \tag{2.8}$$

From (2.8) and under the same conditions, the G-C potential is defined by, see e.g. [30]:

$$\Phi_{L,\kappa}(\beta, z) := -\frac{1}{\beta} \ln(\Xi_{L,\kappa}(\beta, z)) = \frac{1}{\beta} \mathrm{Tr}_{L^2(\Lambda_L^d)}\{\ln(\mathbb{1} - zG_{L,\kappa}(\beta)\}, \tag{2.9}$$

and the operator inside the trace is defined via the Dunford functional calculus, see (2.45)-(2.47). Note that (2.8)-(2.9) are well-defined since the semigroup $\{G_{L,\kappa}(\beta)\}_{\beta>0}$ is trace class on $L^2(\Lambda_L^d)$, see Lemma 2.12. Clearly, $\Phi_{L,\kappa}(\beta, \cdot\,)$ is a $\mathcal{C}^\infty$-function on $(0, \mathrm{e}^{\beta E_{L,\kappa}^{(\mathbf{0})}})$. Actually, one can prove more:

Lemma 2.1. *$\forall d \in \{1,2,3\}$, $\forall L \in (0,\infty)$, $\forall \kappa > 0$ and $\forall \beta > 0$, $\Phi_{L,\kappa}(\beta,\cdot)$ has an analytic continuation to the domain $\mathcal{D} := \mathbb{C} \setminus [\mathrm{e}^{\beta E_{L,\kappa}^{(\mathbf{0})}}, \infty)$. In the following, we denote it by $\hat{\Phi}_{L,\kappa}(\beta,\cdot)$.*

Denote by $\mathcal{B}(r)$ an open ball in $\mathbb{C}$ centered at the origin and having the radius $r > 0$. When restricting to the domain $\mathcal{B}(\mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}}) \subset \mathcal{D}$, one gets a very convenient representation of the analytic continuation of $\Phi_{L,\kappa}(\beta,\cdot)$ involving the semigroup $\{G_{L,\kappa}(\beta)\}_{\beta>0}$. In particular:

Lemma 2.2. *$\forall d \in \{1,2,3\}$, $\forall L \in (0,\infty)$, $\forall \kappa > 0$, $\forall \beta > 0$ and $\forall z \in \mathcal{B}(\mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}})$:*

$$\hat{\Phi}_{L,\kappa}(\beta,z) = -\frac{1}{\beta}\sum_{l=1}^{\infty}\frac{z^l}{l}\mathrm{Tr}_{L^2(\Lambda_L^d)}\{G_{L,\kappa}(l\beta)\}. \tag{2.10}$$

In view of (2.10) and (2.39), introduce $\forall d \in \{1,2,3\}$, $\forall \kappa > 0$, $\forall \beta > 0$ and $\forall z \in \mathcal{B}(\mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}})$:

$$\hat{\Phi}_{\infty,\kappa}(\beta,z) := -\frac{1}{\beta}\sum_{l=1}^{\infty}\frac{z^l}{l}\mathrm{Tr}_{L^2(\mathbb{R}^d)}\{G_{\infty,\kappa}(l\beta)\}. \tag{2.11}$$

Now let us turn to the thermodynamic limit of the G-C potential. Here is the main result:

Theorem 2.3. *$\forall d \in \{1,2,3\}$, $\forall 0 < \kappa_1 < \kappa_2 < \infty$, $\forall 0 < \beta_1 < \beta_2 < \infty$ and for any compact subset $K \subset \mathcal{B}(\mathrm{e}^{\beta_1 E_{\infty,\kappa_1}^{(\mathbf{0})}})$:*

$$\lim_{L\uparrow\infty}\hat{\Phi}_{L,\kappa}(\beta,z) = \hat{\Phi}_{\infty,\kappa}(\beta,z),$$

uniformly in $(\kappa,\beta,z) \in [\kappa_1,\kappa_2]\times[\beta_1,\beta_2]\times K$.

Denoting by $\Phi_{\infty,\kappa}(\beta,\cdot) := \hat{\Phi}_{\infty,\kappa}(\beta,\cdot) \upharpoonright (0, \mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}})$, one has the pointwise convergence:

$$\lim_{L\uparrow\infty}\Phi_{L,\kappa}(\beta,z) = \Phi_{\infty,\kappa}(\beta,z), \quad \kappa > 0,\ \beta > 0,\ z \in (0, \mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}}). \tag{2.12}$$

The proof of Theorem 2.3 leans on the below estimate whose proof lies in Annex, see Sec. B:

Theorem 2.4. *$\forall d \in \{1,2,3\}$ there exists a constant $C_d > 0$ and $\forall 0 < \kappa_0 < 1$ there exists a $\mathcal{L}_{\kappa_0} > 0$ s.t. $\forall L \in [\mathcal{L}_{\kappa_0},\infty)$, $\forall \kappa \in [\kappa_0,\infty)$ and $\forall t > 0$:*

$$\begin{aligned}&\left|\mathrm{Tr}_{L^2(\mathbb{R}^d)}\{G_{\infty,\kappa}(t)\} - \mathrm{Tr}_{L^2(\Lambda_L^d)}\{G_{L,\kappa}(t)\}\right| \\ &\qquad \leq C_d(1+\sqrt{\kappa})(1+\kappa)^d(1+t)^{3(d+\frac{1}{2})}\left(\frac{1}{2\sinh(\frac{\kappa}{2}t)}\right)^d \mathrm{e}^{-\frac{\kappa}{32}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}.\end{aligned} \tag{2.13}$$

Remark 2.5. (i). We mention that the $\mathcal{L}$ in Theorem 2.4 can be chosen uniformly in $\kappa \in [1,\infty)$, for more details see Annex B. Moreover the estimate in (2.13) is given in a suitable form for the proof of Theorem 2.3. Actually we can prove a more general result which can be stated as follows. $\forall d \in \{1,2,3\}$ there exists a constant $C_d > 0$ and a $\mathcal{L} \geq 1$ s.t. $\forall L \in [\mathcal{L},\infty)$, $\forall \kappa > 0$ and $\forall t > 0$:

$$\begin{aligned}&\left|\mathrm{Tr}_{L^2(\mathbb{R}^d)}\{G_{\infty,\kappa}(t)\} - \mathrm{Tr}_{L^2(\Lambda_L^d)}\{G_{L,\kappa}(t)\}\right| \\ &\qquad \leq C_d(1+t)^{2d+\frac{3}{2}}\left\{(1+\sqrt{\kappa})^2\left(\frac{1}{2\sinh(\frac{\kappa}{2}t)}\right)^d + \left(\frac{L}{\sqrt{t}}\right)^{d-1}\right\}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}.\end{aligned} \tag{2.14}$$

(ii). The factor $(2\sinh(\frac{\kappa}{2}t))^{-d}$ in the r.h.s. of (2.13) is nothing but $\mathrm{Tr}_{L^2(\mathbb{R}^d)}\{G_{\infty,\kappa}(t)\}$, see (2.39).
(iii). The powers on the factors $(1+\kappa)$, $(1+t)$ and the numerical constant in the exponential can be optimized. Ditto for the estimate in (2.14).

Proof of Theorem 2.3. $\forall L \in (0,\infty)$ and $\forall(\kappa,\beta,z) \in [\kappa_1,\kappa_2]\times[\beta_1,\beta_2]\times K$, introduce:

$$\forall M \in \mathbb{N}^*, \quad \mathscr{Q}_{L,\kappa,M}(\beta,z) := \frac{1}{\beta}\sum_{l=1}^{M}\frac{z^l}{l}\left|\mathrm{Tr}_{L^2(\mathbb{R}^d)}\{G_{\infty,\kappa}(l\beta)\} - \mathrm{Tr}_{L^2(\Lambda_L^d)}\{G_{L,\kappa}(l\beta)\}\right|.$$

Let $\mathit{L} = \mathit{L}_{\kappa_1}$ s.t. $\forall L \geq \mathit{L}_{\kappa_1}$ the estimate in (2.13) holds. Then $\forall L \in [\mathit{L},\infty)$ and $\forall z \in K$, one has:

$$\mathscr{Q}_{L,\kappa,M}(\beta,z) \leq C_d(1+\sqrt{\kappa})(1+\kappa)^d\frac{(1+\beta)^{3(d+\frac{1}{2})}}{\beta(1-\mathrm{e}^{-\kappa\beta})^d}\mathrm{e}^{-\frac{\kappa}{32}\frac{L^2}{4}\tanh(\frac{\kappa}{2}\beta)}\left(\sum_{l=1}^{M}(|z|\mathrm{e}^{-\beta E_{\infty,\kappa}^{(\mathbf{0})}})^l l^{3d+\frac{1}{2}}\right),$$

for another constant $C_d > 0$. Since $\forall(\kappa,\beta) \in [\kappa_1,\kappa_2]\times[\beta_1,\beta_2]$ one has $\sup_{z\in K}|z|\mathrm{e}^{-\beta E_{\infty,\kappa}^{(\mathbf{0})}} < 1$, then from the above estimate there exists another constant $C_d = C_d(\kappa_1,\kappa_2,\beta_1,\beta_2,K) > 0$ s.t.

$$\lim_{L\uparrow\infty}\sup_{\kappa\in[\kappa_1,\kappa_2]}\sup_{\beta\in[\beta_1,\beta_2]}\sup_{z\in K}\lim_{M\uparrow\infty}\mathscr{Q}_{L,\kappa,M}(\beta,z) \leq C_d\lim_{L\uparrow\infty}\mathrm{e}^{-\frac{\kappa_1}{32}\frac{L^2}{4}\tanh(\frac{\kappa_1}{2}\beta)} = 0. \qquad \square$$

2.2.2 The G-C (rescaled) average number of particles.

From (2.9) and the foregoing, the G-C average number of particles at finite-volume is defined $\forall d \in \{1,2,3\}$, $\forall L \in (0,\infty)$, $\forall\kappa > 0$, $\forall\beta > 0$ and $\forall z \in (0,\mathrm{e}^{\beta E_{L,\kappa}^{(\mathbf{0})}})$ by:

$$\overline{N}_{L,\kappa}(\beta,z) := -\beta z\frac{\partial\Phi_{L,\kappa}}{\partial z}(\beta,z) = \sum_{\mathbf{s}\in\mathbb{N}^d}\frac{z\mathrm{e}^{-\beta E_{L,\kappa}^{(\mathbf{s})}}}{1-z\mathrm{e}^{-\beta E_{L,\kappa}^{(\mathbf{s})}}} = \mathrm{Tr}_{L^2(\Lambda_L^d)}\{\mathfrak{f}_{BE}(\beta,z;H_{L,\kappa})\}, \tag{2.15}$$

where the operator inside the trace is defined in (2.7) via the Dunford functional calculus. Under the conditions of (2.15), we mention that the G-C density of particles is related to $\overline{N}_{L,\kappa}$ by:

$$\rho_{L,\kappa}(\beta,z) := |\Lambda_L^d|^{-1}\overline{N}_{L,\kappa}(\beta,z). \tag{2.16}$$

For the following, it is convenient to introduce another quantity:

Definition 2.6. *$\forall d \in \{1,2,3\}$, $\forall L \in (0,\infty)$, $\forall\kappa > 0$, $\forall\beta > 0$ and $\forall z \in (0,\mathrm{e}^{\beta E_{L,\kappa}^{(\mathbf{0})}})$, we define the G-C rescaled average number of particles from* (2.15) *by setting:*

$$\nu_{L,\kappa}(\beta,z) := \kappa^d\overline{N}_{L,\kappa}(\beta,z). \tag{2.17}$$

Now let us turn to the thermodynamic limits. They follow from Lemma 2.1 along with Theorem 2.3. Indeed, via the Weierstrass theorem in [4, Sec. V.2] one has as corollary of Theorem 2.3:

Corollary 2.7. *$\forall d \in \{1,2,3\}$, $\forall\kappa > 0$ and $\forall\beta > 0$, $z \mapsto \hat{\Phi}_{\infty,\kappa}(\beta,z)$ is analytic on $\mathcal{B}(\mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}})$. Moreover $\forall 0 < \kappa_1 < \kappa_2 < \infty$, $\forall 0 < \beta_1 < \beta_2 < \infty$ and for any compact subset $K \subset \mathcal{B}(\mathrm{e}^{\beta_1 E_{\infty,\kappa_1}^{(\mathbf{0})}})$:*

$$\forall m \in \mathbb{N}^*, \quad \lim_{L\uparrow\infty}\frac{\partial^m\hat{\Phi}_{L,\kappa}}{\partial z^m}(\beta,z) = \frac{\partial^m\hat{\Phi}_{\infty,\kappa}}{\partial z^m}(\beta,z),$$

uniformly in $(\kappa,\beta,z) \in [\kappa_1,\kappa_2]\times[\beta_1,\beta_2]\times K$.

Due to the definition in (2.15), the thermodynamic limit of the G-C average number of particles generically exists. Reminding that $\Phi_{\infty,\kappa}(\beta,\cdot\,) := \hat{\Phi}_{\infty,\kappa}(\beta,\cdot\,) \upharpoonright (0,\mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}})$, one has in particular:

$$\overline{N}_{\infty,\kappa}(\beta,z) := \lim_{L\uparrow\infty}\overline{N}_{L,\kappa}(\beta,z) = -\beta z\frac{\partial\Phi_{\infty,\kappa}}{\partial z}(\beta,z), \quad \kappa > 0,\ \beta > 0,\ z \in (0,\mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}}), \tag{2.18}$$

and the limit $L \uparrow \infty$ commutes with the partial derivative w.r.t. z due to the compact convergence. Under the same conditions, the thermodynamic limit reads from (2.11) followed by (2.43) as:

$$\overline{N}_{\infty,\kappa}(\beta,z) = \sum_{l=1}^{\infty}z^l\mathrm{Tr}_{L^2(\mathbb{R}^d)}\{G_{\infty,\kappa}(l\beta)\} = \mathrm{Tr}_{L^2(\mathbb{R}^d)}\{\mathfrak{f}_{BE}(\beta,z;H_{\infty,\kappa})\}. \tag{2.19}$$

Due to the definition in (2.17), one has similarly to (2.18) the pointwise convergence:

$$\nu_{\infty,\kappa}(\beta,z) := \lim_{L\uparrow\infty}\nu_{L,\kappa}(\beta,z) = \kappa^d\overline{N}_{\infty,\kappa}(\beta,z),\quad \kappa>0,\ \beta>0,\ z\in(0,\mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}}). \tag{2.20}$$

$\nu_{\infty,\kappa}$ stands for the bulk average number of particles (regardless of the boundary effects) weighted with the dimensionless parameter κ^d signature of the 'opening-size' of the harmonic trap.

Hereafter we prefer to see the G-C quantities as a function of the μ-variable instead of z. Next, let us turn to the canonical conditions. In addition with the 'inverse temperature', we assume that the rescaled number of particles ν is fixed. Working with a fixed number of particles as external parameter (instead of the density of particles) is significant in our problem since the (rescaled) average number of particles is finite even in the thermodynamic limit, see (2.19) along with (2.39). Indeed, the thermodynamic limit of the G-C density of particles is identically zero, see (2.16).

Here is the last result of this section:

Lemma 2.8. *$\forall d\in\{1,2,3\}$, $\forall L\in(0,\infty]$, $\forall\kappa>0$, $\forall\beta>0$ and $\forall\nu>0$, there exists a unique $\overline{\mu}_{L,\kappa}=\overline{\mu}_{L,\kappa}(\beta,\nu)<E_{L,\kappa}^{(\mathbf{0})}$ satisfying:*

$$\nu_{L,\kappa}(\beta,\overline{\mu}_{L,\kappa}) = \nu. \tag{2.21}$$

Moreover:

$$\lim_{L\uparrow\infty}\overline{\mu}_{L,\kappa} = \overline{\mu}_{\infty,\kappa}. \tag{2.22}$$

2.3 Appendix 1 - Some results involving the one-parameter semigroup.

Here we collect most of the technical results which we use throughout the paper involving the semigroup generated by $H_{L,\kappa}$. For reader's convenience, all the proves are placed in Sec. 2.3.2.

2.3.1 Semigroup generated by $H_{L,\kappa}$: kernels, estimates and all these things.

For the sake of simplicity, here we use the notation $\Lambda_\infty := \mathbb{R}$. From (2.1)-(2.2), remind that:

$$\forall L\in(0,\infty],\quad H_{L,\kappa} = \frac{1}{2}(-i\nabla_{\mathbf{x}})^2 + \frac{1}{2}\kappa^2|\mathbf{x}|^2\quad \text{in } L^2(\Lambda_L^d),\ d\in\{1,2,3\}. \tag{2.23}$$

Below, we allow the value $\kappa=0$ in (2.23); in that case $H_{L,0}$ is nothing but the Dirichlet Laplacian in Λ_L^d and $H_{\infty,0}$ the free Laplacian on the whole space whose self-adjointness domain is $\mathcal{H}^2(\Lambda_\infty^d)$.

Let us recall some properties on the one-parameter semigroup generated by $H_{L,\kappa}$ in (2.23); we refer to [17, Sec. B]. It is defined $\forall\kappa\geq 0$ and $\forall L\in(0,\infty]$ by $\{G_{L,\kappa}(t) := \mathrm{e}^{-tH_{L,\kappa}} : L^2(\Lambda_L^d)\to L^2(\Lambda_L^d)\}_{t\geq 0}$ and is strongly continuous, i.e. $\lim_{t\to t'}\|G_{L,\kappa}(t)\varphi - G_{L,\kappa}(t')\varphi\|_2 = 0\ \forall\varphi\in L^2(\Lambda_L^d)$ and $\forall t\geq 0$. Moreover $\{G_{L,\kappa}(t)\}_{t\geq 0}$ is a self-adjoint and positive operator on $L^2(\Lambda_L^d)$ by the spectral theorem and the functional calculus. Besides, since $\forall\kappa\geq 0$ and $\forall L\in(0,\infty]$, $\{G_{L,\kappa}(t)\}_{t>0}$ is bounded from $L^2(\Lambda_L^d)$ to $L^\infty(\Lambda_L^d)$, then it is an integral operator by the Dunford-Pettis theorem.

Let us turn to the integral kernel of $\{G_{L,\kappa}(t)\}_{t>0}$ denoted by $G_{L,\kappa}^{(d)}$. $\forall\kappa\geq 0$ and $\forall L\in(0,\infty]$, $G_{L,\kappa}^{(d)}$ is jointly continuous in $(\mathbf{x},\mathbf{y},t)\in\overline{\Lambda_L^d}\times\overline{\Lambda_L^d}\times(0,\infty)$ and vanishes if $\mathbf{x}\in\partial\Lambda_L^d$ or $\mathbf{y}\in\partial\Lambda_L^d$. When $L=\infty$, it is explicitly known. If $\kappa=0$, it is the so-called heat kernel reading for $d=1$ as:

$$\forall(x,y)\in\Lambda_\infty^2,\ \forall t>0,\quad G_{\infty,0}^{(d=1)}(x,y;t) := \frac{1}{\sqrt{2\pi}}\frac{\mathrm{e}^{-\frac{(x-y)^2}{2t}}}{\sqrt{t}}. \tag{2.24}$$

In the case of $\kappa>0$, the one-dimensional kernel is given by the Mehler's formula:

$$\forall(x,y)\in\Lambda_\infty^2,\ \forall t>0,\quad G_{\infty,\kappa}^{(d=1)}(x,y;t) = \sqrt{\frac{\kappa}{2\pi\sinh(\kappa t)}}\mathrm{e}^{-\frac{\kappa}{4}[(x+y)^2\tanh(\frac{\kappa}{2}t)+(x-y)^2\coth(\frac{\kappa}{2}t)]}. \tag{2.25}$$

Note that the multidimensional kernel (i.e. $d = 2, 3$) is directly obtained from (2.24) or (2.25) by:

$$\forall \kappa \geq 0, \quad G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t) := \prod_{j=1}^{d} G_{\infty,\kappa}^{(d=1)}(x_j, y_j; t), \quad \mathbf{x} := \{x_j\}_{j=1}^d,\, \mathbf{y} := \{y_j\}_{j=1}^d. \tag{2.26}$$

When restricting to $L \in (0,\infty)$, the mapping $L \mapsto G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t)$ is positive and monotone increasing. This leads to the following pointwise inequality which holds $\forall \kappa \geq 0$ and $\forall L \in (0,\infty)$:

$$\forall (\mathbf{x},\mathbf{y},t) \in \overline{\Lambda_L^d} \times \overline{\Lambda_L^d} \times (0,\infty), \quad G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t) \leq \sup_{L>0} G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t) = G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t). \tag{2.27}$$

We mention that in the particular case of $\kappa = 0$, the kernel $G_{L,0}^{(d)}$ is explicitly known and reads as:

$$\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d},\, \forall t > 0, \quad G_{L,0}^{(d)}(\mathbf{x},\mathbf{y};t) = \prod_{j=1}^{d} G_{L,0}^{(d=1)}(x_j, y_j; t), \tag{2.28}$$

$$G_{L,0}^{(d=1)}(x,y;t) := \frac{1}{\sqrt{2t}} \sum_{m \in \mathbb{Z}^1} \left\{ \exp\Big(-\frac{(x-y+2mL)^2}{2t}\Big) - \exp\Big(-\frac{(x+y-2mL-L)^2}{2t}\Big)\right\}.$$

In view of (2.25)-(2.26), let us introduce $\forall \kappa > 0$ the new notation:

$$\forall \gamma > 0, \quad G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t,\gamma) := \Big(\frac{\kappa}{2\pi \sinh(\kappa t)}\Big)^{\frac{d}{2}} \prod_{j=1}^{d} \mathrm{e}^{-\frac{\kappa}{4\gamma}[(x_j+y_j)^2 \tanh(\frac{\kappa}{2}t) + (x_j-y_j)^2 \coth(\frac{\kappa}{2}t)]}, \tag{2.29}$$

with the convention: $G_{\infty,\kappa}^{(d)}(\cdot\,,\cdot\,;t) = G_{\infty,\kappa}^{(d)}(\cdot\,,\cdot\,;t,1)$. Here are collected all the needed estimates:

Lemma 2.9. *$\forall d \in \{1,2,3\}$, there exists a constant $C_d > 0$ s.t.*
(i). *$\forall \kappa > 0$, $\forall \gamma > 0$, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_\infty^{2d}$ and $\forall t > 0$:*

$$G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t) \leq \big(\frac{\kappa}{\pi}\big)^{\frac{d}{2}} \frac{\mathrm{e}^{-E_{\infty,\kappa}^{(\mathbf{0})} t}}{(1-\mathrm{e}^{-2\kappa t})^{\frac{d}{2}}}, \quad E_{\infty,\kappa}^{(\mathbf{0})} = \kappa \frac{d}{2}, \tag{2.30}$$

$$G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t,\gamma) \leq \left(\frac{\kappa}{\sinh(\kappa t)}\right)^{\frac{d}{2}} t^{\frac{d}{2}} \gamma^{\frac{d}{2}} G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};\gamma t) \leq \gamma^{\frac{d}{2}} G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};\gamma t) \leq (2\pi t)^{-\frac{d}{2}}, \tag{2.31}$$

$$|\nabla_{\mathbf{x}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t)| \leq C_d \sqrt{\kappa} \sqrt{\coth(\frac{\kappa}{2}t)}\, G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t,2), \tag{2.32}$$

$$|\Delta_{\mathbf{x}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t)| \leq C_d \kappa \coth(\frac{\kappa}{2}t) G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t,2). \tag{2.33}$$

(ii). *$\forall L \in (0,\infty)$, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:*

$$|\nabla_{\mathbf{x}} G_{L,0}^{(d)}(\mathbf{x},\mathbf{y};t)| \leq C_d \frac{(1+t)^d}{\sqrt{t}} G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};2t), \tag{2.34}$$

$$|\Delta_{\mathbf{x}} G_{L,0}^{(d)}(\mathbf{x},\mathbf{y};t)| \leq C_d \frac{(1+t)^d}{t} G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};2t). \tag{2.35}$$

We continue with the following lemma expressing the semigroup property in the kernels sense:

Lemma 2.10. *$\forall d \in \{1,2,3\}$, $\forall \delta > 0$, $\forall t > 0$, $\forall 0 < u < t$:*
(i). *$\forall \kappa \geq 0$, $\forall L \in (0,\infty]$ and $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$:*

$$\int_{\Lambda_L^d} \mathrm{d}\mathbf{z}\, G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{z};\delta(t-u)) G_{L,\kappa}^{(d)}(\mathbf{z},\mathbf{y};\delta u) = G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{y};\delta t). \tag{2.36}$$

(ii). *$\forall \kappa > 0$, $\forall \gamma > 0$ and $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_\infty^{2d}$:*

$$\int_{\Lambda_\infty^d} \mathrm{d}\mathbf{z}\, G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{z};\delta(t-u),\gamma) G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};\delta u,\gamma) = \gamma^{\frac{d}{2}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};\delta t,\gamma). \tag{2.37}$$

Now we give some estimates on the operator and trace norms of the semigroup $\{G_{L,\kappa}(t)\}_{t>0}$. For any $\kappa \geq 0$ and $L \in (0,\infty]$, $\{G_{L,\kappa}(t)\}_{t>0}$ is a contraction semigroup, see e.g. [8, Sec. 1.2]:

Lemma 2.11. *$\forall d \in \{1,2,3\}$, $\forall \kappa \geq 0$ and $\forall t > 0$:*

$$\forall L \in (0,\infty), \quad \|G_{L,\kappa}(t)\| \leq \|G_{\infty,\kappa}(t)\| \leq \begin{cases} (\cosh(\kappa t))^{-\frac{d}{2}} \leq 1 & \text{if } \kappa > 0 \\ 1 & \text{if } \kappa = 0 \end{cases} \tag{2.38}$$

Restricting to $\kappa > 0$, $\forall L \in (0,\infty]$ $\{G_{L,\kappa}(t)\}_{t>0}$ is a Gibbs semigroup, see [8, Sec. 3.1].

Lemma 2.12. *$\forall d \in \{1,2,3\}$, $\forall \kappa > 0$ and $\forall L \in (0,\infty]$, $\{G_{L,\kappa}(t)\}_{t>0}$ is a trace class operator on $L^2(\Lambda_L^d)$. Moreover, for any $L \in (0,\infty)$:*

$$\mathrm{Tr}_{L^2(\Lambda_L^d)}\{G_{L,\kappa}(t)\} \leq \mathrm{Tr}_{L^2(\Lambda_\infty^d)}\{G_{\infty,\kappa}(t)\} = \Big(\frac{1}{2\sinh(\frac{\kappa}{2}t)}\Big)^d = \frac{\mathrm{e}^{-E_{\infty,\kappa}^{(\mathbf{0})}t}}{(1-\mathrm{e}^{-\kappa t})^d}, \quad E_{\infty,\kappa}^{(\mathbf{0})} = d\frac{\kappa}{2}. \tag{2.39}$$

Finally, we end this paragraph by giving some properties on the operator defined in (2.7):

Lemma 2.13. *$\forall d \in \{1,2,3\}$, $\forall L \in (0,\infty]$, $\forall \kappa > 0$, $\forall \beta > 0$:*
(i). *$\forall \mu < E_{L,\kappa}^{(\mathbf{0})}$, $\mathfrak{f}_{BE}(\beta,\mu;H_{L,\kappa})$ is a trace class operator on $L^2(\Lambda_L^d)$.*
(ii). *$\forall \mu < E_{\infty,\kappa}^{(\mathbf{0})}$, $\mathfrak{f}_{BE}(\beta,\mu;H_{L,\kappa})$ has a jointly continuous integral kernel on Λ_L^{2d} reading as:*

$$\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}, \quad (\mathfrak{f}_{BE}(\beta,\mu;H_{L,\kappa}))(\mathbf{x},\mathbf{y}) = \sum_{l=1}^{\infty} \mathrm{e}^{l\beta\mu} G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta). \tag{2.40}$$

2.3.2 Proof of Lemmas 2.9-2.13.

Proof of Lemma 2.9. (2.30) follows from (2.25)-(2.26). From the lower bounds in (A.2)-(A.4), (2.24) is an upper bound for (2.25) what leads to (2.31). (2.32)-(2.33) are obtained by direct calculations. Here we used the key estimate in (A.12). (2.34)-(2.35) follow from [13, Prop. 2]. □

Proof of Lemma 2.10. (i) follows from the semigroup property which reads as: $G_{L,\kappa}(t) = G_{L,\kappa}(t-u)G_{L,\kappa}(u)$ $\forall 0 \leq u \leq t$. The proof of (ii) is based on the following explicit calculation:

$$\forall a,b,c,d > 0, \quad \int_{\mathbb{R}^1} \mathrm{d}z\, \mathrm{e}^{-[a(x+z)^2+b(x-z)^2]}\mathrm{e}^{-[c(z+y)^2+d(z-y)^2]} = \\ \sqrt{\pi}(a+b+c+d)^{-\frac{1}{2}}\mathrm{e}^{-\frac{b(c+d)+a(d+c)+4ab}{a+b+c+d}x^2}\mathrm{e}^{-\frac{b(c+d)+a(d+c)+4cd}{a+b+c+d}y^2}\mathrm{e}^{-2\frac{b(d-c)+a(c-d)}{a+b+c+d}xy}. \tag{2.41}$$

Then set $a_0 := \tanh(\frac{\kappa}{2}\delta u)$, $b_0 := \coth(\frac{\kappa}{2}\delta u)$, $c_0 := \tanh(\frac{\kappa}{2}\delta(t-u))$ and $d_0 := \coth(\frac{\kappa}{2}\delta(t-u))$. From the identities in (A.7)-(A.8) and (A.5): $a_0+b_0+c_0+d_0 = 2\sinh(\kappa\delta t)\{\sinh(\kappa\delta u)\sinh(\kappa\delta(t-u))\}^{-1}$. The rest of the proof consists in using some identities involving hyperbolic functions to simplify each one of the factor in the exponentials of the r.h.s. of (2.41). It is (quite) easy to get:

$$(b_0(c_0+d_0)+a_0(d_0+c_0)+4a_0b_0)(a_0+b_0+c_0+d_0)^{-1} = 2\coth(\kappa\delta t) = \coth(\frac{\kappa}{2}\delta t)+\tanh(\frac{\kappa}{2}\delta t),$$
$$(b_0(d_0-c_0)+a_0(c_0-d_0))(a_0+b_0+c_0+d_0)^{-1} = \tanh(\frac{\kappa}{2}\delta t)-\coth(\frac{\kappa}{2}\delta t).$$
□

