OPTIMAL ŁOJASIEWICZ–SIMON INEQUALITIES AND MORSE–BOTT YANG–MILLS ENERGY FUNCTIONS

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ABSTRACT. For any compact Lie group G, we prove that the Yang–Mills energy function obeys an optimal gradient inequality of Łojasiewicz–Simon type (exponent 1/2) near the critical set of flat connections on a principal G-bundle over a closed Riemannian manifold of dimension $d \geq 2$ and so its gradient flow converges at an exponential rate to that critical set. We establish this optimal Łojasiewicz–Simon gradient inequality by three different methods. Our first proof gives the most general result by direct analysis and relies on our extension of a theorem due to Uhlenbeck [86] that gives existence of a flat connection on a principal G-bundle supporting a connection with $L^{d/2}$-small curvature, existence of a Coulomb gauge transformation, and $W^{1,p}$ Sobolev distance estimates for $p > 1$. Our second proof proceeds by first establishing an optimal Łojasiewicz–Simon gradient inequality for abstract Morse–Bott functions on Banach manifolds, generalizing an earlier result due to the author and Maridakis [31, Theorem 4]. Our third proof establishes the optimal Łojasiewicz–Simon gradient inequality by direct analysis near a given flat connection that is a regular point of the curvature map. We prove similar results for the self-dual Yang–Mills energy function near regular anti-self-dual connections over closed Riemannian four-manifolds and for the full Yang–Mills energy function over closed Riemannian manifolds of dimension $d \geq 2$, when known to be Morse–Bott at a given Yang–Mills connection.

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1. Introduction

Since its discovery by Łojasiewicz in the context of analytic functions on Euclidean spaces [58, Proposition 1, p. 92] and subsequent generalization by Simon to a class of analytic functions on certain Hölder spaces [74, Theorem 3], the *Łojasiewicz–Simon gradient inequality* has played a significant role in analyzing questions such as *a*) global existence, convergence, and analysis of singularities for solutions to nonlinear evolution equations that are realizable as gradient-like systems for an energy function, *b*) uniqueness of tangent cones, and *c*) energy gaps and discreteness of energies. For a survey of applications of the Łojasiewicz–Simon gradient inequality to gradient flows for real analytic functions on Banach spaces, including energy functions arising in applied mathematics, geometric analysis, or mathematical physics, we refer the reader to our article [31] and monograph [26].

In this article, which complements [29, 30], we establish optimal gradient inequalities of Łojasiewicz–Simon type for the Yang–Mills and self-dual Yang–Mills energy functions and for C^2 functions on Banach spaces that are Morse–Bott near a critical point. These inequalities are proved by direct analysis and, in particular, none are proved by reduction to a Łojasiewicz gradient inequality that is known to hold in finite dimensions. Optimal gradient inequalities (exponent $1/2$) are important because they imply that the gradient flow converges at an exponential (rather than power law) rate to the critical set [26].

Our first main result (Theorem 1) is an extension of a theorem due to Uhlenbeck [86, Corollary 4.3] that gives existence of a flat connection on a principal G-bundle supporting a connection with $L^{d/2}$-small curvature, existence of a global Coulomb gauge transformation, and $W^{1,p}$ Sobolev distance estimates, for $p > 1$ and any compact Lie group G. Using Theorem 1 and direct analysis, we prove that the Yang–Mills energy function obeys the optimal Łojasiewicz gradient inequality near the critical set of flat connections on a principal G-bundle over a closed Riemannian manifold of dimension $d \geq 2$ (Theorem 2).

Next, we establish an optimal Łojasiewicz gradient inequality, Theorem 3, for abstract Morse–Bott functions on Banach manifolds, generalizing an earlier result due to the author and Maridakis [31, Theorem 4]. We apply Theorem 3 to prove that the Yang–Mills energy function obeys the optimal Łojasiewicz gradient inequality (Theorem 7) when one restricts to a neighborhood of a flat connection Γ that is a regular point of the curvature map, $A \mapsto F_A$, and hence that the Yang–Mills energy function is Morse–Bott near Γ. We also prove Theorem 7 by direct analysis without appealing to Theorem 3. A surprising conclusion of this analysis is that although the optimal Łojasiewicz gradient inequality for the Yang–Mills energy function near the critical set of flat connections is implied by the assumption of a Morse–Bott condition, that assumption appears not to be necessary even though the proof of the optimal inequality is far easier when the Morse–Bott condition is obeyed. One might speculate that the Yang–Mills energy function always obeys a weaker condition than Morse–Bott near the critical set of flat connections and that a generalization of Theorem 3 holds for such functions. One possible candidate may be the Morse–Bott–Kirwan condition due to Kirwan [47] — see a recent analysis by Holm and Karshon [41] and their statement of the condition in [41, Definitions 2.1 and 2.3].

We also prove an optimal Łojasiewicz gradient inequality for the self-dual Yang–Mills energy function near anti-self-dual connections, over closed Riemannian four-manifolds, that are regular points of the self-dual curvature map, $A \mapsto F_A^+$ (Theorem 6). Finally, we prove an optimal Łojasiewicz gradient inequality for the full Yang–Mills energy function over closed Riemannian manifolds of dimension $d \geq 2$, when known to be Morse–Bott at a given Yang–Mills connection (Theorem 8).

Throughout this article, our conventions and notation are consistent with those of its two predecessors [29, 30] and generally follow those of standard references such as Donaldson and Kronheimer [23], Freed and Uhlenbeck [33], and Friedman and Morgan [34]. We shall not repeat those explanations here but we include a brief summary of our conventions and notation in Section 1.8 for ease of reference.

1.1. **Existence of a flat connection, Coulomb gauge transformation, and Sobolev distance estimate for small curvature in borderline case of critical Sobolev exponents.** Our first main result is a generalization, Theorem 1 below, of Uhlenbeck's [86, Corollary 4.3] from the non-borderline case, $d/2 < p < d$ and L^p-small curvature F_A, to $1 < p < d$ and the *borderline* case of $L^{d/2}$-small curvature. Uhlenbeck's [86, Corollary 4.3] is quoted in this article as Theorem 2.1. The proof of [86, Corollary 4.3] given by Uhlenbeck was brief, so we gave more details in [30, Sections 5 and 6]; our primary concern in [30] was to explain the origin of the key estimate (2.4) more fully. The proof of the remaining items in Theorem 2.1 followed by standard arguments (see [23, 33]), though we included the details in [30] for completeness.

Theorem 1 (Existence of a flat connection on a principal bundle supporting a $W^{1,q}$ connection with $L^{d/2}$-small curvature, Coulomb gauge transformation, and Sobolev distance estimate)**.** *Let (X, g) be a closed, smooth Riemannian manifold of dimension $d \geq 2$, and G be a compact Lie group, and $q \in (d/2, \infty]$ and $p \in (1, d)$ obeying $p \leq q$ and $s_0 > 1$ be constants. Then there are constants, $\varepsilon = \varepsilon(g, G, s_0) \in (0, 1]$ and $C_0 = C_0(g, G, p, s_0) \in [1, \infty)$ and $C_1 = C_1(g, G, p, q, s_0) \in [1, \infty)$ with the following significance. If A is a $W^{1,q}$ connection on a smooth principal G-bundle*

P over X such that[1]

$$\|F_A\|_{L^{s_0}(X)} \leq \varepsilon, \tag{1.1}$$

where $s_0 = d/2$ when $d \geq 3$ or $s_0 > 1$ when $d = 2$, then the following hold.

(1) (Existence of a C^∞ flat connection) *There is a C^∞ flat connection, Γ, on P;*

(2) ($W^{1,p}$-distance estimate) *For $\varepsilon = \varepsilon(g, G, p, s_0) \in (0, 1]$ small enough, Γ obeys*

$$\|A - \Gamma\|_{W^{1,p}_\Gamma(X)} \leq C_0\|F_A\|_{L^p(X)}; \tag{1.2}$$

(3) (Existence of global Coulomb gauge transformation and estimate of Sobolev $W^{1,p}$ distance to the flat connection) *For $\varepsilon = \varepsilon(g, G, p, s_0) \in (0, 1]$ small enough, there is a $W^{2,q}$ gauge transformation, $u \in \mathrm{Aut}(P)$, with*

$$d^*_\Gamma(u(A) - \Gamma) = 0 \quad a.e.\ on\ X; \tag{1.3}$$

$$\|u(A) - \Gamma\|_{W^{1,p}_\Gamma(X)} \leq C_1\|F_A\|_{L^p(X)}. \tag{1.4}$$

We prove Theorem 1 in Section 2.

Remark 1.1 (Existence of flat connections in the case of borderline control over curvature)*.* Our ability in Item (2) to choose $p = 2$ (independent of the value of $d \geq 2$) is of crucial importance in our proof of Theorem 2, but the more novel aspect of Theorem 1 is the sufficiency (when $d \geq 3$) of the borderline hypothesis $\|F_A\|_{L^{d/2}(X)} \leq \varepsilon$ in (1.1) to provide existence in Item (1) of a flat connection Γ on the *same* principal G-bundle P as that supporting the connection A with $L^{d/2}$-small curvature. The well-known argument due to Sedlacek [70] when $d = 4$ would produce a flat connection, Γ, on a possibly different principal G-bundle Q but the classification of principal G-bundles, knowledge of the vector Pontrjagin classes, and the behavior of Sedlacek's obstruction class under weak limits ensures that $Q \cong P$ as continuous principal G-bundles. See the Introduction to Section 2.1 for a discussion of this approach and further details. However, this is not how we prove Item (1).

Instead, recall that Uhlenbeck's [84, Theorem 1.3] gives existence of local Coulomb gauges and *a priori* estimates for local connection one-forms with $L^{d/2}$-small curvature. An application of her [84, Theorem 1.3] to a minimizing sequence of connections yields $W^{1,d/2}$ convergence of local connection one-forms and $W^{2,d/2}$ convergence of local gauge transformations. The Sobolev Embedding [2, Theorem 4.12] implies that $W^{2,p}(X;\mathbb{R}) \subset C^0(X;\mathbb{R})$ is a continuous embedding when $p > d/2$ but not when $p = d/2$ and thus Uhlenbeck's patching arguments do not appear applicable at first glance. However, as we explain in Sections 2.4 and 2.5, the fact that the gauge-transformed local connection one-forms obey a Coulomb gauge condition is sufficient to give us $W^{2,p}$ and thus C^0 control over local gauge transformations with $p > d/2$ and this directly yields the isomorphism $Q \cong P$, without appeal to the classification of principal G-bundles — see Theorems 2.16 and 2.20. Partly related results were proved by Taubes [82, Proposition 4.5 and Lemma A.1] when $d = 4$, using a more difficult method, and by Rivère [67, Theorem IV.1] when $d \geq 4$, using Lorentz spaces rather than the standard Sobolev spaces that we employ throughout this article. See Remark 2.18 for further discussion of the results due to Rivère and Taubes and Remark 2.19 for a discussion of related results due to Isobe [45] and Shevchishin [73].

Remark 1.2 (Application of Theorem 1 to optimal Łojasiewicz–Simon inequalities for the Yang–Mills energy function)*.* Items (1) and (2) are the main ingredients in our application to the proof of

[1] We may choose $s_0 > 1$ arbitrarily close to 1 when $d = 2$ and, in particular, small enough that $|F_A| \in L^{s_0}(X;\mathbb{R})$; for Item (1), the constant ε is independent of p.

Theorem 2, giving the optimal Łojasiewicz–Simon inequalities for the Yang–Mills energy function. We do not need Item (3) for the latter purpose, but we prove Item (3) since those results are of interest for their own sake.

Remark 1.3 (The case $p \geq d$). We only state Theorem 1 for the case $p < d$ because, once $p \geq d$, then the hypothesis (1.1) is no longer sufficient to achieve Items (2) and (3), and must be strengthened, for example[2] to $\|F_A\|_{L^p(X)} \leq \varepsilon$ (see Theorem 2.1 and choose $q = p$ in (2.1)), so Theorem 1 reverts to the non-borderline version due to Uhlenbeck, namely Theorem 2.1.

1.2. **Optimal Łojasiewicz–Simon inequalities for the Yang–Mills energy function.** We define the *Yang–Mills-energy function* by [4, p. 548]

$$\mathscr{E}(A) := \frac{1}{2}\int_X |F_A|^2 \, d\,\mathrm{vol}_g, \tag{1.5}$$

where A is a $W^{1,q}$ connection on P and curvature [23, Equation (2.1.13)],

$$F_A = d_A \circ d_A \in L^2(X; \Lambda^2 \otimes \mathrm{ad}P),$$

where $q \geq \max\{2, 4d/(d+4)\}$. Writing $A = A_1 + a$, for any C^∞ connection A_1 on P, we have [23, Equation (2.1.14)]

$$F_A = F_{A_1} + d_{A_1}a + a \wedge a. \tag{1.6}$$

The constraint $q \geq 2$ ensures that $d_{A_1}a \in L^2(X; \Lambda^2 \otimes \mathrm{ad}P)$ and the constraint $q \geq 4d/(d+4)$ is equivalent to $q^* := dq/(d-q) \geq 4$ and thus $W^{1,q}(X;\mathbb{R}) \subset L^4(X;\mathbb{R})$ when $q < d$ by [2, Theorem 4.12, Part I (C)]. Hence, $a \in L^4(X; \Lambda^1 \otimes \mathrm{ad}P)$ and $a \wedge a \in L^2(X; \Lambda^2 \otimes \mathrm{ad}P)$, which gives $F_A \in L^2(X; \Lambda^2 \otimes \mathrm{ad}P)$, as desired. Note that $d/2 \geq 4d/(d+4) \iff d \geq 4$ and $4d/(d+4) < 2$ only when $d = 2, 3$.

In order to ensure that the energy $\mathscr{E}(A)$ in (1.5) is well-defined for a $W^{1,q}$ connection A and that the action of gauge transformations on P is also well-defined, we shall assume for consistency and simplicity throughout this article that $q \in [2, \infty)$ and obeys $q > d/2$ in this context, even though that condition may be stronger than necessary in some instances.

In writing (1.6), we are slightly abusing notation since in the setting of [23, Section 2.1], for example, a representation, $\rho : G \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$, is assumed and $a \wedge a$ denotes a combination of wedge product of one-forms $a \in \Omega^1(X; \mathrm{End}_{\mathbb{C}}(E))$ and multiplication in $\mathrm{End}_{\mathbb{C}}(E)$, where E is the complex vector bundle, $P \times_\rho \mathbb{C}^n$. Since we view $a \in \Omega^1(X; \mathrm{ad}P)$ and $F_A \in \Omega^2(X; \mathrm{ad}P)$ (as in [6]) rather than $F_A \in \Omega^2(X; \mathrm{End}_{\mathbb{C}}(E))$ (as in [23]), we should more precisely write (see the parenthetical remark just below [23, Equation (2.1.14)] or [6, Lemma 4.5])

$$F_A = F_{A_1} + d_{A_1}a + \frac{1}{2}[a, a], \tag{1.7}$$

where $[a, a](\eta, \zeta) := [a(\eta), a(\zeta)]$ for vector fields $\eta, \zeta \in C^\infty(TX)$ and $[\ ,\ ]$ denotes the Lie bracket on the Lie algebra $\mathfrak{g}$ of G. Compare [49, Theorem II.5.2] or [6, p. 430]. On the other hand, for $a, b \in \Omega^1(X; \mathrm{ad}P)$, the exterior covariant derivative $d_A b$ is expressed in terms of $d_{A_1} b$ when $A = A_1 + a$ by (see [23, Sections 2.1.1 and 2.1.2] or [6, Equations (3.3) and (4.1)])

$$d_A b = d_{A_1 + a} b = d_{A_1} b + [a, b] = d_{A_1} b + 2a \wedge b. \tag{1.8}$$

Normally, these factors of $\frac{1}{2}$ or 2 are immaterial (and in such cases we abuse notation and omit them) but in the proof of Theorem 2, the distinction does matter as we shall see in Section 3.

[2]By analogy with Corollary 2.11, it should be possible to improve this condition slightly, to $\|F_A\|_{L^{\bar{p}}(X)} \leq \varepsilon$, where $\bar{p} = pd/(d+p)$ when $p > d$ and $\bar{p} > d/2$ when $p = d$, but that improvement has limited value in practice.

For $q \in [2, \infty)$ obeying $q > d/2$, let $\mathscr{B}(P) := \mathscr{A}(P)/\operatorname{Aut}(P)$ denote the quotient of the affine space, $\mathscr{A}(P)$, of $W^{1,q}$ connections on P, modulo the action of the group, $\operatorname{Aut}(P)$, of $W^{2,q}$ automorphisms (or gauge transformations) of the principal G-bundle, P. Let

$$M_0(P) := \{A \in \mathscr{A}(P) : F_A = 0\}/\operatorname{Aut}(P) \tag{1.9}$$

denote the moduli space of $W^{1,q}$ *flat connections* on P. We write

$$\operatorname{dist}_{W^{1,2}(X)}\left([A], M_0(P)\right) := \inf_{\substack{u\in\operatorname{Aut}(P),\\ [\Gamma]\in M_0(P)}} \|u(A) - \Gamma\|_{W^{1,2}_\Gamma(X)}. \tag{1.10}$$

Recall that the Yang–Mills energy function, $\mathscr{E} : \mathscr{A}(P) \to \mathbb{R}$, in (1.5) has differential map, $\mathscr{E}' : \mathscr{A}(P) \to T^*\mathscr{A}(P)$, given by

$$\mathscr{E}'(A)(a) = (F_A, d_A a)_{L^2(X)} = (d_A^* F_A, a)_{L^2(X)}, \tag{1.11}$$

for all $A \in \mathscr{A}(P)$ and $a \in T_A\mathscr{A}(P) = W^{1,q}(X; \Lambda^1 \otimes \operatorname{ad}P)$, where $T_A^*\mathscr{A}(P) = W^{-1,q'}(X; \Lambda^1 \otimes \operatorname{ad}P)$ and $q' \in (1, 2]$ is the dual Hölder exponent defined by $1/q + 1/q' = 1$.

