Noncommutativity inspired Black Holes as a Dark Matter Candidate

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We study a black hole with a blurred mass density instead of a singular one, which could be caused by the noncommutativity of 3-space. Depending on its mass, such object has either none, one or two event horizons. It possesses new properties, which become important on a microscopic scale, in particular, the Hawking temperature does not increase indefinitely as the mass goes to zero, but vanishes instead. Such frozen and extremely dense pieces of matter are a good dark matter candidate. In addition, we introduce an object oscillating between a microscopic black hole and a naked (and blurred) singularity, we call it gravimond.

1. Introduction

Noncommutative (NC) theories are built on spaces whose coordinates do not commute and therefore one cannot localize their points with an arbitrary precision. This is believed to be an artifact of quantum gravity and results into a plenty of novel properties, for example a natural UV cutoff. A certain theory of NC quantum mechanics (QM) was studied in[1–4] and this paper aims to analyze its possible physical consequences.

Among the objects sensible to introducing a short scale structure are microscopic black holes. In the classical theory, as they evaporate by Hawking radiation[5] their radius eventually becomes infinitely small and their temperature becomes infinitely high. It is interesting to study how is this behavior affected by introducing a NC structure to 3-space. The appropriate theory to investigate this is the one of quantum gravity, which is yet to be found, or at least properly understood. Therefore we settle for a semiclassical description and implement some of the results from NC QM into the classical theory of gravity.

This method was developed by Nicolini,[6] who dubbed it 'NC inspired'. More details on NC inspired cosmology and gravity could be found,[7–13] the concept of generalized

uncertainty principle in a similar context has been analyzed as well[14, 15].

This paper is organized as follows. In Section 2 we derive a NC delta function (which describes a (nearly) singular matter density), complete it into stress-energy tensor T^μ_ν and solve the Einstein field equations (EFE) for it. This solution is analyzed, focusing mostly on the event horizon(s) and the Hawking temperature, in Section 3. Section 4 contains physical consequences of the model and final conclusions.

2. Noncommutative space, (blurred) delta function and EFE

We will study a model of 3 dimensional rotational invariant NC space described by

$$[\hat{x}_i, \hat{x}_j] = 2i\lambda \hat{x}_k \varepsilon^{ijk} \,, \tag{1}$$

where ε^{ijk} is the Levi-Civita symbol and λ is a constant with the dimension of length, describing the scale of noncommutativity. It is not fixed within our model, but could be expected to be (approximately) the Planck length.

There are several ways how to satisfy (1), see[2, 16–20]. We will employ the bosonic operator approach which was previously used in[1–4] and is well suited for 3 dimensional rotational invariant problems.

Let us define two sets of bosonic creation and annihilation operators satisfying

$$[\hat{a}_\alpha, \hat{a}^+_\beta] = \delta_{\alpha\beta} \,; \, \alpha, \beta = 1, 2 \,, \tag{2}$$

and acting in an auxiliary Fock space $\mathcal{F}$ spanned on normalized states $|n_1, n_2\rangle = \frac{(\hat{a}^+_1)^{n_1}(\hat{a}^+_2)^{n_2}}{\sqrt{n_1! n_2!}}|0, 0\rangle$, where $|0, 0\rangle = |0\rangle$ is the vacuum state annihilated by both $\hat{a}_\alpha$. It is convenient to define their dimensional versions as $\hat{z}_\alpha = \sqrt{\lambda}\hat{a}_\alpha$, $\hat{z}^+_\alpha = \sqrt{\lambda}\hat{a}^+_\alpha$. Using these (and Pauli matrices σ^i), we can define the (Cartesian) coordinates satisfying (1) and the radial coordinate [a] as

$$\hat{x}_i = \sigma^i_{\alpha\beta}\hat{z}^+_\alpha \hat{z}_\beta, \ \hat{r} = \hat{z}^+_\alpha \hat{z}_\alpha + \lambda \,. \tag{3}$$

Fock space states $|n_1, n_2\rangle$ are $\hat{r}$ eigenstates with eigenvalues of $\lambda(n_1 + n_2 + 1)$, the vacuum state $|0, 0\rangle \equiv |0\rangle$ is the state with the minimal eigenvalue, it corresponds to the origin of the coordinate system. Space described by NC coordinates (3) can be looked upon as quantized Euclidean 3-space, see[21].