Proof of Lemma 2.11. The first inequality follows from the fact that the semigroup $\{G_{L,\kappa}(t)\}_{t\geq 0}$ is increasing in L in the sense of [12, Eq. (2.39)]. The Shur-Holmgren criterion provides the estimate on the operator norms. When $\kappa > 0$, we used (2.41) (with $c = 0 = d$) along with (A.6). □

Proof of Lemma 2.12. Let $(\mathfrak{I}_2(L^2(\Lambda_L^d)), \|\cdot\|_{\mathfrak{I}_2})$ and $(\mathfrak{I}_1(L^2(\Lambda_L^d)), \|\cdot\|_{\mathfrak{I}_1})$, $L \in (0,\infty]$ be the Banach space of Hilbert-Schmidt and trace class operators on $L^2(\Lambda_L^d)$ respectively. We start with $d = 1$. Let $\kappa > 0$ and $t > 0$ be fixed. In view of (2.25), from (2.41) (with $c = 0 = d$):

$$\|G_{\infty,\kappa}(t)\|_{\mathfrak{I}_2}^2 = \int_{\Lambda_\infty^1}\mathrm{d}x\int_{\Lambda_\infty^1}\mathrm{d}y\,|G_{\infty,\kappa}^{(d=1)}(x,y;t)|^2 = \frac{1}{\sinh(\kappa t)} < \infty.$$

Therefore $G_{\infty,\kappa}(t)$ is a trace class operator on $L^2(\Lambda_\infty^1)$ since $\|G_{\infty,\kappa}(t)\|_{\mathfrak{I}_1} \leq \|G_{\infty,\kappa}(t/2)\|_{\mathfrak{I}_2}^2 < \infty$. Since $G_{\infty,\kappa}^{(d=1)}(\cdot\,,\cdot\,;t)$ is jointly continuous on Λ_∞^2, from [13, Prop. 9] it follows that:

$$\|G_{\infty,\kappa}(t)\|_{\mathfrak{I}_1} = \int_{\Lambda_\infty^1} \mathrm{d}x\, G_{\infty,\kappa}^{(d=1)}(x,x;t) = \frac{1}{2}\frac{1}{\sinh(\frac{\kappa}{2}t)}, \tag{2.42}$$

where we used the identity (A.5). By positivity of $G_{\infty,\kappa}(t)$, $\|G_{\infty,\kappa}(t)\|_{\mathfrak{I}_1} = \mathrm{Tr}_{L^2(\Lambda_\infty^1)}\{G_{\infty,\kappa}(t)\}$. The rest of the proof leans on the estimate (2.27) which leads to $\|G_{L,\kappa}(t)\|_{\mathfrak{I}_2}^2 \leq \|G_{\infty,\kappa}(t)\|_{\mathfrak{I}_2}^2$. Thus $\forall L \in (0,\infty)$ $G_{L,\kappa}(t)$ is also a trace class operator on $L^2(\Lambda_L^1)$, and by mimicking the above arguments, its trace norm obeys $\|G_{L,\kappa}(t)\|_{\mathfrak{I}_1} = \mathrm{Tr}_{L^2(\Lambda_L^1)}\{G_{L,\kappa}(t)\} \leq \|G_{\infty,\kappa}(t)\|_{\mathfrak{I}_1}$. The case of $d=1$ is done. The generalization to $d = 2,3$ is straightforward due to (2.26). □

Proof of Lemma 2.13. (i). Since $\mathfrak{f}_{BE}(\beta,\mu;H_{L,\kappa})$ is positive, then $\forall L \in (0,\infty]$ and $\forall \mu < E_{L,\kappa}^{(\mathbf{0})}$ its trace-norm obeys: $\|\mathfrak{f}_{BE}(\beta,\mu;H_{L,\kappa})\|_{\mathfrak{I}_1} \leq \mathrm{e}^{\beta\mu}(1-\mathrm{e}^{\beta(\mu - E_{L,\kappa}^{(\mathbf{0})})})^{-1}\mathrm{Tr}_{L^2(\Lambda_L^d)}\{G_{L,\kappa}(\beta)\} < \infty$.
(ii). Let $\mu < E_{\infty,\kappa}^{(\mathbf{0})}$. Expanding $x \mapsto (1-x)^{-1}$, $|x|<1$ in power series, then (2.7) reads as:

$$\forall L \in (0,\infty], \quad \mathfrak{f}_{BE}(\beta,\mu;H_{L,\kappa}) = \sum_{l=1}^{\infty} \mathrm{e}^{l\beta\mu} G_{L,\kappa}(l\beta), \tag{2.43}$$

and (2.43) holds in trace class operators sense due to (2.39). Since $G_{L,\kappa}(t)$ is bounded from $L^2(\Lambda_L^d)$ to $L^\infty(\Lambda_L^d)$ whose norm obeys $\|G_{L,\kappa}(t)\|_{2,\infty} \leq \|G_{\infty,\kappa}(t)\|_{2,\infty} \leq \kappa^{\frac{d}{4}}(2\pi\sinh(2\kappa t))^{-\frac{d}{4}} \leq (4\pi t)^{-\frac{d}{4}}$, then $\forall L \in (0,\infty]$ $\mathfrak{f}_{BE}(\beta,\mu;H_{L,\kappa})$ is an integral operator. Next, $\forall \varphi \in \mathcal{C}_0^\infty(\Lambda_L^d)$ one has:

$$\big(\mathfrak{f}_{BE}(\beta,\mu;H_{L,\kappa})\varphi\big)(\mathbf{x}) = \int_{\Lambda_L^d} \mathrm{d}\mathbf{y} \sum_{l=1}^{\infty} \mathrm{e}^{l\beta\mu} G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta)\varphi(\mathbf{y}) \quad \text{for a.a. } \mathbf{x} \in \Lambda_L^d, \tag{2.44}$$

where we used (2.27) followed by (2.30). Since (2.44) can be extended $\forall \varphi \in L^2(\Lambda_L^d)$, this leads to (2.40). The joint continuity follows from the one of $G_{L,\kappa}^{(d)}(\cdot\,,\cdot\,;t)$ on Λ_L^{2d} and (2.27)-(2.30). □

2.4 Appendix 2 - Proof of Lemmas 2.1, 2.2 and 2.8.

Throughout this section, by ln we understand the principal value of the natural logarithm.

Proof of Lemma 2.1. Let $L \in (0,\infty)$, $\kappa > 0$ and $\beta > 0$ kept fixed. Let $K_0 \subset \mathcal{D}$ s.t. $K_0 \cap \mathbb{R}$ contains a nonempty open interval. For any $z \in K_0$, introduce the following operator:

$$\mathcal{L}_{L,\kappa}(\beta,z) := \frac{i}{2\pi}\int_{\mathscr{C}_{K_0}} \mathrm{d}\xi\, \ln(1-\xi)(zG_{L,\kappa}(\beta)-\xi)^{-1}, \tag{2.45}$$

where $\mathscr{C}_{K_0}$ is a positively oriented simple closed contour defined as follows. If $\sup_{z\in K_0}|z| < \mathrm{e}^{\beta E_{L,\kappa}^{(\mathbf{0})}}$,

$$\mathscr{C}_{K_0} := \{r_{K_0}\mathrm{e}^{i\theta},\, \theta \in [0,2\pi)\}, \quad r_{K_0} := (1 + \sup_{z\in K_0}|z|\mathrm{e}^{-\beta E_{L,\kappa}^{(\mathbf{0})}})/2. \tag{2.46}$$

Otherwise, $\mathscr{C}_{K_0}$ consists of an arc of a circle of radius r_{K_0}, defined as above, which we close in a such way that $\mathscr{C}_{K_0}$ encloses all the eigenvalues of the family of operator $\{zG_{L,\kappa}(\beta),\, z \in K_0\}$ while avoiding the cut $[1,\infty)$ in the complex plane. Since $\mathscr{C}_{K_0}$ is strictly included in the holomorphic domain $\mathbb{C}\setminus[1,\infty)$ of the map $\xi \mapsto \ln(1-\xi)$, then the operator in (2.45) is bounded on $L^2(\Lambda_L^d)$ by the Dunford functional calculus. Moreover, for any $z \in K_0$ one has the identification:

$$\mathcal{L}_{L,\kappa}(\beta,z) = \ln(\mathbb{1} - zG_{L,\kappa}(\beta)). \tag{2.47}$$

Next, let us rewrite the above operator in a more convenient way. To do that, note that:

$$(zG_{L,\kappa}(\beta)-\xi)^{-1} = -\xi^{-1}[\mathbb{1} - (zG_{L,\kappa}(\beta)-\xi)^{-1}zG_{L,\kappa}(\beta)], \quad \xi \in \varrho(zG_{L,\kappa}(\beta)).$$

Since the map $\xi \mapsto \ln(1-\xi)/\xi$ is holomorphic on the domain $\mathbb{C} \setminus [1,\infty)$ (we use its extension by continuity), then from (2.47) along with (2.45), one has $\forall z \in K_0$ the following identiy:

$$\ln(\mathbb{1} - zG_{L,\kappa}(\beta)) = \mathcal{A}_{L,\kappa}(\beta,z)\big(zG_{L,\kappa}(\beta)\big), \quad \mathcal{A}_{L,\kappa}(\beta,z) := \frac{i}{2\pi}\int_{\mathscr{C}_{K_0}} \mathrm{d}\xi\, \frac{\ln(1-\xi)}{\xi}(zG_{L,\kappa} - \xi)^{-1},$$

which holds in the bounded operators sense. Here we used the Cauchy-Goursat theorem. From this representation, it is easy to see that $\{\ln(\mathbb{1} - zG_{L,\kappa}(\beta)), z \in K_0\}$ is a family of trace class operators, and moreover, $K_0 \ni z \mapsto \ln(\mathbb{1} - zG_{L,\kappa}(\beta))$ is analytic in the trace-norm sense. Hence:

$$K_0 \ni z \mapsto \hat{\Phi}_{L,\kappa}^{(0)}(\beta,z) := \frac{1}{\beta}\mathrm{Tr}_{L^2(\Lambda_L^d)}\{\ln(\mathbb{1} - zG_{L,\kappa}(\beta))\}, \tag{2.48}$$

is analytic. It remains to extend this property of analyticity on the whole of $\mathcal{D}$. To do that, consider an increasing sequence of compact subsets $\{K_l\}_l$, with K_0 as above, and in a such way that $\cup_l K_l = \mathcal{D}$. Note that the analytic continuations of $\Phi_{L,\kappa}(\beta,\cdot)$ defined as in (2.48) which correspond respectively to the compact subsets K_{l_p} and K_{l_q}, with $K_{l_p} \subset K_{l_q}$, coincide on K_{l_p}. □

Proof of Lemma 2.2. Let $L \in (0,\infty)$, $\kappa > 0$ and $\beta > 0$ kept fixed. Pick $z \in \mathcal{B}(\mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}})$ and let $K \subset \mathcal{B}(\mathrm{e}^{\beta E_{\infty,\kappa}^{(\mathbf{0})}})$ be a compact subset s.t. $z \in K$. From the proof of Lemma 2.1, one has:

$$\ln(\mathbb{1} - zG_{L,\kappa}(\beta)) = \left(\frac{i}{2\pi}\int_{\mathscr{C}_K} \mathrm{d}\xi\, \frac{\ln(1-\xi)}{\xi}(zG_{L,\kappa}(\beta) - \xi)^{-1}\right) zG_{L,\kappa}(\beta), \tag{2.49}$$

with $\mathscr{C}_K$ as in (2.46). Since $\forall \xi \in \mathscr{C}_K$, $|\xi| < 1$ then for such ξ's we can expand $\xi \mapsto \ln(1-\xi)$ in power series. Thus the operator between brackets can be successively rewritten as:

$$-\sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{i}{2\pi}\int_{\mathscr{C}_K} \mathrm{d}\xi\, \xi^{l-1}(zG_{L,\kappa}(\beta) - \xi)^{-1}\right) = -\sum_{l=1}^{\infty}\frac{(zG_{L,\kappa}(\beta))^{l-1}}{l}, \tag{2.50}$$

where we used in the r.h.s. of (2.50) the Dunford functional calculus. Since $\mu < E_{\infty,\kappa}^{(\mathbf{0})}$, then the series in (2.50) is absolutely convergent in the trace class operators sense due to (2.39). It remains to use the semigroup property $(G_{L,\kappa}(\beta))^l = G_{L,\kappa}(l\beta)$ to conclude. □

Proof of Lemma 2.8. Since $\forall L \in (0,\infty]$, $\forall \kappa > 0$ and $\forall \beta > 0$, the map $\mu \mapsto \overline{N}_{L,\kappa}(\beta,\mu)$ is continuous and strictly increasing on the interval $(-\infty, E_{L,\kappa}^{(\mathbf{0})})$, the first part of the lemma follows. Actually, under the same conditions $\mu \mapsto \overline{N}_{L,\kappa}(\beta,\mu)$ defines a $\mathcal{C}^\infty$-diffeomorphism of $(-\infty, E_{L,\kappa}^{(\mathbf{0})})$ into $(0,\infty)$. Let us now prove (2.22). To achieve that, we show:

$$\overline{\mu}_{\infty,\kappa} \leq \overline{\mu}_{\infty,\kappa}^{\mathrm{inf}} \leq \overline{\mu}_{\infty,\kappa}^{\mathrm{sup}} \leq \overline{\mu}_{\infty,\kappa}, \quad \text{with} \quad \overline{\mu}_{\infty,\kappa}^{\mathrm{inf}} := \liminf_{L\uparrow\infty} \overline{\mu}_{L,\kappa} \ \text{ and } \ \overline{\mu}_{L,\kappa}^{\mathrm{sup}} := \limsup_{L\uparrow\infty} \overline{\mu}_{L,\kappa}. \tag{2.51}$$

Let us prove the first inequality in (2.51). Suppose the contrary, i.e. $\overline{\mu}_{\infty,\kappa}^{\mathrm{inf}} < \overline{\mu}_{\infty,\kappa}$. Then there exists $\eta > 0$ and a divergent sequence $\{L_n\}_{n\geq 1}$ s.t. $\lim_{n\uparrow\infty} \overline{\mu}_{L_n,\kappa} = \overline{\mu}_{\infty,\kappa}^{\mathrm{inf}}$ and $\overline{\mu}_{L_n,\kappa} \leq \overline{\mu}_{\infty,\kappa} - \eta$ $\forall n \geq 1$. Now by using that the map $\mu \mapsto \overline{N}_{L_n,\kappa}(\beta,\mu)$ is increasing on $(-\infty, E_{\infty,\kappa}^{(\mathbf{0})})$, then:

$$\nu = \nu_{L_n,\kappa}(\beta, \overline{\mu}_{L_n,\kappa}) \leq \nu_{L_n,\kappa}(\beta, \overline{\mu}_{\infty,\kappa} - \eta) \quad \forall n \geq 1.$$

Afterwards, since $\{\nu_{L_n,\kappa}(\beta,\cdot)\}_{n\geq 1}$ converges uniformly on compacts w.r.t. μ to $\nu_{\infty,\kappa}(\beta,\cdot)$ as a result of Corollary 2.7, then by using that $\mu \mapsto \nu_{\infty,\kappa}(\beta,\mu)$ is strictly increasing on $(-\infty, E_{\infty,\kappa}^{(\mathbf{0})})$:

$$\nu = \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}^{\mathrm{inf}}) \leq \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa} - \eta) < \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}) = \nu.$$

This contradiction yields $\overline{\mu}_{\infty,\kappa} \leq \overline{\mu}_{\infty,\kappa}^{\mathrm{inf}}$. The last inequality in (2.51) can be proved similarly. □

3 The Bose-Einstein condensation in the *open-trap limit.*

In Sec. 2, we proved that the G-C rescaled average number of particles admits the thermodynamic limit which is a bulk quantity, i.e. independent of the boundary effects. Its introduction allowed to assign to ν the role of canonical parameter in good agreement with the rules of the statistical mechanics. In accordance with the experiments which demonstrate the condensate when the harmonic trap is sufficiently 'open', see e.g. [18, 23, 21, 22], the thermodynamical quantities describing the features of BEC should be approximated at first order by the first-order asymptotic of the corresponding bulk rescaled quantities in the limit $\kappa \downarrow 0$, named *open-trap limit.*

The aim of this section is to figure out what we mean by BEC in the *open-trap limit.* After formulating the concept, we prove that there exists a so-called *open-trap BEC* for $d > 1$ and give the explicit formulas for the critical rescaled average number of particles and rescaled number of particles in the ground-state. Afterwards, we investigate the reduced density matrix to formulate the counterpart of the Penrose-Onsager criterion of BEC in the open-trap limit. Finally, we give a series of results concerning the spatial localization of the condensate/thermal gas. Notice that we often give some comments to relate our results to the well-known results in Physics literature.

3.1 The (critical) open-trap rescaled average number of particles.

Remind that the thermodynamic limit of the G-C rescaled average number of particles is given in (2.20). Seeing $\nu_{\infty,\kappa}$ as a function of the μ-variable, then from (2.19) it can be rewritten as:

$$\forall \kappa > 0,\, \forall \beta > 0,\, \forall \mu \in (-\infty, E_{\infty,\kappa}^{(\mathbf{0})}), \quad \nu_{\infty,\kappa}(\beta,\mu) = \sum_{\mathbf{s}\in\mathbb{N}^d} \frac{\kappa^d}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{s})}-\mu)} - 1}, \quad E_{\infty,\kappa}^{(\mathbf{0})} := d\kappa/2. \tag{3.1}$$

We start by defining the G-C *open-trap* rescaled average number of particles as follows:

Definition 3.1. *Provided that the limit exists in $\mathbb{R}^+$, $\forall d \in \{1,2,3\}$, $\forall \beta > 0$ and $\forall \mu \in (-\infty, 0)$ we define the G-C open-trap rescaled average number of particles as follows:*

$$\nu_{\infty,0}(\beta,\mu) := \lim_{\kappa\downarrow 0} \nu_{\infty,\kappa}(\beta,\mu).$$

For any real $\theta > 0$ introduce:

$$g_\theta(\xi) := \sum_{n=1}^{\infty} \frac{\xi^n}{n^\theta} \qquad \forall \xi \in \mathbb{C} \;\; s.t. \begin{cases} 0 \leq |\xi| < 1, & \text{if } 0 < \theta < 1 \\ 0 \leq |\xi| \leq 1,\, \xi \neq 1, & \text{if } \theta = 1, \\ 0 \leq |\xi| \leq 1, & \text{if } \theta > 1 \end{cases}, \tag{3.2}$$

which is the analytic continuation of the polylogarithm initially defined on the open ball $\mathcal{B}(0,1)$.

From Definition 3.1, we prove:

Proposition 3.2. $\forall d \in \{1,2,3\}$, $\forall \beta > 0$ *and* $\forall \mu < 0$, *one has:*

$$\nu_{\infty,0}(\beta,\mu) = \frac{1}{\Gamma(d)} \int_0^\infty \mathrm{d}\epsilon\, \frac{\epsilon^{d-1}}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1} = \frac{g_d(\mathrm{e}^{\beta\mu})}{\beta^d}. \tag{3.3}$$

Proof. Let $d \in \{1,2,3\}$, $\beta > 0$ and $\mu < 0$ kept fixed. From (3.1), one has:

$$\lim_{\kappa\downarrow 0} \sum_{s_1,\dots,s_d\in\mathbb{N}^d} \frac{\kappa^d}{e^{\beta(\kappa(s_1+\cdots+s_d)+\kappa d/2-\mu)} - 1} = \int_0^\infty \mathrm{d}\epsilon_1 \cdots \int_0^\infty \mathrm{d}\epsilon_d \frac{1}{e^{\beta(\epsilon_1+\cdots+\epsilon_d-\mu)} - 1}, \tag{3.4}$$

where the integrals over $\epsilon_j,\ j = 1,\dots,d$ are obtained by taking the limit $\kappa \downarrow 0$ of the Darboux-Riemann sum in the l.h.s of (3.4). Therefore $\lim_{\kappa\downarrow 0} \nu_{\infty,\kappa}(\beta,\mu)$ exists, and by simple calculations:

$$\nu_{\infty,0}(\beta,\mu) = \frac{1}{\Gamma(d)} \int_0^\infty \mathrm{d}\epsilon\, \frac{\epsilon^{d-1}}{e^{\beta(\epsilon-\mu)} - 1}. \tag{3.5}$$

Afterwards, by expanding $1/(e^{\beta(\epsilon-\mu)}-1)$ in power series and by using the Fubini's theorem:

$$\nu_{\infty,0}(\beta,\mu)=\sum_{l=1}^{\infty}\mathrm{e}^{l\beta\mu}\frac{1}{\Gamma(d)}\int_0^\infty \mathrm{d}\epsilon\,\epsilon^{d-1}\mathrm{e}^{-l\beta\epsilon}=\sum_{l=1}^{\infty}\frac{\mathrm{e}^{l\beta\mu}}{(l\beta)^d}=\frac{g_d(\mathrm{e}^{\beta\mu})}{\beta^d}. \qquad \square$$

By analogy with the perfect Bose gas confined in boxes, let us introduce:

Definition 3.3. *$\forall d\in\{1,2,3\}$ and $\forall\beta>0$, we define the critical open-trap rescaled number of particles for the confined d-dimensional harmonically trapped Bose gas in the G-C ensemble as:*

$$\nu_c(\beta):=\lim_{\mu\uparrow 0}\nu_{\infty,0}(\beta,\mu)=\sup_{\mu<0}\nu_{\infty,0}(\beta,\mu)\in\overline{\mathbb{R}^+}. \tag{3.6}$$

From Proposition 3.2 along with (3.2), one straightforwardly gets:

Proposition 3.4. *$\forall d\in\{1,2,3\}$ and $\forall\beta>0$:*

$$\nu_c(\beta)=\begin{cases}+\infty, & \text{if } d=1; \quad (3.7\text{a})\\ \dfrac{g_d(1)}{\beta^d}<\infty, & \text{if } d=2,3. \quad (3.7\text{b})\end{cases}$$

Subsequently to Proposition 3.4, one has from Proposition 3.2:

Lemma 3.5. *$\forall d\in\{1,2,3\}$, $\forall\beta>0$ and $\forall 0<\nu<\nu_c(\beta)$ defined in (3.7), there exists a unique solution $\overline{\mu}_{\infty,0}=\overline{\mu}_{\infty,0}(\beta,\nu)\in(-\infty,0)$ of the equation $\nu=\nu_{\infty,0}(\beta,\mu)$.*

Lemma 3.5 follows by the same arguments than the ones used to prove Lemma 2.8.

Remark 3.6. In the case of $d=1$, $\forall\beta>0$ and $\forall\nu>0$ the $\overline{\mu}_{\infty,0}$ can be written down explicitly:

$$\nu_{\infty,0}(\beta,\overline{\mu}_{\infty,0})=\frac{g_1(e^{\beta\overline{\mu}_{\infty,0}})}{\beta}=\nu \quad\Longleftrightarrow\quad \overline{\mu}_{\infty,0}=\frac{1}{\beta}\ln(1-\mathrm{e}^{-\beta\nu}),$$

since g_1^{-1}, the inverse function of g_1, reads as: $g_1^{-1}(x)=1-\mathrm{e}^{-x}$ on $\mathbb{R}^+$.

We end this paragraph by giving some definitions to prepare the next section.

Definition 3.7. *$\forall d\in\{1,2,3\}$, $\forall\kappa>0$, $\forall\beta>0$, $\forall\nu>0$ and $\forall\mathbf{s}\in\mathbb{N}^d$, we define the G-C rescaled average number of particles in the $\mathbf{s}$-state as:*

$$\nu_{\infty,\kappa}(\beta,\nu;\mathbf{s})=\nu_{\infty,\kappa}(\beta,\overline{\mu}_{\infty,\kappa};\mathbf{s}):=\frac{\kappa^d}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{s})}-\overline{\mu}_{\infty,\kappa})}-1}, \tag{3.8}$$

where $\overline{\mu}_{\infty,\kappa}=\overline{\mu}_{\infty,\kappa}(\beta,\nu)$ satisfies (2.21).

From Definition 3.7, we introduce similarly to Definition 3.1:

Definition 3.8. *Provided that the limit exists in $\mathbb{R}^+$, $\forall d\in\{1,2,3\}$, $\forall\beta>0$, $\forall\nu>0$ and $\forall\mathbf{s}\in\mathbb{N}^d$, we define the G-C open-trap rescaled average number of particles in the $\mathbf{s}$-state as:*

$$\nu_{\infty,0}(\beta,\nu;\mathbf{s}):=\lim_{\kappa\downarrow 0}\nu_{\infty,\kappa}(\beta,\nu;\mathbf{s})=\lim_{\kappa\downarrow 0}\nu_{\infty,\kappa}(\beta,\overline{\mu}_{\infty,\kappa};\mathbf{s}). \tag{3.9}$$

3.2 The open-trap Bose-Einstein condensation.

We start by giving a definition for the so-called *open-trap BEC* which is analogous to the BEC concept (within the Einstein's formulation) for Bose gas confined in boxes, see e.g. [49]:

Definition 3.9. *Consider a confined d-dimensional harmonically trapped Bose gas, $d\in\{1,2,3\}$ in the G-C ensemble. We say that there exists an open-trap BEC provided that $\forall\beta>0$ the critical open-trap rescaled number of particles is finite: $\nu_c(\beta)<\infty$, and moreover $\forall\nu>\nu_c(\beta)$ the open-trap G-C rescaled average number of particles in the ground state is positive: $\nu_{\infty,0}(\beta,\nu;\mathbf{0})>0$.*

The main result of this paragraph states that the confined d-dimensional harmonically trapped Bose gas manifests an open-trap BEC if $d > 1$:

Theorem 3.10. *Consider a confined d-dimensional harmonically trapped Bose gas, $d \in \{1,2,3\}$ in the G-C ensemble. Then $\forall \beta > 0$ and $\forall \nu > 0$:*
(i). *For $d = 1$, there is no open-trap BEC. Moreover we have $\forall s \in \mathbb{N}$, $\nu_{\infty,0}(\beta,\nu;s) = 0$.*
(ii). *For $d = 2,3$, there exists an open-trap BEC, and the open-trap rescaled number of particles on the ground state satisfies:*

$$\nu_{\infty,0}(\beta,\nu;\mathbf{0}) = \begin{cases} 0, & \text{when } \nu < \nu_c(\beta); & (3.10a) \\ \nu - \nu_c(\beta), & \text{when } \nu \geq \nu_c(\beta). & (3.10b) \end{cases}$$

Here $\nu_c(\beta)$ is defined in (3.6) *and satisfies* (3.7). *Moreover $\forall \mathbf{s} \in (\mathbb{N}^*)^d$, $\nu_{\infty,0}(\beta,\nu;\mathbf{s}) = 0$.*
(iii). *The $\overline{\mu}_{\infty,\kappa} = \overline{\mu}_{\infty,\kappa}(\beta,\nu)$ satisfying* (2.21) *admits the asymptotics in the limit $\kappa \downarrow 0$:*

$$\overline{\mu}_{\infty,\kappa} = \begin{cases} E_{\infty,\kappa}^{(\mathbf{0})} + \overline{\mu}_{\infty,0} + o(1), & \text{when } \nu < \nu_c(\beta) \text{ if } d = 1,2,3; & (3.11a) \\ E_{\infty,\kappa}^{(\mathbf{0})} + o(1), & \text{when } \nu = \nu_c(\beta) \text{ if } d = 2,3; & (3.11b) \\ E_{\infty,\kappa}^{(\mathbf{0})} - \dfrac{\kappa^d}{\beta(\nu - \nu_c(\beta))} + o(\kappa^d), & \text{when } \nu > \nu_c(\beta) \text{ if } d = 2,3. & (3.11c) \end{cases}$$

Here $\overline{\mu}_{\infty,0} = \overline{\mu}_{\infty,0}(\beta,\nu)$ is defined in Lemma 3.5.