Theorem 2 (Optimal Łojasiewicz–Simon inequalities for the Yang–Mills energy function near flat connections)**.** *Let (X, g) be a closed, smooth Riemannian manifold of dimension $d \geq 2$, and G be a compact Lie group, P be a principal G-bundle over X, and $q \in [2, \infty)$ be a constant obeying $q > d/2$. Then there are constants, $C, Z \in (0, \infty)$ and $\varepsilon \in (0, 1]$, depending g, G, and q with the following significance. If A is a $W^{1,q}$ connection on P and its curvature, F_A, obeys the* small $L^{d/2}$-norm condition (1.1)*, that is,*

$$\|F_A\|_{L^{s_0}(X)} \leq \varepsilon,$$

where $s_0 = d/2$ when $d \geq 3$ or $s_0 > 1$ when $d = 2$, then the optimal Łojasiewicz–Simon distance and gradient inequalities *hold for the Yang–Mills energy function $\mathscr{E}$ in* (1.5)*,*

$$\mathscr{E}(A)^{1/2} \geq C \operatorname{dist}_{W^{1,2}}([A], M_0(P)), \tag{1.12}$$

$$\|\mathscr{E}'(A)\|_{W^{-1,2}(X)} \geq Z\, \mathscr{E}(A)^{1/2}. \tag{1.13}$$

We prove Theorem 2 in Section 3.

Remark 1.4 (Other approaches to the proof of the optimal Łojasiewicz–Simon gradient inequality for the Yang–Mills energy function near a *regular* flat connection). The inequalities (1.12) and (1.13) in Theorem 2 are proved in Section 3 by direct geometric analysis as corollaries of Theorem 1. However, as we explain in Section 5.2, the gradient inequality (1.13) may also be proved near a given *regular* flat connection Γ (see Theorem 7) by first establishing that $\mathscr{E}$ is Morse–Bott at Γ (Lemma 5.2) and then appealing to our Theorem 3, which gives the optimal gradient inequality for an abstract Morse–Bott function on a Banach space. The additional hypothesis that Γ is regular yields a relatively simple proof of (1.13), but this hypothesis is strong and will not hold for all flat connections.

Because $W^{1,2}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P) \subset L^2(X; \Lambda^1 \otimes \operatorname{ad}P)$ is a continuous, dense embedding of Sobolev spaces (for any $d \geq 2$), we obtain a continuous embedding of Sobolev spaces,

$$L^2(X; \Lambda^1 \otimes \operatorname{ad}P) \subset W^{-1,2}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P),$$

by duality and so the $W^{-1,2}$ norm in (1.13) can be replaced by the stronger L^2 norm, as convenient in the analysis of the gradient flow equation for $\mathscr{E}$ [26].

1.3. **Łojasiewicz–Simon gradient inequalities for Morse–Bott functions.** In applications to geometry and topology, it is very useful to know when a given energy function is a Morse function (isolated critical points) or more generally a Morse–Bott function (non-isolated critical points). We shall subsequently state a special case (see Theorem 7) of Theorem 2 that admits a much shorter proof than that of Theorem 2 when the Yang–Mills energy function is known to be Morse–Bott near a flat connection.

Definition 1.5 (Morse–Bott function)**.** (See Austin and Braam [8, Section 3.1].) Let $\mathscr{B}$ be a smooth Banach manifold, $\mathscr{E} : \mathscr{B} \to \mathbb{R}$ be a C^2 function, and $\operatorname{Crit}\mathscr{E} := \{x \in \mathscr{B} : \mathscr{E}'(x) = 0\}$. A smooth submanifold $\mathscr{C} \hookrightarrow \mathscr{B}$ is called a *nondegenerate critical submanifold of* $\mathscr{E}$ if $\mathscr{C} \subset \operatorname{Crit}\mathscr{E}$ and

$$(T\mathscr{C})_x = \operatorname{Ker}\mathscr{E}''(x), \quad \forall\, x \in \mathscr{C}, \tag{1.14}$$

where $\mathscr{E}''(x) : (T\mathscr{B})_x \to (T\mathscr{B})_x^*$ is the Hessian of $\mathscr{E}$ at the point $x \in \mathscr{C}$. One calls $\mathscr{E}$ a *Morse–Bott function* if its critical set $\operatorname{Crit}\mathscr{E}$ consists of nondegenerate critical submanifolds.

We say that a C^2 function $\mathscr{E} : \mathscr{B} \to \mathbb{R}$ is *Morse–Bott at a point* $x_0 \in \mathscr{B}$ if there is an open neighborhood $\mathscr{U} \subset \mathscr{B}$ of x_0 such that $\mathscr{U} \cap \operatorname{Crit}\mathscr{E}$ is a relatively open, smooth submanifold of $\mathscr{B}$ and (1.14) holds at x_0.

In Definition 1.5, if we had only assumed that $\mathscr{C} \hookrightarrow \mathscr{B}$ is a smooth submanifold with $\mathscr{C} \subset \operatorname{Crit}\mathscr{E}$, we would still have the inclusion,

$$(T\mathscr{C})_x \subset \operatorname{Ker}\mathscr{E}''(x),$$

for each $x \in \mathscr{C}$. Hence, the key assertion in (1.14) is that *equality* holds and thus each vector $v \in (T\mathscr{B})_x \cap \operatorname{Ker}\mathscr{E}''(x)$ is *integrable*, the tangent vector to a smooth path in $\mathscr{C}$ through x.

Definition 1.5 is a restatement of definitions of a Morse–Bott function on a finite-dimensional manifold, but we omit the condition that $\mathscr{C}$ be compact and connected as in Nicolaescu [63, Definition 2.41] or the condition that $\mathscr{C}$ be compact in Bott [15, Definition, p. 248]. Note that if $\mathscr{B}$ is a Riemannian manifold and $\mathscr{N}$ is the normal bundle of $\mathscr{C} \hookrightarrow \mathscr{B}$, so $\mathscr{N}_x = (T\mathscr{C})_x^\perp$ for all $x \in \mathscr{C}$, where $(T\mathscr{C})_x^\perp$ is the orthogonal complement of $(T\mathscr{C})_x$ in $(T\mathscr{B})_x$, then (1.14) is equivalent to the assertion that the restriction of the Hessian to the fibers of the normal bundle of $\mathscr{C}$,

$$\mathscr{E}''(x) : \mathscr{N}_x \to (T\mathscr{B})_x^*,$$

is *injective* for all $x \in \mathscr{C}$; using the Riemannian metric on $\mathscr{B}$ to identify $(T\mathscr{B})_x^* \cong (T\mathscr{B})_x$, we see that $\mathscr{E}''(x) : \mathscr{N}_x \cong \mathscr{N}_x$ is an isomorphism for all $x \in \mathscr{C}$. In other words, the condition (1.14) is equivalent to the assertion that the Hessian of $\mathscr{E}$ is an isomorphism of the normal bundle $\mathscr{N}$ when $\mathscr{B}$ has a Riemannian metric.

For a development of Morse–Bott theory and a discussion of and references to its numerous applications, we refer to Austin and Braam [8], Banyaga and Hurtubise [9, 10, 11, 12], Nicolaescu [63], and references cited therein.

Definition 1.6 (Gradient map)**.** (See Berger [13, Section 2.5], Huang [42, Definition 2.1.1].) Let $\mathscr{U} \subset \mathscr{X}$ be an open subset of a Banach space, $\mathscr{X}$, and let $\mathscr{Y}$ be a Banach space with continuous embedding, $\mathscr{Y} \subseteq \mathscr{X}^*$. A continuous map, $\mathscr{M} : \mathscr{U} \to \mathscr{Y}$, is called a *gradient map* if there exists a C^1 function, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$, such that

$$\mathscr{E}'(x)v = \langle v, \mathscr{M}(x)\rangle_{\mathscr{X}\times\mathscr{X}^*}, \quad \forall\, x \in \mathscr{U}, \quad v \in \mathscr{X}, \tag{1.15}$$

where $\langle\cdot,\cdot\rangle_{\mathscr{X}\times\mathscr{X}^*}$ is the canonical bilinear form on $\mathscr{X} \times \mathscr{X}^*$. The real-valued function, $\mathscr{E}$, is called a *potential* for the gradient map, $\mathscr{M}$.

When $\mathscr{Y} = \mathscr{X}^*$ in Definition 1.6, then the differential and gradient maps coincide.

Theorem 3 (Łojasiewicz–Simon gradient inequality for C^2 Morse–Bott functions on Banach spaces)**.** *(Compare Feehan and Maridakis* [31, Theorems 3 and 4]*.) Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{G}$, and $\mathscr{H}$ be Banach spaces with continuous embeddings,*

$$\mathscr{X} \subset \mathscr{G} \quad \text{and} \quad \mathscr{Y} \subset \mathscr{H} \subset \mathscr{G}^* \subset \mathscr{X}^*.$$

Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be a C^2 function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, so $\mathscr{E}'(x_\infty) = 0$. Let $\mathscr{M} : \mathscr{U} \to \mathscr{Y}$ be a C^1 gradient map for $\mathscr{E}$ in the sense of Definition 1.6 and require that $\mathscr{E}$ be Morse–Bott at x_∞ in the sense of Definition 1.5, so $\mathscr{U} \cap \operatorname{Crit} \mathscr{E}$ is a relatively open, smooth submanifold of $\mathscr{X}$ and $K := \operatorname{Ker} \mathscr{E}''(x_\infty) = T_{x_\infty} \operatorname{Crit} \mathscr{E}$. Suppose that for each $x \in \mathscr{U}$, the bounded, linear operator,

$$\mathscr{M}'(x) : \mathscr{X} \to \mathscr{Y},$$

has an extension,

$$\mathscr{M}_1(x) : \mathscr{G} \to \mathscr{H},$$

such that the following map is continuous,

$$\mathscr{U} \ni x \mapsto \mathscr{M}_1(x) \in \mathscr{L}(\mathscr{G}, \mathscr{H}).$$

Assume that $K \subset \mathscr{X}$ has a closed complement, $\mathscr{X}_0 \subset \mathscr{X}$, that $\mathscr{K} := \operatorname{Ker} \mathscr{M}_1(x_\infty) \subset \mathscr{G}$ has a closed complement $\mathscr{G}_0 \subset \mathscr{G}$ with $\mathscr{X}_0 \subset \mathscr{G}_0$, and that $\operatorname{Ran} \mathscr{M}_1(x_\infty) \subset \mathscr{H}$ is a closed subspace. Then there are constants, $Z \in (0, \infty)$ and $\sigma \in (0, 1]$, with the following significance. If $x \in \mathscr{U}$ obeys

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{1.16}$$

then

$$\|\mathscr{M}(x)\|_{\mathscr{H}} \geq Z |\mathscr{E}(x) - \mathscr{E}(x_\infty)|^{1/2}. \tag{1.17}$$

We prove Theorem 3 in Section 4.

Remark 1.7 (Previous versions of the Łojasiewicz–Simon gradient inequality for C^2 Morse–Bott functions on abstract Banach spaces)*.* Previous versions of Theorem 3 were proved by Simon [76, Lemma 3.13.1] (for a harmonic map energy function on a Banach space of $C^{2,\alpha}$ sections of a Riemannian vector bundle), Haraux and Jendoubi [38, Theorem 2.1] (for functions on abstract Hilbert spaces) and in more generality by Chill in [19, Corollary 3.12] (for functions on abstract Banach spaces). These authors do not use Morse–Bott terminology but their hypotheses imply this condition — directly in the case of Haraux and Jendoubi and Chill and by a remark due to Simon in [76, p. 80] that his integrability condition [76, Equation (iii), p. 79] is equivalent to a restatement of the Morse–Bott condition. Their gradient inequalities are less general than our Theorem 3. See Feehan [27, Remark 1.16 and Appendix C] for further discussion of the relationship between definitions of integrability, such as those described by Adams and Simon [1], and the Morse–Bott condition.

Remark 1.8 (On the proof of Theorem 3)*.* Special cases of Theorem 3 can be obtained as consequences of suitable Morse–Bott lemmas (see Feehan [27] for a discussion and references). However, proofs of Morse–Bott lemmas require care and it is unclear whether one would hold in the generality provided by Theorem 3. On the other hand, the proof of Theorem 3 provided in Section 4 is quite direct.

Remark 1.9 (Comparison between Inequality (1.17) and other Łojasiewicz-Simon gradient inequalities). In [31, Theorems 1, 2, and 3], Maridakis and the author establish versions of Theorem 3 where the inequality (1.17) is replaced by

$$\|\mathscr{M}(x)\|_{\mathscr{H}} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta, \tag{1.18}$$

for some $\theta \in [1/2, 1)$, the operators $\mathscr{M}'(x_\infty)$ and $\mathscr{M}_1(x_\infty)$ are Fredholm, and $\mathscr{M} : \mathscr{U} \to \mathscr{Y}$ is real analytic. Those results are proved with the aid of a *Lyapunov-Schmidt reduction* of $\mathscr{E}$ (for example, [42, Proposition 5.1]) to a real analytic function on an open neighborhood of the origin in Euclidean space and appealing to Łojasiewicz's gradient inequality [56, 57, 58], with a simplified proof provided by Bierstone and Milman [14, Theorem 6.4 and Remark 6.5]. However, the requirement that the operators $\mathscr{M}'(x_\infty)$ and $\mathscr{M}_1(x_\infty)$ be Fredholm can be restrictive. For example, in the context of Yang–Mills or coupled Yang–Mills energy functions, one must take a quotient of the affine space of all $W^{1,q}$ connections or pairs by the Banach Lie group, $\mathrm{Aut}(P)$, of $W^{2,q}$ gauge transformations and that action can introduce singularities in the quotient space, as we recall in Section 1.4.

Remark 1.10 (Optimal Łojasiewicz-Simon gradient inequalities and exponential convergence of gradient flow). It is of considerable interest to know when the optimal exponent $\theta = 1/2$ is achieved, since in that case one can prove (for example, [26, Theorem 24.21]) that a global solution, $u : [0, \infty) \to \mathscr{X}$, to a gradient system governed by the Łojasiewicz–Simon gradient inequality,

$$\frac{du}{dt} = -\mathscr{E}'(u(t)), \quad u(0) = x_0,$$

has *exponential* rather than mere power-law rate of convergence to the critical point, x_∞. See [26, Section 2.1] for a detailed summary of results of this kind.

Remark 1.11 (Comparison between Theorem 3 and a previous result due to the author and Maridakis). Theorem 3 is a generalization of our previous [31, Theorems 3 and 4], but the advantage of Theorem 3 here is that the operators $\mathscr{M}'(x_\infty)$ and $\mathscr{M}_1(x_\infty)$ are not required to be Fredholm. While that generalization can be established by modifying the proofs of [31, Theorems 3 and 4], we instead give a more direct and much simpler proof in Section 1.3. The latter proof also allows us to slightly relax other hypotheses on the Banach spaces and their embeddings. Of course, when $\mathscr{M}'(x_\infty)$ or $\mathscr{M}_1(x_\infty)$ are Fredholm operators, then their kernels are finite-dimensional and thus have closed complements by [69, Lemma 4.21 (a)], and their ranges are closed.

Remark 1.12 (Choices of the Banach spaces $\mathscr{G}$ and $\mathscr{H}$). In typical applications of Theorem 3 one chooses $\mathscr{G}$ and $\mathscr{H}$ to be Hilbert spaces and that simplifies the statement of the theorem since a closed subspace of a Hilbert space necessarily has a closed (orthogonal) complement [69, Theorem 12.4]. However, the greater generality allows us to quickly infer several corollaries (see the forthcoming Corollaries 4 and 5) analogous to [31, Theorems 1, 2, and 4] and whose statements are shorter and thus more easily understood, but Theorem 3 is the most useful version in applications to proofs of global existence and convergence of gradient flows. For example, Theorem 3 is the only version that yields Simon's [74, Theorem 3] for all dimensions of the base manifold, X, with $\mathscr{X} = C^{2,\alpha}(X; V)$ and $\mathscr{H} = L^2(X; V)$ (where V is a Riemannian vector bundle over X), and, moreover, for a wide variety of alternative choices of Hölder or Sobolev spaces for $\mathscr{X}$; see [31, Remark 1.14].