Coherent states play an important role in the ordinary quantum mechanics and are crucial in NC theories as well[22–26]. A coherent state is well localized wave packet which minimizes the uncertainty relation and is defined as an annihilation operator eigenstate $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Such states can be generated as $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^+}|0\rangle$ and used as an overcomplete sets of states in $\mathcal{F}$,[27]. Overlap of two such states is $\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2+|\beta|^2}{2}+\bar{\alpha}\beta}$. We are interested in a state well localized at the origin, which follows from

$$\tilde{\rho}(z) = |\langle z|0\rangle|^2 = e^{-\frac{|z|^2}{\lambda}} = e^{-\frac{r-\lambda}{\lambda}} \,. \tag{4}$$

Let us pause for a moment to make a few remarks here. First of all, we define taking $\lambda \to 0$ as the commutative limit (RHS of (1) vanishes, as in the ordinary QM). It is easy to see that in this limit the RHS of (4) vanishes everywhere but at the point $r = 0$, it resembles Dirac delta distribution with the source located at the coordinate origin. It is therefore natural to call $\tilde{\rho} \propto e^{-\frac{r}{\lambda}}$ an almost delta distribution or a blurred delta distribution (located at the coordinate origin).

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[a]Their relation is $\hat{r}^2 - \hat{x}^2 = \lambda^2$, as can be easily checked.

Note that $\tilde{\rho}$ in (4) is dimensionless, a dimensional one will be denoted ρ. Since the rest of calculations will be done using the ordinary (not NC) calculus, we will normalize ρ with respect to ordinary integration instead of trace norm, yielding

$$\rho(r) = \frac{M}{8\pi\lambda^3} e^{-\frac{r}{\lambda}} \; . \tag{5}$$

In the paper by P. Nicolini,[6] which served as a main inspiration for ours, a similar line of reasoning was used. The starting point was a two dimensional NC space and the resulting density was generalized into three dimensional only afterwards, yielding $\rho \propto e^{-\frac{r^2}{\lambda^2}}$. As we have shown, a direct three dimensional derivation based on (1) gives different result.

We will take (5) as the matter density of a (blurred) point and complete it into stress-energy tensor. We focus on uncharged nonrotating black holes, so we expect the solution to become Schwarzschild-like in the $\lambda \to 0$ limit. This encourages us to use a diagonal ansatz for the metric tensor with $g_{00} = -g_{rr}^{-1}$, therefore we seek only a single function $f(r)$ such that $g_{\mu\nu} = \mathrm{diag}\left(f, -f^{-1}, r^2, r^2\sin^2\theta\right)$ (with coordinates $(0, r, \theta, \varphi)$ and signature $(-, +, +, +)$. We often set uninteresting constants equal to 1 and omit writing arguments).

For the same reason we are expecting a diagonal T_ν^μ with $T_0^0 = -\rho$ and $T_r^r = T_0^0$ (which follows from the EFE). Because of the chosen ansatz, $T_r^r = T_0^0$ is fixed as well (this also can be seen from the EFE). The other two components follow from the conservation law $T^{\mu\nu}{}_{;\nu} = 0$. For $\mu = \theta$ we get $T_\theta^\theta = T_\varphi^\varphi =: p_\perp$, for $\mu = r$ we get $p_\perp = -\frac{r}{2}(\partial_r\rho + \frac{2}{r}\rho) = -\rho - \frac{r}{2}\partial_r\rho$. Adding this we obtain $T_\nu^\mu = (-\rho, p_r, p_\perp, p_\perp)$, where $p_r = -\rho$ and $p_\perp$ defined above.

Since we are looking for a single function $f(r)$ we only need one of the EFE, let us take $G_0^0 = 8\pi T_0^0$. From it the solution follows as

$$\frac{1 + f(r) + rf'(r)}{r^2} = \frac{M}{\lambda^3} e^{-\frac{r}{\lambda}} \to f(r) = -1 - e^{-\frac{r}{\lambda}}\frac{M}{r}\left(\frac{r^2}{\lambda^2} + \frac{2r}{\lambda} + 2\right) + \frac{C}{r} \; . \tag{6}$$

Recall that $g_{00} = f$, therefore if we want the solution to approach Schwarzschild solution for $r \gg \lambda$, we need to set $C = 2M$. For the rest of this paper we will be needing only the time component of the metric tensor,

$$g_{00}(r; \lambda, M) = -1 + \frac{2M}{r} - e^{-\frac{r}{\lambda}}\frac{M}{r}\left(\frac{r^2}{\lambda^2} + \frac{2r}{\lambda} + 2\right) \; . \tag{7}$$

Let us pause for a brief comment. The stress-energy tensor violates the weak energy condition (see[28]) for $r < 2\lambda$, which signalize a quantum repulsion (preventing the matter from collapsing into a singularity). The strong energy condition is violated between black hole horizons. This might either be an artifact of the semiclassical approach, or a signal of negative energies present. In[29] it is argued that the energy conditions are becoming obsolete, being violated even in classical theory; namely, local conditions are not satisfied for scalar fields with nonminimal coupling to gravity, and even averaged conditions do not always hold if the field reaches transplanckian values.