Proof. Let $\beta, \nu > 0$ kept fixed. We start with the case of $d = 1$. From Definition 3.9 along with (3.7a), then there is no open-trap BEC. Besides, $\overline{\mu}_{\infty,0} = \overline{\mu}_{\infty,0}(\beta,\nu)$ satisfies $\nu = \beta^{-1} g_1(\mathrm{e}^{\beta\overline{\mu}_{\infty,0}})$ from Lemma 3.5. As a result, $\overline{\mu}_{\infty,\kappa} = \overline{\mu}_{\infty,\kappa}(\beta,\nu)$ from Lemma 2.8 has to obey the asymptotic in (3.11a) since by a similar calculus than the one performed in the proof of Proposition 3.2:

$$\lim_{\kappa \downarrow 0} \sum_{s \in \mathbb{N}} \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; s) = \lim_{\kappa \downarrow 0} \sum_{s \in \mathbb{N}} \frac{\kappa}{\mathrm{e}^{\beta(\kappa s - \overline{\mu}_{\infty,0} + o(1))} - 1} = \frac{g_1(\mathrm{e}^{\beta\overline{\mu}_{\infty,0}})}{\beta} = \nu.$$

Moreover, the definition in (3.8) and the asymptotic in (3.11a) lead to $\nu_{\infty,0}(\beta,\nu;s) = 0$ $\forall s \in \mathbb{N}$. Let us turn to the cases of $d = 2,3$. When $\nu < \nu_c(\beta)$ and $\nu = \nu_c(\beta)$, the $\overline{\mu}_{\infty,\kappa} = \overline{\mu}_{\infty,\kappa}(\beta,\nu)$ has to obey the asymptotic in (3.11a) and (3.11b) respectively by the same arguments than the ones used for the case $d = 1$. Hence (3.8) and (3.11a)-(3.11b) together lead to $\nu_{\infty,0}(\beta,\nu;\mathbf{s}) = 0$ $\forall \mathbf{s} \in \mathbb{N}^d$. When $\nu > \nu_c(\beta)$, one has to investigate the open-trap G-C rescaled average number of particles in the ground state to conclude the existence of an open-trap BEC, see Definition 3.9. For such ν, assume that the $\overline{\mu}_{\infty,\kappa} = \overline{\mu}_{\infty,\kappa}(\beta,\nu)$ has the following asymptotic in the limit $\kappa \downarrow 0$:

$$\overline{\mu}_{\infty,\kappa} = E_{\infty,\kappa}^{(\mathbf{0})} - C\kappa^d + o(\kappa^d), \quad C > 0. \tag{3.12}$$

Set $\widetilde{\mu}_{\infty,\kappa} = \overline{\mu}_{\infty,\kappa} - E_{\infty,\kappa}^{(\mathbf{0})}$. We first decompose $\nu_{\infty,\kappa}$ into two contributions:

$$\forall \kappa > 0, \quad \nu = \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}) = \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; \mathbf{0}) + \sum_{\mathbf{s} \in (\mathbb{N}^*)^d} \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; \mathbf{s}). \tag{3.13}$$

By mimicking the arguments leading to (3.4)-(3.5), the second term of the r.h.s. of (3.13) satisfies:

$$\lim_{\kappa \downarrow 0} \sum_{\mathbf{s} \in \mathbb{N}^d \,:\, \sum_{j=1}^d s_j > 0} \frac{\kappa^d}{\mathrm{e}^{\beta(\kappa \sum_{j=1}^d s_j - \widetilde{\mu}_{\infty,\kappa})} - 1} = \frac{1}{\Gamma(d)} \int_0^\infty \mathrm{d}\epsilon \, \frac{\epsilon^{d-1}}{e^{\beta\epsilon} - 1} = \frac{g_d(1)}{\beta^d} = \nu_c(\beta). \tag{3.14}$$

This means that the first term in the r.h.s. of (3.13) satisfies, see the definition in (3.9):

$$\nu_{\infty,0}(\beta,\nu;\mathbf{0}) := \lim_{\kappa \downarrow 0} \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; \mathbf{0}) = \nu - \nu_c(\beta) > 0. \tag{3.15}$$

Finally, from (3.12) one has for $\mathbf{s} = \mathbf{0}$:

$$\nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; \mathbf{0}) = \frac{\kappa^d}{\mathrm{e}^{\beta(C\kappa^d + o(\kappa^d))} - 1} = \frac{1}{\beta C} + o(1) \quad \text{when } \kappa \downarrow 0. \tag{3.16}$$

By gathering (3.16), (3.15) and (3.12) together, we get the asymptotic announced in (3.11c). We stress the point that the asymptotic form that we chose in (3.12) can not be otherwise since it is determined by the limits $\kappa \downarrow 0$ in (3.14) and (3.15). Finally from the foregoing, (3.10b) is proved. This together with Definition 3.9 lead to the existence of an open-trap BEC for $d = 2, 3$. □

Remark 3.11. We emphasize that the results of Theorem 3.10 are based on the *'Einstein formulation'* of the condensation. However, we mention that there exists another kind of condensation named *generalized Bose-Einstein condensation (g-BEC)*. The g-BEC concept was initially introduced in [49] for perfect Bose gas in 'Dirichlet boxes'; for a review of definitions and classifications of g-BEC, see [43]. To our best knowledge, there is no 'rigorous' formulation of the g-BEC in Physics literature for the system we consider here. Based on our concept of *open-trap limit* and by analogy with the van den Berg-Lewis-Pulé formulation of the g-BEC, we can suggest the following definition: *we say that the grand-canonical harmonically trapped Bose gas manifests the g-BEC if:*

$$\lim_{\epsilon \downarrow 0} \lim_{\kappa \downarrow 0} \sum_{\mathbf{s} \in \mathbb{N}^d \,:\, 0 < \sum_{j=1}^d s_j \leq \epsilon/\kappa} \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; \mathbf{s}) > 0. \tag{3.17}$$

Getting back to the proof of Theorem 3.10, the open-trap limit of the sum in the r.h.s. of (3.13) has to be performed slightly differently from (3.14). Indeed, when dealing with the case of $\nu > \nu_c(\beta)$ if $d = 2, 3$, the method consists in decomposing the sum into two contributions:

$$\forall 0 < \epsilon \leq 1, \qquad \sum_{\mathbf{s} \in \mathbb{N}^d \,:\, 0 < \sum_{j=1}^d s_j \leq \epsilon/\kappa} \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; \mathbf{s}) + \sum_{\mathbf{s} \in \mathbb{N}^d \,:\, \sum_{j=1}^d s_j > \epsilon/\kappa} \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; \mathbf{s}),$$

and then in investigating successively the limits $\kappa \downarrow 0$ and $\epsilon \downarrow 0$. One can easily prove that:

$$\lim_{\epsilon \downarrow 0} \lim_{\kappa \downarrow 0} \sum_{\mathbf{s} \in \mathbb{N}^d \,:\, 0 < \sum_{j=1}^d s_j \leq \epsilon/\kappa} \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; \mathbf{s}) = 0. \tag{3.18}$$

It has been shown in [44] that for a specific model of cigar shape harmonic Bose gas, the counterpart of the quantity in (3.18) can be non-zero for $\nu > \nu_c(\beta)$. Some references in Physics literature deal with an analogous phenomenon for the highly elongated trapped Bose gas, see e.g. [31] and also [38] for a discussion on the relationship between the concepts of g-BEC and quasi-condensation.

3.3 The open-trap reduced density matrix and ODLRO.

The reduced density matrix concept was initially introduced by Penrose and Onsager in [39] for investigations on BEC. Involved in their *general criterion* of BEC, it allows to treat the interacting Bose gases (whereas the Einstein criterion was originally formulated for the free Bose gas). There is a huge amount of Physics literature dealing with this criterion for the gas in a box, but to our best knowledge there is no formulation of such a criterion of BEC for the system we consider here. In this section, we rigorously formulate the analogous of Penrose-Onsager criterion for BEC in the open-trap limit and show that it is equivalent to the open-trap BEC criterion in Definition 3.9.

We start by defining the so-called reduced density matrix as well as the local density of particles. Remind that $\forall d \in \{1,2,3\}$, $\forall \kappa > 0$, $\forall \beta > 0$ and $\forall \mu < E_{\infty,\kappa}^{(\mathbf{0})}$, the operator $\mathfrak{f}_{BE}(\beta, \mu; H_{\infty,\kappa})$ is defined in (2.7), and Lemma 2.13 deals with some properties of its integral kernel.

Definition 3.12. $\forall d \in \{1,2,3\}$, $\forall \kappa > 0$, $\forall \beta > 0$ *and* $\forall \nu > 0$, *we define the reduced density matrix as the integral kernel of the operator* $\mathfrak{f}_{BE}(\beta, \overline{\mu}_{\infty,\kappa}; H_{\infty,\kappa})$*:*

$$\forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2d}, \quad \rho_{\infty,\kappa}(\mathbf{x}, \mathbf{y}; \beta, \nu) := (\mathfrak{f}_{BE}(\beta, \overline{\mu}_{\infty,\kappa}; H_{\infty,\kappa}))(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{s} \in \mathbb{N}^d} \frac{\psi_{\infty,\kappa}^{(\mathbf{s})}(\mathbf{x}) \overline{\psi_{\infty,\kappa}^{(\mathbf{s})}(\mathbf{y})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{s})} - \overline{\mu}_{\infty,\kappa})} - 1}, \tag{3.19}$$

where $\overline{\mu}_{\infty,\kappa} = \overline{\mu}_{\infty,\kappa}(\beta,\nu)$ *is defined in Lemma 2.8. The diagonal part of* (3.19) *(obtained by setting* $\mathbf{y} = \mathbf{x}$*) is usually named the local density of particles at the point* $\mathbf{x} \in \mathbb{R}^d$*.*

Remark 3.13. (i). The sum in the r.h.s. of (3.19) comes from the spectral theorem. Without involving directly the eigenfunctions of $H_{\infty,\kappa}$, one has also from Lemma 2.13 (ii) the representation:

$$\forall (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d}, \quad \rho_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu) = \sum_{l=1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta). \tag{3.20}$$

(ii). The reduced density matrix in (3.19) has the dimension of a density of particles since by (2.6) the product of two wave functions has the dimension of the inverse of a volume. Then the local density of particles at $\mathbf{x} \in \mathbb{R}^d$ is interpreted as the number of particles at $\mathbf{x} \in \mathbb{R}^d$ per unit volume.
(iii). From (3.19), (2.6) and (2.4), the reduced density matrix can be rewritten as:

$$\forall (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d}, \quad \rho_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu) = \kappa^{d/2} \sum_{\mathbf{s}\in\mathbb{N}^d} \nu_{\infty,\kappa}(\beta,\overline{\mu}_{\infty,\kappa};\mathbf{s})\psi_{\infty,1}^{(\mathbf{s})}(\mathbf{x}\sqrt{\kappa})\overline{\psi_{\infty,1}^{(\mathbf{s})}(\mathbf{y}\sqrt{\kappa})}. \tag{3.21}$$

The following result shows that the factor $\kappa^{\frac{d}{2}}$ appearing in (3.21) is too large to ensure the convergence of the reduced density matrix in the open-trap limit $\kappa \downarrow 0$ when $\nu > \nu_c(\beta)$:

Proposition 3.14. *Consider a confined d-dimensional harmonically trapped Bose gas,* $d \in \{1,2,3\}$ *in the G-C ensemble. Then* $\forall \beta > 0$, $\forall \nu > 0$ *and* $\forall (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d}$*, the limit:*

$$\rho_{\infty,0}(\mathbf{x},\mathbf{y};\beta,\nu) := \lim_{\kappa\downarrow 0} \rho_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu),$$

exists in $\overline{\mathbb{R}^+}$*. We name it the open-trap reduced density matrix. Moreover, one has on* $\mathbb{R}^{2d}$*:*

$$\rho_{\infty,0}(\mathbf{x},\mathbf{y};\beta,\nu) = \begin{cases} \displaystyle\sum_{l=1}^{\infty} \frac{\mathrm{e}^{l\beta\overline{\mu}_{\infty,0}}}{(2\pi l\beta)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2l\beta}}, & \text{when } \nu < \nu_c(\beta) \text{ if } d = 1,2,3; \quad (3.22\text{a}) \\ +\infty, & \text{when } \nu = \nu_c(\beta) \text{ if } d = 2; \quad (3.22\text{b}) \\ +\infty, & \text{when } \nu > \nu_c(\beta) \text{ if } d = 2,3. \quad (3.22\text{c}) \end{cases}$$

Here $\nu_c(\beta)$ *is defined in* (3.6) *and satisfies* (3.7)*, and* $\overline{\mu}_{\infty,0} < 0$ *is defined in Lemma 3.5.*

Proof. Let $\beta, \nu > 0$ be fixed. We start with the case of $\nu < \nu_c(\beta)$ if $d = 1,2,3$. Consider the representation in (3.20). From (3.11a), there exists a $\kappa_0 > 0$ s.t. $\forall 0 < \kappa \leq \kappa_0$, $\overline{\mu}_{\infty,\kappa} \leq \overline{\mu}_{\infty,0}/2 < 0$. This along with the upper bound in the second inequality of (2.31) lead to:

$$\forall 0 < \kappa \leq \kappa_0, \quad \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta) \leq \mathrm{e}^{l\beta\frac{\overline{\mu}_{\infty,0}}{2}} G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};l\beta) \leq \mathrm{e}^{l\beta\frac{\overline{\mu}_{\infty,0}}{2}} (2\pi l\beta)^{-\frac{d}{2}},$$

uniformly in $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d}$. It follows by standard arguments:

$$\lim_{\kappa\downarrow 0} \sum_{l=1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta) = \sum_{l=1}^{\infty} \lim_{\kappa\downarrow 0} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta) = \sum_{l=1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,0}} G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};l\beta).$$

We continue with the case of $\nu = \nu_c(\beta)$ if $d = 2$. The strategy is to find a lower bound of the sum in (3.20) whose the limit $\kappa \downarrow 0$ diverges. Let us notice that from (3.11b) $\overline{\mu}_{\infty,\kappa} \geq 0$ for $\kappa > 0$ small enough. Then from (2.25)-(2.26), one has $\forall l \in \mathbb{N}^*$, $\forall (\mathbf{x},\mathbf{y}) \in \mathbb{R}^4$ and for $\kappa > 0$ sufficiently small:

$$\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d=2)}(\mathbf{x},\mathbf{y};l\beta) \geq \frac{\kappa}{2\pi\sinh(\kappa l\beta)} \mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}+\mathbf{y}|^2} \mathrm{e}^{-\frac{1}{4}(\kappa+\frac{2}{\beta})|\mathbf{x}-\mathbf{y}|^2}, \tag{3.23}$$

where we used the upper bounds in (A.3)-(A.4). Then under the conditions of (3.23), one has:

$$\rho_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu) \geq -\frac{1}{2\pi\beta} \mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}+\mathbf{y}|^2} \mathrm{e}^{-\frac{1}{4}(\kappa+\frac{2}{\beta})|\mathbf{x}-\mathbf{y}|^2} \ln(\tanh(\frac{\beta\kappa}{2})), \tag{3.24}$$

and the above lower bound diverges in the limit $\kappa \downarrow 0$. To get (3.24) we used an integral comparison to minorize the sum, and then performed explicitly the integral. Let us turn to the case of $\nu > \nu_c(\beta)$ if $d = 2, 3$. If $d = 2$, it is enough to use a similar reasoning than the one leading to (3.24). If $d = 3$, from (2.25)-(2.26), one has $\forall l \in \mathbb{N}^*$, $\forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6$ and for $\kappa > 0$ sufficiently small:

$$\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d=3)}(\mathbf{x}, \mathbf{y}; l\beta) \geq \frac{\kappa^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} \mathrm{e}^{l\beta(\overline{\mu}_{\infty,\kappa} - E_{\infty,\kappa}^{(\mathbf{0})})} \mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}+\mathbf{y}|^2} \mathrm{e}^{-\frac{1}{4}(\kappa + \frac{2}{\beta})|\mathbf{x}-\mathbf{y}|^2},$$

where we used the upper bounds in (A.2)-(A.4). Since $\overline{\mu}_{\infty,\kappa} - E_{\infty,\kappa}^{(\mathbf{0})} < 0$, under the same conditions:

$$\rho_{\infty,\kappa}(\mathbf{x}, \mathbf{y}; \beta, \nu) \geq \frac{1}{(2\pi)^{\frac{3}{2}}\beta} \mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}+\mathbf{y}|^2} \mathrm{e}^{-\frac{1}{4}(\kappa + \frac{2}{\beta})|\mathbf{x}-\mathbf{y}|^2} \kappa^{\frac{3}{2}} \frac{\mathrm{e}^{-\beta(E_{\infty,\kappa}^{(\mathbf{0})} - \overline{\mu}_{\infty,\kappa})}}{E_{\infty,\kappa}^{(\mathbf{0})} - \overline{\mu}_{\infty,\kappa}}, \tag{3.25}$$

where we used again an integral comparison, and then performed explicitly the integral. From the asymptotic in (3.11c), the lower bound in (3.25) diverges in the limit $\kappa \downarrow 0$. □

Remark 3.15. When $\nu \geq \nu_c(\beta)$ if $d = 2$, the divergence comes from logarithmic behavior of the lower bound in (3.24). Hence the asymptotic of the chemical potential can be disregarded. At the contrary, when $\nu > \nu_c(\beta)$ if $d = 3$ the divergence only arises from the asymptotic of the chemical potential in (3.11c). Since the asymptotic form plays an important role when $d = 3$, the case of $\nu = \nu_c(\beta)$ can not be treated without knowing a more refined asymptotic than the one in (3.11b).

In view of (3.22c), a κ-rescaling of the reduced density matrix is needed to make it finite in the open-trap limit when $\nu > \nu_c(\beta)$ while taking into account its local nature. To that purpose:

Definition 3.16. *$\forall d \in \{1, 2, 3\}$, $\forall \kappa > 0$, $\forall \beta > 0$, $\forall \nu > 0$ and $\forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2d}$ we define the rescaled reduced density matrix from* (3.19) *by setting:*

$$r_{\infty,\kappa}(\mathbf{x}, \mathbf{y}; \beta, \nu) := \kappa^{\frac{d}{2}} \rho_{\infty,\kappa}(\mathbf{x}, \mathbf{y}; \beta, \nu) \tag{3.26}$$

$$= \sum_{\mathbf{s} \in \mathbb{N}^d} \nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; \mathbf{s}) \psi_{\infty,1}^{(\mathbf{s})}(\mathbf{x}\sqrt{\kappa}) \overline{\psi_{\infty,1}^{(\mathbf{s})}(\mathbf{y}\sqrt{\kappa})}, \tag{3.27}$$

where $\nu_{\infty,\kappa}(\beta, \overline{\mu}_{\infty,\kappa}; \mathbf{s})$ is the G-C rescaled average number of particles in the $\mathbf{s}$-state in (3.9). *Also we define the rescaled local density of particles at $\mathbf{x} \in \mathbb{R}^d$ from* (3.26) *by setting $\mathbf{y} = \mathbf{x}$.*

We continue by defining the open-trap rescaled reduced density matrix and local density:

Definition 3.17. *Provided that the limit exists in $\mathbb{R}^+$, $\forall d \in \{1, 2, 3\}$, $\forall \beta > 0$ and $\forall \nu > 0$ we define the open-trap rescaled reduced density matrix as:*

$$\forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2d}, \quad r_{\infty,0}(\mathbf{x}, \mathbf{y}; \beta, \nu) := \lim_{\kappa \downarrow 0} r_{\infty,\kappa}(\mathbf{x}, \mathbf{y}; \beta, \nu). \tag{3.28}$$

The open-trap rescaled local density of particles at $\mathbf{x} \in \mathbb{R}^d$ is defined from (3.28) *by setting $\mathbf{y} = \mathbf{x}$.*

From Definition 3.17, the open-trap rescaled reduced density matrix satisfies:

Proposition 3.18. *Consider a confined d-dimensional harmonically trapped Bose gas, $d \in \{1, 2, 3\}$ in the G-C ensemble. Then $\forall \beta > 0$ and $\forall \nu > 0$, one has uniformly in $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2d}$:*

$$r_{\infty,0}(\mathbf{x}, \mathbf{y}; \beta, \nu) = \frac{\nu_{\infty,0}(\beta, \nu; \mathbf{0})}{\pi^{\frac{d}{2}}} = \begin{cases} 0, & \text{when } \nu < \nu_c(\beta) \text{ if } d = 1, 2, 3; \quad (3.29\text{a}) \\ \dfrac{\nu - \nu_c(\beta)}{\pi^{\frac{d}{2}}}, & \text{when } \nu \geq \nu_c(\beta) \text{ if } d = 2, 3. \quad (3.29\text{b}) \end{cases}$$

Here $\nu_c(\beta)$ is defined in (3.6) *and satisfies* (3.7).

Proof. Let $\beta,\nu>0$ be fixed. We start with the case of $\nu<\nu_c(\beta)$ if $d=1,2,3$. From (3.26) with (3.20), then (3.22a) leads to $\lim_{\kappa\downarrow 0} r_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu)=0$ uniformly on $\mathbb{R}^{2d}$. (3.29a) is proved. Let us turn to the case of $\nu\geq\nu_c(\beta)$ if $d=2,3$. The key idea consists in decomposing the quantity defined in (3.26) $\forall 0<\kappa<1$ into two contributions:

$$r_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu)=\kappa^{d/2}\sum_{l=1}^{N_{\kappa,\varepsilon}}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta)+\kappa^{d/2}\sum_{l=N_{\kappa,\varepsilon}+1}^{\infty}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta),\quad(3.30)$$

where $N_{\kappa,\varepsilon}:=\lfloor 1/\kappa^{\varepsilon}\rfloor$ with $0<\varepsilon<d$ for the moment (a limitation will appear when $\nu=\nu_c(\beta)$). Here $\lfloor\cdot\rfloor$ denotes the floor function. Below we prove that the contribution in (3.29b) when $\nu>\nu_c(\beta)$ only arises from the second quantity in the r.h.s. of (3.30). Let us investigate the first term in the r.h.s. of (3.30). From (2.30) followed by the lower bound in (A.10), one has $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2d}$:

$$\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d=2)}(\mathbf{x},\mathbf{y};l\beta)\leq\frac{\kappa}{\pi}\mathrm{e}^{l\beta(\overline{\mu}_{\infty,\kappa}-E_{\infty,\kappa}^{(\mathbf{0})})}+\frac{\mathrm{e}^{l\beta(\overline{\mu}_{\infty,\kappa}-E_{\infty,\kappa}^{(\mathbf{0})})}}{2\pi l\beta},\quad(3.31)$$

$$\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d=3)}(\mathbf{x},\mathbf{y};l\beta)\leq\frac{\kappa^{\frac{3}{2}}}{\pi^{\frac{3}{2}}}\mathrm{e}^{l\beta(\overline{\mu}_{\infty,\kappa}-E_{\infty,\kappa}^{(\mathbf{0})})}\times\begin{cases}1+\left(\frac{1}{l\beta\kappa}+\frac{1}{(2l\beta\kappa)^2}\right), & (3.32\text{a})\\ 1+\left(\frac{3}{\sqrt{2l\beta\kappa}}+\frac{3}{2l\beta\kappa}+\frac{1}{(2l\beta\kappa)^{\frac{3}{2}}}\right). & (3.32\text{b})\end{cases}$$

In (3.32a) and (3.32b) we used that $\forall\alpha>0$ $(1+1/\alpha)^{\frac{3}{2}}\leq(1+1/\alpha)^2$ and $(1+1/\alpha)^{\frac{3}{2}}\leq(1+1/\sqrt{\alpha})^3$ respectively. Since $\overline{\mu}_{\infty,\kappa}-E_{\infty,\kappa}^{(\mathbf{0})}<0$, one has for $\kappa<1$ sufficiently small the upper bounds:

$$\kappa^{\frac{d}{2}-m}\sum_{l=1}^{N_{\kappa,\varepsilon}}\frac{\mathrm{e}^{-l\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}}{l^m}\leq\mathrm{e}^{-\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}\times\begin{cases}2\kappa^{\frac{d}{2}-m-\varepsilon(1-m)}, & \text{if } m\in\{0,\frac{1}{2}\}\\ \kappa^{\frac{d}{2}-1}\ln(\frac{\mathrm{e}}{\kappa^{\varepsilon}}), & \text{if } m=1\\ 3\kappa^{\frac{d}{2}-m}, & \text{if } m\in\{\frac{3}{2},2\}\end{cases}.\quad(3.33)$$

By virtue of the squeeze theorem, when $\nu\geq\nu_c(\beta)$ if $d=2,3$ one has uniformly in $(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2d}$:

$$\forall 0<\varepsilon<d,\quad\lim_{\kappa\downarrow 0}\kappa^{\frac{d}{2}}\sum_{l=1}^{N_{\kappa,\varepsilon}}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta)=0.\quad(3.34)$$

Subsequently let us turn to the second quantity in the r.h.s. of (3.30). Since $\overline{\mu}_{\infty,\kappa}-E_{\infty,\kappa}^{(\mathbf{0})}<0$, then one has $\forall d\in\{2,3\}$, $\forall 0<\varepsilon<d$, $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2d}$ and for $\kappa<1$ sufficiently small:

$$\kappa^{\frac{d}{2}}\sum_{l=N_{\kappa,\varepsilon}+1}^{\infty}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta)\geq\frac{\kappa^d}{\pi^{\frac{d}{2}}}\mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}+\mathbf{y}|^2}\mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}-\mathbf{y}|^2\coth(\frac{\beta}{2}\frac{1}{\kappa^{\varepsilon-1}})}\frac{\mathrm{e}^{-N_{\kappa,\varepsilon}\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}-1}.\quad(3.35)$$

Here we used the upper bounds in (A.2)-(A.3) together with a formula to express the remainder of the geometric series. We now distinguish the case of $\nu>\nu_c(\beta)$ from the one of $\nu=\nu_c(\beta)$. When $\nu>\nu_c(\beta)$, we get from the asymptotic in (3.11c) the existence of a $K_\beta>0$ s.t.

$$\forall 0<\kappa\leq K_\beta,\quad 3\kappa^d\geq 2\beta(\nu-\nu_c(\beta))\{E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa}\}\geq\kappa^d>0.\quad(3.36)$$

By the upper bound in (3.36), one has $\forall 0<\varepsilon<d$, $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2d}$ and $\forall 0<\kappa<\min\{1,K_\beta\}$:

$$\kappa^{\frac{d}{2}}\sum_{l=N_{\kappa,\varepsilon}+1}^{\infty}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta)\geq\frac{\nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})}{\pi^{\frac{d}{2}}}\mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}+\mathbf{y}|^2}\mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}-\mathbf{y}|^2\coth(\frac{\beta}{2}\frac{1}{\kappa^{\varepsilon-1}})}\mathrm{e}^{-\frac{3\kappa^{d-\varepsilon}}{2(\nu-\nu_c(\beta))}}.\quad(3.37)$$

Let us find an upper bound for the l.h.s. of (3.37). If $d=2$, from the upper bound in (3.31) then $\forall 0<\varepsilon<2$, $\forall (\mathbf{x},\mathbf{y})\in\mathbb{R}^4$ and for $\kappa<\min\{1,K_\beta\}$ sufficiently small:

$$\kappa \sum_{l=N_{\kappa,\varepsilon}+1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d=2)}(\mathbf{x},\mathbf{y};l\beta) \leq \frac{\kappa^2}{\pi}\sum_{l=1}^{\infty}\mathrm{e}^{-l\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})} + \frac{\kappa}{2\pi\beta}\Gamma_0(\beta N_{\kappa,\varepsilon}(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa}))$$
$$\leq \frac{\nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})}{\pi} + \frac{\kappa}{2\pi\beta}\Gamma_0\Big(\frac{\kappa^2(\kappa^{-\varepsilon}-1)}{2(\nu-\nu_c(\beta))}\Big), \tag{3.38}$$

where $\Gamma_0(\cdot)$ is the incomplete Gamma function [1] (below γ stands for the Euler constant):

$$\forall x>0,\quad \Gamma_0(x) := \int_x^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-t}}{t} = -\gamma - \ln(x) - \sum_{k=1}^{\infty}(-1)^k\frac{x^k}{k(k!)}. \tag{3.39}$$

If $d=3$, from the bound in (3.32a) then $\forall 0<\varepsilon<3$, $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^6$ and for $\kappa<1$ small enough:

$$\kappa^{\frac{3}{2}} \sum_{l=N_{\kappa,\varepsilon}+1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d=3)}(\mathbf{x},\mathbf{y};l\beta)$$
$$\leq \frac{\nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})}{\pi^{\frac{3}{2}}} + \frac{\kappa^2}{\beta\pi^{\frac{3}{2}}}\Gamma_0(\beta N_{\kappa,\varepsilon}(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})) + \frac{1}{(2\beta)^2\pi^{\frac{3}{2}}}\frac{\kappa}{N_{\kappa,\varepsilon}}, \tag{3.40}$$

where we used some integral comparisons. Since the r.h.s. of (3.37) and (3.38)-(3.40) converge $\forall 0<\varepsilon<d$ and uniformly in $(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2d}$ to the same value when $\kappa\downarrow 0$, then $\lim_{\kappa\downarrow 0} r_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu)$ exists by the squeeze theorem and is equal to (3.29b). Note that the result strongly relies on the asymptotic form of the chemical potential in (3.11c). When $\nu=\nu_c(\beta)$, we use a similar method but the upper bounds in (3.38) and (3.40) have to be replaced with some independent of the difference $E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa}$. Indeed, our asymptotic in (3.11b) does not allow us to conclude from the bounds in (3.38) and (3.40) due to the presence of the ln in (3.39). In the case of $d=2$, from (3.35) and (3.31), then one has $\forall 0<\varepsilon<2$, $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^4$ and for $\kappa<1$ sufficiently small:

$$\frac{\nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})}{\pi}\mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}+\mathbf{y}|^2}\mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}-\mathbf{y}|^2\coth(\frac{\beta}{2}\frac{1}{\kappa^{\varepsilon-1}})}\mathrm{e}^{-\frac{\beta}{2}\frac{1}{\kappa^{\varepsilon}}}$$
$$\leq \kappa\sum_{l=N_{\kappa,\varepsilon}+1}^{\infty}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d=2)}(\mathbf{x},\mathbf{y};l\beta) \leq \frac{\nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})}{\pi}\Big\{1+\frac{1}{2\beta}\frac{1}{\kappa N_{\kappa,\varepsilon}}\Big\}. \tag{3.41}$$

To get the upper bound, we minorized the l in the denominator of the second term in the r.h.s. of (3.31) before extending the sum up to $l=1$. In order to apply the squeeze theorem (remind that $\lim_{\kappa\downarrow 0}\nu_{\infty,\kappa}(\beta,\nu_c(\beta);\mathbf{0})=0$, see (3.10b)), we require the limiting condition $d>\varepsilon>1$. From (3.32) and by using similar arguments, the same limitation is required if $d=3$. □

Now define the *open-trap ODLRO* criterion analogous to the one of Penrose-Onsager in [39]:

Definition 3.19. *Consider a confined d-dimensional harmonically trapped Bose gas, $d\in\{1,2,3\}$ in the G-C ensemble. For any $\beta>0$ and $\nu>0$, we say that there exists an open-trap ODLRO if:*

$$r_{\infty,0}(\beta,\nu) := \lim_{|\mathbf{x}-\mathbf{y}|\uparrow\infty} r_{\infty,0}(\mathbf{x},\mathbf{y};\beta,\nu) > 0. \tag{3.42}$$

Subsequently to Definition 3.19 together with Proposition 3.18, we straightforwardly get:

Theorem 3.20. *Consider a confined d-dimensional harmonically trapped Bose gas, $d\in\{1,2,3\}$ in the G-C ensemble. Then $\forall\beta>0$ and $\forall\nu>0$, there is no open-trap-ODLRO if $d=1$, otherwise if $d=2,3$ there is an open-trap OLDRO if and only if $\nu>\nu_c(\beta)$.*

As a consequence of Theorem 3.20, there is equivalence between both Definitions 3.9 and 3.19, i.e. there is an open-trap BEC if and only if there is an open-trap ODLRO.

The following Corollary interprets the last results:

Corollary 3.21. *Consider a confined d-dimensional harmonically trapped Bose gas, $d \in \{1,2,3\}$ in the G-C ensemble. Then $\forall \beta > 0$, $\forall \nu > 0$ and $\forall (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d}$:*
(i). *For the open-trap rescaled reduced density matrix, one has:*

$$r_{\infty,0}(\mathbf{x},\mathbf{y};\beta,\nu) = \lim_{\kappa\downarrow 0} \nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})\psi^{(\mathbf{0})}_{\infty,1}(\mathbf{x}\sqrt{\kappa})\overline{\psi^{(\mathbf{0})}_{\infty,1}(\mathbf{y}\sqrt{\kappa})}. \tag{3.43}$$

(ii). *For the open-trap density matrix without the ground-state, one has:*

$$\lim_{\kappa\downarrow 0} \sum_{\mathbf{s}\in(\mathbb{N}^*)^d} \frac{\psi^{(\mathbf{s})}_{\infty,0}(\mathbf{x})\psi^{(\mathbf{s})}_{\infty,0}(\mathbf{y})}{\mathrm{e}^{\beta(E^{(\mathbf{s})}_{\infty,\kappa}-\overline{\mu}_{\infty,\kappa})}-1} = \begin{cases} \displaystyle\sum_{l=1}^{\infty} \frac{\mathrm{e}^{l\beta\overline{\mu}_{\infty,0}}}{(2\pi l\beta)^{\frac{d}{2}}}\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2l\beta}}, & \textit{when } \nu < \nu_c(\beta) \textit{ if } d \geq 1 \quad (3.44a)\\ +\infty, & \textit{when } \nu \geq \nu_c(\beta) \textit{ if } d = 2 \quad (3.44b)\\ \displaystyle\sum_{l=1}^{\infty} \frac{1}{(2\pi l\beta)^{\frac{3}{2}}}\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2l\beta}}, & \textit{when } \nu > \nu_c(\beta) \textit{ if } d = 3 \quad (3.44c)\end{cases}$$

Here is the meaning: the long range order is due to the condensate on the ground state, and the finite part of the reduced density matrix is due to the *thermal gas* (=*non-condensate gas*).