Remark 1.13 (Harmonic map energy function for maps from a Riemann surface into a closed Riemannian manifold). For the harmonic map energy function, an optimal Łojasiewicz–Simon gradient inequality,

$$\|\mathscr{E}'(f)\|_{L^p(S^2)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^{1/2},$$

has been obtained by Kwon [53, Theorem 4.2] for maps $f : S^2 \to N$, where N is a closed Riemannian manifold and f is close to a harmonic map f_∞ in the sense that

$$\|f - f_\infty\|_{W^{2,p}(S^2)} < \sigma,$$

where p is restricted to the range $1 < p \leq 2$, and f_∞ is assumed to be *integrable* in the sense of [53, Definitions 4.3 or 4.4 and Proposition 4.1]. Her [53, Proposition 4.1] quotes results of Simon [75, pp. 270–272] and Adams and Simon [1]. The [55, Lemma 3.3] due to Liu and Yang is another example of an optimal Łojasiewicz–Simon gradient inequality for the harmonic map energy function, but restricted to the setting of maps $f : S^2 \to N$, where N is a Kähler manifold of complex dimension $n \geq 1$ and nonnegative bisectional curvature, and the energy $\mathscr{E}(f)$ is sufficiently small. The result of Liu and Yang generalizes that of Topping [83, Lemma 1], who assumes that $N = S^2$.

Remark 1.14 (Yamabe function for Riemannian metrics on a closed manifold). For the Yamabe function, an optimal Łojasiewicz–Simon gradient inequality, has been obtained by Carlotto, Chodosh, and Rubinstein [18] under the hypothesis that the critical point is *integrable* in the sense of their [18, Definition 8], a condition that they observe in [18, Lemma 9] (quoting [1, Lemma 1] due to Adams and Simon) is equivalent to a function on Euclidean space given by the *Lyapunov-Schmidt reduction* of $\mathscr{E}$ being constant on an open neighborhood of the critical point.

Remark 1.15 (Yang–Mills energy function over a Riemann surface). For the Yang–Mills energy function for connections on a principal $\mathrm{U}(n)$-bundle over a closed Riemann surface, an optimal Łojasiewicz–Simon gradient inequality, has been obtained by Råde [68, Proposition 7.2] when the Yang–Mills connection is *irreducible*.

Remark 1.16 (F-function on the space of hypersurfaces in Euclidean space). Colding and Minicozzi [20, 21] have directly proved Łojasiewicz–Simon gradient and distance inequalities [22, Equations (5.9) and (5.10)] that do not involve Lyapunov-Schmidt reduction to a finite-dimensional gradient inequality. Their gradient inequality applies to the F function [22, Section 2.4] on the space of hypersurfaces $\Sigma \subset \mathbb{R}^{d+1}$ and is analogous to (1.18) with $\theta = 2/3$. Their cited articles contain detailed technical statements of their inequalities while their article with Pedersen [22] contains a less technical summary of some of their main results.

If $\mathscr{G} = \mathscr{X}$ and $\mathscr{H} = \mathscr{Y}$, then the statement of Theorem 3 simplifies to give the following generalization of [31, Theorems 2 and 4].

Corollary 4 (Łojasiewicz–Simon gradient inequality for C^2 Morse–Bott functions on Banach spaces). *(Compare Feehan and Maridakis* [31, Theorems 2 and 4]*.) Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces with a continuous embedding, $\mathscr{Y} \subset \mathscr{X}^*$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be a C^2 function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, so $\mathscr{E}'(x_\infty) = 0$. Let $\mathscr{M} : \mathscr{U} \to \mathscr{Y}$ be a C^1 gradient map for $\mathscr{E}$ in the sense of Definition 1.6 and require that $\mathscr{E}$ be Morse–Bott at x_∞ in the sense of Definition 1.5, so $\mathscr{U} \cap \operatorname{Crit}\mathscr{E}$ is a relatively open, smooth submanifold of $\mathscr{X}$ and $K := \operatorname{Ker}\mathscr{E}''(x_\infty) = T_{x_\infty}\operatorname{Crit}\mathscr{E}$. Assume that $K \subset \mathscr{X}$ has a closed complement, $\mathscr{X}_0 \subset \mathscr{X}$, and that $\operatorname{Ran}\mathscr{M}'(x_\infty) \subset \mathscr{Y}$ is a closed subspace. Then there are constants, $Z \in (0,\infty)$ and $\sigma \in (0,1]$, with the following significance. If $x \in \mathscr{U}$ obeys*

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{1.19}$$

then

$$\|\mathscr{M}(x)\|_{\mathscr{Y}} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^{1/2}. \tag{1.20}$$

For example, Corollary 4 yields a version of Simon's [74, Theorem 3] when X has dimension $d = 2$ or 3 and choose $\mathscr{X} = W^{1,p}(X; V)$ and $\mathscr{Y} = W^{-1,p}(X; V)$, where $p > d$ is small enough that $L^2(X; V) \subset W^{-1,p}(X; V)$; see [31, Remark 1.15].

If in addition $\mathscr{Y} = \mathscr{X}^*$, then the statement of Theorem 3 simplifies further to give the following generalization of [31, Theorems 1 and 4].

Corollary 5 (Łojasiewicz–Simon gradient inequality for C^2 Morse–Bott functions on Banach spaces)**.** *(Compare Feehan and Maridakis* [31, Theorems 1 and 4]*.) Let $\mathscr{X}$ be a Banach space, $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be a C^2 function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, so $\mathscr{E}'(x_\infty) = 0$. Require that $\mathscr{E}$ be Morse–Bott at x_∞ in the sense of Definition 1.5, so $\mathscr{U} \cap \operatorname{Crit}\mathscr{E}$ is a relatively open, smooth submanifold of $\mathscr{X}$ and $K := \operatorname{Ker}\mathscr{E}''(x_\infty) = T_{x_\infty}\operatorname{Crit}\mathscr{E}$. Assume that $K \subset \mathscr{X}$ has a closed complement, $\mathscr{X}_0 \subset \mathscr{X}$, and that $\operatorname{Ran}\mathscr{E}''(x_\infty) \subset \mathscr{X}^*$ is a closed subspace. Then there are constants, $Z \in (0, \infty)$ and $\sigma \in (0, 1]$, with the following significance. If $x \in \mathscr{U}$ obeys*

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{1.21}$$

then

$$\|\mathscr{E}'(x)\|_{\mathscr{X}^*} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^{1/2}. \tag{1.22}$$

For example, Corollary 4 yields Råde's Łojasiewicz-Simon gradient inequality for the Yang–Mills energy function when the base manifold has dimension $d = 2$ or 3 and our version of the same inequality [26, Theorem 23.17] when $d = 4$, for $\mathscr{X} = W^{1,2}(X; \operatorname{ad}P)$, but not $d \geq 5$, nor does it yield any version of Simon's [74, Theorem 3].

1.4. **Optimal Łojasiewicz–Simon inequalities and Morse–Bott properties for the self-dual Yang–Mills energy function near anti-self-dual connections.** We refer the reader to Donaldson and Kronheimer [23, Section 4.2] or Freed and Uhlenbeck [33, Chapter 3] for constructions of a smooth Banach manifold structure on the quotient, $\mathscr{B}^*(P) := \mathscr{A}^*(P)/\operatorname{Aut}(P)$, by the Banach Lie group, $\operatorname{Aut}(P)$, of $W^{2,q}$ gauge transformations of P, where $\mathscr{A}^*(P) \subset \mathscr{A}(P)$ is by definition the open subset consisting of $W^{1,q}$ connections on P whose isotropy group is minimal, namely the center of G [23, p. 132].

We now restrict our attention to the case of X of dimension $d = 4$. For a C^∞ connection, A, on P we recall the splitting [23, Equation (2.1.25)],

$$F_A = F_A^+ + F_A^+ \in \Omega^2(X; \operatorname{ad}P) = \Omega^+(X; \operatorname{ad}P) \oplus \Omega^-(X; \operatorname{ad}P), \tag{1.23}$$

corresponding to the splitting, $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, into positive and negative eigenspaces, $\Lambda^\pm$, of the Hodge star operator $*$ on $\Lambda^2 = \Lambda^2(T^*X)$, defined by the metric, g, so $\Omega^\pm(X; \operatorname{ad}P) = C^\infty(X; \Lambda^\pm \otimes \operatorname{ad}P)$ and [81, Equation (1.3)]

$$F_A^\pm = \frac{1}{2}(1 \pm *)F_A \in \Omega^\pm(X; \operatorname{ad}P). \tag{1.24}$$

Rather than consider the full Yang–Mills energy function (1.5) for which it appears difficult to show has the Morse–Bott property at critical points that are not flat connections, we shall consider the *self-dual Yang–Mills energy function*, $\mathscr{E}_+ : \mathscr{A}(P) \to \mathbb{R}$, on the affine space of $W^{1,q}$ connections A on P (with $q \geq 2$),

$$\mathscr{E}_+(A) := \frac{1}{2}\int_X |F_A^+|^2 \, d\operatorname{vol}_g. \tag{1.25}$$

Our definition (1.25) is partly motivated by the fact that when, for example, $G = \mathrm{SU}(n)$ and the second Chern number of P is non-negative, $c_2(P)[X] \geq 0$, the energy function, $\mathscr{E} : \mathscr{A}(P) \to \mathbb{R}$, achieves its absolute minimum value at a connection A if and only if A is *anti-self-dual*, so $F_A^+ = 0$, and $\mathscr{E}(A) = 16\pi^2 c_2(P)[X]$, a constant that depends only on the topology of the principal G-bundle P; see [23, Equation (2.1.33)] for $G = \mathrm{SU}(n)$ and [26, Section 10] for more general formulae for the energies of anti-self-dual connections in the case of compact Lie groups. Our definition (1.25) of $\mathscr{E}_+$ effectively subtracts this topological constant from $\mathscr{E}$ in (1.5).

Proceeding as in the case of the full Yang–Mills energy function, we see that $\mathscr{E}_+ : \mathscr{A}(P) \to \mathbb{R}$ has differential map, $\mathscr{E}_+' : \mathscr{A}(P) \to T^*\mathscr{A}(P)$, given by

$$\mathscr{E}_+'(A)(a) = (F_A^+, d_A^+ a)_{L^2(X)} = (d_A^{+,*} F_A^+, a)_{L^2(X)}, \tag{1.26}$$

for all $a \in T_A\mathscr{A}(P) = W^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P)$.

We denote the finite-dimensional subvariety of gauge-equivalence classes of solutions to the *anti-self-dual* equation with respect to g by

$$M_+(P, g) := \left\{ [A] \in \mathscr{B}(P) : F_A^+ = 0 \quad \text{a.e. on } X \right\}. \tag{1.27}$$

As usual [23, Section 2.3.1], one denotes $d_A^+ = \frac{1}{2}(1 + *)d_A : \Omega^1(X; \mathrm{ad}P) \to \Omega^+(X; \mathrm{ad}P)$ and $H_A^2 = \operatorname{Coker} d_A^+$ [23, Equation (4.2.27)]. We recall from [23, Section 4.2.5] that if $H_A^2 = 0$ then

$$\tilde{M}_+(P, g) := \{B \in \mathscr{A}(P) : F_B^+ = 0 \quad \text{a.e. on } X\}$$

is a smooth manifold near A and

$$M_+^*(P, g) := M_+(P, g) \cap \mathscr{B}^*(P)$$

is a smooth manifold near $[A]$. The Generic Metrics Theorem [23, Corollary 4.3.18] due to Freed and Uhlenbeck implies that $H_A^2 = 0$ for all $[A] \in M_+^*(P, g)$ if $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$ and g is suitably generic.

If $F_A^+ = 0$, then $\mathscr{E}_+'(A) \equiv 0$ by (1.26) and A is a critical point of $\mathscr{E}_+ : \mathscr{A}(P) \to \mathbb{R}$, so that

$$\tilde{M}_+(P, g) \subset \widetilde{\mathrm{Crit}}(\mathscr{E}_+) \cap \mathscr{A}(P),$$

where $\widetilde{\mathrm{Crit}}(\mathscr{E}_+)$ denotes the critical set of $\mathscr{E}_+ : \mathscr{A}(P) \to \mathbb{R}$. Conversely, if $A \in \widetilde{\mathrm{Crit}}(\mathscr{E}_+)$ and $\operatorname{Coker} d_A^+ = 0$ then (1.26) implies that $F_A^+ = 0$ and $A \in \tilde{M}_+(P, g)$.

By gauge invariance, the self-dual Yang–Mills energy function is well-defined on the quotient, $\mathscr{E}_+ : \mathscr{B}^*(P) \to \mathbb{R}$ (with $q > 2$), and we have the inclusion,

$$M_+^*(P, g) \subset \mathrm{Crit}(\mathscr{E}_+) \cap \mathscr{B}^*(P),$$

where $\mathrm{Crit}(\mathscr{E}_+)$ denotes the critical set of $\mathscr{E}_+ : \mathscr{B}^*(P) \to \mathbb{R}$. Conversely, if $[A] \in \mathrm{Crit}(\mathscr{E}_+)$ and $\operatorname{Coker} d_A^+ = 0$, then $[A] \in M_+^*(P, g)$.

We have the following analogue of [32, Theorem 3] for the (coupled) Yang–Mills energy function, but with the improvement that $\theta = 1/2$, the optimal Łojasiewicz–Simon exponent.

Theorem 6 (Optimal Łojasiewicz–Simon inequalities for the self-dual Yang–Mills energy function)**.** *Let (X, g) be a closed, four-dimensional, smooth Riemannian manifold, G be a compact Lie group, P be a smooth principal G-bundle over X, and $q > 2$ be a constant. If A_∞ is a $W^{1,q}$ anti-self-dual Yang–Mills connection on P that is* regular*,*

$$H_{A_\infty}^2 := \operatorname{Coker}\left(d_{A_\infty}^+ : W^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P) \to L^q(X; \Lambda^+ \otimes \mathrm{ad}P) \right) = 0, \tag{1.28}$$

then $\mathscr{E}_+ : \mathscr{A}(P) \to \mathbb{R}$ *is a Morse–Bott function at* A_∞ *and there are constants* $C, Z \in (0,\infty)$ *and* $\sigma \in (0,1]$, *depending on* A_∞, g, *and* G, *with the following significance. If* A *is a* $W^{1,q}$ *connection on* P *obeying the* Łojasiewicz–Simon neighborhood *condition,*

$$\|A - A_\infty\|_{L^4(X)} < \sigma, \tag{1.29}$$

then the self-dual Yang–Mills energy function (1.25) *obeys the* optimal Łojasiewicz–Simon distance and gradient inequalities*,*

$$\mathscr{E}_+(A)^{1/2} \geq C\|A - A_\infty\|_{W^{1,2}_{A_\infty}(X)}, \tag{1.30}$$

$$\|\mathscr{E}_+'(A)\|_{W^{-1,2}_{A_\infty}(X)} \geq Z|\mathscr{E}_+(A)|^{1/2}. \tag{1.31}$$

Moreover, if the isotropy group of A_∞ *in* $\operatorname{Aut}(P)$ *is minimal (the center of* G*), then* $\mathscr{E}_+ : \mathscr{B}^*(P) \to \mathbb{R}$ *is a Morse–Bott function at* $[A_\infty]$*.*

We prove Theorem 6 in Section 5.

Remark 1.17 (Small self-dual Yang–Mills energy hypothesis). A more sophisticated (and more difficult) analysis would allow us to replace the small L^4 distance hypothesis (1.29) by a small self-dual Yang–Mills energy hypothesis,

$$\mathscr{E}_+(A) < \varepsilon,$$

for a fixed constant $\varepsilon = \varepsilon(g, G) \in (0,1]$, and replace the inequality (1.30) by

$$\mathscr{E}_+(A)^{1/2} \geq C \operatorname{dist}_{W^{1,2}}([A], M_+(P,g)),$$

both of which are more appropriate for Morse–Bott theory and giving an analogue of Theorem 2 for $\mathscr{E}_+$ in place of $\mathscr{E}$ when $d = 4$. We shall describe this refinement elsewhere.

Remark 1.18 (Two approaches to the proof of the optimal Łojasiewicz–Simon distance and gradient inequality for the self-dual Yang–Mills energy function near a regular anti-self-dual connection). Theorem 6 is proved in Section 5.1. As we explain there, the gradient inequality (1.31) may be proved in two different ways: *a*) by direct geometric analysis using methods of Yang–Mills gauge theory, and *b*) by first establishing that $\mathscr{E}_+$ is Morse–Bott at an anti-self-dual connection that is regular in the sense of (1.28) (see Lemma 5.1) and then appealing to our Theorem 3, giving the optimal gradient inequality for an abstract Morse–Bott function on a Banach space.

1.5. Optimal Łojasiewicz–Simon inequalities and Morse–Bott properties for the Yang–Mills energy function near flat connections.

We return to the case where X is a manifold of arbitrary dimension $d \geq 2$. While the forthcoming Theorem 7 is weaker in several respects than Theorem 2, it is easier to prove due to the additional hypothesis (1.32).