The particular form of the stress-energy tensor followed from the postulate of T_{00} and from the requirement of Schwarzschild-like form of $g_{\mu\nu}$. However since we will be using only the g_{00} component our results hold also for different completions of $T_{\mu\nu}$, which might be free of (strong condition) violations.

3. Event horizon(s) and Hawking radiation

Event horizons are solutions of the equation $g_{00}(r) = 0$. For an ordinary Schwarzschild black hole there is only a single solution $r = 2M$, now there are two, one or zero solutions,

depending on the value of M (see Figure 1). In the case there are two, let us denote them r_-, r_+ $(r_- < r_+)$.

When the mass is large $(M \gg \lambda)$, there are two horizons, one near the singularity $(r_- \approx 0)$ and the other near the classical horizon $(r_+ \approx 2M)$. As M gets smaller, these two surfaces move towards each other and meet for $M =: M_0$ at $r =: r_0$. We call a black hole with the mass M_0 and a single horizon at r_0 minimal, since for any smaller M there is no horizon at all, minimal black hole is the smallest (and lightest) possible black hole. The values of M_0, r_0 can be obtained numerically

$$M_0 \doteq 2.57\lambda, \; r_0 \doteq 3.38\lambda \,. \tag{8}$$

The Hawking temperature of a minimal black hole is zero. This follows from the fact that it is proportional to the surface gravity at the (outer) horizon $\kappa = -\frac{g'_{00}(r_0)}{2}$, which has to be zero, since $g_{00}(r)$ reaches its maximum there. The black hole becomes frozen and evaporation ceases when the minimal mass $M = M_0$ is reached.

Note that infinite temperatures are avoided (Figure 2). From a dimensional analysis we can see that the maximal reached temperature (denoted T_m) is proportional to λ^{-1}. To find the constant of proportionality, let us first factorize out the mass from g_{00}

$$g_{00}(r; \lambda, M) = -1 + M\tilde{g}(r; \lambda) \,, \tag{9}$$

where $\tilde{g}(r)$ does not depend on M. At the (outer) horizon $\tilde{g}(r_+) = \frac{1}{M}$, and

$$g'_{00}(r_+) = M\tilde{g}'(r_+) = \frac{\tilde{g}'(r_+)}{\tilde{g}(r_+)} \,. \tag{10}$$

This is, up to a multiplicative constant, equal to the Hawking temperature. To find r_+, for which this achieves extremum we need to solve $\partial_{r_+} g'_{00}(r_+) = 0$. This can be, again, done numerically (choosing $\lambda = 1$), finding that the extremum is reached as $g'_{00}(r_+ \doteq 6.54) \doteq -0.12$. Plugging this into the relation for the temperature we obtain

$$T_m \doteq \frac{\hbar c}{4\pi k_B} \frac{0.12}{\lambda}, \; \frac{\hbar c}{4\pi k_B} \doteq 0.18 \times 10^{-3} mK. \tag{11}$$

It can be observed in Figure 2 that the temperature grows very rapidly for $M \gtrsim M_0$. It is therefore interesting to investigate what happens after adding a small mass $\delta M \ll M_0$ into a minimal black hole.
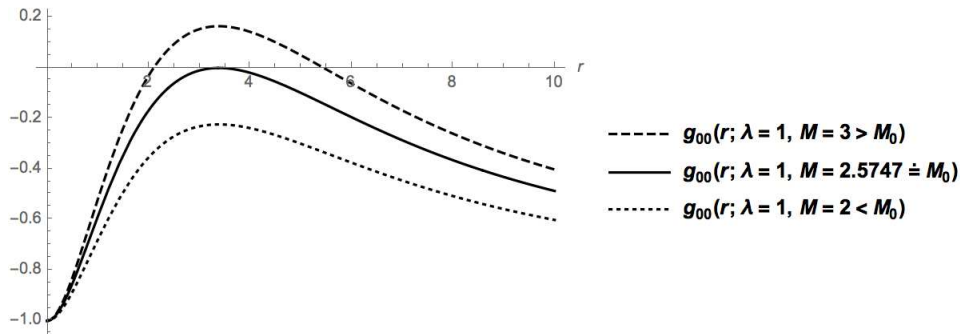

Fig. 1. $g_{00}(r)$ for $\lambda = 1$ and different values of M.