Proof. Let $\beta, \nu > 0$ be fixed. From (2.6)-(2.4) along with (3.10), one directly gets:

$$\lim_{\kappa\downarrow 0} \nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})\psi^{(\mathbf{0})}_{\infty,1}(\mathbf{x}\sqrt{\kappa})\psi^{(\mathbf{0})}_{\infty,1}(\mathbf{y}\sqrt{\kappa}) = \lim_{\kappa\downarrow 0} \nu_{\infty,\kappa}(\beta,\nu;\mathbf{0}) \frac{\mathrm{e}^{-\frac{\kappa}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)}}{\pi^{\frac{d}{2}}} = \frac{\nu_{\infty,0}(\beta,\nu)}{\pi^{d/2}}.$$

From Proposition 3.18, the r.h.s. is nothing but $r_{\infty,0}(\mathbf{x},\mathbf{y};\beta,\nu)$ which is independent of $\mathbf{x},\mathbf{y} \in \mathbb{R}^{2d}$. We continue with (ii). Let us mention that the reduced density matrix can be rewritten as:

$$\forall (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d}, \quad \rho_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu) = \frac{\psi^{(\mathbf{0})}_{\infty,\kappa}(\mathbf{x})\overline{\psi^{(\mathbf{0})}_{\infty,\kappa}(\mathbf{y})}}{\mathrm{e}^{\beta(E^{(\mathbf{0})}_{\infty,\kappa}-\overline{\mu}_{\infty,\kappa})}-1} + \sum_{\mathbf{s}\in(\mathbb{N}^*)^d} \frac{\psi^{(\mathbf{s})}_{\infty,\kappa}(\mathbf{x})\overline{\psi^{(\mathbf{s})}_{\infty,\kappa}(\mathbf{y})}}{\mathrm{e}^{\beta(E^{(\mathbf{s})}_{\infty,\kappa}-\overline{\mu}_{\infty,\kappa})}-1}. \tag{3.45}$$

When $\nu < \nu_c(\beta)$ if $d = 1,2,3$, the first quantity in the r.h.s. of (3.45) vanishes in the limit $\kappa \downarrow 0$. Then (3.44a) follows from (3.22a). We turn to the cases of $\nu \geq \nu_c(\beta)$ if $d = 2$, $\nu > \nu_c(\beta)$ if $d = 3$. Similarly to (3.30), we decompose $\forall 0 < \kappa < 1$ the reduced density matrix into two contributions:

$$\rho_{\infty,\kappa}(\mathbf{x},\mathbf{y};\beta,\nu) = \sum_{l=1}^{N_{\kappa,\varepsilon}} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};l\beta) + \sum_{l=N_{\kappa,\varepsilon}+1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};l\beta), \tag{3.46}$$

where $N_{\kappa,\varepsilon} = \lfloor 1/\kappa^{\varepsilon} \rfloor$ with $0 < \varepsilon < d$ for the moment. When $\nu \geq \nu_c(\beta)$ if $d = 2$, the strategy consists in finding a lower bound for the l.h.s. of (3.44) involving the first quantity in the r.h.s. of (3.46). $\forall (\mathbf{x},\mathbf{y}) \in \mathbb{R}^4$, $\forall 0 < \varepsilon < 2$ and for $\kappa < 1$ sufficiently small, one has from (3.19)-(3.20):

$$\begin{aligned}\sum_{l=1}^{N_{\kappa,\varepsilon}} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G^{(d=2)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};l\beta) &= \sum_{l=1}^{N_{\kappa,\varepsilon}} \sum_{\mathbf{s}\in\mathbb{N}^2} \mathrm{e}^{l\beta(\overline{\mu}_{\infty,\kappa}-E^{(\mathbf{s})}_{\infty,\kappa})}\psi^{(\mathbf{s})}_{\infty,\kappa}(\mathbf{x})\overline{\psi^{(\mathbf{s})}_{\infty,\kappa}(\mathbf{y})} \\ &\leq \frac{\kappa}{\pi} \frac{1-\mathrm{e}^{-\beta N_{\kappa,\varepsilon}(E^{(\mathbf{0})}_{\infty,\kappa}-\overline{\mu}_{\infty,\kappa})}}{\mathrm{e}^{\beta(E^{(\mathbf{0})}_{\infty,\kappa}-\overline{\mu}_{\infty,\kappa})}-1}\mathrm{e}^{-\frac{\kappa}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)} + \sum_{\mathbf{s}\in(\mathbb{N}^*)^2} \frac{\psi^{(\mathbf{s})}_{\infty,\kappa}(\mathbf{x})\overline{\psi^{(\mathbf{s})}_{\infty,\kappa}(\mathbf{y})}}{\mathrm{e}^{\beta(E^{(\mathbf{s})}_{\infty,\kappa}-\overline{\mu}_{\infty,\kappa})}-1},\end{aligned} \tag{3.47}$$

where we separated the case $\mathbf{s}=\mathbf{0}$ from the sum over $\mathbf{s}$ before extending to ∞ the sum over l in the second term of (3.47). Under the same conditions and since $\overline{\mu}_{\infty,\kappa}\geq 0$ for $\kappa<1$ small enough:

$$\begin{aligned}\sum_{\mathbf{s}\neq\mathbf{0}}\frac{\psi_{\infty,\kappa}^{(\mathbf{s})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{s})}(\mathbf{y})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{s})}-\overline{\mu}_{\infty,\kappa})}-1}&\geq\sum_{l=1}^{N_{\kappa,\varepsilon}}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d=2)}(\mathbf{x},\mathbf{y};l\beta)-\frac{\kappa}{\pi}\frac{1-\mathrm{e}^{-\beta N_{\kappa,\varepsilon}(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}-1}\mathrm{e}^{-\frac{\kappa}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)}\\&\geq\frac{1}{2\beta\pi}\left(\mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}+\mathbf{y}|^2}\mathrm{e}^{-\frac{1}{4}(\kappa+\frac{2}{\beta})|\mathbf{x}-\mathbf{y}|^2}\ln\left(\frac{\kappa^{-\varepsilon}-1}{1+\beta\kappa^{1-\varepsilon}}\right)-2\beta\kappa^{1-\varepsilon}\mathrm{e}^{-\frac{\kappa}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)}\right),\end{aligned}\tag{3.48}$$

and the above lower bound diverges when $\kappa\downarrow 0$ $\forall 0<\varepsilon<1$ and $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^4$. In the l.h.s. of the second inequality, we majorized the second term by the lower and upper bound in (A.11) and (A.10) respectively, then we minorized the sum by an integral (as we did in (3.24) from (3.23)) and used (A.4). Now we treat the case of $\nu>\nu_c(\beta)$ if $d=3$. The strategy consists in showing that the limit $\kappa\downarrow 0$ of the l.h.s. of (3.44) equals the one of the first quantity in the r.h.s. of (3.46) for some suitable ε. $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^6$, $\forall 0<\varepsilon<3$ and $\forall 0<\kappa<\min\{1,K_\beta\}$ (see (3.36)), one has:

$$\begin{aligned}&\sum_{l=N_{\kappa,\varepsilon}+1}^{\infty}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d=3)}(\mathbf{x},\mathbf{y};l\beta)-\frac{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{y})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}-1}\\&\geq\frac{\kappa^{\frac{3}{2}}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}-1}\frac{\mathrm{e}^{-\frac{\kappa}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)}}{\pi^{\frac{3}{2}}}\left(\mathrm{e}^{-\frac{\kappa}{4}[\coth(\frac{\beta}{2}\kappa^{1-\varepsilon})-1]|\mathbf{x}-\mathbf{y}|^2}\mathrm{e}^{-\beta N_{\kappa,\varepsilon}(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}-1\right)\\&\geq-\frac{1}{2}\frac{\nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})}{\kappa^{\frac{3}{2}}}\frac{\mathrm{e}^{-\frac{\kappa}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)}}{\pi^{\frac{3}{2}}}\left(\kappa\frac{\mathrm{e}^{-\beta\kappa^{1-\varepsilon}}}{1-e^{-\beta\kappa^{1-\varepsilon}}}|\mathbf{x}-\mathbf{y}|^2+3\frac{\kappa^{3-\varepsilon}}{(\nu-\nu_c(\beta))}\right),\end{aligned}\tag{3.49}$$

and the above lower bound vanishes when $\kappa\downarrow 0$ $\forall 1<\varepsilon<\frac{3}{2}$ and $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^6$. To get the r.h.s. of the second inequality from the l.h.s. we used the lower bound in (A.11). Here the exponential decay in $\kappa^{1-\varepsilon}$ arises from the difference $\coth(\beta\kappa^{1-\varepsilon}/2)-1$. Under the same conditions than (3.49):

$$\begin{aligned}\sum_{l=N_{\kappa,\varepsilon}+1}^{\infty}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d=3)}(\mathbf{x},\mathbf{y};l\beta)-\frac{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{y})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}-1}\leq\frac{\kappa^{\frac{3}{2}}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}-1}\frac{\mathrm{e}^{-\frac{\kappa}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)}}{\pi^{\frac{3}{2}}}\times\\\times\left(\mathrm{e}^{\frac{\kappa}{4}(1-\tanh(\frac{\kappa}{2}\beta\kappa^{1-\varepsilon}))|\mathbf{x}+\mathbf{y}|^2}-1\right)+\frac{\kappa^{\frac{3}{2}}}{\pi^{\frac{3}{2}}}\sum_{l=N_{\kappa,\varepsilon}+1}^{\infty}\left(\frac{1}{\kappa l\beta}+\frac{1}{(2l\beta\kappa)^2}\right)\mathrm{e}^{-l\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})},\end{aligned}\tag{3.50}$$

where we used (3.32a). Afterwards, by using the upper bound in (A.11) and the argument leading to the estimate in (3.40), under the conditions of (3.49) the r.h.s. of (3.50) is less than:

$$\begin{aligned}\frac{\nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})}{\kappa^{\frac{3}{2}}}\frac{\mathrm{e}^{-\frac{\kappa}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)}}{\pi^{\frac{3}{2}}}\frac{\kappa}{2}\frac{\mathrm{e}^{-\beta\kappa^{1-\varepsilon}}}{1+\mathrm{e}^{-\beta\kappa^{1-\varepsilon}}}|\mathbf{x}+\mathbf{y}|^2\mathrm{e}^{\frac{\kappa}{2}\mathrm{e}^{-\beta\kappa^{1-\varepsilon}}|\mathbf{x}+\mathbf{y}|^2}+\\+\frac{\sqrt{\kappa}}{\beta\pi^{\frac{3}{2}}}\Gamma_0\left(\frac{\kappa^3(\kappa^{-\varepsilon}-1)}{2(\nu-\nu_c(\beta))}\right)+\frac{1}{(2\beta)^2\pi^{\frac{3}{2}}}\frac{\kappa^{\varepsilon-\frac{1}{2}}}{1-\kappa^{\varepsilon}},\end{aligned}\tag{3.51}$$

where the exponential decay in $\kappa^{1-\varepsilon}$ arises from the difference $1-\tanh(\beta\kappa^{1-\varepsilon}/2)-1$. Since (3.51) vanishes when $\kappa\downarrow 0$ $\forall 1<\varepsilon<3$ and $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^6$, we conclude from (3.49) by the squeeze theorem:

$$\forall 1<\varepsilon<\frac{3}{2},\ \forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^6,\quad\lim_{\kappa\downarrow 0}\left(\sum_{l=N_{\kappa,\varepsilon}+1}^{\infty}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d=3)}(\mathbf{x},\mathbf{y};l\beta)-\frac{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{y})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}-1}\right)=0.$$

In view of (3.45), (3.46) and the foregoing, to prove (3.44c) it remains to show:

$$\forall 1<\varepsilon<\frac{3}{2},\ \forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^6,\quad\lim_{\kappa\downarrow 0}\sum_{l=1}^{N_{\kappa,\varepsilon}}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}}G_{\infty,\kappa}^{(d=3)}(\mathbf{x},\mathbf{y};l\beta)=\sum_{l=1}^{\infty}\frac{1}{(2\pi l\beta)^{\frac{3}{2}}}\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2l\beta}}.\tag{3.52}$$

On the one hand, $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^6$, $\forall 0<\varepsilon<3$ and $\forall 0<\kappa<\min\{1,K_\beta\}$ (see (3.36)), one has:

$$\sum_{l=1}^{N_{\kappa,\varepsilon}} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d=3)}(\mathbf{x},\mathbf{y};l\beta) \geq \mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}+\mathbf{y}|^2}\mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}-\mathbf{y}|^2}\mathrm{e}^{-\frac{3}{2}\frac{\kappa^{3-\varepsilon}}{\nu-\nu_c(\beta)}} \sum_{l=1}^{N_{\kappa,\varepsilon}} \frac{\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2\beta l}}}{(2\pi\beta l)^{\frac{3}{2}}}, \tag{3.53}$$

where we used the upper bounds in (A.10) and the ones in (A.3)-(A.4). On the other hand, from (3.32b) together with (3.33), one has under the same conditions than (3.53):

$$\sum_{l=1}^{N_{\kappa,\varepsilon}} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d=3)}(\mathbf{x},\mathbf{y};l\beta) \leq \sum_{l=1}^{N_{\kappa,\varepsilon}} \frac{\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2\beta l}}}{(2\pi l\beta)^{\frac{3}{2}}} + 2\frac{\mathrm{e}^{-\frac{\kappa^{3-\varepsilon}}{2(\nu-\nu_c(\beta))}}}{\pi^{\frac{3}{2}}}\left(\kappa^{\frac{3}{2}-\varepsilon} + \frac{3}{\sqrt{2\beta}}\kappa^{1-\frac{\varepsilon}{2}} + \frac{3}{4\beta}\sqrt{\kappa}\ln(\frac{\mathrm{e}}{\kappa^{\varepsilon}})\right). \tag{3.54}$$

Along with (3.53), the squeeze theorem leads to (3.52). □

Remark 3.22. We showed in (3.48) that $\forall\beta>0$ and $\forall\nu>0$, the open-trap limit of the non-condensate part of the reduced density matrix diverges when $\nu\geq\nu_c(\beta)$ if $d=2$. Actually, by using more refined arguments one can prove that the growth is logarithmic in κ^{-1} when $\kappa\downarrow 0$ uniformly in $(\mathbf{x},\mathbf{y})\in\mathbb{R}^4$. Indeed, one has for $\kappa<1$ small enough the following lower bound:

$$\begin{aligned}
&\sum_{l=1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d=2)}(\mathbf{x},\mathbf{y};l\beta) - \frac{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{y})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}-1} \\
&\geq \frac{\kappa}{\pi}\sum_{l=1}^{\infty}\mathrm{e}^{-l\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}\left(\frac{\mathrm{e}^{-\frac{\kappa}{4}|\mathbf{x}+\mathbf{y}|^2}\mathrm{e}^{-\frac{\kappa}{4}\coth(\frac{\beta}{2}\kappa)|\mathbf{x}-\mathbf{y}|^2}-1}{1-\mathrm{e}^{-2\kappa l\beta}} + \frac{\mathrm{e}^{-2\kappa l\beta}}{1-\mathrm{e}^{-2\kappa l\beta}}\right) \\
&\geq -\frac{\kappa^2}{4\pi}\left\{|\mathbf{x}+\mathbf{y}|^2\sum_{l=1}^{\infty}\mathrm{e}^{-l\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})} + \coth(\frac{\beta}{2}\kappa)|\mathbf{x}-\mathbf{y}|^2\sum_{l=1}^{\infty}\mathrm{e}^{-l\beta E_{\infty,\kappa}^{(\mathbf{0})}}\right\} + \frac{1}{2\pi\beta}\sum_{l=1}^{\infty}\frac{\mathrm{e}^{-3l\beta E_{\infty,\kappa}^{(\mathbf{0})}}}{l} \\
&\geq -\frac{\nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})}{4\pi}|\mathbf{x}+\mathbf{y}|^2 - \frac{\kappa^2}{4\pi}|\mathbf{x}-\mathbf{y}|^2\coth(\frac{\beta}{2}\kappa)\int_1^{\infty}\mathrm{d}t\,\mathrm{e}^{-\beta\kappa t} + \frac{1}{2\pi\beta}\int_1^{\infty}\mathrm{d}t\,\frac{\mathrm{e}^{-3\kappa\beta t}}{t}.
\end{aligned}$$

To get the first inequality, we bounded the term $\mathrm{e}^{-\frac{\kappa}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)}$ coming from $\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{y})}$ by 1. To derive the second inequality, we used that $1-\mathrm{e}^{-\alpha}\leq 1$ $\forall\alpha>0$ for the first term between the parenthesis in the first inequality, followed by the upper bound in (A.10) to minorize its numerator. The two last terms in the third inequality come from an integral comparison. On the other hand:

$$\begin{aligned}
&\sum_{l=1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d=2)}(\mathbf{x},\mathbf{y};l\beta) - \frac{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{y})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}-1} \\
&\qquad\leq \frac{\kappa}{\pi}\mathrm{e}^{-\frac{\kappa}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)}\sum_{l=1}^{\infty}\mathrm{e}^{-l\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}\left(\frac{\mathrm{e}^{\frac{\kappa}{4}[1-\tanh(\frac{\beta}{2}\kappa l)]|\mathbf{x}+\mathbf{y}|^2}-1}{1-\mathrm{e}^{-2\kappa l\beta}} + \frac{\mathrm{e}^{-2\kappa l\beta}}{1-\mathrm{e}^{-2\kappa l\beta}}\right),
\end{aligned}$$

and from the arguments leading to (3.50), the above r.h.s. is less than:

$$\begin{aligned}
&\frac{\kappa}{\pi}\sum_{l=1}^{\infty}\mathrm{e}^{-l\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}\left(\frac{\kappa}{2}\frac{\mathrm{e}^{-\kappa\beta l}}{1+\mathrm{e}^{-\kappa\beta l}}|\mathbf{x}+\mathbf{y}|^2\mathrm{e}^{\frac{\kappa}{2}\frac{\mathrm{e}^{-\kappa\beta l}}{1+\mathrm{e}^{-\kappa\beta l}}|\mathbf{x}+\mathbf{y}|^2} + \mathrm{e}^{-2\kappa\beta l}\right)\left(1+\frac{1}{2\kappa\beta l}\right) \\
&\qquad\leq \frac{\nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})}{2\pi}|\mathbf{x}+\mathbf{y}|^2\mathrm{e}^{\frac{\kappa}{2}|\mathbf{x}+\mathbf{y}|^2} + \frac{\kappa}{4\pi\beta}|\mathbf{x}+\mathbf{y}|^2\mathrm{e}^{\frac{\kappa}{2}|\mathbf{x}+\mathbf{y}|^2}\int_0^{\infty}\mathrm{d}t\,\mathrm{e}^{-\kappa\beta t}+ \\
&\qquad\qquad + \frac{\kappa}{\pi}\int_0^{\infty}\mathrm{d}t\,\mathrm{e}^{-2\kappa\beta t} + \frac{1}{2\pi\beta}\mathrm{e}^{-2\kappa\beta} + \frac{1}{2\pi\beta}\int_1^{\infty}\mathrm{d}t\,\frac{\mathrm{e}^{-2\kappa\beta t}}{t}.
\end{aligned}$$

From the lower and upper bounds derived above together with the identity in (3.39), then when $\nu\geq\nu_c(\beta)$ if $d=2$ the squeeze theorem provides us with:

$$\sum_{\mathbf{s}\in(\mathbb{N}^*)^2}\frac{\psi_{\infty,\kappa}^{(\mathbf{s})}(\mathbf{x})\overline{\psi_{\infty,\kappa}^{(\mathbf{s})}(\mathbf{y})}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{s})}-\overline{\mu}_{\infty,\kappa})}-1} \sim \frac{1}{2\pi\beta}\ln(\frac{1}{\kappa}) \quad \text{when } \kappa\downarrow 0 \text{ uniformly in } (\mathbf{x},\mathbf{y})\in\mathbb{R}^4.$$

3.4 Localization of the condensate/thermal gas in the open-trap limit.

Here we focus on the diagonal part of the (rescaled) reduced density matrix, interpreted as the (rescaled) local density of particles, in the open-trap limit. By introducing a scaling of the spacial variable, initially introduced by van den Berg *et al.* in [49] in the so-called *barometric formula*, we state some results in regards to the spatial localization of the condensate/thermal gas in the open-trap limit. We can relate our statements with some well-known results in Physics literature concerning the shape of the condensate/thermal gas in the space. This will be discuss in Sec. 4.

Theorem 3.23. *Consider a confined d-dimensional harmonically trapped Bose gas, $d \in \{1,2,3\}$ in the G-C ensemble. Let $\nu_c(\beta)$ be the critical density of particles in* (3.6) *satisfying* (3.7). *Then $\forall d \in \{1,2,3\}$, $\forall \beta > 0$, $\forall \nu > 0$, $\forall \mathbf{x} \in (\mathbb{R}^*)^d$ and $\forall 0 \leq \delta \leq 1$ the two following limits exist in $\overline{\mathbb{R}^+}$:*

$$\rho_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu) := \lim_{\kappa\downarrow 0} \rho_{\infty,\kappa}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};\beta,\nu), \quad r_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu) := \lim_{\kappa\downarrow 0} r_{\infty,\kappa}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};\beta,\nu).$$

With g_θ, $\theta > 0$ defined in (3.2) *and $\overline{\mu}_{\infty,0} = \overline{\mu}_{\infty,0}(\beta,\nu)$ in Lemma 3.5, one has more precisely:*
(i). *$\forall d \in \{1,2,3\}$ and $\forall \nu < \nu_c(\beta)$:*

$$\rho_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu) = \begin{cases} \dfrac{g_{\frac{d}{2}}(\mathrm{e}^{\beta\overline{\mu}_{\infty,0}})}{(2\pi\beta)^{\frac{d}{2}}}, & \text{if } 0 \leq \delta < 1; \quad (3.55a) \\ \dfrac{g_{\frac{d}{2}}(\mathrm{e}^{\beta(\overline{\mu}_{\infty,0}-\frac{1}{2}|\mathbf{x}|^2)})}{(2\pi\beta)^{\frac{d}{2}}}, & \text{if } \delta = 1. \quad (3.55b) \end{cases}$$

$$r_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu) = 0, \quad \text{if } \delta \geq 0. \tag{3.56}$$

(ii). *$\forall d \in \{2,3\}$ and $\forall \nu \geq \nu_c(\beta)$:*

$$\text{If } d=2, \quad \rho_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu) = \begin{cases} +\infty, & \text{if } 0 \leq \delta < 1; \quad (3.57a) \\ \dfrac{g_1(\mathrm{e}^{-\frac{1}{2}\beta|\mathbf{x}|^2})}{2\pi\beta}, & \text{if } \delta = 1. \quad (3.57b) \end{cases}$$

$$\text{If } d=3, \quad \rho_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu) = \begin{cases} +\infty, & \text{if } 0 \leq \delta \leq \frac{1}{2} \text{ and } \nu \neq \nu_c(\beta); \quad (3.58a) \\ \dfrac{g_{\frac{3}{2}}(1)}{(2\pi\beta)^{\frac{3}{2}}}, & \text{if } \frac{1}{2} < \delta < 1; \quad (3.58b) \\ \dfrac{g_{\frac{3}{2}}(\mathrm{e}^{-\frac{1}{2}\beta|\mathbf{x}|^2})}{(2\pi\beta)^{\frac{3}{2}}}, & \text{if } \delta = 1. \quad (3.58c) \end{cases}$$

$$r_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu) = \begin{cases} \dfrac{\nu - \nu_c(\beta)}{\pi^{\frac{d}{2}}}, & \text{if } 0 \leq \delta < \frac{1}{2}; \quad (3.59a) \\ \dfrac{\nu - \nu_c(\beta)}{\pi^{\frac{d}{2}}}\mathrm{e}^{-|\mathbf{x}|^2}, & \text{if } \delta = \frac{1}{2}; \quad (3.59b) \\ 0, & \text{if } \frac{1}{2} < \delta \leq 1. \quad (3.59c) \end{cases}$$

Remark 3.24. In Theorem 3.23, we restrict to $\mathbf{x} \in (\mathbb{R}^*)^d$ since the case of $\mathbf{x} = \mathbf{0}$ is covered by Proposition 3.14 and Theorem 3.20.

Proof. Let $\beta, \nu > 0$ be fixed.

- Case of $\nu < \nu_c(\beta)$ if $d = 1,2,3$ - Proof of (3.55)-(3.56)

At first, let us notice that from (3.11a) and (2.25)-(2.26), one has $\forall l \in \mathbb{N}^*$ and $\forall \mathbf{x} \in \mathbb{R}^d$:

$$\lim_{\kappa\downarrow 0} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};l\beta) = \frac{\mathrm{e}^{l\beta\overline{\mu}_{\infty,0}}}{(2\pi l\beta)^{\frac{d}{2}}} \times \begin{cases} 1, & \text{if } 1 > \delta \geq 0 \\ \mathrm{e}^{-\frac{1}{2}l\beta|\mathbf{x}|^2}, & \text{if } \delta = 1 \end{cases}, \tag{3.60}$$

where $\overline{\mu}_{\infty,0} = \overline{\mu}_{\infty,0}(\beta,\nu) < 0$ is defined in Lemma 3.5. Here we used the following:

$$\forall l \in \mathbb{N}^*, \quad \lim_{\kappa\downarrow 0}\left(\frac{\kappa}{2\pi\sinh(\kappa l\beta)}\right)^{\frac{d}{2}} = \frac{1}{(2\pi l\beta)^{\frac{d}{2}}}, \quad \lim_{\kappa\downarrow 0}\kappa^{1-2\delta}\tanh(\frac{\kappa}{2}l\beta) = \begin{cases} 0, & \text{if } 1-2\delta > -1 \\ \frac{l\beta}{2}, & \text{if } 1-2\delta = -1 \end{cases}.$$

Subsequently, from (3.11a) there exists a $\kappa_0 > 0$ s.t. $\forall 0 < \kappa \leq \kappa_0$, $\overline{\mu}_{\infty,\kappa} \leq \overline{\mu}_{\infty,0}/2 < 0$. Then by using the rough upper bound in (2.31), one has $\forall 0 \leq \delta \leq 1$ and uniformly on $\mathbb{R}^d$:

$$\forall 0 < \kappa \leq \kappa_0, \quad \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};l\beta) \leq \mathrm{e}^{l\beta\frac{\overline{\mu}_{\infty,0}}{2}}/(2\pi l\beta)^{\frac{d}{2}}.$$

By standard arguments, it follows:

$$\lim_{\kappa\downarrow 0}\sum_{l=1}^{\infty}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};l\beta) = \sum_{l=1}^{\infty}\lim_{\kappa\downarrow 0}\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};l\beta),$$

what proves (3.55) by virtue of (3.60). From (3.26), (3.56) results directly from (3.55).

- Case of $\nu \geq \nu_c(\beta)$ if $d = 2,3$ - Proof of (3.57)-(3.58).