Theorem 7 (Optimal Łojasiewicz–Simon inequalities and Morse–Bott properties for the Yang–Mills energy function near regular flat connections)**.** *Let* (X, g) *be a closed, smooth Riemannian manifold of dimension* $d \geq 2$, *and* G *be a compact Lie group,* P *be a smooth principal* G*-bundle over* X, *and* $q \in [2,\infty)$ *obeying* $q > d/2$ *and* $r_0 > 2$ *be constants. If* Γ *is a* $W^{1,q}$ *flat connection on* P *that is* regular *in the sense that*

$$\begin{aligned} H^2_\Gamma := \operatorname{Ker}\Big(d_\Gamma &: L^q(X;\Lambda^2 \otimes \operatorname{ad}P) \to W^{-1,q}(X;\Lambda^3 \otimes \operatorname{ad}P)\Big) \\ &/ \operatorname{Ran}\Big(d_\Gamma : W^{1,q}(X;\Lambda^1 \otimes \operatorname{ad}P) \to L^q(X;\Lambda^2 \otimes \operatorname{ad}P)\Big) = 0, \end{aligned} \tag{1.32}$$

then $\mathscr{E} : \mathscr{A}(P) \to \mathbb{R}$ *is a Morse–Bott function at* Γ *and there are constants* $C, Z \in (0,\infty)$ *and* $\sigma \in (0,1]$*, depending on* Γ*,* g*,* G*, and* r_0 *with the following significance. If* A *is a* $W^{1,q}$ *connection on* P *obeying the* Łojasiewicz–Simon neighborhood *condition,*

$$\|A - \Gamma\|_{L^{r_0}(X)} < \sigma, \tag{1.33}$$

where $r_0 = d$ *when* $d \geq 3$ *and* $r_0 > 2$ *when* $d = 2$*, then the Yang–Mills energy function* (1.5) *obeys the* optimal Łojasiewicz–Simon distance and gradient inequalities*,*

$$\mathscr{E}(A)^{1/2} \geq C\|A - \Gamma\|_{W^{1,2}_\Gamma(X)}, \tag{1.34}$$

$$\|\mathscr{E}'(A)\|_{W^{-1,2}_\Gamma(X)} \geq Z|\mathscr{E}(A)|^{1/2}. \tag{1.35}$$

Moreover, if the isotropy group of Γ *in* $\mathrm{Aut}(P)$ *is minimal (the center of* G*), then* $\mathscr{E} : \mathscr{B}^*(P) \to \mathbb{R}$ *is a Morse–Bott function at* $[\Gamma]$*.*

We prove Theorem 7 in Section 5.

When X has dimension two, then Poincaré duality (for example, see [40, Lemma 2.1]) implies that $H^2_\Gamma \cong H^0_\Gamma$ (this observation is used in [62, p. 189]), where

$$H^0_\Gamma := \mathrm{Ker}\left(d_\Gamma : W^{2,q}(X; \mathrm{ad}P) \to W^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P)\right).$$

Recall [23, p. 132] that H^0_Γ is isomorphic to the tangent space to the isotropy group of Γ in $\mathrm{Aut}(P)$. In the special case that $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$, then a connection A on P is *irreducible* if and only if the isotropy of A in $\mathrm{Aut}(P)$ is the center of G [23, p. 133]. Thus, if $G = \mathrm{SU}(2)$ (with center $\mathbb{Z}/2\mathbb{Z}$) and $\dim X = 2$ (with $\mathrm{genus}(X) \geq 1$) and Γ is an irreducible flat connection, then $H^0_\Gamma = 0$ and consequently $H^2_\Gamma = 0$, so Γ is a smooth point of $M_0(P)$. In particular, the moduli space of gauge equivalence classes of irreducible flat connections on P, namely $M_0^*(P) := M_0(P) \cap \mathscr{B}^*(P)$, is a smooth manifold (of dimension $6\,\mathrm{genus}(X) - 6$ [72]) and therefore $\mathscr{E} : \mathscr{A}^*(P) \to \mathbb{R}$ is a Morse–Bott function near the critical set,

$$\tilde{M}_0^*(P) := \{A \in \mathscr{A}^*(P) : F_A = 0\},$$

and $\mathscr{E} : \mathscr{B}^*(P) \to \mathbb{R}$ is a Morse–Bott function near the critical set $M_0^*(P)$, in the sense of Definition 1.5 for both cases. When X is a Riemann surface, there is a vast literature devoted to the study of $M_0(P)$ from many different perspectives, often in the context of its interpretation as a moduli space, $\mathrm{Hom}(\pi_1(X), G)/G$, of representations of the fundamental group, $\pi_1(X)$, in G [23, Proposition 2.2.3] or in the context of the symplectic structure on $\mathscr{A}(P)$ and interpretation of (a multiple of) the map $A \mapsto F_A$ as a moment map. We refer to Atiyah and Bott [4] and the many articles that cite [4] for further details.

When X has dimension three and is a circle bundle over a closed Riemann surface, the geometry of $\mathrm{Hom}(\pi_1(X), \mathrm{SU}(2))$ is described by Morgan, Mrowka, and Ruberman in [62, Chapter 13].

1.6. **Optimal Łojasiewicz–Simon inequalities and Morse–Bott properties for the Yang–Mills energy function near arbitrary Yang–Mills connections.** One calls A a *Yang-Mills connection* if it is a critical point of the Yang–Mills energy function (1.5) on the affine space of $W^{1,q}$ connections, $\mathscr{E} : \mathscr{A}(P) \to \mathbb{R}$ on P, that is, $\mathscr{E}'(A) = 0$ and so by (1.11), obeys

$$d_A^* F_A = 0 \tag{1.36}$$

in a sense that depends on the regularity of A, weakly if A is $W^{1,q}$ or strongly if A is $W^{2,q}$, with $q \in [2,\infty)$ obeying $q > d/2$. We define

$$\mathrm{Crit}(\mathscr{E}) := \{A \in \mathscr{A}(P) : \mathscr{E}'(A) = 0\}. \tag{1.37}$$

The Yang–Mills energy function, $\mathscr{E} : \mathscr{A}(P) \to \mathbb{R}$, in (1.5) has the Hessian operator, $\mathscr{E}(A) : T_A\mathscr{A}(P) \to T_A^*\mathscr{A}(P)$, at $A \in \mathscr{A}(P)$ given by

$$\mathscr{E}''(A)(a)b = (d_Aa, d_Ab)_{L^2(X)} + (F_A, a \wedge b)_{L^2(X)}, \tag{1.38}$$

for all $a, b \in W^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P)$. We now state a partial generalization of Theorem 7.

Theorem 8 (Optimal Łojasiewicz–Simon inequalities when the Yang–Mills energy function is Morse–Bott near a Yang–Mills connection)**.** *Let (X, g) be a closed, smooth Riemannian manifold of dimension $d = 2$, 3, or 4, and G be a compact Lie group, P be a smooth principal G-bundle over X, and $q \in [2, \infty)$ obeying $q > d/2$ be a constant. Let A_∞ be a $W^{1,q}$ Yang–Mills connection on P and assume that $\mathscr{E} : \mathscr{A}(P) \to \mathbb{R}$ is a Morse–Bott function at A_∞ in the sense of Definition 1.5, so $\mathrm{Crit}(\mathscr{E}) \cap \mathscr{U}_{A_\infty}$ is a C^∞ relatively open submanifold of the space $\mathscr{A}(P)$ of $W^{1,q}$ connections on P for some open neighborhood $\mathscr{U}_{A_\infty}$ of A_∞ and*

$$\mathrm{Ker}\,\mathscr{E}''(A_\infty) \cap W^{1,q}_{A_\infty}(X; \Lambda^1 \otimes \mathrm{ad}P) = T_{A_\infty}\,\mathrm{Crit}(\mathscr{E}).$$

Then there are constants $Z \in (0, \infty)$ and $\sigma \in (0, 1]$, depending on A_∞, g, and G with the following significance. If A is a $W^{1,q}$ connection on P that obeys the Łojasiewicz–Simon neighborhood *condition,*

$$\|A - A_\infty\|_{W^{1,2}_{A_\infty}(X)} < \sigma, \tag{1.39}$$

then the Yang–Mills energy (1.5) *obeys the* optimal Łojasiewicz–Simon gradient inequality*,*

$$\|\mathscr{E}'(A)\|_{W^{-1,2}_{A_\infty}(X)} \geq Z|\mathscr{E}(A)|^{1/2}. \tag{1.40}$$

We prove Theorem 8 in Section 5.

Remark 1.19 (Optimal Łojasiewicz–Simon gradient inequality for the Yang–Mills energy function over Riemann surfaces)*.* Råde has shown (see [68, Proposition 7.2]) that if $d = 2$ and $G = \mathrm{U}(n)$ and A_∞ is an irreducible Yang–Mills connection, then inequality (1.40) in Theorem 8 holds. His proof (see [68, Section 10]) is very different from our proof of (1.40) and does not proceed by showing that $\mathscr{E}$ is Morse–Bott at A_∞. As far as we are aware, it is not known whether $\mathscr{E}$ is necessarily Morse–Bott at an irreducible Yang–Mills $\mathrm{U}(n)$ connection over a Riemann surface and it would be interesting to try to show this.

In Theorem 6, we noted that the self-dual Yang–Mills energy function $\mathscr{E}$ is Morse–Bott at an anti-self-dual connection A_∞ that is a regular point of the map,

$$W^{1,q}_{A_\infty}(X; \Lambda^1 \otimes \mathrm{ad}P) \ni A \mapsto F_A^+ \in L^q(X; \Lambda^+ \otimes \mathrm{ad}P),$$

or, equivalently, that the map

$$W^{1,q}_{A_\infty}(X; \Lambda^1 \otimes \mathrm{ad}P) \ni A \mapsto F_A \in L^q(X; \Lambda^2 \otimes \mathrm{ad}P),$$

is transverse to the subspace $L^q(X; \Lambda^- \otimes \mathrm{ad}P)$. Similarly, the final conclusion of Theorem 7 may be rephrased as the assertion that the Yang–Mills energy function $\mathscr{E}$ is Morse–Bott at a flat connection Γ that is a regular point of the map,

$$W^{1,q}_\Gamma(X; \Lambda^1 \otimes \mathrm{ad}P) \ni A \mapsto \Pi_\Gamma F_A \in \mathrm{Ker}\,d_\Gamma \cap L^q(X; \Lambda^2 \otimes \mathrm{ad}P),$$

where $\Pi_\Gamma : L^2(X; \Lambda^2 \otimes \mathrm{ad}P) \to \mathrm{Ker}\,d_\Gamma \cap L^2(X; \Lambda^2 \otimes \mathrm{ad}P)$ is L^2-orthogonal projection or, equivalently, that the map

$$W^{1,q}_\Gamma(X; \Lambda^1 \otimes \mathrm{ad}P) \ni A \mapsto \Pi_\Gamma F_A \in L^q(X; \Lambda^2 \otimes \mathrm{ad}P),$$

is transverse to the subspace $\operatorname{Ran} d_\Gamma \cap L^q(X; \Lambda^2 \otimes \operatorname{ad} P)$. The gauge-theoretic concept of Γ as a *regular point* of the map $A \mapsto F_A$ is based on the elliptic complex containing $d_\Gamma : \Omega^1(X; \operatorname{ad} P) \to \Omega^2(X; \operatorname{ad} P)$ while the gauge-theoretic concept of A_∞ as a *regular point* of the map $A \mapsto F_A^+$ is based on the elliptic complex containing $d_{A_\infty}^+ : \Omega^1(X; \operatorname{ad} P) \to \Omega^+(X; \operatorname{ad} P)$.

At an arbitrary critical point A_∞ for the Yang–Mills energy function $\mathscr{E}$ over a manifold X of dimension $d \geq 2$, there is no deformation theory that is exactly analogous to those just described for flat or anti-self-dual connections. Koiso [50, Lemma 1.5] proposes employing the observation that the following "Dual Bianchi Identity" (see [23, p. 235] or [64, p. 577]),

$$d_A^* d_A^* F_A = 0,$$

which holds for *any* connection A, be used to define an elliptic complex for the Yang–Mills equation, perhaps by analogy with viewing the Bianchi Identity, $d_A F_A = 0$, as motivation for the concept of a regular point in the zero locus of the map $A \mapsto F_A$. However, as Koiso himself seems to suggest [50, p. 156], this does not appear to yield a useful deformation theory for arbitrary solutions A_∞ to the Yang–Mills equation, except possibly when the formal dimension of the critical set is zero at the gauge-equivalence class $[A_\infty]$ [50, Corollary 2.11]. In the case of flat connections on principal G-bundles over Riemannian manifolds, Ho, Wilkin and Wu [40] compare concepts of regular points from the perspectives of gauge theory and representation varieties.

1.7. **Morse–Bott functions and moment maps.** We briefly note the well-known relationship between Morse–Bott functions and moment maps and recall the following result due to Atiyah.

Theorem 1.20 (Moment maps and Morse–Bott functions)**.** *(See Atiyah* [3] *or Nicolaescu* [63, Theorem 3.52]*.) Let (M, ω) be a compact symplectic manifold equipped with a Hamiltonian action of the torus $T = S^1 \times \cdots \times S^1$ (ν times for $\nu \geq 1$). Let $\mu : M \to \mathfrak{t}^*$ be the moment map of this action, where $\mathfrak{t}$ denotes the Lie algebra of T. Then, for every $\xi \in \mathfrak{t}$, the function*

$$\phi_\xi : M \ni x \mapsto \langle \xi, \mu(x) \rangle_{\mathfrak{t} \times \mathfrak{t}^*} \in \mathbb{R} \tag{1.41}$$

is Morse–Bott. The critical submanifolds are T-invariant symplectic submanifolds of M and all the Morse indices and co-indices are even.

If we define $\mathscr{E} : M \to \mathbb{R}$ by setting $\mathscr{E}(x) = \frac{1}{2}\|\mu(x)\|^2$, then $\mathscr{E}'(x)\xi = \langle \xi, \mu(x) \rangle$ and $\operatorname{Crit}(\mathscr{E}) = \mu^{-1}(0)$ and $\mathscr{E}''(x)(\eta)\xi = \langle \xi, \mu'(x)\eta \rangle$. If $x_0 \in M$ is a regular point in the zero-locus of the moment map μ, then $\mu^{-1}(0) \cap U$ is a relatively open, smooth submanifold of M and x_0 is a critical point of $\mathscr{E} : M \to \mathbb{R}$ and $T_{x_0}(\mu^{-1}(0) \cap U) = \operatorname{Ker} \mathscr{E}''(x_0) \cap T_{x_0}M$. In other words, $\mathscr{E}$ is Morse–Bott at regular points $x_0 \in \mu^{-1}(0)$. Some aspects of Atiyah's Theorem have been extended, at least formally, to more general finite and infinite-dimensional settings and we refer to [23, Section 6.5.1–3] for a discussion of moment maps and a survey of examples. The instances most relevant to this article include the *a*) affine space $\mathscr{A}(P)$ of $W^{1,q}$ connections on a principal G-bundle P over a Riemann surface X and moment map $A \mapsto F_A$ for the Banach Lie group $\operatorname{Aut}(P)$ of gauge transformations [4]; and more generally, the *b*) affine space $\mathscr{A}(P)$ of $W^{1,q}$ connections on a principal G-bundle P over a *symplectic* manifold (X, ω) of dimension $2n$ and moment map $A \mapsto F_A \wedge \omega^{n-1}$ for $\operatorname{Aut}(P)$ [23, Proposition 6.5.8]. Donaldson and Kronheimer also point out that the Atiyah–Hitchin– Drinfel′d–Manin (ADHM) description of instantons over $\mathbb{R}^4$ [5], [23, Section 3.3.2] may be viewed as the zero-locus of a suitably defined moment map [23, p. 250].

For further discussion of Morse–Bott functions, moment maps, and gradient flows in symplectic geometry, we refer to Donaldson and Kronheimer [23, Section 6.5], Kirwan [47, 46], Lerman [54], Swoboda [78], the references cited therein, and to Atiyah and Bott [4] and wealth of articles citing [4].

1.8. **Notation.** For the notation of function spaces, we follow Adams and Fournier [2]. If V is a Riemannian vector bundle with orthogonal, smooth connection A over a smooth Riemannian manifold X, we let $W_A^{k,p}(X;V)$ denote its Sobolev space of sections with up to k covariant derivatives in L^p. We write $W^{k,p}(X;V)$ if the connection is unimportant for the context or if the Sobolev space is defined using standard definitions for functions on Euclidean space from [2, Chapter 3] and choices of local coordinate charts for X and local trivializations for V. To define Sobolev norms of maps from a manifold into a compact Lie group, G, we choose a faithful unitary representation, $G \hookrightarrow \mathrm{End}(\mathbb{C}^N)$.

If G is compact Lie group and P is a principal G-bundle over a manifold X, we let $\mathrm{ad}P := P \times_{\mathrm{ad}} \mathfrak{g}$ denote the real vector bundle associated to P by the adjoint representation of G on its Lie algebra, $\mathrm{Ad} : G \ni u \to \mathrm{Ad}_u \in \mathrm{Aut}\,\mathfrak{g}$. We fix a G-invariant inner product on the Lie algebra $\mathfrak{g}$ and thus define a fiber metric on $\mathrm{ad}P$. (When G is semi-simple, one may use the Killing form to define a G-invariant inner product $\mathfrak{g}$.) When X is a smooth manifold, we denote $\Lambda^l := \Lambda^l(T^*X)$ for integers $l \geq 1$ and $\Lambda^0 := X \times \mathbb{R}$ when X is equipped with a smooth Riemannian metric, we let $\mathrm{Inj}(X, g)$ denote the injectivity radius of (X, g) and, when X also has an orientation, denote the corresponding volume form by $d\,\mathrm{vol}_g$. Unless stated otherwise, all manifolds are assumed to be compact and without boundary (closed), connected, orientable, and smooth.