To answer this question we use the decomposition (9). Let us denote the increment in radius δr, horizon condition after adding a small mass δM reads

$$-1 + (M_0 + \delta M)\tilde{g}(r_0 + \delta r; \lambda) = 0. \tag{12}$$

Truncating the Taylor expansion of (12) we obtain

$$\tilde{g}(r_0 + \delta r; \lambda) \doteq \underbrace{\tilde{g}(r_0; \lambda)}_{M_0^{-1}} + \delta r \underbrace{\partial_r \tilde{g}(r_0; \lambda)}_{0} + \frac{1}{2}\delta r^2 \partial_r^2 \tilde{g}(r_0; \lambda). \tag{13}$$

Inserting this back into (12) we arrive to

$$\delta r \doteq \pm\sqrt{\frac{-2\delta M}{M_0^2 \partial_r^2 \tilde{g}(r_0; \lambda)}}. \tag{14}$$

Evaluating for M_0 and r_0 as given in (8) yields $\delta r \doteq \pm 2.54\sqrt{\lambda \delta M}$ (there are two symmetric solutions because we have truncated the Taylor expansion after the quadratic term).

We can now determine the temperature of the resulting black hole

$$T(r_0 + \delta r) \doteq \overbrace{T(r_0)}^{0} + \partial_r T(r_0)\delta r \tag{15}$$

$$= -\frac{M_0 \tilde{g}''(r_0)}{4\pi}\delta r \doteq \frac{1}{4\pi}\frac{1}{M_0}\frac{2\delta M}{\delta r} \doteq \frac{1}{2\pi}\frac{\sqrt{\delta M}}{6.53 \, \lambda^{3/2}},$$

Recovering constants for a moment

$$T(M_0 + \delta M) \doteq \frac{\sqrt{\delta M}}{41.01 \, \lambda^{3/2}}\frac{\hbar c}{k_B}. \tag{16}$$

It is useful to express this with respect to the maximal temperature

$$\frac{T(M_0 + \delta M)}{T_m} \doteq 2.55\sqrt{\frac{\delta M}{\lambda}} \doteq 4.09\sqrt{\frac{\delta M}{M_0}}. \tag{17}$$
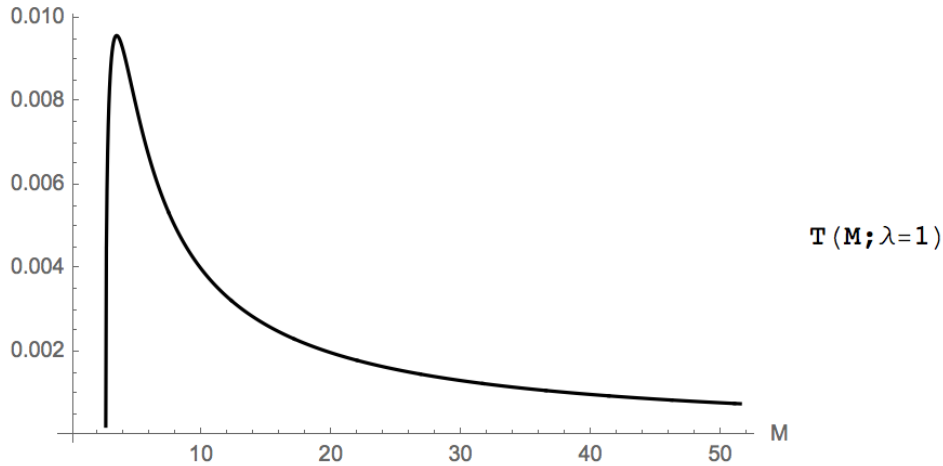

Fig. 2.   The Hawking temperature as a function of black hole's mass.

We can see that for $\delta M \ll M_0$ the black hole does not reach its maximum temperature, only a small fraction of it (which is, absolutely speaking, still huge).

The last question of this section will be whether does the temperature reach the maximum value if we merge two minimal black holes together? When we have $M = 2M_0$ we are in a region where we can safely take $r_+ = 2M = 4M_0 \doteq 10.28\lambda$. This is larger than the value $r_+ = 6.54\lambda$ for which the temperature reaches maximum, therefore the maximum will be reached, when the new black hole evaporates from the radius of 10.28λ to 6.54λ.

4. Physical consequences and conclusion

To be able to evaluate physical consequences let us assume that $\lambda \sim l_{\text{Planck}} \doteq 1.62 \times 10^{-35}m$, as is usually done (scaling rules for a different choice will be included). Most sensitive to introducing the noncommutativity are microscopic black holes, with a radius of the order of a few λ. The most important case is the minimal black holes, let us denote it mBH.