Let us start with (3.57a)-(3.58a). We look for a lower bound of the sum in (3.20) whose the limit $\kappa \downarrow 0$ diverges. If $d = 2$, from (2.25)-(2.26), then $\forall l \in \mathbb{N}^*$, $\forall \mathbf{x} \in \mathbb{R}^2$ and for $\kappa > 0$ small enough:

$$\mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d=2)}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};l\beta) \geq \frac{1}{l\beta}\frac{\kappa l\beta}{2\pi\sinh(\kappa l\beta)}\mathrm{e}^{-\frac{1}{2}\kappa^{2-2\delta}|\mathbf{x}|^2 l\beta} \geq \frac{1}{2\pi}\frac{1}{l\beta}\mathrm{e}^{-\kappa l\beta}\mathrm{e}^{-\frac{1}{2}\kappa^{2-2\delta}|\mathbf{x}|^2 l\beta}.$$

In the first inequality we used (A.4) and $\overline{\mu}_{\infty,\kappa} \geq 0$ for $\kappa > 0$ small enough, see (3.11b)-(3.11c). In the second one, we used the expansion in power series of the sinh which yields:

$$\frac{\kappa l\beta}{\sinh(\kappa l\beta)} = \frac{\kappa l\beta}{\sum_{m=0}^{\infty}\frac{(\kappa l\beta)^{2m+1}}{(2m+1)!}} = \Big(\sum_{m=0}^{\infty}\frac{(\kappa l\beta)^{2m}}{(2m+1)!}\Big)^{-1} \geq (\cosh(\kappa l\beta))^{-1} \geq \mathrm{e}^{-\kappa l\beta}. \tag{3.61}$$

Then we have for $\kappa > 0$ sufficiently small and $\forall \mathbf{x} \in \mathbb{R}^2$:

$$\rho_{\infty,\kappa}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};\beta,\nu) \geq \frac{1}{2\pi\beta}\int_1^{\infty}\mathrm{d}t\,\frac{\mathrm{e}^{-\beta(\kappa+\frac{1}{2}\kappa^{2-2\delta}|\mathbf{x}|^2)t}}{t} = \Gamma_0(\beta(\kappa+\frac{1}{2}\kappa^{2-2\delta}|\mathbf{x}|^2)),$$

where Γ_0 is the incomplete Gamma function in (3.39). In the limit $\kappa \downarrow 0$, the above lower bound diverges $\forall 0 \leq \delta \leq 1$ if $\mathbf{x} = \mathbf{0}$, $\forall 0 \leq \delta < 1$ otherwise. If $d = 3$, from (2.25)-(2.26) and by mimicking the arguments leading to (3.25), then $\forall l \in \mathbb{N}^*$, $\forall \mathbf{x} \in \mathbb{R}^3$ and for $\kappa > 0$ small enough:

$$\rho_{\infty,\kappa}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};\beta,\nu) \geq \frac{1}{(2\pi)^{\frac{3}{2}}\beta}\mathrm{e}^{-\kappa^{1-2\delta}|\mathbf{x}|^2}\frac{\kappa^{\frac{3}{2}}}{E_{\infty,\kappa}^{(\mathbf{0})} - \overline{\mu}_{\infty,\kappa}}\mathrm{e}^{-\beta(E_{\infty,\kappa}^{(\mathbf{0})} - \overline{\mu}_{\infty,\kappa})}.$$

Due to (3.11c), the above lower bound diverges in the limit $\kappa \downarrow 0$ $\forall 0 \leq \delta \leq 1$ if $\mathbf{x} = \mathbf{0}$, $\forall 0 \leq \delta \leq \frac{1}{2}$ otherwise. Next, let us prove (3.57b) and (3.58b)-(3.58c). To do so, let us firstly give a lower bound of the quantity $\rho_{\infty,\kappa}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};\beta,\nu)$ when $\delta = 1$ if $d = 2$ and when $\frac{1}{2} < \delta \leq 1$ if $d = 3$. If $d = 2$ and $\delta = 1$, for any $\mathbf{x} \in (\mathbb{R}^*)^2$ and for $\kappa > 0$ small enough:

$$\rho_{\infty,\kappa}(\mathbf{x}\kappa^{-1},\mathbf{x}\kappa^{-1};\beta,\nu) \geq \frac{1}{2\pi\beta}\sum_{l=1}^{\infty}\frac{\mathrm{e}^{-l\beta[(E_{\infty,\kappa}^{(\mathbf{0})} - \overline{\mu}_{\infty,\kappa}) + \frac{1}{2}|\mathbf{x}|^2]}}{l} = \frac{g_1(\mathrm{e}^{-\beta[(E_{\infty,\kappa}^{(\mathbf{0})} - \overline{\mu}_{\infty,\kappa}) + \frac{1}{2}|\mathbf{x}|^2]})}{2\pi\beta}, \tag{3.62}$$

where we used (3.61). If $d = 3$ and $1 \geq \delta > \frac{1}{2}$, $\forall \mathbf{x} \in (\mathbb{R}^*)^3$ and for $\kappa > 0$ small enough:

$$\rho_{\infty,\kappa}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};\beta,\nu) \geq \frac{g_{\frac{3}{2}}(\mathrm{e}^{-\beta[(E_{\infty,\kappa}^{(\mathbf{0})} - \overline{\mu}_{\infty,\kappa}) + \frac{1}{2}|\mathbf{x}|^2\kappa^{2-2\delta}]})}{(2\pi\beta)^{\frac{3}{2}}}. \tag{3.63}$$

In the limit $\kappa \downarrow 0$, the lower bounds in (3.62) and (3.63) converge to (3.57b) and (3.58b)-(3.58c) respectively following the values of δ. Secondly, let us give an upper bound of $\rho_{\infty,\kappa}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};\beta,\nu)$ when $\delta=1$ if $d=2$ and when $\frac{1}{2}<\delta\leq 1$ if $d=3$ whose the limit $\kappa\downarrow 0$ reduced to the announced results. Under these conditions, introduce $\forall 0<\kappa<1$ and $\forall \mathbf{x}\in(\mathbb{R}^*)^d$ the decomposition:

$$\rho_{\infty,\kappa}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};\beta,\nu) = \left\{ \sum_{l=1}^{N_{\kappa,\varepsilon}} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} + \sum_{l=N_{\kappa,\varepsilon}+1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} \right\} G_{\infty,\kappa}^{(d)}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};l\beta), \tag{3.64}$$

where $N_{\kappa,\varepsilon}:=\lfloor 1/\kappa^{\varepsilon}\rfloor$ with $0<\varepsilon<d$ for the moment. Let us give an upper bound for the second sum in the r.h.s. of (3.64). From (3.31)-(3.32a) and by mimicking the arguments leading to (3.38) and (3.40), one has $\forall d\in\{2,3\}$, $\forall\mathbf{x}\in(\mathbb{R}^*)^d$, $\forall 0<\varepsilon<d$ and for $\kappa<1$ sufficiently small:

$$\sum_{l=N_{\kappa,\varepsilon}+1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};l\beta) \leq \mathrm{e}^{-\frac{1}{2}\beta|\mathbf{x}|^2\frac{\kappa^{2-2\delta-\varepsilon}}{1+\beta\kappa^{1-\varepsilon}}} \times$$
$$\times \frac{1}{\beta\pi^{\frac{d}{2}}}\left\{ \kappa^{\frac{d}{2}} \frac{\mathrm{e}^{-\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}}{E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa}} + \frac{\kappa^{\frac{d}{2}-1}}{2^{3-d}}\Gamma_0[N_{\kappa,\varepsilon}\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})] + \frac{d-2}{4\beta}\frac{1}{\sqrt{\kappa}N_{\kappa,\varepsilon}} \right\}, \tag{3.65}$$

where we used that $\forall l\geq N_{\kappa,\varepsilon}+1$, $\tanh(\frac{\kappa\beta}{2}l)\geq\tanh(\frac{\beta}{2}\kappa^{1-\varepsilon})$ followed by the upper bound in (A.4). Note that from (3.11c), the upper bound in (3.65) vanishes in the limit $\kappa\downarrow 0$ $\forall 2-2\delta<\varepsilon<d$ if $d=2,3$ and $\forall\mathbf{x}\in(\mathbb{R}^*)^d$. Let us give an upper bound for the first sum in the r.h.s. of (3.64). From (3.31), (3.32b) and (3.33), $\forall d\in\{2,3\}$, $\forall\mathbf{x}\in(\mathbb{R}^*)^d$, $\forall 0<\varepsilon<d$ and for $\kappa<1$ small enough:

$$\sum_{l=1}^{N_{\kappa,\varepsilon}} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};l\beta)$$
$$\leq \sum_{l=1}^{\infty} \frac{\mathrm{e}^{-l\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}}{(2\pi l\beta)^{\frac{d}{2}}} \mathrm{e}^{-\frac{1}{2}l\beta|\mathbf{x}|^2\frac{\kappa^{2-2\delta}}{1+\beta\kappa^{1-\varepsilon}}} - \sum_{l=N_{\kappa,\varepsilon}+1}^{\infty} \frac{\mathrm{e}^{-l\beta[(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})+\frac{|\mathbf{x}|^2}{2}\frac{\kappa^{2-2\delta}}{1+\beta\kappa^{1-\varepsilon}}]}}{(2\pi l\beta)^{\frac{d}{2}}} +$$
$$+ 2\frac{\mathrm{e}^{-\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}}{\pi^{\frac{d}{2}}}\left(\kappa^{\frac{d}{2}-\varepsilon} + \frac{3(d-2)}{\sqrt{2\beta}}\kappa^{1-\frac{\varepsilon}{2}} + \frac{3(d-2)}{4\beta}\sqrt{\kappa}\ln(\frac{\mathrm{e}}{\kappa^{\varepsilon}}) \right). \tag{3.66}$$

From the asymptotic in (3.11c), the last term in the r.h.s. of (3.66) vanishes in the limit $\kappa\downarrow 0$ $\forall 0<\varepsilon<\frac{d}{2}$ if $d=2,3$. By an integral comparison, one can prove that the second term vanishes in the limit $\kappa\downarrow 0$ $\forall 2-2\delta<\varepsilon<d$ if $d=2,3$ and $\forall\mathbf{x}\in(\mathbb{R}^*)^d$. Finally by standard arguments, one has $\forall 0<\varepsilon<1$ if $d=2$, $\forall 0<\varepsilon<d$ if $d=3$ and $\forall\mathbf{x}\in(\mathbb{R}^*)^d$:

$$\lim_{\kappa\downarrow 0}\sum_{l=1}^{\infty} \frac{\mathrm{e}^{-l\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}}{(2\pi l\beta)^{\frac{d}{2}}} \mathrm{e}^{-\frac{1}{2}l\beta|\mathbf{x}|^2\frac{\kappa^{2-2\delta}}{1+\beta\kappa^{1-\varepsilon}}} = \sum_{l=1}^{\infty}\lim_{\kappa\downarrow 0} \frac{\mathrm{e}^{-l\beta(E_{\infty,\kappa}^{(\mathbf{0})}-\overline{\mu}_{\infty,\kappa})}}{(2\pi l\beta)^{\frac{d}{2}}} \mathrm{e}^{-\frac{1}{2}l\beta|\mathbf{x}|^2\frac{\kappa^{2-2\delta}}{1+\beta\kappa^{1-\varepsilon}}}. \tag{3.67}$$

By adding the r.h.s. of (3.65) and (3.66), we found an upper bound for the l.h.s. of (3.64) which converges when $\delta=1$ to (3.57b)-(3.58c) if $d=2,3$ $\forall 0<\varepsilon<\frac{d}{2}$, when $\frac{1}{2}<\delta<1$ to (3.58b) if $d=3$ $\forall 1<\varepsilon<\frac{3}{2}$. Along with (3.62)-(3.63), the squeeze theorem lead to (3.57b) and (3.58b)-(3.58c).

- Case of $\nu\geq\nu_c(\beta)$ if $d=2,3$ - Proof of (3.59)

Similarly to (3.30), the starting-point is a decomposition of the quantity defined in (3.20) $\forall 0\leq\delta\leq 1$, $\forall d\in\{2,3\}$ and $\forall 0<\kappa<1$ into two contributions:

$$r_{\infty,\kappa}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};\beta,\nu) = \kappa^{d/2}\left\{ \sum_{l=1}^{N_{\kappa,\varepsilon}} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} + \sum_{l=N_{\kappa,\varepsilon}+1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} \right\} G_{\infty,\kappa}^{(d)}(\mathbf{x}\kappa^{-\delta},\mathbf{x}\kappa^{-\delta};l\beta), \tag{3.68}$$

where $N_{\kappa,\varepsilon} := \lfloor 1/\kappa^{\varepsilon} \rfloor$ with $0 < \varepsilon < d$ for the moment (a limitation will appear if $\delta = \frac{1}{2}$ when $\nu > \nu_c(\beta)$ or when $\nu = \nu_c(\beta)$). From (3.31)-(3.33), one has $\forall 0 \leq \delta \leq 1$, $\forall d \in \{2,3\}$ and $\forall \mathbf{x} \in \mathbb{R}^d$:

$$\forall 0 < \varepsilon < d, \quad \lim_{\kappa \downarrow 0} \kappa^{\frac{d}{2}} \sum_{l=1}^{N_{\kappa,\varepsilon}} \mathrm{e}^{l\beta \overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x}\kappa^{-\delta}, \mathbf{x}\kappa^{-\delta}; l\beta) = 0.$$

Let us investigate the second term in the r.h.s. of (3.68). We distinguish the case of $\nu > \nu_c(\beta)$ from the one of $\nu = \nu_c(\beta)$. When $\nu > \nu_c(\beta)$, by using the same arguments leading to (3.37) and (3.38)-(3.40), one has $\forall 0 \leq \delta \leq 1$, $\forall d \in \{2,3\}$, $\forall \mathbf{x} \in (\mathbb{R}^*)^d$, $\forall 0 < \varepsilon < d$ and $\forall 0 < \kappa < \min\{1, K_\beta\}$:

$$\frac{\nu_{\infty,\kappa}(\beta,\nu;\mathbf{0})}{\pi^{\frac{d}{2}}} \mathrm{e}^{-\kappa^{1-2\delta}|\mathbf{x}|^2} \mathrm{e}^{-\frac{3\kappa^{d-\varepsilon}}{2(\nu-\nu_c(\beta))}} \leq \kappa^{\frac{d}{2}} \sum_{l=N_{\kappa,\varepsilon}+1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x}\kappa^{-\delta}, \mathbf{x}\kappa^{-\delta}; l\beta)$$
$$\leq \mathrm{e}^{-\kappa^{1-2\delta}|\mathbf{x}|^2 \tanh(\frac{\beta}{2}\frac{1}{\kappa^{\varepsilon}-1})} \times \begin{cases} \text{r.h.s. of (3.38)}, & \text{if } d = 2 \\ \text{r.h.s. of (3.40)}, & \text{if } d = 3 \end{cases}$$

The above l.h.s. and r.h.s. tend to (3.59a) if $0 \leq \delta < \frac{1}{2}$ $\forall 0 < \varepsilon < d$, (3.59b) if $\delta = \frac{1}{2}$ $\forall 1 < \varepsilon < d$, and 0 if $\frac{1}{2} < \delta \leq 1$ $\forall 0 < \varepsilon < d$. Then the squeeze theorem leads to (3.59). The case of $\nu = \nu_c(\beta)$ can be treated by using the same arguments than the ones used at the end of the proof of Theorem 3.20, see e.g. (3.41). By requiring the limiting condition $d > \varepsilon > 1$, (3.59) follows. □

Corollary 3.25. *Consider a confined d-dimensional harmonically trapped Bose gas, $d \in \{1,2,3\}$ in the G-C ensemble. Then $\forall \beta > 0$, $\forall \nu > 0$ and $\forall \mathbf{x} \in (\mathbb{R}^*)^d$:*
(i). *For the open-trap rescaled local density of particles, one has:*

$$\forall 0 \leq \delta \leq 1, \; r_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu) = \lim_{\kappa \downarrow 0} \nu_{\infty,\kappa}(\beta,\nu;\mathbf{0}) |\psi_{\infty,1}^{(\mathbf{0})}(\mathbf{x}\kappa^{-\delta}\sqrt{\kappa})|^2, \quad \textit{if } d = 1,2,3. \tag{3.69}$$

(ii). *For the open-trap local density of particles, one has:*

$$\lim_{\kappa \downarrow 0} \sum_{\mathbf{s} \in (\mathbb{N}^*)^d} \frac{|\psi_{\infty,\kappa}^{(\mathbf{s})}(\mathbf{x}\kappa^{-\delta})|^2}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{s})} - \overline{\mu}_{\infty,\kappa})} - 1} =$$

$$\begin{cases} \rho_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu), & \textit{when } \nu < \nu_c(\beta) \textit{ if } 0 \leq \delta \leq 1 \textit{ and } d = 1,2,3; & \text{(3.70a)} \\ \rho_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu), & \textit{when } \nu \geq \nu_c(\beta) \textit{ if } 0 \leq \delta \leq 1 \textit{ and } d = 2; & \text{(3.70b)} \\ \dfrac{g_{\frac{3}{2}}(\mathrm{e}^{\beta\overline{\mu}_{\infty,0}})}{(2\pi\beta)^{\frac{3}{2}}}, & \textit{when } \nu > \nu_c(\beta) \textit{ if } 0 \leq \delta \leq \frac{1}{2} \textit{ and } d = 3; & \text{(3.70c)} \\ \rho_{\infty,0}^{(\delta)}(\mathbf{x};\beta,\nu), & \textit{when } \nu \geq \nu_c(\beta) \textit{ if } \frac{1}{2} < \delta \leq 1 \textit{ and } d = 3. & \text{(3.70d)} \end{cases}$$

Remark 3.26. We restricted to $\mathbf{x} \in (\mathbb{R}^*)^d$ since the case of $\mathbf{x} = \mathbf{0}$ is covered by Corollary 3.21.

Proof. Let $\beta, \nu > 0$ be fixed. In view of (3.56) and (3.59), (3.69) follows by direct calculations from (3.10) and (2.6)-(2.4). Let us turn to (ii). By setting $\mathbf{x} = \mathbf{y}$ in (3.45) after dilating the spacial variables by $\kappa^{-\delta}$, then from (2.6)-(2.4) the first term of the r.h.s. can be rewritten as:

$$\forall x \in \mathbb{R}^*, \quad \frac{|\phi_{\infty,\kappa}^{(0)}(x\kappa^{-\delta})|^2}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(0)} - \overline{\mu}_{\infty,\kappa})} - 1} = \frac{\sqrt{\kappa}}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(0)} - \overline{\mu}_{\infty,\kappa})} - 1} \frac{\mathrm{e}^{-x^2\kappa^{1-2\delta}}}{\sqrt{\pi}}, \tag{3.71}$$

$$\forall d \in \{2,3\}, \forall \mathbf{x} \in (\mathbb{R}^*)^d, \quad \frac{|\psi_{\infty,\kappa}^{(\mathbf{0})}(\mathbf{x}\kappa^{-\delta})|^2}{\mathrm{e}^{\beta(E_{\infty,\kappa}^{(\mathbf{0})} - \overline{\mu}_{\infty,\kappa})} - 1} = \nu_{\infty,\kappa}(\beta,\nu;\mathbf{0}) \frac{\mathrm{e}^{-|\mathbf{x}|^2\kappa^{1-2\delta}}}{(\kappa\pi)^{\frac{d}{2}}}. \tag{3.72}$$

From (3.11a), the r.h.s. of (3.71) vanishes in the limit $\kappa \downarrow 0$ $\forall 0 \leq \delta \leq 1$. From (3.10), the r.h.s. of (3.72) vanishes in the limit $\kappa \downarrow 0$ $\forall \frac{1}{2} < \delta \leq 1$ and $\forall d \in \{2,3\}$. This proves (3.70d) in particular. Let us now turn to the case of $0 \leq \delta \leq \frac{1}{2}$ if $d = 2,3$. To do that, it is enough to mimic the arguments used in the proof of Corollary 3.21. By setting $\mathbf{x} = \mathbf{y}$ in (3.45) and (3.46) after dilating the spacial variable by $\kappa^{-\delta}$, then since $1 - 2\delta \geq 0$ $\forall 0 \leq \delta < \frac{1}{2}$, the conclusions obtained from (3.47)-(3.48) and from (3.49)-(3.51) still hold true $\forall \mathbf{x} \in (\mathbb{R}^*)^2$ and $\forall \mathbf{x} \in (\mathbb{R}^*)^3$ respectively. □

4 Concluding remarks and perspectives.

4.1 Conclusions & discussions.

(D1) *Homogeneous versus inhomogeneous systems.*

We stress the point that our results obtained in the open-trap limit for the harmonically trapped Bose gas differ from the ones stated in [51, 48] for the perfect Bose gas confined in 'Dirichlet boxes' (commonly referred to as 'homogeneous systems'), and from the ones in [45, 47, 53] stated for the free Bose gas in a weak harmonic trap model. The main difference concerns the critical density. From a rescaling of the average number of particles (see Definition 2.6), we find that the critical rescaled number of particles in the open-trap limit for the harmonically trapped Bose gas is finite if $d = 2, 3$ and proportional to $g_d(1)$, see Proposition 3.4. This result contrasts with the case of homogeneous systems for which the bulk critical density of particles (by bulk we mean in the thermodynamic limit) is finite if $d = 3$ and proportional to $g_{d/2}(1)$, and also with the case of the weak harmonic trap model for which it is finite if $d = 2, 3$ and proportional to $\int_0^1 du\, g_{d/2}(\mathrm{e}^{-\beta \frac{u^2}{2}})$.

However, let us notice that our expressions for the non-condensate part of the open-trap reduced density matrix for the harmonically trapped Bose gas (see Corollary 3.21) are exactly the same than the ones for the non-condensate part of the bulk reduced density matrix for homogeneous systems in any dimension, and so it diverges if $d = 2$ and converges if $d = 3$. Even at the scales $0 < \delta < 1$, the non-condensate part of the open-trap scaled local density (see Corollary 3.25) remains equal to the non-condensate part of the bulk local density for homogeneous systems. This means that the non-condensate bosons do not feel the trap for those scales and behave like free particles in the whole space $\mathbb{R}^d$. But at the scale $\delta = 1$ and when $\nu \geq \nu_c(\beta)$ if $d = 2, 3$, the open-trap scaled local density has a gaussian decay whereas the bulk scaled local density (with the scaling $\mathbf{u} := \mathbf{x}/L$, $\mathbf{u} \in [-1, 1]^d$) for homogeneous systems is constant on $(-1, 1)^d$ and vanishes on the boundaries, see e.g. [50]. Concerning the condensate part of the open-trap rescaled density function for the harmonically trapped Bose gas, it is constant at the scales $0 \leq \delta < 1/2$ and has a gaussian decay at the scale $\delta = 1/2$ when $\nu \geq \nu_c(\beta)$ if $d = 2, 3$ (see Corollary 3.25), whereas the condensate part of the bulk scaled density function for homogenous systems oscillates on $[-1, 1]^d$.

(D2) *Localization of the non-condensate versus condensate gas.*

From (3.57)-(3.58) and (3.59), the localization range of the condensate is not the same than the localization range of the *thermal gas* (i.e. non-condensate gas). This means that they do not coexist at the same scale of spatial distances as it is stated in [41, Eq. (10.28), pp. 153]. To understand more properly, let us define $\forall d \in \{2, 3\}$, $\forall \beta > 0$ and $\forall \nu > 0$ the large scale (i.e. $\delta = 1$) average square radius in the j-th direction of the open-trap reduced local density function as:

$$\langle x_j^2 \rangle_{\infty,0}^{(T)}(\beta, \nu) := \frac{\displaystyle\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\, x_j^2\, \rho_{\infty,0}^{(\delta=1)}(\mathbf{x}; \beta, \nu)}{\displaystyle\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\, \rho_{\infty,0}^{(\delta=1)}(\mathbf{x}; \beta, \nu)}, \quad j = 1, \cdots, d. \tag{4.1}$$

Similarly, let us define under the same conditions the medium scale (i.e. $\delta = 1/2$) average square radius in the j-th direction of the open-trap rescaled reduced local density function as:

$$\langle x_j^2 \rangle_{\infty,0}^{(0)}(\beta, \nu) := \frac{\displaystyle\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\, x_j^2\, r_{\infty,0}^{(\delta=\frac{1}{2})}(\mathbf{x}; \beta, \nu)}{\displaystyle\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\, r_{\infty,0}^{(\delta=\frac{1}{2})}(\mathbf{x}; \beta, \nu)}, \quad j = 1, \cdots, d. \tag{4.2}$$

Let us mention that from (3.57b)-(3.58c) and (3.59b) the quantities in (4.1)-(4.2) are well-defined. By a direct calculus, we get $\forall d \in \{2, 3\}$, $\forall \beta > 0$ and $\forall \nu > 0$ the following ratio:

$$\frac{\langle x_j^2 \rangle_{\infty,0}^{(T)}(\beta, \nu)}{\langle x_j^2 \rangle_{\infty,0}^{(0)}(\beta, \nu)} = \frac{2\zeta(d+1)}{\beta\zeta(d)}, \quad j = 1, \cdots, d, \tag{4.3}$$

where $\zeta(\alpha) := g_\alpha(1)\ \forall \alpha > 1$. The r.h.s. of (4.3) is j-independent since the trap is homogeneous. Note that (4.3) is the open-trap counterpart of [41, Eq. (10.28)] for $d = 3$ (here β has to be replaced with $\beta\hbar\Omega_0$). (4.3) can be interpreted as follows: the density profile of the thermal gas is much more spread out than the rescaled density profile of the condensate (large vs medium scale).

Nevertheless, the local density of particles in the condensate is of the order of $\kappa^{-3/2}$ for $d = 3$ and so is infinite in the open-trap limit (see (3.22c)), whereas the local density of particles in the thermal gas is finite in the open-trap limit (see (3.44c)). Hence, one can talk about a *spatial Bose-Einstein condensation* since a very large number of particles is localized in a small region of the space compared with the region where the thermal gas is spread. Note that the first experimental demonstrations of the condensate is based on this latter feature since it is enough to 'take pictures' of the gas to bring out the spatial density of the particles distribution, see e.g. [18, 23].

(D3) *A 'semiclassical formula' for the local density.*

Another remark relates to the expression of the open-trap local density in (3.57b) and (3.58c). By reintroducing all the physical constants: m, $\hbar$ and Ω_0 (see (1.6)), then $\forall d \in \{2, 3\}$, $\forall \beta > 0$, $\forall \nu \geq \nu_c(\beta)$ and $\forall \mathbf{x} \in (\mathbb{R}^*)^d$ the open-trap local density at the scale $\delta = 1$ can be rewritten as:

$$\rho_{\infty,0}^{(\delta=1)}(\mathbf{x}; \beta, \nu) = \frac{g_{\frac{d}{2}}(\mathrm{e}^{-\beta v(\mathbf{x})})}{\lambda_\beta^d} = \int_{\mathbb{R}^d} \frac{\mathrm{d}\mathbf{p}}{(2\pi\hbar)^d} \frac{1}{\mathrm{e}^{\beta(\frac{\mathbf{p}^2}{2m} + v(\mathbf{x}))} - 1}, \tag{4.4}$$

where $\lambda_\beta = \sqrt{2\pi\beta\hbar^2/m}$ stands for the *de Broglie thermal wave length*, and $v(\mathbf{x}) := m\Omega_0^2|\mathbf{x}|^2/2$.

In Physics literature, (4.4) is often referred to as the 'semi-classical formula' for the local density, see e.g. [41, Eqs. (10.25)-(10.27), pp. 152–153] and also [40, Eqs. (2.45)-(2.50), pp. 28]. To make a link with the semiclassics, let us introduce the unitary transformation on $L^2(\mathbb{R}^d)$:

$$(\mathcal{U}(\kappa)\varphi)(\mathbf{x}) := \kappa^{-\frac{d}{2}}\varphi(\mathbf{x}/\kappa), \quad \kappa > 0,\ \varphi \in L^2(\mathbb{R}^d).$$

Under the above transformation, the operator in (1.6) is unitary equivalent to:

$$\mathcal{U}(\kappa)H_{\infty,\kappa}\mathcal{U}^{-1}(\kappa) = \tilde{H}_{\infty,\kappa}, \quad \tilde{H}_{\infty,\kappa} := -\frac{h_\kappa^2}{2m}\Delta_{\mathbf{x}} + v(\mathbf{x}) \ \text{ with } \ h_\kappa := \hbar\kappa. \tag{4.5}$$

From the definition in (2.7), (4.5) implies in the kernels sense on $\mathbb{R}^{2d}$:

$$\forall \kappa > 0, \quad (\mathfrak{f}_{BE}(\beta, \overline{\mu}_{\infty,\kappa}; H_{\infty,\kappa}))(\mathbf{x}\kappa^{-1}, \mathbf{y}\kappa^{-1}) = \kappa^d(\mathfrak{f}_{BE}(\beta, \overline{\mu}_{\infty,\kappa}; \tilde{H}_{\infty,\kappa}))(\mathbf{x}, \mathbf{y}),$$

Then it follows from the definition in (3.19) together with (4.4):

$$(\mathfrak{f}_{BE}(\beta, \overline{\mu}_{\infty,\kappa}; \tilde{H}_{\infty,\kappa}))(\mathbf{x}, \mathbf{y}) \sim \int_{\mathbb{R}^d} \frac{\mathrm{d}\mathbf{p}}{(2\pi h_\kappa)^d} \frac{1}{\mathrm{e}^{\beta(\frac{\mathbf{p}^2}{2m} + v(\mathbf{x}))} - 1} \quad \text{when } \kappa \downarrow 0.$$

Hence (4.4) coincides with the leading term of the asymptotic expansion of the local density in the semiclassical limit when the dynamics of the system is determined by the Hamiltonian in (4.5).

4.2 Conclusions & perspectives.

(P1) *Generalized BEC (g-BEC).*

In Remark 3.11, we showed that the g-BEC term in the sense of (3.17) vanishes, see (3.18). However, for some model of anisotropic harmonic traps recently studied in [44, 38], and analogously to the anisotropic boxes [48, 46, 49, 50], it has been shown that the g-BEC term is non-zero. For instance, if one chooses $\Omega_j = \kappa_j\Omega_0$, $j = 1, 2, 3$ with $\kappa_1 = \kappa\exp(-\kappa_0^2/\kappa^2)$ and $\kappa_2 = \kappa = \kappa_3$ (here $\kappa_0 > 0$ is a given constant), one gets that the g-BEC term in the sense of (3.17) is positive when $\nu > \nu_c(\beta)$. But the usual condensate on the ground state is non-zero only when (see e.g. [44, 38]):

$$\nu > \nu_m(\beta, \kappa_0) := \nu_c(\beta) + \frac{\kappa_0}{\hbar\beta\Omega_0}.$$

Giving a rigorous proof of these results and investigating the local effects of g-BEC (from the reduced density matrix and scale local density function) are two perspectives. In other words, does g-BEC give some additional contributions to the non-condensate part of the reduced density matrix in (3.44), and does it change the scale of the localization of the condensate in (3.70)?

(P2) *Local generalized BEC (local g-BEC) and loop path approach.*

In this paper, our approach essentially leans on the representation (3.20) of the reduced density matrix by the kernel of the Bose-Einstein function of the operator $H_{\infty,\kappa}$ in (2.7), see Lemma 2.13. From such a representation, investigating the behavior in the open-trap limit of the (rescaled) reduced density matrix/local density function requires some sharp estimates on the integral kernel of the semigroup generated by $H_{\infty,\kappa}$ for small values of κ. Since this kernel is explicitly known (see (2.25)-(2.26)), then our approach turns out to be more robust than the one based on the representation in (3.19) involving directly the eigenfunctions of $H_{\infty,\kappa}$. Indeed, the control of the sum in (3.19) for small values of κ is made difficult by the behavior of the Hermite polynomials which oscillate especially as the s-index summation gets larger. As reviewed in Introduction, our approach is well-known in Physics literature (it was introduced by R. Feynman in [26]), and it is used in numerical methods like the Path-Integral Quantum Monte Carlo, see e.g. [33, 35].