We let $\mathbb{N} := \{1, 2, 3, \dots\}$ denote the set of positive integers. We use $C = C(*, \dots, *)$ to denote a constant which depends at most on the quantities appearing on the parentheses. In a given context, a constant denoted by C may have different values depending on the same set of arguments and may increase from one inequality to the next. We emphasize that a constant ε (respectively, C) may need to be chosen sufficiently small (respectively, large) by writing $\varepsilon \in (0, 1]$ (respectively, $C \in [1, \infty)$).

For notation in functional analysis, we follow Brezis [16] and Rudin [69]. If $\mathscr{X}, \mathscr{Y}$ is a pair of Banach spaces, then $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ denotes the Banach space of all continuous linear operators from $\mathscr{X}$ to $\mathscr{Y}$. We denote the continuous dual space of $\mathscr{X}$ by $\mathscr{X}^* = \mathscr{L}(\mathscr{X}, \mathbb{R})$. We write $\alpha(x) = \langle x, \alpha \rangle_{\mathscr{X} \times \mathscr{X}^*}$ for the canonical pairing between $\mathscr{X}$ and its dual space, where $x \in \mathscr{X}$ and $\alpha \in \mathscr{X}^*$. If $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, then its range and kernel are denoted by $\mathrm{Ran}\,T$ and $\mathrm{Ker}\,T$, respectively.

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2. Existence of a flat connection for critical exponents, Coulomb gauge transformation, and Sobolev distance estimate

Our goal in this section is to prove Theorem 1, thus extending a result [86, Corollary 4.3] due to Uhlenbeck, quoted as the forthcoming Theorem 2.1, in two ways. First, we relax the forthcoming curvature hypothesis (2.1), that is,

$$\|F_A\|_{L^q(X)} \leq \varepsilon,$$

to the weaker condition (1.1), namely,

$$\|F_A\|_{L^{s_0}(X)} \leq \varepsilon,$$

where $s_0 = d/2$ when $d \geq 3$ or $s_0 > 1$ when $d = 2$. Second, in the forthcoming bound (2.4), we allow any $p \in (1, \infty)$ that obeys[3] $p \leq q$, where $q > d/2$ and A is a $W^{1,q}$ connection on a principal G-bundle P over a closed, smooth Riemannian manifold, X, of dimension $d \geq 2$.

In Section 2.1, we recall the statement of Theorem 2.1, together with remarks on its hypotheses. In Section 2.3, we establish an extension of Uhlenbeck's [84, Theorem 1.3] on existence of a local Coulomb gauge and *a priori* $W^{1,p}$ estimate for connections with $L^{d/2}$-small curvature over a ball to include the range $1 < p < d/2$ (when $d \geq 3$). Our principal goal in Section 2.4 is to prove Theorem 2.16, which yields a continuous isomorphism between principal G-bundles that support Sobolev connections whose local connection one-forms in Coulomb gauge are L^d-close and whose corresponding transition functions are L^p-close for some $p \in (d/2, d)$. In Section 2.5 we establish Theorem 2.20, verifying Item (1) in Theorem 1, giving existence of a C^∞ flat connection Γ on a principal bundle supporting a $W^{1,q}$ connection A with $L^{d/2}$-small curvature F_A for $d \geq 3$ or L^{s_0}-small curvature for $d = 2$ and $s_0 > 1$. Next, in Section 2.6, we establish Item (2) in Theorem 1, giving an *a priori* $W^{1,p}$ estimate for $A - \Gamma$ in terms of the L^p norm of F_A when $1 < p < d$. In Section 2.7, we discuss a slight enhancement of [32, Theorem 9] that gives existence of $W^{2,q}$ Coulomb gauge transformations $u \in \mathrm{Aut}(P)$ for $W^{1,q}$ connections A that are L^d close to a $W^{1,q}$ reference connection A_0. Finally, in Section 2.8 we establish the existence of a $W^{2,q}$ gauge transformation $u \in \mathrm{Aut}(P)$ bringing A into Coulomb gauge relative to Γ and thus prove Item (3) in Theorem 1.

2.1. **Existence of a flat connection for supercritical exponents, Coulomb gauge transformation, and Sobolev distance estimate.** In [86], Uhlenbeck establishes the

Theorem 2.1 (Existence of a flat connection on a principal bundle supporting a $W^{1,q}$ connection with L^q-small curvature, Coulomb gauge transformation, and Sobolev distance estimate)**.** *(See Uhlenbeck* [86, Corollary 4.3] *and* [30, Theorem 5.1]*.) Let (X, g) be a closed, smooth Riemannian manifold of dimension $d \geq 2$, and G be a compact Lie group, and $q \in (d/2, \infty]$. Then there is a constant, $\varepsilon = \varepsilon(g, G, q) \in (0, 1]$ and, for $p \in (1, \infty)$ obeying*[4] *$d/2 \leq p \leq q$, there are constants $C_i = C_i(g, G, p) \in [1, \infty)$ for $i = 0, 1$ with the following significance. Let A be a $W^{1,q}$ connection on a smooth principal G-bundle P over X. If*

$$\|F_A\|_{L^q(X)} \leq \varepsilon, \tag{2.1}$$

then the following hold:

(1) (Existence of a flat connection) *There is a C^∞ flat connection, Γ, on P;*
(2) ($W^{1,p}$-distance estimate) *The flat connection, Γ, satisfies*

$$\|A - \Gamma\|_{W^{1,p}_\Gamma(X)} \leq C_0 \|F_A\|_{L^p(X)}; \tag{2.2}$$

[3] Although when $p \geq d$, we will need to impose the stronger hypothesis (2.1).

[4] The restriction $p > 1$ should be included in the statements of [30, Theorem 5.1] and [86, Corollary 4.3] since the bound (2.4) ultimately follows from an *a priori* L^p estimate for an elliptic system that is apparently only valid when $1 < p < \infty$.

(3) (Existence of a Coulomb gauge transformation and estimate of Sobolev $W^{1,p}$ distance to the flat connection) *There exists a $W^{2,q}$ gauge transformation, $u \in \mathrm{Aut}(P)$, such that*

$$d_\Gamma^*(u(A) - \Gamma) = 0 \quad \text{a.e. on } X; \tag{2.3}$$

$$\|u(A) - \Gamma\|_{W^{1,p}_\Gamma(X)} \leq C_1 \|F_A\|_{L^p(X)}. \tag{2.4}$$

Remark 2.2 (On the proof of Theorem 2.1)*.* The proof of Theorem 2.1 given in [86] was brief, so we provided a more detailed exposition in our proof of this result as [30, Theorem 5.1]. Our proof in [30] of the estimate (2.4) in Theorem 2.1 was more complicated than we would like, but the complications appear unavoidable in the absence of further hypotheses. A global proof of the estimate (2.4) with the aid of an application of the Implicit Function Theorem is possible when Γ is a regular point in the sense of (1.32) — see the proof of Theorem 7 in Section 5.2.

Remark 2.3 (On the statement of Theorem 2.1)*.* Our statement of Theorem 2.1 makes explicit the fact that the flat connection, Γ, is C^∞ (not merely $W^{1,q}$) and is $W^{1,p}$-close to A in the sense of (2.2), clarifying those two points in [30, Theorem 5.1].

As we noted in [30, p. 563], the bound (2.4) is similar to a Łojasiewicz–Simon inequality. Indeed, when $2 \leq d \leq 4$ so we can choose $p = 2 \geq d/2$, then (2.4) precisely yields the *Łojasiewicz–Simon distance inequality* (1.12) (compare [14, Remark 6.5]),

$$\mathscr{E}(A) \geq C \operatorname{dist}_{W^{1,2}(X)}\left([A], M_0(P)\right)^2,$$

using the expression (1.5) for the Yang–Mills energy $\mathscr{E}(A)$ and expression (1.10) for the $W^{1,2}$ distance function. However, to obtain the analogous Łojasiewicz–Simon distance inequality when $d \geq 5$, we shall need to extend the bound (2.4) to allow $1 < p < d/2$ (and $p = 2$ in particular).

2.2. **Flat bundles.** We recall the equivalent characterizations of *flat bundles* [48, Section 1.2], that is, bundles admitting a flat connection. Let G be a Lie group and P be a smooth principal G-bundle over a smooth manifold, X. Let $\{U_\alpha\}_{\alpha \in \mathscr{I}}$ be an open cover of X with local sections, $\sigma_\alpha : U_\alpha \to P$ and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ be the family of transition functions defined by $\{U_\alpha, \sigma_\alpha\}$. A *flat structure* in P is given by $\{U_\alpha, \sigma_\alpha\}_{\alpha \in \mathscr{I}}$ such that the $g_{\alpha\beta}$ are all constant maps. A connection in P is said to be *flat* if its curvature vanishes identically.

Proposition 2.4 (Characterizations of flat principal bundles)**.** *(See* [48, Proposition 1.2.6]*.) For a smooth principal G-bundle P over a smooth manifold, X, the following conditions are equivalent:*

(1) P admits a flat structure,
(2) P admits a flat connection,
(3) P is defined by a representation $\pi_1(X) \to G$.

Given a flat structure on P, we may construct a flat connection, Γ, on P using the zero local connection one-forms, $\gamma_\alpha \equiv 0$ on U_α, for each α as in [48, Equation (1.2.1′)] and observing that the compatibility conditions [48, Equation (1.1.16)],

$$0 = \gamma_\beta = g_{\alpha\beta}^{-1} \gamma_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} = 0 \quad \text{on } U_\alpha \cap U_\beta,$$

are automatically obeyed.

2.3. **An extension of Uhlenbeck's Theorem on existence of a local Coulomb gauge.** We shall need to extend Uhlenbeck's Theorem on existence of a local Coulomb gauge to include the range $1 < p < d/2$ when $d \geq 3$ (and in particular, $p = 2$, when $d \geq 5$) as well as $d/2 \leq p < d$. The required extension is given by Corollary 2.10. We first recall the original statement of Uhlenbeck's Theorem (with a clarification due to Wehrheim).

Theorem 2.5 (Existence of a local Coulomb gauge and *a priori* estimate for a Sobolev connection with $L^{d/2}$-small curvature)**.** *(See Uhlenbeck* [84, Theorem 1.3 or Theorem 2.1 and Corollary 2.2] *or Wehrheim* [88, Theorem 6.1]*.) Let $d \geq 2$, and G be a compact Lie group, and $p \in (1,\infty)$ obeying $d/2 \leq p < d$ and $s_0 > 1$ be constants. Then there are constants, $\varepsilon = \varepsilon(d,G,p,s_0) \in (0,1]$ and $C = C(d,G,p,s_0) \in [1,\infty)$, with the following significance. For $q \in [p,\infty)$, let A be a $W^{1,q}$ connection on $B \times G$ such that*

$$\|F_A\|_{L^{s_0}(B)} \leq \varepsilon, \tag{2.5}$$

where $B \subset \mathbb{R}^d$ is the unit ball with center at the origin and $s_0 = d/2$ when $d \geq 3$ and $s_0 > 1$ when $d = 2$. Then there is a $W^{2,q}$ gauge transformation, $u : B \to G$, such that the following holds. If $A = \Theta + a$, where Θ is the product connection on $B \times G$, and $u(A) = \Theta + u^{-1}au + u^{-1}du$, then

$$\begin{aligned} d^*(u(A) - \Theta) &= 0 \quad \text{a.e. on } B, \\ (u(A) - \Theta)(\vec{n}) &= 0 \quad \text{on } \partial B, \end{aligned}$$

where $\vec{n}$ is the outward-pointing unit normal vector field on ∂B, and

$$\|u(A) - \Theta\|_{W^{1,p}(B)} \leq C\|F_A\|_{L^p(B)}. \tag{2.6}$$

Remark 2.6 (Restriction of p to the range $1 < p < \infty$)*.* The restriction $p \in (1,\infty)$ should be included in the statements of [84, Theorem 1.3 or Theorem 2.1 and Corollary 2.2] since the bound (2.6) ultimately follows from an *a priori* L^p estimate for an elliptic system that is apparently only valid when $1 < p < \infty$. Wehrheim makes a similar observation in her [88, Remark 6.2 (d)]. This is also the reason that when $d = 2$, we require $s_0 > 1$ in (2.5).

Remark 2.7 (Dependencies of the constants in Theorem 2.5)*.* (See [30, Remark 4.2].) The statements of [84, Theorem 1.3 or Theorem 2.1 and Corollary 2.2] imply that the constants, ε in (2.5) and C in (2.6), only depend the dimension, d. However, their proofs suggest that these constants may also depend on G and p through the appeal to an elliptic estimate for $d+d^*$ in the verification of [84, Lemma 2.4] and arguments immediately following.

Remark 2.8 (Construction of a $W^{k+1,q}$ transformation to Coulomb gauge)*.* (See [30, Remark 4.3].) We note that if A is of class $W^{k,q}$, for an integer $k \geq 1$ and $q \geq 2$, then the gauge transformation, u, in Theorem 2.5 is of class $W^{k+1,q}$; see [84, page 32], the proof of [84, Lemma 2.7] via the Implicit Function Theorem for smooth functions on Banach spaces, and our proof of [28, Theorem 1.1] — a global version of Theorem 2.5.

Remark 2.9 (Non-flat Riemannian metrics)*.* Theorem 2.5 continues to hold for geodesic unit balls in a manifold X endowed a non-flat Riemannian metric, g. The only difference in this more general situation is that the constants C and ε will depend on bounds on the Riemann curvature tensor, R. See Wehrheim [88, Theorem 6.1].

We now provide an extension of Theorem 2.5 to include the range $1 < p < d/2$ (and in particular, $p = 2$, when $d \geq 5$).

Corollary 2.10 (Existence of a local Coulomb gauge and *a priori* $W^{1,p}$ estimate for a Sobolev connection with $L^{d/2}$-small curvature when $p < d/2$)**.** *Assume the hypotheses of Theorem 2.5, but allow any $p \in (1,\infty)$ obeying $p < d/2$ when $d \geq 3$. Then the estimate* (2.6) *holds for $1 < p < d/2$.*

Proof. The proof of Theorem 2.5 by Uhlenbeck in [84, Section 2] makes use of the hypothesis $d/2 \leq p < d$ through her appeal to a Hölder inequality and a Sobolev embedding. However, an alternative Hölder inequality and Sobolev embedding apply for the case $1 < p < d/2$, as we now explain. Write $a := u(A) - \Theta \in W^{1,q}(B; \Lambda^1 \otimes \mathfrak{g})$ for brevity and observe that by ellipticity of the first-order operator $d+d^* : \Omega^1(B;\mathfrak{g}) \to \Omega^2(B;\mathfrak{g}) \oplus \Omega^0(B;\mathfrak{g})$ with its Neumann boundary condition, we have the *a priori* global estimate (see [88, Theorem 5.1 and p. 102, last paragraph]),

$$\|a\|_{W^{1,p}(B)} \leq C\|(d+d^*)a\|_{L^p(B)}, \tag{2.7}$$

for $C = C(d,G,p) \in [1,\infty)$. Using $d^*a = 0$ and $F_{u(A)} = F_{\Theta+a} = F_\Theta + da + a \wedge a = da + a \wedge a$ and $|F_{u(A)}| = |F_A|$ a.e. on B, the preceding bound yields

$$\|a\|_{W^{1,p}(B)} \leq C\left(\|F_A\|_{L^p(B)} + \|a \wedge a\|_{L^p(B)}\right).$$

We can estimate $\|a \wedge a\|_{L^p(B)}$ by writing, for a constant $c = c(d,G) \in [1,\infty)$,

$$\|a \wedge a\|_{L^p(B)} \leq c\|a\|_{L^s(B)}\|a\|_{L^d(B)},$$

where $s > p$ is defined by $1/p = 1/s + 1/d$, that is, $1/s = (d-p)/dp$ or $s = dp/(d-p)$. Recall from [2, Theorem 4.12, Part I (C)] that there is a continuous embedding of Sobolev spaces, $W^{1,p}(B;\mathbb{R}) \subset L^{p^*}(B;\mathbb{R})$, when $1 \leq p < d$ (by hypothesis, we have $1 < p < d/2$) and $p^* = dp/(d-p) = s$. Hence, noting that $(d/2)^* = d(d/2)/(d-(d/2)) = d$, we obtain[5]

$$\|a\|_{L^d(B)} \leq C\|a\|_{W^{1,d/2}(B)},$$
$$\|a\|_{L^{p^*}(B)} \leq C\|a\|_{W^{1,p}(B)},$$

for $C = C(d)$ or $C = C(d,p) \in [1,\infty)$, respectively. Therefore,

$$\|a \wedge a\|_{L^p(B)} \leq c\|a\|_{L^{p^*}(B)}\|a\|_{L^d(B)} \leq C\|a\|_{W^{1,p}(B)}\|a\|_{W^{1,d/2}(B)},$$

for $C = C(d,G,p) \in [1,\infty)$. The estimate (2.4) (with $p = d/2$) from Theorem 2.1 yields

$$\|a\|_{W^{1,d/2}(B)} \leq C\|F_A\|_{L^{d/2}(B)},$$

for $C = C(d,G) \in [1,\infty)$. But $\|F_A\|_{L^{d/2}(B)} \leq \varepsilon$ by hypothesis (2.5) of Theorem 2.1, so we may combine the preceding inequalities to give, for $C = C(d,G,p) \in [1,\infty)$,

$$\|a \wedge a\|_{L^p(B)} \leq C\varepsilon\|a\|_{W^{1,p}(B)}.$$

Consequently,

$$\begin{aligned}\|a\|_{W^{1,p}(B)} &\leq C\left(\|F_A\|_{L^p(B)} + \|a \wedge a\|_{L^p(B)}\right)\\ &\leq C\left(\|F_A\|_{L^p(B)} + \varepsilon\|a\|_{W^{1,p}(B)}\right).\end{aligned}$$

Hence, for small enough $\varepsilon = \varepsilon(d,G,p) \in (0,1]$, we may use rearrangement to find

$$\|a\|_{W^{1,p}(B)} \leq C\|F_A\|_{L^p(B)},$$

and this yields (2.6) when $p \in (1,d/2)$. □

For completeness, we shall also include the following extension of Theorem 2.5 (and slight improvement of our [30, Corollary 4.4]) to include the range $d \leq p < \infty$, although this extension will not be needed in this article.