According to (8) a mBH should have a radius $r_0 \doteq 5.48 \times 10^{-35}m$ (we can take the cross section to be $\sigma = \pi r_0^2 \doteq 9.43 \times 10^{-69}m^2$) and a mass $M_0 \doteq 5.59 \times 10^{-8}kg$ (r_0, M_0 scale as λ). Furthermore the maximal temperature T_{m} is $1.33 \times 10^{30}K$, which is two orders below the Planck temperature (this scales as λ^{-1}).

Considering these numbers, mBH (or microscopic black holes in general) are possible cold dark matter constituents. They are cold and dark (since their radiation froze out), have extremely small cross section and are heavy enough so only a small concentration $n_{mBH} \doteq 4.25 \times 10^{-20}m^{-3}$ is needed to make up for the observed dark matter mass density $\rho_{DM} \doteq 2.38 \times 10^{-27}kg\,m^{-3}$ (this scales as λ^{-1}). Dark matter density is uniform only on cosmological scales, there is more of it in galaxies (by factor $10^5 - 10^6$, see,[30] possibly even more within solar systems). The idea of mBH as dark matter candidates has appeared as a brief comment in,[31] with the question of their creation left open. We propose that they might have been formed shortly after the Big Bang (perhaps from primordial inhomogeneities[32]), cooling down afterwards, until they eventually froze out.

The cross section of mBH is small enough for them not to interact with each other, however it is still possible for them to be hit by another particle. Let us assume that a mBH gets hit by a proton and absorbs it, what would happen? Since the mass of the proton is significantly smaller than M_0 we can use eq. (17), for this example $\frac{\delta M}{M_0} \doteq 2.98 \times 10^{-20}$. The resulting microscopic black hole will warm up to 7.06×10^{-10} of T_m, which is $9.39 \times 10^{20}K$ (this scales as $\lambda^{-\frac{3}{2}}$), two orders below the energy of ultra-high-energy cosmic rays which are being observed. Had the λ been shorter than the Planck length, a radiation of a microscopic black hole after consuming a proton could account for such rays. It should be noted here that it might be more correct to consider mBH-electron or mBH-quark collision instead[b], since the proton is significantly larger than mBH.

It is important to note that in the considered case the energy of radiation exceeds the energy of the consumed particle. The possible scenario is that the energy will be radiated in one or two quanta and the resulting object will end with $M < M_0$, it will have no horizons and stops being a black hole. Then it will be moving through the space as an extremely dense lump of matter and collect additional mass until it reaches mass M_0 and becomes mBH again.

Such object, let us name it *gravimond*, lives in cycles: first it is a mBH with mass M_0. Then, after it absorbs a particle its radiation is reignited as $M > M_0$. Shortly after it stops being a black hole, since so much energy has been radiated that $M < M_0$, it becomes an

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[b]Interesting questions about the confinement arise in that case.

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extremely dense object (almost a black hole), which needs to capture additional mass to become mBH again. The period of these cycles is unknown and largely depends on the location of such object (how often does it get to interact with other matter).

Conclusion: The paper analyzed (microscopic) black holes with a blurred mass density, instead of a singular one. Such matter density originated from considering a NC structure of 3-space, yet the following calculations have been done using the ordinary calculus and the general theory relativity[c]. There are many cases in the history of physics advocating for a semiclassical approach, just recall the Bohr's derivation of the Rydberg's formula. We do not expect our results to be as exact, but merely to give a hint of what to expect from a proper quantum theory of gravity. Since some of the features persist also in full NC approach (for example existence of the minimal possible event horizon radius can be compared to the minimal event horizon area $A \approx 4\pi l_{\mathrm{Planck}}^2$ in[33]) it is plausible that other features will hold in a full NC approach as well.

The emphasis of our analysis was on the radius of event horizon(s) and the Hawking temperature of such black holes. The principal results of this paper are:

- Investigating the effects of blurring the matter singularity on the behavior of (microscopic) black holes. Showing the existence of a minimal possible black hole radius $r_0 \doteq 3.38\lambda$ and $M_0 \doteq 2.57\lambda$.

- Derivating the temperature dependency on the black hole mass, which is close to the ordinary case for $M \gg M_0$, drops to zero for $M = M_0$ and grows rapidly for $M \gtrsim M_0$.

- Conjecture that microscopic black holes could be dark matter constituents (and are also capable of generating high energy radiation).

Acknowledgment

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[c]This is why the objects in question are sometimes referred to as NC-inspired black holes instead of just NC black holes.

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