Along this approach, the key idea involved in the proof of Proposition 3.18, Corollary 3.21 and Theorem 3.23 consists in decomposing the sum in (3.20) into two contributions:

$$\sum_{l=1}^{N_{\kappa,\varepsilon}} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta) + \sum_{l=N_{\kappa,\varepsilon}+1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};l\beta),$$

where $N_{\kappa,\varepsilon} := \lfloor 1/\kappa^{\varepsilon} \rfloor$, with some well-chosen $0 < \varepsilon < d$. Actually, what is hidden behind such a decomposition is the short/long cycles concept involved in the description of g-BEC, see e.g. [43]. We already noticed in Remark 3.11 that the grand-canonical harmonically trapped Bose gas does not manifest the g-BEC in the sense of the definition given in (3.17) (in a way, this is a *global* formulation). But some well-chosen decomposition of the sum in (3.20) allow us to investigate the possible existence of *local g-BEC* (even though the global g-BEC is absent) in the sense given below. For instance, one can decompose the local density into three distinct contributions:

$$\rho_{\infty,\kappa}(\mathbf{x};\beta,\nu) = \Big\{ \sum_{l=1}^{N_{\kappa,\varepsilon}} + \sum_{l=N_{\kappa,\varepsilon}+1}^{N_{\kappa,\eta}} \Big\} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{x};l\beta) + \sum_{l=N_{\kappa,\eta}+1}^{\infty} \mathrm{e}^{l\beta\overline{\mu}_{\infty,\kappa}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{x};l\beta). \quad (4.6)$$

Here the first sum stands for the sum over the 'short loops', the second one for the sum over the 'mesoscopic loops' and the third one for the sum over the 'macroscopic loops'. For some $0 < \varepsilon, \eta$ well-chosen, performing the open-trap limit in (4.6) can provide some information on the local g-BEC (it is present if the open-trap limit of the sum over the 'mesoscopic loops' is positive). For instance, in the case of $d = 3$ when $\nu > \nu_c(\beta)$, we showed in the proof of Corollary 3.21 that for $1 < \varepsilon < \eta < \frac{3}{2}$, the sum over the 'macroscopic loops' (i.e. $l > N_{\kappa,\eta}$) gives the density on the ground state while the sum over the 'mesoscopic loops' (i.e. $N_{\kappa,\eta} \geq l > N_{\kappa,\varepsilon}$) vanishes.

(P3) *G-BEC in two-dimensions and in anisotropic three-dimensional traps.*

We showed in Remark 3.22 that, when $\nu \geq \nu_c(\beta)$ if $d = 2$, the non-condensate part of the reduced density matrix growths logarithmically in $1/\kappa$ in a neighborhood of $\kappa = 0$. We mention that a similar behavior has been pointed out in [34]. However, it is possible to regularize this two-dimensional log-divergence for a three-dimensional anisotropic harmonic trap model mimicking the two-dimensional properties for certain values of $\nu > \nu_c(\beta)$. To do so, it is enough to consider that in each direction we have the pulsation $\Omega_j = \kappa_j \Omega_0$, $j = 1, 2, 3$ with $\kappa_1 = \kappa \exp(-\kappa_0/\kappa) = \kappa_2$ and $\kappa_3 = \kappa$ (here $\kappa_0 > 0$ is a given constant). This means that for small values of κ the characteristic length $\sqrt{\hbar/(m\Omega_j)}$ along the x_j-direction, $j = 1, 2$ is very large compared to the one along the x_3-direction. This anisotropic model is inspired by a model introduced in [45] for homogeneous systems, and was recently studied in [44, 38]. We emphasize that the problem of BEC in very anisotropic harmonic traps is a crucial problem encountered in cold atom Physics, see e.g. [29]. Hence the exponential anisotropic harmonic model is given as a perspective. We think that a decomposition of type (4.6) allows to show the existence of local g-BEC following similar technics developed in [43] for the perfect Bose gas in Casimir boxes with periodic boundary conditions (it has been proved that there exist two equivalent classifications: g-BEC of type I,II,III versus macroscopic/mesoscopic loops). An additional motivation to tackle the anisotropic harmonic trap problem is to investigate the shape of the local density in the regime $\nu = \nu_c(\beta)$ if $d = 3$.

Finally, we mention that the possibility of existence of g-BEC for the two-dimensional interacting Bose gas has been considered in Physics literature in the last decade, see e.g. [37] and also [38]. Finding a model of homogeneous/inhomogeneous interactions for the two-dimensional Bose gas which can exhibit simultaneously the presence of g-BEC (in a sense to define) along with the absence of condensation in the ground-state, remains a challenging problem.

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A Annex - Some useful identities/inequalities.

Here we collect some miscellaneous inequalities/identities involving the hyperbolic functions that we used in this paper. Most of them can be found in [1, Sec. 4.5]. For any real $\alpha \geq 0$:

$$1 \leq \cosh(\alpha) \leq \mathrm{e}^{\alpha}, \tag{A.1}$$

$$\alpha \leq \sinh(\alpha) \leq \frac{1}{2}\mathrm{e}^{\alpha}, \tag{A.2}$$

$$0 \leq \tanh(\alpha) \leq 1, \tag{A.3}$$

$$\frac{1}{\alpha} \leq \coth(\alpha) := \frac{1}{\tanh(\alpha)} \leq \frac{1+\alpha}{\alpha}, \quad \alpha > 0. \tag{A.4}$$

For any reals $\alpha > 0$ and $t > s > 0$:

$$\sinh(\alpha t) = 2\sinh(\frac{\alpha}{2}t)\cosh(\frac{\alpha}{2}t), \tag{A.5}$$

$$\coth(\alpha t) = \frac{1}{2}\coth(\frac{\alpha}{2}t) + \frac{1}{2}\tanh(\frac{\alpha}{2}t), \tag{A.6}$$

$$\coth(\alpha s) + \coth(\alpha(t-s)) = \frac{\sinh(\alpha t)}{\sinh(\alpha s)\sinh(\alpha(t-s))}, \tag{A.7}$$

$$\tanh(\alpha s) + \tanh(\alpha(t-s)) = \frac{\sinh(\alpha t)}{\cosh(\alpha s)\cosh(\alpha(t-s))}, \tag{A.8}$$

$$\tanh(\alpha s) + \tanh(\alpha(t-s)) = \tanh(\alpha t)\{1 + \tanh(\alpha s)\tanh(\alpha(t-s))\} \geq \tanh(\alpha t). \tag{A.9}$$

For any reals $x \geq 0$ and $\mu, \nu > 0$:

$$\frac{x}{1+x} < 1 - \mathrm{e}^{-x} < x, \tag{A.10}$$

$$x \leq \mathrm{e}^{x} - 1 \leq x\mathrm{e}^{x}, \tag{A.11}$$

$$x^{\mu}\mathrm{e}^{-\nu x} \leq \left(\frac{2\mu}{\mathrm{e}\nu}\right)^{\mu} \mathrm{e}^{-\frac{\nu}{2}x}. \tag{A.12}$$

B Annex – Proof of Theorem 2.4.

B.1 The beginning.

The starting-point consists in rewriting the difference between the two traces involving the difference between the semigroup kernels. Since $\forall L \in (0,\infty]$ and $\forall \kappa > 0$ the semigroup $\{G_{L,\kappa}(t)\}_{t>0}$ is a trace class operator with a jointly continuous integral kernel (see Appendix 2.3) then:

$$\mathrm{Tr}_{L^2(\mathbb{R}^d)}\{G_{\infty,\kappa}(t)\} - \mathrm{Tr}_{L^2(\Lambda_L^d)}\{G_{L,\kappa}(t)\} = \mathscr{Y}_{L,\kappa}^{(d)}(t) + \mathscr{Z}_{L,\kappa}^{(d)}(t)$$

with $\forall d \in \{1,2,3\}$, $\forall L \in (0,\infty)$, $\forall \kappa > 0$ and $\forall t > 0$:

$$\mathscr{Y}_{L,\kappa}^{(d)}(t) := \int_{\Lambda_L^d} \mathrm{d}\mathbf{x}\, \{G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{x};t) - G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{x};t)\}, \tag{B.1}$$

$$\mathscr{Z}_{L,\kappa}^{(d)}(t) := \int_{\mathbb{R}^d \setminus \Lambda_L^d} \mathrm{d}\mathbf{x}\, G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{x};t). \tag{B.2}$$

We firstly establish:

Lemma B.1. *$\forall d \in \{1,2,3\}$, $\forall L \in (0,\infty)$, $\forall \kappa > 0$ and $\forall t > 0$:*

$$\mathscr{Z}_{L,\kappa}^{(d)}(t) \leq \Big(\frac{1}{2\sinh(\frac{\kappa}{2}t)}\Big)^{d} \mathrm{e}^{-d\kappa\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}. \tag{B.3}$$

Proof of Lemma B.1. Let $\beta > 0$ and $\kappa > 0$ be fixed. By virtue of (2.26), it is enough to treat only the case of $d = 1$. From (2.25) and by setting $x = y$, one has by direct computations:

$$\forall L \in (0,\infty),\, \forall t > 0, \quad \mathscr{Z}_{L,\kappa}^{(d=1)}(t) = \frac{1}{\sqrt{2\sinh(\kappa t)\tanh(\frac{\kappa}{2}t)}}\mathrm{erfc}\Big(\sqrt{\kappa\tanh(\frac{\kappa}{2}t)}\frac{L}{2}\Big),$$

where erfc denotes the complementary error function defined in [1, Eq. (7.1.2)]. From the Chernoff inequality reading as: $\forall \alpha \geq 0$, $\mathrm{erfc}(\alpha) \leq \mathrm{e}^{-\alpha^2}$ along with the identity (A.5):

$$\forall L \in (0,\infty),\, \forall t > 0, \quad \mathscr{Z}_{L,\kappa}^{(d=1)}(t) \leq \frac{1}{2\sinh(\frac{\kappa}{2}t)}\mathrm{e}^{-\kappa\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}. \qquad \square$$

In view of (B.1), we need an expression for the difference between the two semigroup kernels:

Lemma B.2. *$\forall L \in (0,\infty)$, $\forall \kappa > 0$ and $\forall t > 0$:*

$$\forall (x,y) \in \Lambda_L^2, \quad G_{\infty,\kappa}^{(1)}(x,y;t) - G_{L,\kappa}^{(1)}(x,y;t) =$$
$$-\frac{1}{2}\int_0^t \mathrm{d}s\, \{G_{\infty,\kappa}^{(1)}(x,-\frac{L}{2};s)(\partial_z G_{L,\kappa}^{(1)})(-\frac{L}{2},y;t-s) - G_{\infty,\kappa}^{(1)}(x,\frac{L}{2};s)(\partial_z G_{L,\kappa}^{(1)})(\frac{L}{2},y;t-s)\}, \tag{B.4}$$

and in the case of $d = 2,3$:

$$\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}, \quad G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t) - G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t) =$$
$$-\frac{1}{2}\int_0^t \mathrm{d}s \int_{\partial\Lambda_L^d} \mathrm{d}\sigma(\mathbf{z})\, G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{z};s)[\mathbf{n}_{\mathbf{z}} \cdot \nabla_{\mathbf{z}} G_{L,\kappa}^{(d)}(\mathbf{z},\mathbf{y};t-s)], \tag{B.5}$$

where $\mathrm{d}\sigma(\mathbf{z})$ is the measure on $\partial\Lambda_L^d$ and $\mathbf{n}_{\mathbf{z}}$ is the outer normal to $\partial\Lambda_L^d$ at $\mathbf{z}$.

The proof of the above Lemma in the case of $d = 3$ can be found in [10, Lem. 4.2]. Since the generalization to $d = 1, 2$ can be easily obtained by the same arguments, we do not give a proof.

In view of (B.1) together with the expressions in Lemma B.2, the actual problem boils down to establishing a sufficiently sharp estimate on the gradient of the finite-volume semigroup kernel (remind that the kernel $G_{\infty,\kappa}^{(d)}$ is explicitly known) allowing to bring out an exponential decay in L^2 for the quantity in (B.1). To achieve that, we need to establish a quite sharp approximation of the semigroup $G_{L,\kappa}(t)$, $L \in (0, \infty)$ from $G_{\infty,\kappa}(t)$. This is the aim of the following section.

B.2 Approximation via a geometric perturbation theory (GPT).

The key idea consists in isolating in Λ_L^d the region close to the boundary from the bulk where the semigroup $G_{\infty,\kappa}(t)$ will act. The underlying difficulty is to keep a good control of the remainder terms arising from this approximation. This will be achieved by the use of well-chosen cutoff functions. We mention that our method is borrowed from [14]; see also [15, 16].

For any $0 < \eta < 1$, $0 < \vartheta \leq 1000$, $d \in \{1, 2, 3\}$ and $L \in (0, \infty)$ define:

$$\Theta_{L,\eta}(\vartheta) := \{\mathbf{x} \in \overline{\Lambda_L^d} : \mathrm{dist}(\mathbf{x}, \partial\Lambda_L^d) \leq \vartheta L^\eta\}. \tag{B.6}$$

For L sufficiently large, $\Theta_{L,\eta}(\vartheta)$ models a 'thin' compact subset of Λ_L^d near the boundary with Lebesgue-measure $|\Theta_{L,\eta}(\vartheta)|$ of order $\mathcal{O}(L^{(d-1)+\eta})$. For any $0 < \eta < 1$, let $L_0 = L_0(\eta) \geq 1$ s.t.

$$\Theta_{L_0,\eta}(1000) \subsetneq \Lambda_{L_0}^d, \quad L_0 - L_0^\eta \geq L_0/\sqrt{2}, \tag{B.7}$$

and L_0 large enough. Let us now introduce some well-chosen family of smooth cutoff functions. Let $f_{L,\eta}$ and $f_{L,\eta}^c$, $L \in [L_0(\eta), \infty)$ be a partition of the unity of Λ_L^d satisfying:

$$f_{L,\eta} + f_{L,\eta}^c = 1 \quad \text{on } \Lambda_L^d;$$
$$\mathrm{Supp}(f_{L,\eta}) \subset (\Lambda_L^d \setminus \Theta_{L,\eta}(\frac{1}{16})), \quad f_{L,\eta} = 1 \text{ when } \mathbf{x} \in (\Lambda_L^d \setminus \Theta_{L,\eta}(\frac{1}{8})), \quad 0 \leq f_{L,\eta} \leq 1;$$
$$\mathrm{Supp}(f_{L,\eta}^c) \subset \Theta_{L,\eta}(\frac{1}{8}), \quad f_{L,\eta}^c = 1 \text{ when } \mathbf{x} \in \Theta_{L,\eta}(\frac{1}{16}).$$

Moreover there exists a constant $C > 0$ s.t.

$$\forall L \geq L_0(\eta), \quad \|D^\sigma f_{L,\eta}\|_\infty \leq C L^{-|\sigma|\eta}, \quad \forall |\sigma| \leq 2,\ |\sigma| = \sigma_1 + \cdots + \sigma_d.$$

Also let $\hat{f}_{L,\eta}$ and $\hat{\hat{f}}_{L,\eta}$, $L \in [L_0(\eta), \infty)$ satisfying:

$$\mathrm{Supp}(\hat{f}_{L,\eta}) \subset (\Lambda_L^d \setminus \Theta_{L,\eta}(\frac{1}{64})), \quad \hat{f}_{L,\eta} = 1 \text{ when } \mathbf{x} \in (\Lambda_L^d \setminus \Theta_{L,\eta}(\frac{1}{32})), \quad 0 \leq \hat{f}_{L,\eta} \leq 1;$$
$$\mathrm{Supp}(\hat{\hat{f}}_{L,\eta}) \subset \Theta_{L,\eta}(\frac{1}{2}), \quad \hat{\hat{f}}_{L,\eta} = 1 \text{ when } \mathbf{x} \in \Theta_{L,\eta}(\frac{1}{4}), \quad 0 \leq \hat{\hat{f}}_{L,\eta} \leq 1.$$

Moreover there exists another constant $C > 0$ s.t.

$$\forall L \geq L_0(\eta), \quad \max\{\|D^\sigma \hat{f}_{L,\eta}\|_\infty, \|D^\sigma \hat{\hat{f}}_{L,\eta}\|_\infty\} \leq C L^{-|\sigma|\eta}, \quad \forall |\sigma| \leq 2.$$

With these properties, one straightforwardly gets:

$$\hat{f}_{L,\eta} f_{L,\eta} = f_{L,\eta}; \tag{B.8}$$
$$\mathrm{dist}(\mathrm{Supp}(D^\sigma \hat{f}_{L,\eta}), \mathrm{Supp}(D^\tau f_{L,\eta})) \geq C L^\eta, \quad \forall 1 \leq |\sigma| \leq 2,\ \forall 0 \leq |\tau| \leq 2; \tag{B.9}$$
$$\hat{\hat{f}}_{L,\eta} f_{L,\eta}^c = f_{L,\eta}^c; \tag{B.10}$$
$$\mathrm{dist}(\mathrm{Supp}(D^\sigma \hat{\hat{f}}_{L,\eta}), \mathrm{Supp}(D^\tau f_{L,\eta}^c)) \geq C L^\eta, \quad \forall 1 \leq |\sigma| \leq 2,\ \forall 0 \leq |\tau| \leq 2, \tag{B.11}$$

for some L-independent constants $C > 0$.

Afterwards, let us define $\forall 0 < \eta < 1$, $\forall L \in [L_0(\eta), \infty)$ (see (B.7)) and $\forall \kappa > 0$ on $\mathcal{C}_0^\infty(\Lambda_L^d)$:

$$h_{L,\kappa,\eta} := \frac{1}{2}(-i\nabla_{\mathbf{x}})^2 + \frac{1}{2}\kappa^2 V_{L,\eta}(\mathbf{x}), \quad V_{L,\eta}(\mathbf{x}) := \begin{cases} |\mathbf{x}|^2 & \text{if } \mathbf{x} \in \mathrm{Supp}(\hat{\hat{f}}_{L,\eta}), \\ \frac{1}{4}(L - L^\eta)^2 & \text{otherwise} \end{cases} . \tag{B.12}$$

By standard arguments, (B.12) extends to a family of self-adjoint and semi-bounded operators for any $L \in [L_0(\eta), \infty)$, denoted again by $h_{L,\kappa,\eta}$, with domain $D(h_{L,\kappa,\eta}) = \mathcal{H}_0^1(\Lambda_L^d) \cap \mathcal{H}^2(\Lambda_L^d)$. $\forall 0 < \eta < 1$, $\forall L \in [L_0(\eta), \infty)$ and $\forall \kappa > 0$, let $\{g_{L,\kappa,\eta}(t) := \mathrm{e}^{-th_{L,\kappa,\eta}} : L^2(\Lambda_L^d) \to L^2(\Lambda_L^d)\}_{t \geq 0}$ be the strongly-continuous one-parameter semigroup generated by $h_{L,\kappa,\eta}$. It is an integral operator with an integral kernel jointly continuous in $(\mathbf{x}, \mathbf{y}, t) \in \overline{\Lambda_L^d} \times \overline{\Lambda_L^d} \times (0, \infty)$. We denote it by $g_{L,\kappa,\eta}^{(d)}$.

Next, introduce $\forall 0 < \eta < 1$, $\forall L \in [L_0(\eta), \infty)$ and $\forall \kappa > 0$ the following operators on $L^2(\Lambda_L^d)$:

$$\forall t > 0, \quad \mathcal{G}_{L,\kappa,\eta}(t) := \hat{f}_{L,\eta} G_{\infty,\kappa}(t) f_{L,\eta} + \hat{\hat{f}}_{L,\eta} g_{L,\kappa,\eta}(t) f_{L,\eta}^c, \tag{B.13}$$

$$\begin{aligned} \mathcal{W}_{L,\kappa,\eta}(t) : = & -\{\frac{1}{2}(\Delta \hat{f}_{L,\eta}) + i(\nabla \hat{f}_{L,\eta}) \cdot (-i\nabla)\} G_{\infty,\kappa}(t) f_{L,\eta} \\ & - \{\frac{1}{2}(\Delta \hat{\hat{f}}_{L,\eta}) + i(\nabla \hat{\hat{f}}_{L,\eta}) \cdot (-i\nabla)\} g_{L,\kappa,\eta}(t) f_{L,\eta}^c. \end{aligned} \tag{B.14}$$

Sometimes we will use the notations:

$$\forall t > 0, \quad \mathcal{G}_{L,\kappa,\eta}^{(p)}(t) := \hat{f}_{L,\eta} G_{\infty,\kappa}(t) f_{L,\eta}, \quad \mathcal{G}_{L,\kappa,\eta}^{(r)}(t) := \hat{\hat{f}}_{L,\eta} g_{L,\kappa,\eta}(t) f_{L,\eta}^c. \tag{B.15}$$

Below we establish a Duhamel-like formula involving the operators defined in (B.13)-(B.14):

Proposition B.3. *$\forall d \in \{1, 2, 3\}$, $\forall 0 < \eta < 1$, $\forall L \in [L_0(\eta), \infty)$ and $\forall \kappa > 0$, it takes place in the bounded operators sense on $L^2(\Lambda_L^d)$:*

$$\forall t > 0, \quad G_{L,\kappa}(t) = \mathcal{G}_{L,\kappa,\eta}(t) - \int_0^t \mathrm{d}s \, G_{L,\kappa}(t - s) \mathcal{W}_{L,\kappa,\eta}(s). \tag{B.16}$$

This result leans on [13, Prop. 3] that we reproduce here for reader's convenience:

Proposition B.4. *Let $\mathscr{H}$ be a separable Hilbert space and H be a self-adjoint and positive operator having the domain $D \subset \mathscr{H}$. Fix $t_0 > 0$. Assume that there exists an application $(0, t_0] \ni t \mapsto S(t) \in \mathfrak{B}(\mathscr{H})$ (the algebra of bounded operators on $\mathscr{H}$) with the following properties:*
(A). *$\sup_{0 < t \leq t_0} \|S(t)\| \leq c_1 < \infty$;*
(B). *It is strongly differentiable, $\mathrm{Ran}(S(t)) \subset D$ and $s - \lim_{t \downarrow 0} S(t) = \mathbb{1}$;*
(C). *There exists an application $(0, t_0] \ni t \mapsto R(t) \in \mathfrak{B}(\mathscr{H})$ continuous in the operator-norm sense s.t. $\|R(t)\| \leq c_2 t^{-\alpha}$ where $0 \leq \alpha < 1$, and:*

$$\frac{\partial S}{\partial t}(t)\phi + HS(t)\phi = R(t)\phi.$$

Then the following two statements are true:
(i). *The sequence of bounded operators $(n > [1/t])$:*

$$T_n(t) := \int_{\frac{1}{n}}^{t - \frac{1}{n}} \mathrm{d}s \, \exp[-(t - s)H] R(s),$$

converges in norm; let $T(t)$ be its limit;
(ii). *The following equality takes place on $\mathfrak{B}(\mathscr{H})$: $\exp(-tH) = S(t) - T(t)$.*

Before giving the proof of Proposition B.3, we need a series of estimates related with the kernel of the semigroup generated by the operator defined in (B.12). The proof of the below Lemma can be found in the appendix of this section, see Sec. B.5.

Lemma B.5. *$\forall d\in\{1,2,3\}$ there exists a constant $C_d>0$ s.t. $\forall 0<\eta<1$, $\forall L\in[L_0(\eta),\infty)$, $\forall \kappa>0$, $\forall(\mathbf{x},\mathbf{y})\in\Lambda_L^{2d}$ and $\forall t>0$:*

$$g_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t)\leq C_d \mathrm{e}^{-\frac{\kappa^2}{4}\frac{L^2}{4}t}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};t), \tag{B.17}$$

$$|\nabla_{\mathbf{x}} g_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t)|\leq C_d \frac{(1+t)^d}{\sqrt{t}}\mathrm{e}^{-\frac{\kappa^2}{8}\frac{L^2}{4}t}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};2t), \tag{B.18}$$

$$|\Delta_{\mathbf{x}} g_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t)|\leq C_d \frac{(1+t)^{2d}}{t}\mathrm{e}^{-\frac{\kappa^2}{16}\frac{L^2}{4}t}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};2t). \tag{B.19}$$

Here $G_{\infty,0}^{(d)}(\cdot\,,\cdot;t)$ denotes the heat kernel which is defined in (2.24)-(2.26).

Remark B.6. One can derive the following upper bounds on the operator norms. $\forall d\in\{1,2,3\}$ there exist two constants $C_d,c>0$ s.t. $\forall 0<\eta<1$, $\forall L\in[L_0(\eta),\infty)$, $\forall\kappa>0$ and $\forall t>0$:

$$\|\mathcal{G}_{L,\kappa,\eta}(t)\|\leq \|\mathcal{G}_{L,\kappa,\eta}^{(p)}(t)\|+\|\mathcal{G}_{L,\kappa,\eta}^{(r)}(t)\|\leq (\cosh(\kappa t))^{-\frac{d}{2}}+C_d\mathrm{e}^{-\frac{\kappa^2}{16}L^2 t}, \tag{B.20}$$

$$\|\mathcal{W}_{L,\kappa,\eta}(t)\|\leq C_d\sqrt{1+\kappa}\frac{\sqrt{1+t}}{\sqrt{t}}\mathrm{e}^{-c\frac{L^{2\eta}}{t}}\{1+(1+t)^{d-\frac{1}{2}}\mathrm{e}^{-\frac{\kappa^2}{8}\frac{L^2}{4}t}\}, \tag{B.21}$$

(B.20) comes from (2.38) and (B.17). The rough estimate in (B.21) is derived from Lemmas B.5 and 2.9 together with the properties (B.9)-(B.11).

Proof of Proposition B.3. The only thing we have to do is verify the assumptions of Proposition B.4 in which $\mathcal{G}_{L,\kappa,\eta}(t)$ plays the role of $S(t)$. Let $0<\eta<1$, $L\in[L_0(\eta),\infty)$ and $\kappa>0$ kept fixed. (A) From (B.20), $\mathcal{G}_{L,\kappa,\eta}(t)$ is uniformly bounded in t by some constant $C_d>0$. (B) By using that $s-\lim_{t\downarrow 0}G_{\infty,\kappa}(t)=\mathbb{1}$ and $s-\lim_{t\downarrow 0}g_{L,\kappa,\eta}(t)=\mathbb{1}$ in the kernels sense, then:

$$\forall\phi\in L^2(\Lambda_L^d),\quad \lim_{t\downarrow 0}\mathcal{G}_{L,\kappa,\eta}\phi=\{\hat{f}_{L,\eta}f_{L,\eta}+\hat{\hat{f}}_{L,\eta}f_{L,\eta}^c\}\phi=\{f_{L,\eta}+f_{L,\eta}^c\}\phi=\phi,$$

where we used (B.8) and (B.10). Next, let us investigate the strong differentiability. From (B.15):

$$\begin{aligned}\forall\phi\in L^2(\Lambda_L^d),\quad &\frac{1}{\delta t}\{(\mathcal{G}_{L,\kappa,\eta}^{(p)}(t+\delta t)\phi)(\cdot\,)-(\mathcal{G}_{L,\kappa,\eta}^{(p)}(t)\phi)(\cdot\,)\}\\ &=\hat{f}_{L,\eta}(\cdot\,)\frac{1}{\delta t}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{y}\Big\{\int_{\mathbb{R}^d}\mathrm{d}\mathbf{z}\,G_{\infty,\kappa}^{(d)}(\cdot\,,\mathbf{z};t)G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};\delta t)-G_{\infty,\kappa}^{(d)}(\cdot\,,\mathbf{y};t)\Big\}f_{L,\eta}(\mathbf{y})\phi(\mathbf{y}).\end{aligned}$$

Since $G_{\infty,\kappa}(t)L^2(\mathbb{R}^d)\to D(H_{\infty,\kappa})$, then the Stone theorem (in the kernels sense) provides:

$$\begin{aligned}\lim_{\delta t\downarrow 0}\frac{1}{\delta t}\Big\{&\int_{\mathbb{R}^d}\mathrm{d}\mathbf{y}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{z}\,G_{\infty,\kappa}^{(d)}(\cdot\,,\mathbf{z};t)G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};\delta t)f_{L,\eta}(\mathbf{y})\phi(\mathbf{y})+\\ &-\int_{\mathbb{R}^d}\mathrm{d}\mathbf{y}\,G_{\infty,\kappa}^{(d)}(\cdot\,,\mathbf{y};t)f_{L,\eta}(\mathbf{y})\phi(\mathbf{y})\Big\}=-H_{\infty,\kappa}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{y}\,G_{\infty,\kappa}^{(d)}(\cdot\,,\mathbf{y};t)f_{L,\eta}(\mathbf{y})\phi(\mathbf{y}).\end{aligned}$$

By using similar arguments to treat the contribution coming from $\mathcal{G}_{L,\kappa,\eta}^{(r)}(\cdot\,)$, we therefore obtain:

$$\begin{aligned}&\lim_{\delta t\downarrow 0}\frac{1}{\delta t}\{\mathcal{G}_{L,\kappa,\eta}(t+\delta t)\phi-\mathcal{G}_{L,\kappa,\eta}(t)\phi\}\\ &\qquad=-\hat{f}_{L,\eta}H_{\infty,\kappa}G_{\infty,\kappa}(t)f_{L,\eta}\phi-\hat{\hat{f}}_{L,\eta}h_{L,\kappa,\eta}g_{L,\kappa,\eta}(t)f_{L,\eta}^c\phi=:\frac{\partial\mathcal{G}_{L,\kappa,\eta}}{\partial t}(t)\phi.\end{aligned}$$

(C) Let $D_0:=\{\phi\in\mathcal{C}^1(\overline{\Lambda_L^d})\cap\mathcal{C}^2(\Lambda_L^d),\phi|_{\partial\Lambda_L^d}=0,\Delta\phi\in L^2(\Lambda_L^d)\}$ be the domain on which $H_{L,\kappa}$ is essentially self-adjoint. In the weak sense for any $\varphi\in D_0$, $\psi\in\mathcal{C}_0^\infty(\Lambda_L^d)$ and $t>0$:

$$l_L(\varphi,\psi):=\langle H_{L,\kappa}\varphi,\mathcal{G}_{L,\kappa,\eta}(t)\psi\rangle_{L^2(\Lambda_L^d)}=-\langle\varphi,\frac{\partial\mathcal{G}_{L,\kappa,\eta}}{\partial t}(t)\psi\rangle_{L^2(\Lambda_L^d)}+\langle\varphi,\mathcal{W}_{L,\kappa,\eta}(t)\psi\rangle_{L^2(\Lambda_L^d)},$$

where the operator $\mathcal{W}_{L,\kappa,\eta}(t)$ is defined in (B.14). Let us mention that the second equality is obtained by performing some integration by parts, and afterwards by using the following identities:

$$H_{L,\kappa}\hat{f}_{L,\eta} = H_{\infty,\kappa}\hat{f}_{L,\eta} = [H_{\infty,\kappa}, \hat{f}_{L,\eta}] + \hat{f}_{L,\eta}H_{\infty,\kappa},$$

as well as (remind that the potential V_L in (B.12) satisfies $V_L(\mathbf{x}) = |\mathbf{x}|^2$ on $\mathrm{Supp}(\hat{\hat{f}}_{L,\eta})$):

$$H_{L,\kappa}\hat{\hat{f}}_{L,\eta} = h_{L,\kappa,\eta}\hat{\hat{f}}_{L,\eta} = [h_{L,\kappa,\eta}, \hat{\hat{f}}_{L,\eta}] + \hat{\hat{f}}_{L,\eta}h_{L,\kappa,\eta}.$$

Since $l_L(\varphi,\cdot\,)$ is a bounded linear functional $\forall\varphi \in D_0$ then $\mathcal{C}_0^\infty(\Lambda_L^d) \ni \psi \mapsto l_L(\varphi,\psi)$ can be extended in a linear and bounded functional on $L^2(\Lambda_L^d)$ by the B.L.T. theorem. As well since $l_L(\cdot\,,\psi)$ is a bounded linear functional $\forall\psi \in L^2(\Lambda_L^d)$ then $\varphi \mapsto l_L(\varphi,\psi)$ can be extended on the self-adjointness domain $D(H_{L,\kappa})$. This means that $\forall t > 0$, $\mathrm{Ran}\big(\mathcal{G}_{L,\kappa,\eta}(t)\big) \subset D(H_{L,\kappa})$. Hence:

$$\langle\varphi, H_{L,\kappa}\mathcal{G}_{L,\kappa,\eta}(t)\psi\rangle_{L^2(\Lambda_L^d)} = -\langle\varphi, \frac{\partial\mathcal{G}_{L,\kappa,\eta}}{\partial t}(t)\psi\rangle_{L^2(\Lambda_L^d)} + \langle\varphi, \mathcal{W}_{L,\kappa,\eta}(t)\psi\rangle_{L^2(\Lambda_L^d)}.$$

By a density argument we then deduce:

$$\forall t > 0, \quad \frac{\partial\mathcal{G}_{L,\kappa,\eta}}{\partial t}(t)\psi + H_{L,\kappa}\mathcal{G}_{L,\kappa,\eta}(t)\psi = \mathcal{W}_{L,\kappa,\eta}(t)\psi.$$

Finally, from (B.21) $\|\mathcal{W}_{L,\kappa,\eta}(t)\| \leq Ct^{-\frac{1}{2}}$ $\forall 0 < t \leq 1$. Hence $\|\mathcal{W}_{L,\kappa,\eta}(t)\|$ is integrable in $t \sim 0$. $\square$

B.3 The crucial estimate from the GPT.