[5] Throughout this article, we apply the pointwise Kato Inequality [33, Equation (6.20)] to pass from a Sobolev inequality for scalar functions to a Sobolev inequality with the same constant for sections of a vector bundle.

Corollary 2.11 (Existence of a local Coulomb gauge and *a priori* $W^{1,p}$ estimate for a Sobolev connection one-form with $L^{\bar{p}}$-small curvature when $p \geq d$)**.** *Assume the hypotheses of Theorem 2.5, but consider $d \leq p < \infty$ and strengthen* (2.5) *to*[6]

$$\|F_A\|_{L^{\bar{p}}(B)} \leq \varepsilon, \tag{2.8}$$

where $\bar{p} = dp(d+p)$ when $p > d$ and $\bar{p} > d/2$ when $p = d$. Then the estimate (2.6) *holds for $d \leq p < \infty$ and constant $C = C(d, p, \bar{p}, G) \in [1, \infty)$.*

Proof. We modify the proof of Corollary 2.10 and separately consider the cases $d < p < \infty$ and $p = d$. When $p > d$, then [2, Theorem 4.12, Part I (A)] provides a continuous embedding of Sobolev spaces, $W^{1,p}(B;\mathbb{R}) \subset L^\infty(B;\mathbb{R})$. Also, [2, Theorem 4.12, Part I (C)] provides a continuous embedding of Sobolev spaces, $W^{1,\bar{p}}(B;\mathbb{R}) \subset L^p(B;\mathbb{R})$ when $p = \bar{p}^* := d\bar{p}/(d - \bar{p}) \in (d, \infty)$, that is, $\bar{p} = dp/(d+p) \in (d/2, d)$. Thus,

$$\|a \wedge a\|_{L^p(B)} \leq c\|a\|_{L^p(B)}\|a\|_{L^\infty(B)} \leq C\|a\|_{W^{1,\bar{p}}(B)}\|a\|_{W^{1,p}(B)},$$

for $c = c(d, G) \in [1, \infty)$ and $C = C(d, G, p) \in [1, \infty)$. Because $\bar{p} \in (d/2, d)$, Theorem 2.5 applies to give

$$\|a\|_{W^{1,\bar{p}}(B)} \leq C\|F_A\|_{L^{\bar{p}}(B)}.$$

Thus, for F_A obeying (2.8) with $\bar{p} = dp/(d+p)$, the proof of Corollary 2.10 yields estimate (2.6).

When $p = d$, choose $s \in (d, \infty)$ and define $t \in (d, \infty)$ by $1/d = 1/s + 1/t$, so that

$$\|a \wedge a\|_{L^p(B)} \leq c\|a\|_{L^s(B)}\|a\|_{L^t(B)}.$$

For $\bar{s} = ds/(d+s) \in (d/2, d)$, we have a continuous embedding of Sobolev spaces, $W^{1,\bar{s}}(B;\mathbb{R}) \subset L^s(B;\mathbb{R})$. Also, [2, Theorem 4.12, Part I (B)] provides a continuous embedding of Sobolev spaces, $W^{1,d}(B;\mathbb{R}) \subset L^t(B;\mathbb{R})$. Therefore, applying these embeddings to the preceding inequality yields

$$\|a \wedge a\|_{L^p(B)} \leq C\|a\|_{W^{1,\bar{s}}(B)}\|a\|_{W^{1,d}(B)},$$

for $C = C(d, G, p, s) \in [1, \infty)$. Because $\bar{s} \in (d/2, d)$, Theorem 2.5 again applies to give

$$\|a\|_{W^{1,\bar{s}}(B)} \leq C\|F_A\|_{L^{\bar{s}}(B)}.$$

Thus, for F_A obeying (2.8) with $\bar{p} = \bar{s}$, the proof of Corollary 2.10 again yields estimate (2.6). □

2.4. **Continuous principal bundles.** Our principal goal in this subsection is to prove the forthcoming Theorem 2.16, which yields a continuous isomorphism between principal G-bundles that support Sobolev connections whose local connection one-forms in Coulomb gauge are L^d-small and whose corresponding transition functions are L^p-close for some $p \in (d/2, d)$.

Recall [44, Theorem 5.3.2] that a continuous principal G-bundle P_g over X is uniquely defined up to isomorphism by a collection of maps, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, corresponding to a covering $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$ of X by open subsets, that obeys the cocycle condition,

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \mathrm{id}_G \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma, \tag{2.9}$$

for all $\alpha, \beta, \gamma \in \mathscr{I}$ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$. The condition (2.9) implies that $g_{\alpha\alpha} = \mathrm{id}_G$ on U_α and $g_{\alpha\beta}^{-1} = g_{\beta\alpha}$ on $U_\alpha \cap U_\beta$. Moreover, according to [44, Proposition 5.2.5], a bundle P_g is isomorphic to $P_h = (\{h_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}, \{U_\alpha\}_{\alpha\in\mathscr{I}})$ if and only if there exist continuous maps, $\rho_\alpha : U_\alpha \to G$ for all $\alpha \in \mathscr{I}$, such that

$$h_{\alpha\beta} = \rho_\alpha^{-1}g_{\alpha\beta}\rho_\beta \quad \text{on } U_\alpha \cap U_\beta, \tag{2.10}$$

[6] In [30, Corollary 4.4], we assumed the still stronger condition, $\|F_A\|_{L^p(B)} \leq \varepsilon$.

for all $\alpha, \beta \in \mathscr{I}$ such that $U_\alpha \cap U_\beta \neq \varnothing$. The result below due to Uhlenbeck allows us to replace the equality (2.10) by the forthcoming estimate (2.12).

Proposition 2.12 (Isomorphisms of principal bundles with sufficiently close transition functions)**.** *(See Uhlenbeck* [84, Proposition 3.2]*, Wehrheim* [88, Lemma 7.2 (i)]*.) Let G be a compact Lie group and X be a compact manifold of dimension $d \geq 2$ endowed with a Riemannian metric, g. Let $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ be two sets of continuous transition functions with respect to a finite open cover, $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$, of X. Then there exist constants, $\varepsilon = \varepsilon(g, G, \mathscr{U}) \in (0, 1]$ and $C = C(g, G, \mathscr{U}) \in [1, \infty)$, with the following significance. If*

$$\delta := \sup_{\substack{x\in U_\alpha\cap U_\beta,\\ \alpha,\beta\in\mathscr{I}}} |g_{\alpha\beta}(x) - h_{\alpha\beta}(x)| \leq \varepsilon, \tag{2.11}$$

then there exists a finite open cover, $\mathscr{V} = \{V_\alpha\}_{\alpha\in\mathscr{I}}$, of X, with $V_\alpha \subset U_\alpha$ and a set of continuous maps, $\rho_\alpha : V_\alpha \to G$, such that

$$\rho_\alpha g_{\alpha\beta} \rho_\beta^{-1} = h_{\alpha\beta} \quad \text{on } V_\alpha \cap V_\beta$$

and

$$\sup_{\substack{x\in V_\alpha,\\ \alpha\in\mathscr{I}}} |\rho_\alpha(x) - \mathrm{id}_G| \leq C\delta. \tag{2.12}$$

In particular, the principal G-bundle defined by $\{g_{\alpha\beta}\}$ is isomorphic to the principal G-bundle defined by $\{h_{\alpha\beta}\}$.

Remark 2.13 (Dependencies of the constants ε and C in Proposition 2.12). The dependencies of the constants ε and C in [84, Proposition 3.2] are not explicitly labeled, but those in our quotation, Proposition 2.12, are inferred from its proof in [84].

Next, we have the

Theorem 2.14 ($W^{2,p}$ bounds on transition functions for continuous principal bundles with L^d-small local connection one-forms in Coulomb gauge)**.** *Let (X, g) be a closed, smooth Riemannian manifold of dimension $d \geq 2$, and G be a compact Lie group, $q > d/2$ be a constant, P be a $W^{2,q}$ principal G-bundle over X, and $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$ be a finite cover of X by open subsets. Let A be a $W^{1,d/2}$ connection on P and $\sigma_\alpha : U_\alpha \to P$ be $W^{2,q}$ local sections such that the local connection one-forms,*

$$a_\alpha := \sigma_\alpha^* A \in W^{1,d/2}(U_\alpha; \Lambda^1 \otimes \mathfrak{g}),$$

obey, for each $\alpha \in \mathscr{I}$,

$$d^{*_g} a_\alpha = 0 \quad \text{a.e. on } U_\alpha.$$

Let $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$ be the corresponding set of transition functions in $W^{2,q}(U_\alpha \cap U_\beta; G)$ for each $\alpha, \beta \in \mathscr{I}$ such that $U_\alpha \cap U_\beta \neq \varnothing$. If $p \leq q$ obeys[7] *$1 < p < d$ and $\mathscr{V} = \{V_\alpha\}_{\alpha\in\mathscr{I}}$ is a finite cover of X by open subsets such that $V_\alpha \Subset U_\alpha$, then there are constants $C = C(g, G, p, \mathscr{U}, \mathscr{V}) \in [1, \infty)$ and $\varepsilon = \varepsilon(g, G, p, \mathscr{U}, \mathscr{V}) \in (0, 1]$ with the following significance. If*

$$\max_{\alpha\in\mathscr{I}} \|a_\alpha\|_{L^d(U_\alpha\cap U_\beta)} \leq \varepsilon, \tag{2.13}$$

then

$$\|g_{\alpha\beta}\|_{W^{2,p}(V_\alpha\cap V_\beta)} \leq C. \tag{2.14}$$

[7]By analogy with Corollary 2.11 or [30, Corollary 4.4], the condition $p < d$ could be relaxed to $p \leq q$ at the expense in the hypothesis (2.15) of replacing the L^d norm by an L^p norm when $p > d$.

Remark 2.15 (Uniform Hölder norm bounds on transition functions for continuous principal bundles with L^d-small local connection one-forms in Coulomb gauge). Recall from [2, Theorem 4.12, Part II] that there is a continuous embedding,

$$W^{2,p}(U;\mathbb{R}) \subset C^\delta(\bar{U};\mathbb{R}),$$

where for an open subset $U \subset \mathbb{R}^d$ (obeying an interior cone condition) and *a*) $0 < \delta \leq 2 - (d/p)$ if $p < d < 2p$, or *b*) $0 < \delta < 1$ if $p = d$, and so the transition functions $g_{\alpha\beta}$ in Theorem 2.14 obey a uniform $C^\delta(\bar{V}_\alpha \cap \bar{V}_\beta; G)$ bound.

The proof of Theorem 2.14 is very similar to the proof of the forthcoming Theorem 2.16 and so is omitted: one simply uses $b_\alpha = a_\alpha$ and $h_{\alpha\beta} = g_{\alpha\beta}$ in the proof of Theorem 2.16 and notes that because G is compact, $\|g_{\alpha\beta}\|_{L^\infty(U_\alpha\cap U_\beta)} \leq C$. Our proof of the forthcoming Theorem 2.20 relies on the following generalization of Proposition 2.12.

Theorem 2.16 (Continuous principal bundles with L^d-small local connection one-forms in Coulomb gauge)**.** *Let (X,g) be a closed, smooth Riemannian manifold of dimension $d \geq 2$, and G be a compact Lie group, $q > d/2$ be a constant, P and Q be $W^{2,q}$ principal G-bundles over X, and $\{U_\alpha\}_{\alpha\in\mathscr{I}}$ be a finite cover of X by open subsets. Let A and B be $W^{1,d/2}$ connections on P and Q, respectively, and $\sigma_\alpha, \varsigma_\alpha : U_\alpha \to P$ be $W^{2,q}$ local sections such that the local connection one-forms,*

$$a_\alpha := \sigma_\alpha^* A \in W^{1,d/2}(U_\alpha;\Lambda^1\otimes\mathfrak{g}) \quad \text{and} \quad b_\alpha := \varsigma_\alpha^* B \in W^{1,d/2}(U_\alpha;\Lambda^1\otimes\mathfrak{g}),$$

obey, for each $\alpha \in \mathscr{I}$,

$$d^{*_g} a_\alpha = 0 = d^{*_g} b_\alpha \quad \text{a.e. on } U_\alpha.$$

Let $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$ and $\{h_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$ be the corresponding sets of transition functions in $W^{2,q}(U_\alpha \cap U_\beta; G)$ for each $\alpha,\beta \in \mathscr{I}$ such that $U_\alpha \cap U_\beta \neq \varnothing$. If $p \leq q$ obeys[8] *$1 < p < d$ and $\{V_\alpha\}_{\alpha\in\mathscr{I}}$ is a finite cover of X by open subsets such that $V_\alpha \Subset U_\alpha$, then there are constants $C = C(g,G,p,\mathscr{U},\mathscr{V}) \in [1,\infty)$ and $\varepsilon = \varepsilon(g,G,p) \in (0,1]$ with the following significance. If*

$$\max_{\alpha\in\mathscr{I}} \|a_\alpha\|_{L^d(U_\alpha\cap U_\beta)} \leq \varepsilon \quad \text{and} \quad \max_{\alpha\in\mathscr{I}} \|b_\alpha\|_{L^d(U_\alpha\cap U_\beta)} \leq \varepsilon, \tag{2.15}$$

then

$$\begin{aligned}\|g_{\alpha\beta} - h_{\alpha\beta}\|_{W^{2,p}(V_\alpha\cap V_\beta)} &\leq C\|g_{\alpha\beta} - h_{\alpha\beta}\|_{L^p(U_\alpha\cap U_\beta)} \\ &\quad + C\left(\|a_\alpha - b_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta - b_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right).\end{aligned} \tag{2.16}$$

Moreover, if

$$\max_{\alpha,\beta\in\mathscr{I}} \|g_{\alpha\beta} - h_{\alpha\beta}\|_{L^p(U_\alpha\cap U_\beta)} \leq \varepsilon, \tag{2.17}$$

then

$$\max_{\alpha,\beta\in\mathscr{I}} \|g_{\alpha\beta} - h_{\alpha\beta}\|_{W^{2,p}(V_\alpha\cap V_\beta)} \leq C\varepsilon. \tag{2.18}$$

Finally, if $p > d/2$ and $\varepsilon = \varepsilon(g,G,p,\mathscr{U},\mathscr{V}) \in (0,1]$ is sufficiently small, then P is isomorphic to Q as a continuous principal G-bundle.

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[8] By analogy with Corollary 2.11 or [30, Corollary 4.4], the condition $p < d$ could be relaxed to $p \leq q$ at the expense in the hypothesis (2.15) of replacing the L^d norm by an L^p norm when $p > d$.

Proof. We proceed by simplifying Taubes' proof of his [82, Lemma A.1] (where $d = 4$) and Rivière's proof of his [67, Theorem IV.1] (where $d \geq 4$). Let us observe that $d^{*_g} = -*_g\, d *_g : \Omega^1(X; \operatorname{End}(\mathfrak{g})) \to \Omega^0(X; \operatorname{End}(\mathfrak{g}))$ by [87, Section 6.1], where $* = *_g : \Omega^l(X; \mathbb{R}) \to \Omega^{d-l}(X; \mathbb{R})$ (for integers $0 \leq l \leq d$) is the Hodge $*$-operator for the Riemannian metric g on X and we write $d^* = d^{*_g}$ for brevity in the remainder of the proof. Because $d^* a_\alpha = 0$ on U_α for all $\alpha \in \mathscr{I}$, then the identity

$$dg_{\alpha\beta} = g_{\alpha\beta} a_\beta + a_\alpha g_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta, \tag{2.19}$$

yields

$$\begin{aligned} d^* dg_{\alpha\beta} &= -*(dg_{\alpha\beta} \wedge *a_\beta) + g_{\alpha\beta} d^* a_\beta + (d^* a_\alpha) g_{\alpha\beta} + *((*a_\alpha) \wedge dg_{\alpha\beta}) \\ &= -*(dg_{\alpha\beta} \wedge *a_\beta) + *((*a_\alpha) \wedge dg_{\alpha\beta}) \quad \text{on } U_\alpha \cap U_\beta. \end{aligned}$$

Similarly, we have

$$\begin{aligned} dh_{\alpha\beta} &= h_{\alpha\beta} b_\beta + b_\alpha h_{\alpha\beta}, \\ d^* dh_{\alpha\beta} &= -*(dh_{\alpha\beta} \wedge *b_\beta) + *((*b_\alpha) \wedge dh_{\alpha\beta}) \quad \text{on } U_\alpha \cap U_\beta. \end{aligned}$$

For brevity, define

$$f_{\alpha\beta} := g_{\alpha\beta} - h_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta, \quad \forall\, \alpha, \beta \in \mathscr{I},$$

and observe that, by subtracting the corresponding the equations for $dh_{\alpha\beta}$ and $d^* dh_{\alpha\beta}$ from those for $dg_{\alpha\beta}$ and $d^* dg_{\alpha\beta}$, we obtain

$$\begin{aligned} df_{\alpha\beta} &= f_{\alpha\beta} a_\beta + a_\alpha f_{\alpha\beta} + h_{\alpha\beta}(a_\beta - b_\beta) + (a_\alpha - b_\alpha) h_{\alpha\beta}, \\ d^* df_{\alpha\beta} &= -*(df_{\alpha\beta} \wedge *a_\beta) + *((*a_\alpha) \wedge df_{\alpha\beta}) - *(dh_{\alpha\beta} \wedge *(a_\beta - b_\beta)) + *((*(a_\alpha - b_\alpha)) \wedge dh_{\alpha\beta}). \end{aligned}$$

If $\varphi_{\alpha_\beta} \in C_0^\infty(U_\alpha \cap U_\beta; \mathbb{R})$, then

$$\begin{aligned} d(\varphi_{\alpha_\beta} f_{\alpha\beta}) &= (d\varphi_{\alpha_\beta}) f_{\alpha\beta} + \varphi_{\alpha_\beta} df_{\alpha\beta}, \\ d^* d(\varphi_{\alpha_\beta} f_{\alpha\beta}) &= (d^* d\varphi_{\alpha_\beta}) f_{\alpha\beta} + 2\langle \operatorname{grad} \varphi_{\alpha_\beta}, \operatorname{grad} f_{\alpha\beta} \rangle + \varphi_{\alpha_\beta} d^* df_{\alpha\beta}. \end{aligned}$$

Therefore, writing $c_\alpha := a_\alpha - b_\alpha$ for $\alpha \in \mathscr{I}$ for brevity, we have

$$\begin{aligned} d^* d(\varphi_{\alpha_\beta} f_{\alpha\beta}) = &-*(df_{\alpha\beta} \wedge *a_\beta)\varphi_{\alpha_\beta} + *((*a_\alpha) \wedge df_{\alpha\beta})\varphi_{\alpha_\beta} \\ &- *(dh_{\alpha\beta} \wedge *c_\beta)\varphi_{\alpha_\beta} + *((*c_\alpha) \wedge dh_{\alpha\beta})\varphi_{\alpha_\beta} \\ &+ (d^* d\varphi_{\alpha_\beta}) f_{\alpha\beta} + 2\langle \operatorname{grad} \varphi_{\alpha_\beta}, \operatorname{grad} f_{\alpha\beta} \rangle, \end{aligned}$$

which gives

$$\begin{aligned} d^* d(\varphi_{\alpha_\beta} f_{\alpha\beta}) = &-*(d(\varphi_{\alpha_\beta} f_{\alpha\beta}) \wedge *a_\beta) + *((*a_\alpha) \wedge d(\varphi_{\alpha_\beta} f_{\alpha\beta})) \\ &+ *((d\varphi_{\alpha_\beta}) f_{\alpha\beta} \wedge *a_\beta) - *((*a_\alpha) \wedge (d\varphi_{\alpha_\beta}) f_{\alpha\beta}) \\ &- *(dh_{\alpha\beta} \wedge *c_\beta)\varphi_{\alpha_\beta} + *((*c_\alpha) \wedge dh_{\alpha\beta})\varphi_{\alpha_\beta} \\ &+ (d^* d\varphi_{\alpha_\beta}) f_{\alpha\beta} + 2\langle \operatorname{grad} \varphi_{\alpha_\beta}, \operatorname{grad} f_{\alpha\beta} \rangle. \end{aligned}$$

Assume that $\operatorname{supp} \varphi_\alpha \subset U'_\alpha$, where $U'_\alpha \Subset U_\alpha$ is an open subset (obeying an interior cone condition) for each $\alpha \in \mathscr{I}$. Thus, for $d \geq 2$ and $p \in [1, d)$ and $p^* = dp/(d-p) \in [d/(d-1), \infty)$, so

$1/p = 1/p^* + 1/d$, we have (provided $q \geq p$), for $c = c(g,G) \in [1,\infty)$,

$$\begin{aligned}
\|d^*d(\varphi_{\alpha_\beta} f_{\alpha\beta})\|_{L^p(X)} &\leq c\|d(\varphi_{\alpha_\beta} f_{\alpha\beta})\|_{L^{p^*}(X)} \left(\|a_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right) \\
&\quad + c\|f_{\alpha\beta}\|_{L^{p^*}(U'_\alpha\cap U'_\beta)} \left(\|a_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right) \\
&\quad + c\|dh_{\alpha\beta}\|_{L^{p^*}(U_\alpha\cap U_\beta)} \left(\|a_\alpha - b_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta - b_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right), \\
&\quad + c\left(\|d^*d\varphi_{\alpha_\beta}\|_{L^\infty(X)}\|f_{\alpha\beta}\|_{L^p(U'_\alpha\cap U'_\beta)} + \|d\varphi_{\alpha_\beta}\|_{L^\infty(X)}\|df_{\alpha\beta}\|_{L^p(U'_\alpha\cap U'_\beta)}\right).
\end{aligned}$$

Using $W^{1,p}(U;\mathbb{R}) \subset L^{p^*}(U;\mathbb{R})$, the continuous embedding of Sobolev spaces given by [2, Theorem 4.12, Part I, Case C], for an open subset $U \subset \mathbb{R}^d$ (obeying an interior cone condition), we obtain

$$\begin{aligned}
\|d^*d(\varphi_{\alpha_\beta} f_{\alpha\beta})\|_{L^p(X)} &\leq c\|\varphi_{\alpha_\beta} f_{\alpha\beta}\|_{W^{2,p}(X)} \left(\|a_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right) \\
&\quad + C\|f_{\alpha\beta}\|_{W^{1,p}(U'_\alpha\cap U'_\beta)} \left(\|a_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right) \\
&\quad + C\|h_{\alpha\beta}\|_{W^{2,p}(U_\alpha\cap U_\beta)} \left(\|a_\alpha - b_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta - b_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right) \\
&\quad + c\left(\|d^*d\varphi_{\alpha_\beta}\|_{L^\infty(X)}\|f_{\alpha\beta}\|_{L^p(U'_\alpha\cap U'_\beta)} + \|d\varphi_{\alpha_\beta}\|_{L^\infty(X)}\|df_{\alpha\beta}\|_{L^p(U'_\alpha\cap U'_\beta)}\right),
\end{aligned}$$

for a constant $C = C(g,G,p,\mathscr{U}) \in [1,\infty)$. For any $p \in (1,\infty)$, we have the *a priori* L^p global elliptic estimate (see [26, Theorem 14.60] or [35, Theorem 9.11]),

$$\|\varphi_{\alpha_\beta} f_{\alpha\beta}\|_{W^{2,p}(X)} \leq C\left(\|d^*d(\varphi_{\alpha_\beta} f_{\alpha\beta})\|_{L^p(X)} + \|\varphi_{\alpha_\beta} f_{\alpha\beta}\|_{L^p(X)}\right),$$

for $C = C(g,G,p) \in [1,\infty)$. Hence, for a_α obeying (2.15) and choosing $\varepsilon = \varepsilon(g,G,p) \in [1,\infty)$ sufficiently small, rearrangement gives

$$\begin{aligned}
\|d^*d(\varphi_{\alpha_\beta} f_{\alpha\beta})\|_{L^p(X)} &\leq C\|f_{\alpha\beta}\|_{W^{1,p}(U'_\alpha\cap U'_\beta)} \left(\|a_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right) \\
&\quad + C\|h_{\alpha\beta}\|_{W^{2,p}(U_\alpha\cap U_\beta)} \left(\|a_\alpha - b_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta - b_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right) \\
&\quad + C\left(\|d^*d\varphi_{\alpha_\beta}\|_{L^\infty(X)}\|f_{\alpha\beta}\|_{L^p(U'_\alpha\cap U'_\beta)} + \|d\varphi_{\alpha_\beta}\|_{L^\infty(X)}\|df_{\alpha\beta}\|_{L^p(U'_\alpha\cap U'_\beta)}\right),
\end{aligned}$$

and thus

$$\begin{aligned}
\|\varphi_{\alpha_\beta} f_{\alpha\beta}\|_{W^{2,p}(X)} &\leq C\|h_{\alpha\beta}\|_{W^{2,p}(U_\alpha\cap U_\beta)} \left(\|a_\alpha - b_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta - b_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right) \\
&\quad + C\left(1 + \|d^*d\varphi_{\alpha_\beta}\|_{L^\infty(X)}\right)\|f_{\alpha\beta}\|_{L^p(U'_\alpha\cap U'_\beta)} \\
&\quad + C\left(1 + \|d\varphi_{\alpha_\beta}\|_{L^\infty(X)}\right)\|df_{\alpha\beta}\|_{L^p(U'_\alpha\cap U'_\beta)}.
\end{aligned} \tag{2.20}$$

We could now choose $\varphi_{\alpha\beta}$ obeying $\varphi_{\alpha\beta} = 1$ on $V_\alpha \cap V_\beta$, so the preceding inequality and the $W^{2,p}$ bounds (2.14) for $h_{\alpha\beta}$ (with $g_{\alpha\beta}$ and V_α replaced by $h_{\alpha\beta}$ and U_α, respectively) would yield

$$\begin{aligned}
\|g_{\alpha\beta} - h_{\alpha\beta}\|_{W^{2,p}(V_\alpha\cap V_\beta)} &\leq C\|g_{\alpha\beta} - h_{\alpha\beta}\|_{W^{1,p}(U_\alpha\cap U_\beta)} \\
&\quad + C\left(\|a_\alpha - b_\alpha\|_{L^d(U_\alpha\cap U_\beta)} + \|a_\beta - b_\beta\|_{L^d(U_\alpha\cap U_\beta)}\right).
\end{aligned}$$

However, a more refined application of the inequality (2.20) in conjunction with Krylov's approach to derivation of *a priori* interior L^p estimates for an elliptic, linear, scalar, second-order partial differential operator on a bounded open subset, $U \Subset \mathbb{R}^d$, and the interpolation inequality [35, Theorem 7.28] (for $C = C(U) \in [1,\infty)$ and any $\delta > 0$),

$$\|\nabla f_{\alpha\beta}\|_{L^p(U)} \leq \delta\|\nabla^2 f_{\alpha\beta}\|_{L^p(U)} + C\delta^{-1}\|f_{\alpha\beta}\|_{L^p(U)},$$

as in Krylov's proofs of [51, Theorems 7.1.1 or 8.11.1] or [52, Theorem 9.4.1] (noting that we can assume without loss of generality that the open subsets $U_\alpha \subset X$ are geodesic balls), allows us to also eliminate the term $\|df_{\alpha\beta}\|_{L^p(U'_\alpha \cap U'_\beta)}$ from the right-hand side of (2.20) and obtain the desired estimate (2.16). Moreover, the estimate (2.16) and bounds (2.15) and (2.17) (with $p \leq d$) gives the estimate (2.18).

If $p > d/2$, then $W^{2,p}(U;\mathbb{R}) \subset C^0(U;\mathbb{R})$ is a continuous embedding of Sobolev spaces by [2, Theorem 4.12, Part I, Case A], for an open subset $U \subset \mathbb{R}^d$ (obeying an interior cone condition), so (2.18) yields

$$\max_{\alpha,\beta \in \mathscr{I}} \|g_{\alpha\beta} - h_{\alpha\beta}\|_{C^0(\bar{V}_\alpha \cap \bar{V}_\beta)} \leq C\varepsilon,$$

for $C = C(g, G, p, \mathscr{U}, \mathscr{V}) \in [1, \infty)$. We now appeal to Proposition 2.12 to give the desired isomorphism between P and Q to complete the proof of Theorem 2.16. □

A slight variant of the derivation of (2.16) in Theorem 2.16 yields

Corollary 2.17 (Continuous principal bundles with L^d-small local connection one-forms in Coulomb gauge)**.** *Assume the hypotheses of Theorem 2.16. Then*

$$\|g_{\alpha\beta} - h_{\alpha\beta}\|_{W^{2,p}(V_\alpha \cap V_\beta)} \leq C\|g_{\alpha\beta} - h_{\alpha\beta}\|_{L^p(U_\alpha \cap U_\beta)} + C\left(\|a_\alpha - b_\alpha\|_{L^p(U_\alpha \cap U_\beta)} + \|a_\beta - b_\beta\|_{L^p(U_\alpha \cap U_\beta)}\right), \tag{2.21}$$

where $C = C(g, G, p, \mathscr{U}, \mathscr{V}) \in [1, \infty)$.

Remark 2.18 (Related results due to Rivère and Taubes)*.* Theorem 2.16 is essentially equivalent to Rivère's [67, Theorem IV.1] and that in turn may be viewed as a generalization of part of Taubes' [82, Proposition 4.5 and Lemma A.1] from the case of $d = 4$ to arbitrary $d \geq 4$. (It is likely that [67, Theorem IV.1] also holds for $d = 3$ and possibly even $d = 2$, but that would require carefully checking that all of the results used in the proof involving Lorentz spaces when $d \geq 4$ (as implicit throughout [67]) also hold for $d = 2$ or 3.) In [67, Theorem IV.1], Rivère does not explicitly state that the local sections are continuous, but this appears to be implied by the proof.

In [67], Rivière uses *Lorentz spaces* to obtain the necessary L^∞ control over transition functions in this case of borderline $W^{1,d/2}$ strong convergence of local connection one-forms in Coulomb gauge. References for key results on Lorentz spaces employed by Rivière include Brezis and Wainger [17], Lorentz [59, 60], Peetre [65, 66], Stein and Weiss [77], Tartar [79, 80], and Grafakos [37] for a more recent exposition. Our proof of Theorem 2.16 is considerably simpler than those of [67, Theorem IV.1] or [82, Lemma A.1] since it only requires standard results on Sobolev spaces [2] and an *a priori* L^p estimate for the Laplace operator [35].

Remark 2.19 (Related results due to Isobe and Shevchishin). Isobe has shown that any two C^0 principal G-bundles that are sufficiently close to each other in the $W^{1,d}(X)$-norm are necessarily isomorphic (see [45, Theorem 1.1 and Proposition 3.1]); compare Shevchishin [73, Theorem 2.6] for a related result. However, the proofs of [45, Theorem 1.1 and Proposition 3.1] are quite involved whereas the proof of Theorem 2.16 is direct and the result more than adequate for our application.

2.5. **Existence of a flat connection for the critical exponent.** In this subsection, we establish the forthcoming Theorem 2.20 — an extension of Item (1) in Theorem 2.1 — giving existence of a C^∞ flat connection on a principal bundle supporting a $W^{1,q}$ connection with $L^{d/2}$-small curvature for $d \geq 3$ or L^{s_0}-small curvature for $d = 2$ and $s_0 > 1$. This will also verify Item (1) in Theorem 1.

Suppose temporarily that X is a closed, *four*-dimensional, oriented, *topological* manifold and that G is a compact *simple* Lie group. We recall from [70, Appendix], [81, Propositions A.1 and A.2] that a topological principal G-bundle, P over X, is classified up to isomorphism by a cohomology class $\eta(P) \in H^2(X;\pi_1(G))$ and its *first Pontrjagin class*, $p_1(P) \in H^4(X;\mathbb{Z})$, or equivalently, *first Pontrjagin degree*, $\langle p_1(P), [X]\rangle \in \mathbb{Z}$, where $[X] \in H_4(X;\mathbb{Z})$ denotes the fundamental class of X. The topological invariant, $\eta \in H^2(X;\pi_1(G))$, is the *obstruction* to the existence of a principal G-bundle, P over X, with a specified Pontrjagin degrees.

In his Ph.D. thesis [71] and its published version [70], Sedlacek applied the direct minimization method to the Yang–Mills energy function (1.5) on the affine space of $W^{1,q}$ connections on a smooth principal G-bundle P over a closed, four-dimensional, smooth Riemannian manifold (X, g) to prove existence of a C^∞ Yang–Mills connection, A_∞, on a smooth principal G-bundle, P_∞, over X, where $\eta(P_\infty) = \eta(P)$ and $p_1(P_\infty)[X] \geq p_1(P)[X]$ (see [70, Theorems 4.3, 5.5, and 7.1 and Corollary 5.6]). Here, $\eta(P)$ is the obstruction class (see [70, Section 2]), $p_1(P) \in H^4(X;\mathbb{Z})$ is the Pontrjagin class of P, and $p_1(P)[X] \in \mathbb{Z}$ is the Pontrjagin degree for P. The case $p_1(P_\infty)[X] > p_1(P)[X]$ arises due to the phenomenon of energy bubbling, as explained in [70, Sections 5 and 7]. In his proof of [70, Theorems 4.1 and 4.3 and Proposition 4.2], Sedlacek considers a sequence of C^∞ connections, $\{A^i\}_{i=1}^\infty$, on P such that

$$\mathscr{E}(A^i) \searrow m(\eta), \quad \text{as } i \to \infty,$$

where $m(\eta) := \inf\{\mathscr{E}(A) : A \text{ is a } C^\infty \text{ connection on a smooth principal } G\text{-bundle } P' \text{ such that } \eta(P') = \eta\}$ and finds a C^∞ Yang–Mills connection A_∞ on a smooth principal G-bundle P_∞ with $\eta(P_\infty) = \eta(P)$ by [70, Theorem 5.6]. If P supports a C^∞ connection A obeying the condition (1.1) with $d = 4$, namely

$$\|F_A\|_{L^2(X)} \leq \varepsilon,$$

then the Chern–Weil representation of characteristic classes [61] implies that $p_1(\mathrm{ad}P)[X] = 0$ for small enough $\varepsilon = \varepsilon(g, G, k) \in (0, 1]$ (where $k = p_1(\mathrm{ad}P)[X] \in \mathbb{Z}$). (Arguing along these lines, Sedlacek obtains his [70, Theorem 7.1].) But $\mathscr{E}(A_\infty) \leq \mathscr{E}(A)$ and thus also $p_1(\mathrm{ad}P_\infty)[X] = 0$. Hence, P is isomorphic to P_∞ as a continuous principal G-bundle, at least when G is simple, by the preceding remarks on their classification.