From the approximation obtained in the previous section (see Proposition B.3 along with Remark B.6), we are in the position to give a sufficiently sharp estimate on the gradient of the kernel of the finite-volume semigroup involved in (B.4)-(B.5). In particular, one has:

Proposition B.7. $\forall d \in \{1,2,3\}$ *there exists a constant* $C_d > 0$ *and* $\forall \frac{1}{4} < \eta < 1$, $\forall 0 < \kappa_0 < 1$ *there exists a* $\mathcal{L}_{\kappa_0}(\eta) > 0$ *s.t.* $\forall L \in [\mathcal{L}_{\kappa_0}(\eta), \infty)$, $\forall \kappa \in [\kappa_0,\infty)$, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ *and* $\forall t > 0$*:*

$$|\nabla_{\mathbf{x}} G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t)| \leq C_d\{\mathcal{P}_{\infty,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) + \mathcal{R}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t)\}, \tag{B.22}$$

$$\mathcal{P}_{\infty,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) := (1+\sqrt{\kappa})(1+t)^{\frac{5}{2}}\sqrt{\coth(\frac{\kappa}{2}t)}\,G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t,8); \tag{B.23}$$

$$\mathcal{R}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) := \kappa^{\frac{d}{2}}(1+\kappa)^{\frac{d}{2}}\frac{t^{\frac{d-1}{2}}}{(\sinh(\kappa t))^{\frac{d}{2}}}(1+t)^{\frac{5d}{2}+1}\mathrm{e}^{-\frac{\kappa}{16}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};4t). \tag{B.24}$$

Here $G_{\infty,\kappa}^{(d)}(\cdot\,,\cdot\,;t,\gamma)$, $\kappa > 0$ *and* $\gamma > 0$ *is defined in* (2.29) *and* $G_{\infty,0}^{(d)}(\cdot\,,\cdot\,;t)$ *in* (2.24)-(2.26).

Remark B.8. In the estimate (B.24), we artificially made appear the singularity $(\sinh(\kappa t))^{\frac{d}{2}}$ in the denominator for the need of the proof of Theorem 2.3. The price to pay is that for small values of κ, the estimate holds for L large enough chosen accordingly. When $\kappa \geq 1$ the estimate holds for $L \geq L_0(\eta)$ large enough uniformly in κ. In great generality, one can prove: $\forall d \in \{1,2,3\}$ there exists a $C_d > 0$ s.t. $\forall \frac{1}{4} < \eta < 1$, $\forall L \in [L_0(\eta),\infty)$, $\forall \kappa > 0$, $\forall(\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$|\nabla_{\mathbf{x}} G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t)| \leq C_d\{\mathcal{P}_{\infty,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) + \hat{\mathcal{R}}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t)\},$$

with $\mathcal{P}_{\infty,\kappa,\eta}^{(d)}$ defined in (B.23) and:

$$\hat{\mathcal{R}}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) := \frac{(1+t)^{2d+1}}{\sqrt{t}}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};4t).$$

The starting-point in the proof of Proposition B.7 is the Duhamel-like formula in (B.16). Taking its adjoint, one has $\forall d \in \{1,2,3\}$, $\forall 0<\eta<1$, $\forall L \in [L_0(\eta),\infty)$ and $\forall \kappa>0$ on $L^2(\Lambda_L^d)$:

$$\forall t>0, \quad G_{L,\kappa}(t) = \mathcal{G}_{L,\kappa,\eta}^*(t) - \int_0^t \mathrm{d}s\, \mathcal{W}_{L,\kappa,\eta}^*(s) G_{L,\kappa}(t-s), \tag{B.25}$$

where the adjoint operator of $\mathcal{G}_{L,\kappa,\eta}(t)$ and $\mathcal{W}_{L,\kappa,\eta}(t)$ reads respectively as, see (B.13)-(B.14):

$$\mathcal{G}_{L,\kappa,\eta}^*(t) = f_{L,\eta} G_{\infty,\kappa}(t) \hat{f}_{L,\eta} + f_{L,\eta}^c g_{L,\kappa,\eta}(t) \hat{\hat{f}}_{L,\eta}, \tag{B.26}$$

$$\begin{aligned}\mathcal{W}_{L,\kappa,\eta}^*(t) = &-f_{L,\eta} G_{\infty,\kappa}(t) \frac{1}{2}(\Delta \hat{f}_{L,\eta}) - i f_{L,\eta}\{(-i\nabla) G_{\infty,\kappa}(t) - [(-i\nabla), G_{\infty,\kappa}(t)]\}(\nabla \hat{f}_{L,\eta}) \\ &- f_{L,\eta}^c g_{L,\kappa,\eta}(t) \frac{1}{2}(\Delta \hat{\hat{f}}_{L,\eta}) - i f_{L,\eta}^c\{(-i\nabla) g_{L,\kappa,\eta}(t) - [(-i\nabla), g_{L,\kappa,\eta}(t)]\}(\nabla \hat{\hat{f}}_{L,\eta}),\end{aligned} \tag{B.27}$$

where in the bounded operators sense:

$$[(-i\nabla), G_{\infty,\kappa}(t)] = -\int_0^t \mathrm{d}s\, G_{\infty,\kappa}(t-s)[(-i\nabla), H_{\infty,\kappa}] G_{\infty,\kappa}(s), \tag{B.28}$$

$$[(-i\nabla), g_{L,\kappa,\eta}(t)] = -\int_0^t \mathrm{d}s\, g_{L,\kappa,\eta}(t-s)[(-i\nabla), h_{L,\kappa,\eta}] g_{L,\kappa,\eta}(s). \tag{B.29}$$

Writing (B.25) in the kernels sense, it follows this identity which holds $\forall(\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t>0$:

$$\nabla_{\mathbf{x}} G_{L,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t) = \nabla_{\mathbf{x}} (\mathcal{G}_{L,\kappa,\eta}^*)^{(d)}(\mathbf{x},\mathbf{y};t) - \int_0^t \mathrm{d}s \int_{\Lambda_L^d} \mathrm{d}\mathbf{z}\, \nabla_{\mathbf{x}} (\mathcal{W}_{L,\kappa,\eta}^*)^{(d)}(\mathbf{x},\mathbf{z};s) G_{L,\kappa}^{(d)}(\mathbf{z},\mathbf{y};t-s). \tag{B.30}$$

Next we need the following estimates whose proves lie in the appendix, see Sec. B.5:

Lemma B.9. $\forall d \in \{1,2,3\}$ *there exist two constants* $c, C_d > 0$ *s.t.:*
(i) $\forall 0<\eta<1$, $\forall L \in [L_0(\eta),\infty)$, $\forall \kappa>0$, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ *and* $\forall t>0$*:*

$$|\nabla_{\mathbf{x}} (\mathcal{G}_{L,\kappa,\eta}^*)^{(d)}(\mathbf{x},\mathbf{y};t)| \leq C_d\{P_{\infty,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) + R_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t)\}, \tag{B.31}$$

$$P_{\infty,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) := (1+\sqrt{\kappa})\sqrt{\coth(\frac{\kappa}{2}t)}\, G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t,2), \tag{B.32}$$

$$R_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) := \frac{(1+t)^d}{\sqrt{t}} \mathrm{e}^{-\frac{\kappa^2}{8}\frac{L^2}{4}t} G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};2t). \tag{B.33}$$

(ii). $\forall \frac{1}{4}<\eta<1$, $\forall L \in [L_0(\eta),\infty)$, $\forall \kappa>0$, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ *and* $\forall t>0$*:*

$$|\nabla_{\mathbf{x}} (\mathcal{W}_{L,\kappa,\eta}^*)^{(d)}(\mathbf{x},\mathbf{y};t)| \leq C_d\{r_{\infty,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) + r_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t)\}, \tag{B.34}$$

$$r_{\infty,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) := (1+\sqrt{\kappa})\sqrt{\coth(\frac{\kappa}{2}t)}(1+t)\mathrm{e}^{-c\kappa L^{2\eta}\coth(\frac{\kappa}{2}t)} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t,8), \tag{B.35}$$

$$r_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) := \frac{(1+t)^{2d}}{\sqrt{t}} \mathrm{e}^{-\frac{\kappa^2}{16}\frac{L^2}{4}t} \mathrm{e}^{-c\frac{L^{2\eta}}{t}} \chi_{\Theta_{L,\eta}(\frac{1}{8})}(\mathbf{x}) G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};4t) \chi_{\Theta_{L,\eta}(\frac{1}{2})}(\mathbf{y}). \tag{B.36}$$

Here $\chi_{\Theta_{L,\eta}(\vartheta)}$*,* $\vartheta>0$ *denotes the indicator function associated with* $\Theta_{L,\eta}(\vartheta)$ *defined in* (B.6).

Remark B.10. We restricted the η to $(\frac{1}{4},1)$ in (ii) only to make the estimates more elegant.

Proof of Proposition B.7. In view of the second term in the r.h.s. of (B.30), (B.34) with (B.35)-(B.36) and (2.27), we need to estimate the following two quantities:

$$\mathcal{Q}_{\infty,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) := \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}\mathbf{z}\, r_{\infty,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{z};s) G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};t-s,1), \tag{B.37}$$

$$\mathcal{Q}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) := \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}\mathbf{z}\, r_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{z};s) G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};t-s,1). \tag{B.38}$$

Let $d \in \{1,2,3\}$ and $\frac{1}{4} < \eta < 1$ kept fixed. For the moment, let $L \geq L_0(\eta)$ defined in (B.7). We start with (B.37). From (B.35) followed by (2.37), then $\forall \kappa > 0$, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$\mathcal{Q}_{\infty,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) \leq C(1+\sqrt{\kappa})(1+t)\frac{\sqrt{\coth(\frac{\kappa}{2}t)}}{\sqrt{\coth(\frac{\kappa}{2}t)}} G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t,8)\int_0^t \mathrm{d}s\,\sqrt{\coth(\frac{\kappa}{2}s)}\mathrm{e}^{-c\kappa L^{2\eta}\coth(\frac{\kappa}{2}s)}, \tag{B.39}$$

for some constants $c, C > 0$. By using (A.12) to get rid of the coth in the integral w.r.t. s, along with the lower bound in (A.4) in the r.h.s. of (B.39), then the upper bound (B.23) follows by gathering the above estimate and (B.32) together. Let us turn to the quantity in (B.38). From (B.36), one has $\forall \kappa > 0$, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$\mathcal{Q}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) \leq (1+t)^{2d}\int_0^t \mathrm{d}s\,\frac{\mathrm{e}^{-\frac{\kappa^2}{16}\frac{L^2}{4}s}\mathrm{e}^{-c\frac{L^{2\eta}}{s}}}{\sqrt{s}}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{z}\,G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{z};4s)\chi_{\Theta_{L,\eta}(\frac{1}{2})}(\mathbf{z})G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};t-s,1).$$

We want to make appear from the integration over $\mathbb{R}^d$ an exponential decay in L^2 and having the argument $t-s$. We emphasize that the presence of the indicator function $\chi_{\Theta_{L,\eta}(\frac{1}{2})}$ in this integral plays a crucial role. Indeed let us remark that on $\mathbb{R}^{2d}$, for any $0 < s < t$:

$$\begin{aligned}
G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};t-s,1) &= \Big(\prod_{j=1}^d \mathrm{e}^{-\frac{\kappa}{8}(z_j+y_j)^2\tanh(\frac{\kappa}{2}(t-s))}\mathrm{e}^{-\frac{\kappa}{8}(z_j-y_j)^2\coth(\frac{\kappa}{2}(t-s))}\Big)G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};t-s,2)\\
&\leq \Big(\prod_{j=1}^d \mathrm{e}^{-2\frac{\kappa}{8}(z_j^2+y_j^2)\tanh(\frac{\kappa}{2}(t-s))}\Big)G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};t-s,2)\\
&\leq \mathrm{e}^{-\frac{\kappa}{4}|\mathbf{z}|^2\tanh(\frac{\kappa}{2}(t-s))}G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};t-s,2).
\end{aligned} \tag{B.40}$$

To get the second inequality, we expanded the squares and using that $ab \leq \frac{1}{2}(a^2+b^2)$ combined with the fact that $\coth(\alpha) - \tanh(\alpha) \geq 0$ $\forall \alpha > 0$. Since $\forall \mathbf{z} \in \Theta_{L,\eta}(\frac{1}{2})$, $|\mathbf{z}| \geq \frac{1}{2}(L - L^\eta)$ (remind that $L \geq L_0(\eta)$, and $L_0(\eta)$ satisfies (B.7)) then one has $\forall \kappa > 0$, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$\begin{aligned}
\mathcal{Q}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) \leq (1+t)^{2d}\int_0^t \mathrm{d}s\,\frac{s^{\frac{d-1}{2}}}{s^{\frac{d-1}{2}}}\frac{\mathrm{e}^{-\frac{\kappa^2}{16}\frac{L^2}{4}s}\mathrm{e}^{-c\frac{L^{2\eta}}{s}}}{\sqrt{s}}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}(t-s))}\times\\
\times\int_{\Theta_{L,\eta}(\frac{1}{2})}\mathrm{d}\mathbf{z}\,G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{z};4s)G_{\infty,\kappa}^{(d)}(\mathbf{z},\mathbf{y};t-s,2).
\end{aligned}$$

From the upper bound in the first inequality of (2.31), then extending the integration to $\mathbb{R}^d$ followed by (2.36), one gets under the same conditions:

$$\mathcal{Q}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) \leq C\kappa^d(1+t)^{2d}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};4t)\int_0^t \mathrm{d}s\,\frac{s^{\frac{d-1}{2}}(t-s)^{\frac{d}{2}}\,\mathrm{e}^{\frac{d}{2}\kappa s}\mathrm{e}^{-\frac{\kappa^2}{16}\frac{L^2}{4}s}}{\{\sinh(\kappa s)\sinh(\kappa(t-s))\}^{\frac{d}{2}}}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}(t-s))},$$

for another $C > 0$. Here we used the lower bound in (A.4) and the upper bound in (A.1). For the need, we artificially made appear a $(\sinh(\kappa s))^{\frac{d}{2}}$ under the integration w.r.t. s. The price to pay is the appearance of the term $\mathrm{e}^{\frac{d}{2}\kappa s}$ in the numerator. When $\kappa \geq 1$, we can get rid of it via $\mathrm{e}^{-\frac{\kappa^2}{16}\frac{L^2}{4}s}$ for $L \geq L_0(\eta)$ large enough since $\kappa \leq \kappa^2$. When $0 < \kappa < 1$, we have to choose L large enough accordingly to κ (i.e. $L \geq cste/\sqrt{\kappa}$). Thus, let $\kappa_0 > 0$. Let $\mathcal{L} = \mathcal{L}_{\kappa_0}(\eta) \geq L_0(\eta)$ s.t. $\forall L \geq \mathcal{L}_{\kappa_0}(\eta)$, $\mathrm{e}^{-\frac{\kappa_0^2}{2}(\frac{1}{8}\frac{L^2}{4}-\frac{3}{\kappa_0})s} \leq \mathrm{e}^{-\frac{\kappa_0^2}{32}\frac{L^2}{4}s}$. Then $\forall L \in [\mathcal{L}_{\kappa_0}(\eta),\infty)$, $\forall \kappa \in [\kappa_0,\infty)$, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$\mathcal{Q}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) \leq C\kappa^d(1+t)^{2d}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};4t)\int_0^t \mathrm{d}s\,\frac{s^{\frac{d-1}{2}}(t-s)^{\frac{d}{2}}\,\mathrm{e}^{-\frac{\kappa^2}{32}\frac{L^2}{4}s}}{\{\sinh(\kappa s)\sinh(\kappa(t-s))\}^{\frac{d}{2}}}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}(t-s))}, \tag{B.41}$$

for some constant $C > 0$. The rest of the proof leans on two estimates. The first one is:

$$\mathrm{e}^{-\frac{\kappa^2}{32}\frac{L^2}{4}s}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}(t-s))} \leq \mathrm{e}^{-\frac{\kappa}{16}\frac{L^2}{4}\tanh(\frac{\kappa}{2}s)}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}(t-s))} \leq \mathrm{e}^{-\frac{\kappa}{16}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)},\quad 0 < s < t,$$

justified by the lower bound in (A.4) along with (A.9). The second one is (A.7). Gathering the above estimates together, $\forall L \in [\mathscr{L}_{\kappa_0}(\eta),\infty)$, $\forall \kappa \in [\kappa_0,\infty)$, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$\mathscr{Q}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) \leq C_d\kappa^d\frac{(1+t)^{2d}}{(\sinh(\kappa t))^{\frac{d}{2}}}\mathrm{e}^{-\frac{\kappa}{16}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};4t)\times \\ \times\int_0^t \mathrm{d}s\, s^{\frac{d-1}{2}}(t-s)^{\frac{d}{2}}\{(\coth(\kappa s))^{\frac{d}{2}} + (\coth(\kappa(t-s)))^{\frac{d}{2}}\},$$

for some constant $C_d > 0$. Here we used that $(a+b)^\delta \leq 2^\delta(a^\delta + b^\delta)$ $\forall a,b,\delta > 0$. To conclude this estimate it remains to use that there exists another constant $C > 0$ s.t. $\forall t > 0$ and $\forall \kappa > 0$:

$$\max\{\int_0^t \mathrm{d}s\, s^{\frac{d-1}{2}}(t-s)^{\frac{d}{2}}(\coth(\kappa s))^{\frac{d}{2}}, \int_0^t \mathrm{d}s\, s^{\frac{d-1}{2}}(t-s)^{\frac{d}{2}}(\coth(\kappa(t-s)))^{\frac{d}{2}}\} \leq C\frac{(1+\kappa)^{\frac{d}{2}}}{\kappa^{\frac{d}{2}}}(1+t)^{\frac{d}{2}}t^{\frac{d+1}{2}}. \tag{B.42}$$

To get (B.24), we have to modify the upper bound in (B.33) by mimicking the same method than the one used above to make appear the singularity $(\sinh(\kappa t))^{\frac{d}{2}}$ in the denominator (instead of $\sqrt{t}$). □

B.4 End of the proof.

To conclude the proof of Theorem 2.4, one establishes this last result in which we give an upper bound for the quantity defined in (B.1):

Proposition B.11. *$\forall d \in \{1,2,3\}$ there exists a constant $C_d > 0$ and $\forall \frac{1}{4} < \eta < 1$, $\forall 0 < \kappa_0 < 1$ there exists a $\mathscr{L}_{\kappa_0}(\eta) > 0$ s.t. $\forall L \in [\mathscr{L}_{\kappa_0}(\eta),\infty)$, $\forall \kappa \in [\kappa_0,\infty)$ and $\forall t > 0$:*

$$|\mathscr{Y}_{L,\kappa}^{(d)}(t)| \leq C_d(1+\sqrt{\kappa})(1+\kappa)^d(1+t)^{3(d+\frac{1}{2})}\Big(\frac{1}{2\sinh(\frac{\kappa}{2}t)}\Big)^d\mathrm{e}^{-\frac{\kappa}{32}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}. \tag{B.43}$$

Remark B.12. We stress the point that the estimate in Proposition B.11 is given in a suitable form for the proof of Theorem 2.3. Actually, in great generality one can prove: $\forall d \in \{1,2,3\}$ there exists a $C_d > 0$ s.t. $\forall \frac{1}{4} < \eta < 1$, $\forall L \in [L_0(\eta),\infty)$, $\forall \kappa > 0$ and $\forall t > 0$:

$$|\mathscr{Y}_{L,\kappa}^{(d)}(t)| \leq C_d(1+t)^{2d+\frac{3}{2}}\Big\{(1+\sqrt{\kappa})^2\Big(\frac{1}{2\sinh(\frac{\kappa}{2}t)}\Big)^d + \Big(\frac{L}{\sqrt{t}}\Big)^{d-1}\Big\}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}.$$

Gathering Lemma B.1 and Proposition B.11 together, then Theorem 2.4 follows by setting $\eta = 1/2$ for instance in Proposition B.11. Let us end this annex by the proof of Proposition B.11.

Proof of Proposition B.11. Let $\frac{1}{4} < \eta < 1$ and $\kappa_0 > 0$ kept fixed. Let us start with the case of $d = 1$. Let us denote $\varsigma_L = \pm\frac{L}{2}$. In view of (B.4) and (B.22), we need to estimate $\forall L \in [\mathscr{L}_{\kappa_0}(\eta),\infty)$:

$$\forall t > 0,\quad \mathscr{Y}_{L,\kappa,\eta}^{(d=1),1}(t) := \frac{1}{2}\int_0^t \mathrm{d}s\int_{\Lambda_L^1}\mathrm{d}x\, G_{\infty,\kappa}^{(d=1)}(x,\varsigma_L;s,1)\mathscr{P}_{\infty,\kappa,\eta}^{(d=1)}(\varsigma_L,x;t-s), \tag{B.44}$$

$$\mathscr{Y}_{L,\kappa,\eta}^{(d=1),2}(t) := \frac{1}{2}\int_0^t \mathrm{d}s\int_{\Lambda_L^1}\mathrm{d}x\, G_{\infty,\kappa}^{(d=1)}(x,\varsigma_L;s,1)\mathscr{R}_{L,\kappa,\eta}^{(d=1)}(\varsigma_L,x;t-s). \tag{B.45}$$

Here we have commuted the two integrals, this will be justify by what follows. Let us start with the quantity in (B.44). In view of (2.25) and (B.23), then from (2.37) $\forall L \in [\mathscr{L}_{\kappa_0}(\eta),\infty)$, $\forall \kappa \in [\kappa_0,\infty)$:

$$\forall t > 0,\quad \mathscr{Y}_{L,\kappa,\eta}^{(d=1),1}(t) \leq C\sqrt{\kappa}(1+\sqrt{\kappa})(1+t)^{\frac{5}{2}}\frac{1}{\sqrt{2\sinh(\kappa t)}}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}\int_0^t \mathrm{d}s\sqrt{\coth(\frac{\kappa}{2}(t-s))}, \tag{B.46}$$

for some constant $C > 0$. By using the upper bound in (A.4) along with the inequality:

$$\frac{1}{\sqrt{2\sinh(\kappa t)}} = \frac{\sqrt{\tanh(\frac{\kappa}{2}t)}}{\sqrt{2\sinh(\kappa t)\tanh(\frac{\kappa}{2}t)}} = \frac{\sqrt{\tanh(\frac{\kappa}{2}t)}}{2\sinh(\frac{\kappa}{2}t)} \leq \frac{1}{2\sinh(\frac{\kappa}{2}t)}, \tag{B.47}$$

justified by (A.5), this leads $\forall L \in [\mathcal{L}_{\kappa_0}(\eta), \infty)$, $\forall \kappa \in [\kappa_0, \infty)$ to:

$$\forall t > 0, \quad \mathscr{Y}_{L,\kappa,\eta}^{(d=1),1}(t) \leq C(1+\sqrt{\kappa})\sqrt{1+\kappa}\frac{(1+t)^{\frac{7}{2}}}{2\sinh(\frac{\kappa}{2}t)}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}, \tag{B.48}$$

for another L-independent constant $C > 0$. Next, let us estimate the quantity in (B.45). In view of (B.24) and (2.25), one has $\forall L \in [\mathcal{L}_{\kappa_0}(\eta), \infty)$, $\forall \kappa \in [\kappa_0, \infty)$ and $\forall t > 0$:

$$\mathscr{Y}_{L,\kappa,\eta}^{(d=1),2}(t) \leq \sqrt{\kappa}\sqrt{1+\kappa}(1+t)^{\frac{7}{2}} \int_0^t \frac{\mathrm{d}s}{\sqrt{\sinh(\kappa(t-s))}}\mathrm{e}^{-\frac{\kappa}{16}\frac{L^2}{4}\tanh(\frac{\kappa}{2}(t-s))} \times \\ \times \int_{\Lambda_L^1} \mathrm{d}x\, G_{\infty,\kappa}^{(d=1)}(x, \varsigma_L; s, 1) G_{\infty,0}^{(d=1)}(\varsigma_L, x; 4(t-s)).$$

Now from (B.40) $\forall x \in \mathbb{R}^1$ and $\forall s > 0$:

$$G_{\infty,\kappa}^{(d=1)}(x, \varsigma_L; s, 1) \leq \mathrm{e}^{-\frac{\kappa}{4}(|x|^2+\varsigma_L^2)\tanh(\frac{\kappa}{2}s)} G_{\infty,\kappa}^{(d=1)}(x, \varsigma_L; s, 2), \tag{B.49}$$

then by using the upper bound in the first inequality of (2.31) along with (2.36), one has $\forall L \in [\mathcal{L}_{\kappa_0}(\eta), \infty)$, $\forall \kappa \in [\kappa_0, \infty)$ and $\forall t > 0$:

$$\mathscr{Y}_{L,\kappa,\eta}^{(d=1),2}(t) \leq C\kappa\sqrt{1+\kappa}\frac{(1+t)^{\frac{7}{2}}}{\sqrt{t}} \int_0^t \mathrm{d}s\, \sqrt{s}\frac{\mathrm{e}^{-\frac{\kappa}{16}\frac{L^2}{4}\tanh(\frac{\kappa}{2}(t-s))}\mathrm{e}^{-\frac{\kappa}{4}\frac{L^2}{4}\tanh(\frac{\kappa}{2}s)}}{\sqrt{\sinh(\kappa s)\sinh(\kappa(t-s))}},$$

for some $C > 0$. It remains to use successively (A.9), (A.7), (B.42) and (B.47) which together lead $\forall L \in [\mathcal{L}_{\kappa_0}(\eta), \infty)$, $\forall \kappa \in [\kappa_0, \infty)$ and $\forall t > 0$ to:

$$\mathscr{Y}_{L,\kappa,\eta}^{(d=1),2}(t) \leq C\sqrt{\kappa}(1+\kappa)\sqrt{t}\frac{(1+t)^4}{2\sinh(\frac{\kappa}{2}t)}\mathrm{e}^{-\frac{\kappa}{16}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}, \tag{B.50}$$

for another $C > 0$. Gathering (B.48)-(B.50), we get (B.43) in the case of $d = 1$.
Let us turn to the case of $d = 2$. The quantity in (B.5) being made up of four terms (the integration is over the boundaries), then the same holds for the quantity in (B.1). Since these terms have exactly the same structure, it is enough to treat only one of them. In view of (B.22) and (B.23), we need to estimate $\forall L \in [\mathcal{L}_{\kappa_0}(\eta), \infty)$, $\forall \kappa \in [\kappa_0, \infty)$ and $\forall t > 0$:

$$\mathcal{Y}_{L,\kappa,\eta}^{(d=2),1}(t) := \frac{1}{2}\int_0^t \mathrm{d}s \int_{\Lambda_L^2} \mathrm{d}\mathbf{x} \int_{\Lambda_L^1} \mathrm{d}z_1\, G_{\infty,\kappa}^{(d=2)}(\mathbf{x}, (z_1, \varsigma_L); s, 1)\mathcal{P}_{\infty,\kappa,\eta}^{(d=2)}((z_1, \varsigma_L), \mathbf{x}; t-s), \tag{B.51}$$

$$\mathcal{Y}_{L,\kappa,\eta}^{(d=2),2}(t) := \frac{1}{2}\int_0^t \mathrm{d}s \int_{\Lambda_L^2} \mathrm{d}\mathbf{x} \int_{\Lambda_L^1} \mathrm{d}z_1\, G_{\infty,\kappa}^{(d=2)}(\mathbf{x}, (z_1, \varsigma_L); s, 1)\mathcal{R}_{L,\kappa,\eta}^{(d=2)}((z_1, \varsigma_L), \mathbf{x}; t-s). \tag{B.52}$$

Here we have commuted the two integrals, this will be justify by what follows. The strategy consists in using the property (2.26) in order to use the results stated in the case of $d = 1$. Let us start with the quantity in (B.51). In view of (2.25)-(2.26) and (B.23), then from (2.37):

$$\int_{\Lambda_L^2} \mathrm{d}\mathbf{x} \int_{\Lambda_L^1} \mathrm{d}z_1\, G_{\infty,\kappa}^{(d=2)}(\mathbf{x}, (z_1, \varsigma_L); s, 1) G_{\infty,\kappa}^{(d=2)}((z_1, \varsigma_L), \mathbf{x}; t-s, 8) \\ \leq C \int_{\mathbb{R}^1} \mathrm{d}x_1\, G_{\infty,\kappa}^{(d=1)}(x_1, x_1; t, 8) \int_{\mathbb{R}^1} \mathrm{d}x_2\, G_{\infty,\kappa}^{(d=1)}(x_2, \varsigma_L; s, 1) G_{\infty,\kappa}^{(d=1)}(\varsigma_L, x_2; t-s, 8),$$

for some constant $C>0$. From (2.42), the first integral in the above r.h.s. is nothing but the trace (multiplied by a constant). Then $\forall L \in [\mathcal{L}_{\kappa_0}(\eta),\infty)$, $\forall \kappa \in [\kappa_0,\infty)$ and $\forall t>0$ we arrive at:

$$\Upsilon_{L,\kappa,\eta}^{(d=2),1}(t) \leq C(1+\sqrt{\kappa})\frac{(1+t)^{\frac{5}{2}}}{2\sinh(\frac{\kappa}{2}t)}\times \\ \times \int_0^t \mathrm{d}s\, \sqrt{\coth(\frac{\kappa}{2}(t-s))}\int_{\mathbb{R}^1}\mathrm{d}x_2\, G_{\infty,\kappa}^{(d=1)}(x_2,\varsigma_L;s,1)G_{\infty,\kappa}^{(d=1)}(\varsigma_L,x_2;t-s,8),$$

for another constant $C>0$. The above calculus is exactly what we did in the case of $d=1$, see (B.46). By mimicking the arguments leading to (B.48), $\forall L \in [\mathcal{L}_{\kappa_0}(\eta),\infty)$ and $\forall \kappa \in [\kappa_0,\infty)$:

$$\forall t>0,\quad \Upsilon_{L,\kappa,\eta}^{(d=2),1}(t) \leq C(1+\sqrt{\kappa})\sqrt{1+\kappa}\frac{(1+t)^{\frac{7}{2}}}{(2\sinh(\frac{\kappa}{2}t))^2}\mathrm{e}^{-\frac{\kappa}{8}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}, \tag{B.53}$$

for another L-independent constant $C>0$. Next let us estimate the quantity in (B.52). In view of (2.25)-(2.26) and (B.24), then from (2.31) followed by (2.37):

$$\int_{\Lambda_L^2}\mathrm{d}\mathbf{x}\int_{\Lambda_L^1}\mathrm{d}z_1\, G_{\infty,\kappa}^{(d=2)}(\mathbf{x},(z_1,\varsigma_L);s,1)G_{\infty,0}^{(d=2)}((z_1,\varsigma_L),\mathbf{x};4(t-s)) \leq C\sqrt{\frac{\kappa}{\sinh(\kappa s)}}\sqrt{s}\times \\ \times \int_{\Lambda_L^1}\mathrm{d}x_1\, G_{\infty,0}^{(d=1)}(x_1,x_1;4t)\int_{\mathbb{R}^1}\mathrm{d}x_2\, G_{\infty,\kappa}^{(d=1)}(x_2,\varsigma_L;s,1)G_{\infty,0}^{(d=1)}(\varsigma_L,x_2;4(t-s)),$$

for some constant $C>0$. Note that the integrand in the first integral of the above r.h.s. is nothing but a constant. This will make appear a factor L, but we will get rid of it at the end. Ergo, in view of (B.24) and (2.25), one has $\forall L \in [\mathcal{L}_{\kappa_0}(\eta),\infty)$, $\forall \kappa \in [\kappa_0,\infty)$ and $\forall t>0$:

$$\Upsilon_{L,\kappa,\eta}^{(d=2),2}(t) \leq C\kappa^{\frac{3}{2}}(1+\kappa)L\frac{(1+t)^6}{\sqrt{t}}\int_0^t \mathrm{d}s\,\frac{\sqrt{s}\sqrt{t-s}}{\sinh(\kappa(t-s))\sqrt{\sinh(\kappa s)}}\mathrm{e}^{-\frac{\kappa}{16}\frac{L^2}{4}\tanh(\frac{\kappa}{2}(t-s))}\times \\ \times \int_{\mathbb{R}^1}\mathrm{d}x_2\, G_{\infty,\kappa}^{(d=1)}(x_2,\varsigma_L;s,1)G_{\infty,0}^{(d=1)}(\varsigma_L,x_2;4(t-s)),$$

for another L-independent constant $C>0$. The rest of the proof follows the strategy we used for the case of $d=1$. From (B.49), then by using the upper bound in the first inequality of (2.31) along with (2.36), one has $\forall L \in [\mathcal{L}_{\kappa_0}(\eta),\infty)$, $\forall \kappa \in [\kappa_0,\infty)$ and $\forall t>0$:

$$\Upsilon_{L,\kappa,\eta}^{(d=2),2}(t) \leq C\kappa^2(1+\kappa)L\frac{(1+t)^6}{t}\int_0^t \mathrm{d}s\,\frac{s\sqrt{t-s}}{\sinh(\kappa(t-s))\sinh(\kappa s)}\mathrm{e}^{-\frac{\kappa}{16}\frac{L^2}{4}\tanh(\frac{\kappa}{2}(t-s))}\mathrm{e}^{-\frac{\kappa}{4}\frac{L^2}{4}\tanh(\frac{\kappa}{2}s)}.$$

By using successively (A.9), (A.7) and (B.42), this together leads to:

$$\Upsilon_{L,\kappa,\eta}^{(d=2),2}(t) \leq C\kappa(1+\kappa)^2L\sqrt{t}\frac{(1+t)^7}{2\sinh(\kappa t)}\mathrm{e}^{-\frac{\kappa}{16}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)},$$

for another L-independent $C>0$. It remains to use (A.12) to get rid of the factor L:

$$\frac{L}{2\sinh(\kappa t)}\mathrm{e}^{-\frac{\kappa}{32}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)} \leq \frac{C}{\sqrt{\kappa}}\frac{1}{2\sinh(\kappa t)\sqrt{\tanh(\frac{\kappa}{2}t)}} \leq \frac{C}{\sqrt{\kappa}}\frac{\sqrt{\tanh(\frac{\kappa}{2}t)}}{2\sinh(\kappa t)\tanh(\frac{\kappa}{2}t)} \leq \frac{C}{\sqrt{\kappa}}\frac{1}{(2\sinh(\frac{\kappa}{2}t))^2}.$$

Gathering the above estimates together, one has $\forall L \in [\mathcal{L}_{\kappa_0}(\eta),\infty)$, $\forall \kappa \in [\kappa_0,\infty)$ and $\forall t>0$:

$$\Upsilon_{L,\kappa,\eta}^{(d=2),2}(t) \leq C\sqrt{\kappa}(1+\kappa)^2\sqrt{t}\frac{(1+t)^7}{(2\sinh(\frac{\kappa}{2}t))^2}\mathrm{e}^{-\frac{\kappa}{32}\frac{L^2}{4}\tanh(\frac{\kappa}{2}t)}, \tag{B.54}$$

for another constant $C>0$. Adding (B.53) and (B.54), we then obtain (B.43) in the case of $d=2$. The case of $d=3$ can be deduced by similar arguments; we do ont give further details. □

B.5 Appendix: proof of intermediary results.

Proof of Lemma B.5. (B.17) follows from the Feynman-Kac formula in [6, Thm. X.68] together with (2.27) and the definition of the L_0 in (B.7) leading to $(L - L^\eta)^2 \geq L^2/2$ $\forall L \in [L_0(\eta), \infty)$. Next, let us turn to the proof of (B.18)-(B.19). To do that let us introduce an operator of reference. $\forall d \in \{1, 2, 3\}$, $\forall 0 < \eta < 1$, $\forall L \in (0, \infty)$ and $\forall \kappa > 0$, define on $\mathcal{C}_0^\infty(\Lambda_L^d)$:

$$\tilde{h}_{L,\kappa,\eta} := \frac{1}{2}(-i\nabla_{\mathbf{x}})^2 + \frac{\kappa^2}{2}\tilde{V}_{L,\eta}(\mathbf{x}), \quad \tilde{V}_{L,\eta}(\mathbf{x}) := \frac{1}{4}(L - L^\eta)^2. \tag{B.55}$$

By standard arguments, (B.55) extends to a family of self-adjoint and semi-bounded operators for any $L \in (0, \infty)$, denoted again by $\tilde{h}_{L,\kappa,\eta}$, with domain $D(\tilde{h}_{L,\kappa,\eta}) = \mathcal{H}_0^1(\Lambda_L^d) \cap \mathcal{H}^2(\Lambda_L^d)$. This definition corresponds to choose Dirichlet boundary conditions on $\partial\Lambda_L^d$. $\forall 0 < \eta < 1$, $\forall L \in (0, \infty)$ and $\forall \kappa > 0$, let $\{\tilde{g}_{L,\kappa,\eta}(t) := \mathrm{e}^{-t\tilde{h}_{L,\kappa,\eta}} : L^2(\Lambda_L^d) \to L^2(\Lambda_L^d)\}_{t\geq 0}$ be the strongly-continuous one-parameter semigroup generated by $\tilde{h}_{L,\kappa,\eta}$. It possesses an integral kernel jointly continuous in $(\mathbf{x}, \mathbf{y}, t) \in \overline{\Lambda_L^d} \times \overline{\Lambda_L^d} \times (0, \infty)$. Denoting it by $\tilde{g}_{L,\kappa,\eta}^{(d)}$, it is explicitly known and reads as:

$$\forall (\mathbf{x}, \mathbf{y}) \in \Lambda_L^{2d},\ \forall t > 0, \quad \tilde{g}_{L,\kappa,\eta}^{(d)}(\mathbf{x}, \mathbf{y}; t) = \mathrm{e}^{-\frac{\kappa^2}{8}(L-L^\eta)^2 t} G_{L,0}^{(d)}(\mathbf{x}, \mathbf{y}; t), \tag{B.56}$$

where $G_{L,0}^{(d)}$ is the kernel of the semigroup generated by the Dirichlet Laplacian defined in Λ_L^d, see (2.28). Note that (B.56) directly follows from the Feynman-Kac formula. The starting-point of the proof of (B.18)-(B.19) is a Duhamel-like formula to express the semigroup $\{g_{L,\kappa,\eta}(t)\}_{t>0}$ in terms of $\{\tilde{g}_{L,\kappa,\eta}(t)\}_{t>0}$ whose integral kernel is given in (B.56). Let $0 < \eta < 1$, $L \in [L_0(\eta), \infty)$ (see (B.7)) and $\kappa > 0$ be fixed. In the bounded operators sense on $L^2(\Lambda_L^d)$, it holds:

$$\forall t > 0, \quad g_{L,\kappa,\eta}(t) = \tilde{g}_{L,\kappa,\eta}(t) - \int_0^t \mathrm{d}s\, \tilde{g}_{L,\kappa,\eta}(s)\{h_{L,\kappa,\eta} - \tilde{h}_{L,\kappa,\eta}\} g_{L,\kappa,\eta}(t - s), \tag{B.57}$$

where we used the self-adjointness of the semigroups $\{g_{L,\kappa,\eta}(t)\}_{t\geq 0}$, $\{\tilde{g}_{L,\kappa,\eta}(t)\}_{t\geq 0}$.

Proof of (B.18). From (B.57), it follows in the kernels sense:

$$\forall (\mathbf{x}, \mathbf{y}) \in \Lambda_L^{2d},\ \forall t > 0, \quad \nabla_{\mathbf{x}} g_{L,\kappa,\eta}^{(d)}(\mathbf{x}, \mathbf{y}; t) = \nabla_{\mathbf{x}} \tilde{g}_{L,\kappa,\eta}^{(d)}(\mathbf{x}, \mathbf{y}; t) - \frac{1}{2}\mathfrak{q}_{L,\kappa,\eta}^{(d)}(\mathbf{x}, \mathbf{y}; t), \tag{B.58}$$

$$\mathfrak{q}_{L,\kappa,\eta}^{(d)}(\mathbf{x}, \mathbf{y}; t) := \kappa^2 \int_0^t \mathrm{d}s \int_{\Lambda_L^d} \mathrm{d}\mathbf{z}\, \nabla_{\mathbf{x}} \tilde{g}_{L,\kappa,\eta}^{(d)}(\mathbf{x}, \mathbf{z}; s)\{V_{L,\eta}(\mathbf{z}) - \tilde{V}_{L,\eta}(\mathbf{z})\} g_{L,\kappa,\eta}^{(d)}(\mathbf{z}, \mathbf{y}; t - s). \tag{B.59}$$

Remind that $V_{L,\eta}(\mathbf{z}) - \tilde{V}_{L,\eta}(\mathbf{z}) = |\mathbf{z}|^2 - \frac{1}{4}(L - L^\eta)^2$ on $\mathrm{Supp}(\hat{\hat{f}}_{L,\eta})$, 0 otherwise. Let us estimate the first kernel in the r.h.s. of (B.58). From (2.34) and (B.56), there exists a constant $C_d > 0$ s.t.

$$\forall (\mathbf{x}, \mathbf{y}) \in \Lambda_L^{2d},\ \forall t > 0, \quad |\nabla_{\mathbf{x}} \tilde{g}_{L,\kappa,\eta}^{(d)}(\mathbf{x}, \mathbf{y}; t)| \leq C_d \frac{(1 + t)^d}{\sqrt{t}} \mathrm{e}^{-\frac{\kappa^2}{4}\frac{L^2}{4}t} G_{\infty,0}^{(d)}(\mathbf{x}, \mathbf{y}; 2t). \tag{B.60}$$

Subsequently, we turn to the quantity in (B.59). From (B.60) along with (B.17), there exists another constant $C_d > 0$ s.t. $\forall (\mathbf{x}, \mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$\begin{aligned} |\mathfrak{q}_{L,\kappa,\eta}^{(d)}(\mathbf{x}, \mathbf{y}; t)| &\leq C_d \kappa^2 L^2 (1 + t)^d \mathrm{e}^{-\frac{\kappa^2}{4}\frac{L^2}{4}t} \int_0^t \frac{\mathrm{d}s}{\sqrt{s}} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{z}\, G_{\infty,0}^{(d)}(\mathbf{x}, \mathbf{z}; 2s) G_{\infty,0}^{(d)}(\mathbf{z}, \mathbf{y}; 2(t - s)) \\ &\leq C_d \kappa^2 L^2 (1 + t)^d \mathrm{e}^{-\frac{\kappa^2}{4}\frac{L^2}{4}t} G_{\infty,0}^{(d)}(\mathbf{x}, \mathbf{y}; 2t) \int_0^t \frac{\mathrm{d}s}{\sqrt{s}}, \end{aligned}$$

where we used in the last inequality (2.36). Finally use (A.12) to get rid of the L^2 which leads to:

$$|\mathfrak{q}_{L,\kappa,\eta}^{(d)}(\mathbf{x}, \mathbf{y}; t)| \leq C_d \frac{(1 + t)^d}{\sqrt{t}} \mathrm{e}^{-\frac{\kappa^2}{8}\frac{L^2}{4}t} G_{\infty,0}^{(d)}(\mathbf{x}, \mathbf{y}; 2t), \tag{B.61}$$

for another constant $C_d > 0$. It remains to gather (B.60) and (B.61) together.
Proof of (B.19). Starting from the below identity which holds in the bounded operators sense:

$$\forall t > 0, \quad [(-i\nabla), \tilde{g}_{L,\kappa,\eta}(t)] = -\int_0^t \mathrm{d}s\, \tilde{g}_{L,\kappa,\eta}(t-s)[(-i\nabla), \tilde{h}_{L,\kappa,\eta}]\tilde{g}_{L,\kappa,\eta}(s),$$

then by remarking that $[(-i\nabla), \tilde{h}_{L,\kappa,\eta}] = -i\frac{\kappa^2}{2}(\nabla\tilde{V}_{L,\eta}) = 0$, from (B.57) we get on $L^2(\Lambda_L^d)$:

$$\forall t > 0, \quad (-i\nabla)g_{L,\kappa,\eta}(t) = (-i\nabla)\tilde{g}_{L,\kappa,\eta}(t) - \int_0^t \mathrm{d}s\, \tilde{g}_{L,\kappa,\eta}(s)(-i\nabla)\{h_{L,\kappa,\eta} - \tilde{h}_{L,\kappa,\eta}\}g_{L,\kappa,\eta}(t-s).$$

It follows in the kernels sense:

$$\forall(\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}, \forall t > 0, \quad \Delta_{\mathbf{x}} g_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) = \Delta_{\mathbf{x}}\tilde{g}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) - \frac{1}{2}\mathfrak{u}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t), \tag{B.62}$$

with:

$$\mathfrak{u}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t) = \mathfrak{u}_{L,\kappa,\eta}^{(d),1}(\mathbf{x},\mathbf{y};t) + \mathfrak{u}_{L,\kappa,\eta}^{(d),2}(\mathbf{x},\mathbf{y};t),$$
$$\mathfrak{u}_{L,\kappa,\eta}^{(d),1}(\mathbf{x},\mathbf{y};t) := \kappa^2 \int_0^t \mathrm{d}s \int_{\Lambda_L^d} \mathrm{d}\mathbf{z}\, \nabla_{\mathbf{x}}\tilde{g}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{z};s)(\nabla_{\mathbf{z}}V_{L,\eta})(\mathbf{z})g_{L,\kappa,\eta}^{(d)}(\mathbf{z},\mathbf{y};t-s),$$
$$\mathfrak{u}_{L,\kappa,\eta}^{(d),2}(\mathbf{x},\mathbf{y};t) := \kappa^2 \int_0^t \mathrm{d}s \int_{\Lambda_L^d} \mathrm{d}\mathbf{z}\, \nabla_{\mathbf{x}}\tilde{g}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{z};s)\{\tilde{V}_{L,\eta}(\mathbf{z}) - V_{L,\eta}(\mathbf{z})\}\nabla_{\mathbf{z}} g_{L,\kappa,\eta}^{(d)}(\mathbf{z},\mathbf{y};t-s).$$

From (2.35) and (B.56), there exists a constant $C_d > 0$ s.t.

$$\forall(\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}, \forall t > 0, \quad |\Delta_{\mathbf{x}}\tilde{g}_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t)| \leq C_d \frac{(1+t)^d}{t} \mathrm{e}^{-\frac{\kappa^2}{4}\frac{L^2}{4}t} G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};2t).$$

Subsequently, by mimicking the method leading to (B.61), there exists another $C_d > 0$ s.t.

$$\forall(\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}, \ \forall t > 0, \quad |\mathfrak{u}_{L,\kappa,\eta}^{(d),1}(\mathbf{x},\mathbf{y};t)| \leq C_d \frac{(1+t)^d}{\sqrt{t}} \mathrm{e}^{-\frac{\kappa^2}{8}\frac{L^2}{4}t} G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};2t).$$

By the same method again, but replacing the estimate (B.17) with (B.18), we have:

$$|\mathfrak{u}_{L,\kappa,\eta}^{(d),2}(\mathbf{x},\mathbf{y};t)| \leq C_d \frac{(1+t)^{2d}}{t} \mathrm{e}^{-\frac{\kappa^2}{16}\frac{L^2}{4}t} G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};2t) \int_0^t \frac{\mathrm{d}s}{\sqrt{s}\sqrt{t-s}}.$$

Gathering the three above estimates together, then the proof of (B.19) is over. □

Proof of Lemma B.9. Let $d \in \{1,2,3\}$, $0 < \eta < 1$, $L \in [L_0(\eta), \infty)$ and $\kappa > 0$ kept fixed.
(i). From (B.26) written in the kernels sense, then $\forall(\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$\nabla_{\mathbf{x}}(\mathcal{G}_{L,\kappa,\eta}^{*})^{(d)}(\mathbf{x},\mathbf{y};t) = (\nabla f_{L,\eta})(\mathbf{x})G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t)\hat{f}_{L,\eta}(\mathbf{y}) + f_{L,\eta}(\mathbf{x})\nabla_{\mathbf{x}}G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t)\hat{f}_{L,\eta}(\mathbf{y}) +$$
$$+ (\nabla f_{L,\eta}^c)(\mathbf{x})g_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t)\hat{\hat{f}}_{L,\eta}(\mathbf{y}) + f_{L,\eta}^c(\mathbf{x})\nabla_{\mathbf{x}}g_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{y};t)\hat{\hat{f}}_{L,\eta}(\mathbf{y}).$$

(B.32) is an upper bound for the two first kernels in the above r.h.s. obtained from (2.25)-(2.26) and (2.32). (B.33) is an upper bound for the two last kernels obtained from (B.17) and (B.18).
(ii). From (B.27) written in the kernels sense, then $\forall(\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$\nabla_{\mathbf{x}}(\mathcal{W}_{L,\kappa,\eta}^{*})^{(d)}(\mathbf{x},\mathbf{y};t) = \sum_{m=1}^{4} Q_{L,\kappa,\eta}^{(d),m}(\mathbf{x},\mathbf{y};t), \quad \text{with:}$$

$$Q^{(d),1}_{L,\kappa,\eta}(\mathbf{x},\mathbf{y};t) := -(\nabla f_{L,\eta})(\mathbf{x}) G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t)\frac{1}{2}(\Delta \hat{f}_{L,\eta})(\mathbf{y})+ \\ -(\nabla f_{L,\eta})(\mathbf{x})\nabla_{\mathbf{x}} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t)(\nabla \hat{f}_{L,\eta})(\mathbf{y}) + i(\nabla f_{L,\eta})(\mathbf{x})[(-i\nabla), G_{\infty,\kappa}(t)](\mathbf{x},\mathbf{y})(\nabla \hat{f}_{L,\eta})(\mathbf{y}), \tag{B.63}$$

$$Q^{(d),2}_{L,\kappa,\eta}(\mathbf{x},\mathbf{y};t) := -f_{L,\eta}(\mathbf{x})\nabla_{\mathbf{x}} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t)\frac{1}{2}(\Delta \hat{f}_{L,\eta})(\mathbf{y})+ \\ -f_{L,\eta}(\mathbf{x})\Delta_{\mathbf{x}} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t)(\nabla \hat{f}_{L,\eta})(\mathbf{y}) + i f_{L,\eta}(\mathbf{x})\nabla_{\mathbf{x}}[(-i\nabla), G_{\infty,\kappa}(t)](\mathbf{x},\mathbf{y})(\nabla \hat{f}_{L,\eta})(\mathbf{y}), \tag{B.64}$$

$$Q^{(d),3}_{L,\kappa,\eta}(\mathbf{x},\mathbf{y};t) := -(\nabla f^c_{L,\eta})(\mathbf{x}) g^{(d)}_{L,\kappa,\eta}(\mathbf{x},\mathbf{y};t)\frac{1}{2}(\Delta \hat{\hat{f}}_{L,\eta})(\mathbf{y})+ \\ -(\nabla f^c_{L,\eta})(\mathbf{x})\nabla_{\mathbf{x}} g^{(d)}_{L,\kappa,\eta}(\mathbf{x},\mathbf{y};t)(\nabla \hat{\hat{f}}_{L,\eta})(\mathbf{y}) + i(\nabla f^c_{L,\eta})(\mathbf{x})[(-i\nabla), g_{L,\kappa,\eta}(t)](\mathbf{x},\mathbf{y})(\nabla \hat{\hat{f}}_{L,\eta})(\mathbf{y}), \tag{B.65}$$

$$Q^{(d),4}_{L,\kappa,\eta}(\mathbf{x},\mathbf{y};t) := -f^c_{L,\eta}(\mathbf{x})\nabla_{\mathbf{x}} g^{(d)}_{L,\kappa,\eta}(\mathbf{x},\mathbf{y};t)\frac{1}{2}(\Delta \hat{\hat{f}}_{L,\eta})(\mathbf{y})+ \\ -f^c_{L,\eta}(\mathbf{x})\Delta_{\mathbf{x}} g^{(d)}_{L,\kappa,\eta}(\mathbf{x},\mathbf{y};t)(\nabla \hat{\hat{f}}_{L,\eta})(\mathbf{y}) + i f^c_{L,\eta}(\mathbf{x})\nabla_{\mathbf{x}}[(-i\nabla), g_{L,\kappa,\eta}(t)](\mathbf{x},\mathbf{y})(\nabla \hat{\hat{f}}_{L,\eta})(\mathbf{y}), \tag{B.66}$$

Unlike the proof of (i), here we will use the properties (B.9) to get rid of the growth with L coming from the commutators in (B.63)-(B.64). Let us first estimate (B.63). Clearly (B.32) is an upper bound for the first two terms in the r.h.s. of (B.63). For the last term in (B.63), use (B.28) in the kernels sense. Then there exists a constant $C_d > 0$ s.t. $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$\frac{\kappa^2}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}\mathbf{z}\, G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{z};t-s)(\nabla_{\mathbf{z}}|\mathbf{z}|^2) G^{(d)}_{\infty,\kappa}(\mathbf{z},\mathbf{y};s) \leq C_d(\kappa^2 L + \kappa^{\frac{3}{2}}) t G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t,2). \tag{B.67}$$

Here we used that $|\mathbf{z}| \leq |\mathbf{x}-\mathbf{z}| + |\mathbf{x}|$, then (A.12) to get rid of the factor $|\mathbf{x}-\mathbf{z}|$ and (2.37), and finally the inequality $\coth(\alpha) \geq 1\ \forall \alpha > 0$. Now we use the property (B.11) to get rid of the factor κ in (B.67) via (A.12). Then there exist two other constants $c, C_d > 0$ s.t. on Λ_L^{2d}:

$$|i(\nabla f_{L,\eta})(\mathbf{x})[(-i\nabla), G_{\infty,\kappa}(t)](\mathbf{x},\mathbf{y})(\nabla \hat{f}_{L,\eta})(\mathbf{y})| \leq C_d L^{-3\eta}(1+L^{1-\eta}) t \mathrm{e}^{-c\kappa L^{2\eta}\coth(\frac{\kappa}{2}t)} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t,4).$$

Restricting to $1 > \eta > \frac{1}{4}$, and gathering the estimates together, then there exist two other constants $c, C_d > 0$ s.t. $\forall L \in [L_0(\eta),\infty)$ and $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$:

$$\forall t > 0, \quad |Q^{(d),1}_{L,\kappa,\eta}(\mathbf{x},\mathbf{y};t)| \leq C_d(1+\sqrt{\kappa})\sqrt{\coth(\frac{\kappa}{2}t)}(1+t)\mathrm{e}^{-c\kappa L^{2\eta}\coth(\frac{\kappa}{2}t)} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t,4). \tag{B.68}$$

Subsequently, let us turn to (B.64). From (2.32) and (2.33) together with the property (B.9), then there exist two other constants $c, C_d > 0$ s.t. $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$|f_{L,\eta}(\mathbf{x})\nabla_{\mathbf{x}} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t)\frac{1}{2}(\Delta \hat{f}_{L,\eta})(\mathbf{y}) + f_{L,\eta}(\mathbf{x})\Delta_{\mathbf{x}} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t)(\nabla \hat{f}_{L,\eta})(\mathbf{y})| \\ \leq C_d\sqrt{\kappa}\sqrt{\coth(\frac{\kappa}{2}t)}\mathrm{e}^{-c\kappa L^{2\eta}\coth(\frac{\kappa}{2}t)} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t,4).$$

Here the property (B.9) is essential to remove a $\sqrt{\coth(\frac{\kappa}{2}t)}$ in the numerator of (2.33). For the last term of (B.64), we use the same reasoning leading to (B.67) combined with the property (B.9). Then there exist two other constants $c, C_d > 0$ s.t. $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$\frac{\kappa^2}{2}\left| f_{L,\eta}(\mathbf{x}) \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}\mathbf{z}\, \nabla_{\mathbf{x}} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{z};t-s)(\nabla_{\mathbf{z}}|\mathbf{z}|^2) G^{(d)}_{\infty,\kappa}(\mathbf{z},\mathbf{y};s)(\nabla \hat{f}_{L,\eta})(\mathbf{y})\right| \\ \leq C_d(1+\sqrt{\kappa})L^{-4\eta}(1+L)(1+t)\mathrm{e}^{-c\kappa L^{2\eta}\coth(\frac{\kappa}{2}t)} G^{(d)}_{\infty,\kappa}(\mathbf{x},\mathbf{y};t,8).$$

Restricting to $1 > \eta > \frac{1}{4}$, and gathering the above estimates together, then there exist two other constants $c, C_d > 0$ s.t. $\forall L \in [L_0(\eta), \infty)$, $\forall(\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$ and $\forall t > 0$:

$$|Q_{L,\kappa,\eta}^{(d),2}(\mathbf{x},\mathbf{y};t)| \leq C_d(1+\sqrt{\kappa})\sqrt{\coth(\frac{\kappa}{2}t)}(1+t)\mathrm{e}^{-c\kappa L^{2\eta}\coth(\frac{\kappa}{2}t)}G_{\infty,\kappa}^{(d)}(\mathbf{x},\mathbf{y};t,8). \tag{B.69}$$

Adding estimates (B.68) and (B.69) together, this leads to (B.35).
We continue with (B.65). (B.33) is an upper bound for the first two terms in the r.h.s. of (B.65). From (B.29) in the kernels sense, then by (A.12) there exist two other constants $c, C_d > 0$ s.t.

$$\frac{\kappa^2}{2}\int_0^t \mathrm{d}s \int_{\Lambda_L^d} \mathrm{d}\mathbf{z}\, g_{L,\kappa,\eta}^{(d)}(\mathbf{x},\mathbf{z};t-s)(\nabla_{\mathbf{z}}V_{L,\eta})(\mathbf{z})g_{L,\kappa,\eta}^{(d)}(\mathbf{z},\mathbf{y};s) \leq C_d\mathrm{e}^{-\frac{\kappa^2}{8}\frac{L^2}{4}t}\mathrm{e}^{-c\frac{L^{2\eta}}{t}}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};2t).$$

We conclude that there exist two other constants $c, C_d > 0$ s.t. $\forall(\mathbf{x},\mathbf{y}) \in \Lambda_L^{2d}$:

$$\forall t > 0, \quad |Q_{L,\kappa,\eta}^{(d),3}(\mathbf{x},\mathbf{y};t)| \leq C_d\frac{(1+t)^d}{\sqrt{t}}\mathrm{e}^{-\frac{\kappa^2}{8}\frac{L^2}{4}t}\mathrm{e}^{-c\frac{L^{2\eta}}{t}}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};4t). \tag{B.70}$$

Concerning (B.66), one can prove that there exist two other constants $c, C_d > 0$ s.t. on Λ_L^{2d}:

$$\forall t > 0, \quad |Q_{L,\kappa,\eta}^{(d),4}(\mathbf{x},\mathbf{y};t)| \leq C_d\frac{(1+t)^{2d}}{\sqrt{t}}\mathrm{e}^{-\frac{\kappa^2}{16}\frac{L^2}{4}t}\mathrm{e}^{-c\frac{L^{2\eta}}{t}}G_{\infty,0}^{(d)}(\mathbf{x},\mathbf{y};4t). \tag{B.71}$$

We mention that we used property (B.11) combined with (A.12) to get rid of a $\sqrt{t}$ in the denominator of (B.19). Adding estimates (B.70) and (B.71) together, this leads to (B.36). □

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