While Sedlacek confines his attention to manifolds X of dimension $d = 4$, his argument employs Uhlenbeck's Theorem 2.5, which is valid for the unit ball $B \subset \mathbb{R}^d$ of any dimension $d \geq 2$. As we discuss here, it is therefore not difficult to modify his proof to yield a version of his [70, Theorem 4.3] which is also valid for X of any dimension $d \geq 2$. Furthermore, that generalization to $d \geq 2$ of [70, Theorem 4.3] from $d = 4$ will yield the desired enhancement (from $p > d/2$ to $p = d/2$) of Item (1) in Theorem 2.1 (existence of a C^∞ flat connection, Γ, on a principal G-bundle P supporting a $W^{1,q}$ connection, A, with $L^p(X)$-small curvature F_A).

Theorem 2.20 (Existence of a C^∞ flat connection on a principal bundle supporting a $W^{1,q}$ connection with $L^{d/2}$-small curvature)**.** *Let (X, g) be a closed, smooth Riemannian manifold of dimension $d \geq 2$, and G be a compact Lie group, and $s_0 > 1$ be a constant. Then there is a constant, $\varepsilon = \varepsilon(g, G, s_0) \in (0, 1]$, with the following significance. If $q \in (d/2, \infty]$ and A is a $W^{1,q}$ connection on a smooth principal G-bundle P over X whose curvature obeys* (1.1)*, namely*

$$\|F_A\|_{L^{s_0}(X)} \leq \varepsilon,$$

where $s_0 = d/2$ when $d \geq 3$ or $s_0 > 1$ when $d = 2$, then there is a C^∞ flat connection, Γ, on P.

Proof. Suppose the conclusion is false, so we may select a sequence $\{A^i\}_{i=1}^\infty$ of $W^{1,q}$ connections on P such that $\|F_{A^i}\|_{L^{d/2}(X)} \to 0$ as $i \to \infty$ but P does not admit a C^∞ flat connection. Choose a finite cover of X by geodesic balls, $B^\alpha = B_\varrho(x_\alpha) \subset X$ with centers $x_\alpha \in X$ and radius $\varrho \in (0, \mathrm{Inj}(X,g))$, for all $\alpha \in \mathscr{I}$. With the aid of geodesic normal coordinates, one sees that the Riemannian metric, g, is C^1-close to a flat metric in a small enough open neighborhood of x_α (see Aubin [7, Definition 1.24, Proposition 1.25, and Corollary 1.32]). Choose $\varepsilon \in (0,1]$ small enough that we can apply Theorem 2.5. Hence, there are a sequence of $W^{2,q}$ local sections, $\sigma_\alpha^i : B_\alpha \to P$, and $W^{2,q}$ transition functions, $g_{\alpha\beta}^i : B_\alpha \cap B_\beta \to G$, and local connection one-forms, $a_\alpha^i = (\sigma_\alpha^i)^* A^i \in W^{1,q}(B_\alpha; \Lambda^1 \otimes \mathfrak{g})$, such that for all $i \in \mathbb{N}$ and $\alpha, \beta, \gamma \in \mathscr{I}$,

$$
\begin{aligned}
d^{*_g} a_\alpha^i &= 0 \quad \text{on } B_\alpha, \\
\|a_\alpha^i\|_{W^{1,d/2}(B_\alpha)} &\leq c\|F_{A^i}\|_{L^{d/2}(B_\alpha)}, \\
g_{\alpha\beta}^i g_{\beta\gamma}^i g_{\gamma\alpha}^i &= \mathrm{id}_G \quad \text{on } B_\alpha \cap B_\beta \cap B_\gamma, \\
\nabla g_{\alpha\beta}^i = d g_{\alpha\beta}^i &= g_{\alpha\beta}^i a_\beta^i + a_\alpha^i g_{\alpha\beta}^i \quad \text{on } B_\alpha \cap B_\beta,
\end{aligned}
$$

where $c = c(g,G) \in [1,\infty)$. Hence, for all $\alpha, \beta \in \mathscr{I}$ and $i \to \infty$, we have

$$
\begin{aligned}
a_\alpha^i &\to 0 \quad \text{in } W^{1,d/2}(B_\alpha; \Lambda^1 \otimes \mathfrak{g}), \\
\nabla g_{\alpha\beta}^i &\to 0 \quad \text{in } L^d(B_\alpha \cap B_\beta; G),
\end{aligned}
$$

since, using the continuous embedding of Sobolev spaces, $W^{1,d/2}(B;\mathbb{R}) \subset L^d(B;\mathbb{R})$ by [2, Theorem 4.12, Part I (C)] for any ball $B \Subset \mathbb{R}^d$,

$$
\|\nabla g_{\alpha\beta}^i\|_{L^d(B_\alpha \cap B_\beta)} \leq c\left(\|a_\alpha^i\|_{W^{1,d/2}(B_\alpha)} + \|a_\beta^i\|_{W^{1,d/2}(B_\beta)}\right),
$$

and the fact that G is compact, so $\|g_{\alpha\beta}^i\|_{L^\infty(B_\alpha \cap B_\beta)} \leq c_0$, where $c_0 = c_0(G)$ and $c = c(g,G,\varrho) \in [1,\infty)$. Moreover, because

$$
\nabla^2 g_{\alpha\beta}^i = \nabla g_{\alpha\beta}^i \otimes a_\beta^i + g_{\alpha\beta}^i \nabla a_\beta^i + (\nabla a_\alpha^i) g_{\alpha\beta}^i + a_\alpha^i \otimes \nabla g_{\alpha\beta}^i,
$$

and $L^d(B_\alpha \cap B_\beta) \times L^d(B_\alpha \cap B_\beta) \to L^{d/2}(B_\alpha \cap B_\beta)$ is a continuous Sobolev multiplication map and $W^{1,d/2}(B;\mathbb{R}) \subset L^d(B;\mathbb{R})$ is a continuous Sobolev embedding, we see that

$$
\nabla^2 g_{\alpha\beta}^i \to 0 \quad \text{in } L^{d/2}(B_\alpha \cap B_\beta; G),
$$

for all $\alpha, \beta \in \mathscr{I}$, as $i \to \infty$. In particular, the sequence $\{g_{\alpha\beta}^i\}_{i=1}^\infty$ is uniformly bounded in $W^{2,d/2}(B_\alpha \cap B_\beta; G)$ and because the Sobolev embedding, $W^{2,d/2}(B;\mathbb{R}) \Subset W^{1,r}(B;\mathbb{R})$ for $r \in [1,d)$, is compact by the Rellich-Kondrachov Theorem (see [2, Theorem 6.3]), thus, after passing to a subsequence, there is a collection of maps, $h_{\alpha\beta} : B_\alpha \cap B_\beta \to G$ such that $\nabla h_{\alpha\beta} = 0$ on $B_\alpha \cap B_\beta$ and

$$
g_{\alpha\beta}^i \to h_{\alpha\beta} \quad \text{in } L^d(B_\alpha \cap B_\beta; G), \quad i \to \infty,
$$

for all $\alpha, \beta \in \mathscr{I}$. Hence, the sequence $\{g_{\alpha\beta}^i\}_{i=1}^\infty$ of $W^{2,q}$ transition functions, defining a sequence of $W^{2,q}$ principal G-bundles, P_i, isomorphic to P (as continuous principal bundles) converges in $W^{1,r}(B_\alpha \cap B_\beta; G)$ to a collection of constant maps, $\{h_{\alpha\beta}\}_{\alpha,\beta \in \mathscr{I}}$, obeying the cocycle condition,

$$
h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} = \mathrm{id}_G \quad \text{on } B_\alpha \cap B_\beta \cap B_\gamma, \quad \forall\, \alpha, \beta, \gamma \in \mathscr{I}.
$$

Therefore, by Proposition 2.4 the collection $\{h_{\alpha\beta}\}_{\alpha,\beta \in \mathscr{I}}$ defines a C^∞ flat connection, Γ, on a C^∞ principal G-bundle, Q, over X with local connection one-forms, $b_\alpha = 0$ on B_α, for all $\alpha \in \mathscr{I}$. But Theorem 2.16 implies that the sequence $\{g_{\alpha\beta}^i\}_{i=1}^\infty$ actually converges to $h_{\alpha\beta}$ in $W^{2,p}_{\mathrm{loc}}(B_\alpha \cap B_\beta; G)$, for any $p \leq q$ obeying $1 < p < d$ and all $\alpha, \beta \in \mathscr{I}$, and that Q is isomorphic to P as a continuous principal bundle. This contradicts our initial assumption and thus proves Theorem 2.20. $\square$

Remark 2.21 (Alternative proof of convergence of transition functions). Rather than apply the Rellich-Kondrachov Theorem in the proof of Theorem 2.20, we may instead observe that the difference between the average $h^i_{\alpha\beta} := (g^i_{\alpha\beta})_{B_\alpha \cap B_\beta} \in G$ of $g^i_{\alpha\beta}$ on $B_\alpha \cap B_\beta$,

$$(g_{\alpha\beta})_{B_\alpha \cap B_\beta} := \frac{1}{\mathrm{vol}(B_\alpha \cap B_\beta)} \int_{B_\alpha \cap B_\beta} g_{\alpha\beta}\, d\,\mathrm{vol}, \quad \forall\, \alpha, \beta \in \mathscr{I},$$

and $g^i_{\alpha\beta}$ may be estimated via the Poincaré Inequality [24, Theorem 5.8.1],

$$\|g^i_{\alpha\beta} - h^i_{\alpha\beta}\|_{L^p(B_\alpha \cap B_\beta)} \leq C\|dg^i_{\alpha\beta}\|_{L^p(B_\alpha \cap B_\beta)}, \quad \forall\, \alpha, \beta \in \mathscr{I}, \quad i \in \mathbb{N}. \tag{2.22}$$

But G is compact and thus, after passing to a subsequence and relabelling, we may suppose that the sequence, $\{h^i_{\alpha\beta}\}_{i=}^\infty$, converges to a limit $h_{\alpha\beta} \in G$ and consequently the sequence, $\{g^i_{\alpha\beta}\}_{i=1}^\infty$, converges in $W^{2,p}(B_\alpha \cap B_\beta; G)$ to a limit $h_{\alpha\beta}$.

The proof of Item (2) in Theorem 1 will take up most of the remainder of Section 2. However, the proof of Theorem 2.20 already shows that the flat connection, Γ, that it provides is close to A in the following sense.

Corollary 2.22 (Comparison between flat connection and $W^{1,q}$ connection with $L^{d/2}$-small curvature)**.** *Continue the assumptions of Theorem 2.20 and let $\{B_\alpha\}_{\alpha \in \mathscr{I}}$ be a finite cover of X by open geodesic balls of radius $\varrho \in (0, \mathrm{Inj}(X,g))$, let $\mathscr{V} = \{V_\alpha\}_{\alpha \in \mathscr{I}}$ be a finite cover of X by open subsets such that $V_\alpha \Subset B_\alpha$, and let $p \leq q$ be a constant obeying $1 < p < d$. Then there are a constant $C = C(g, G, p, \mathscr{V}, \varrho) \in [1, \infty)$, a set $\{\sigma_\alpha\}_{\alpha \in \mathscr{I}}$ of $W^{2,q}$ local sections, $\sigma_\alpha : B_\alpha \to P$, a corresponding set $\{g_{\alpha\beta}\}_{\alpha,\beta \in \mathscr{I}}$ of $W^{2,q}$ transition maps, $g_{\alpha\beta} : B_\alpha \cap B_\beta \to G$, defined by*

$$\sigma_\beta = \sigma_\alpha g_{\alpha\beta} \quad \text{on } B_\alpha \cap B_\beta,$$

a set of local connection one-forms, $a_\alpha = \sigma^*_\alpha A \in W^{1,q}(B_\alpha; \Lambda^1 \otimes \mathfrak{g})$, a set $\{\varsigma_\alpha\}_{\alpha \in \mathscr{I}}$ of C^∞ local sections, $\varsigma_\alpha : B_\alpha \to P$, and a corresponding set $\{h_{\alpha\beta}\}_{\alpha,\beta \in \mathscr{I}}$ of constant transition maps, $h_{\alpha\beta} : B_\alpha \cap B_\beta \to G$, defined by

$$\varsigma_\beta = \varsigma_\alpha h_{\alpha\beta} \quad \text{on } B_\alpha \cap B_\beta,$$

and a set of local connection one-forms, $b_\alpha = \varsigma^*_\alpha \Gamma \in C^\infty(B_\alpha; \Lambda^1 \otimes \mathfrak{g})$ such that the following hold:

$$d^{*_g} a_\alpha = 0 \quad \text{a.e. on } B_\alpha \quad \text{and} \quad b_\alpha = 0 \quad \text{a.e. on } B_\alpha, \tag{2.23a}$$

$$\|a_\alpha\|_{W^{1,p}(B_\alpha)} \leq C\|F_A\|_{L^p(B_\alpha)}, \tag{2.23b}$$

$$\|g_{\alpha\beta} - h_{\alpha\beta}\|_{W^{2,p}(V_\alpha \cap V_\beta)} \leq C\varepsilon, \tag{2.23c}$$

where ε is as in Theorem 2.20.

Proof. The conclusions (2.23) are provided by the proof of Theorem 2.20. By Husemoller [44, Theorem 5.3.2], a set of transition maps, $\{h_{\alpha\beta}\}_{\alpha,\beta \in \mathscr{I}}$, with respect to a finite cover, $\{B_\alpha\}_{\alpha \in \mathscr{I}}$, of X by open subsets, defines a principal G-bundle P over X (up to isomorphism) and a set of local sections, $\{\varsigma_\alpha\}_{\alpha \in \mathscr{I}}$, with respect to which the maps, $h_{\alpha\beta} : B_\alpha \cap B_\beta \to G$, are the corresponding transition functions. Indeed, one begins by defining $\varsigma_\alpha : B_\alpha \to B_\alpha \times G$ by setting $\varsigma_\alpha(x) = (x, \mathrm{id}_G)$ for all $x \in B_\alpha$ and obtaining P from the local coordinate descriptions, $B_\alpha \times G$, as a quotient via the equivalence relation defined by the transition maps, $h_{\alpha\beta}$. $\square$

Remark 2.23 (Distance between flat connection and $W^{1,q}$ connection with $L^{d/2}$-small curvature). At first glance, it might seem that Corollary 2.22 would almost immediately yield the estimates (1.2) or (1.4) asserted by Theorem 1, however that is not so since $\|a_\alpha\|_{W^{1,p}(B_\alpha)}$ does not directly measure the distance between A or $u(A)$ and Γ with respect to the *same* local section. Indeed, $\varsigma_\alpha^*(A-\Gamma) = \varsigma_\alpha^* A$ since $\varsigma_\alpha^*\Gamma = 0$ but $\varsigma_\alpha^* A \neq a_\alpha$, while $\sigma_\alpha^*(A-\Gamma) = a_\alpha - \sigma_\alpha^*\Gamma$ since $\sigma_\alpha^* A = a_\alpha$ but $\sigma_\alpha^*\Gamma \neq 0$. The analogous comments apply to $u(A)-\Gamma$. Instead, our strategy will be to show that $\|\sigma_\alpha^*\Gamma\|_{W^{1,p}(V_\alpha)}$ may be estimated in terms of $\|F_A\|_{L^p(B_\alpha)}$, for a finite cover, $\{V_\alpha\}_{\alpha\in\mathscr{I}}$, of X by open subsets $V_\alpha \Subset B_\alpha$.

2.6. **Estimate of Sobolev $W^{1,p}$ distance to the flat connection for $1 < p \leq q$.** In the forthcoming Theorem 2.24, we will establish an extension of Item (2) in Theorem 2.1, by *a*) relaxing the condition (2.1) (that F_A is $L^q(X)$-small) to (1.1) (that F_A is $L^{s_0}(X)$-small, where $s_0 = d/2$ when $d \geq 3$ or $s_0 > 1$ when $d = 2$), and *b*) relaxing the condition that $p \in (1,\infty)$ obey $d/2 \leq p \leq q$ to obeying $1 < p < d$. (The case $d \leq p \leq q$ is covered by Item (2) in Theorem 2.1 under the hypothesis (2.1).) In particular, this will prove Item (2) in Theorem 1.

Theorem 2.24 (Estimate of Sobolev $W^{1,p}$ distance to the flat connection for $1 < p < d$)**.** *Let (X,g) be a closed, smooth Riemannian manifold of dimension $d \geq 2$, and G be a compact Lie group, and $q \in (d/2,\infty]$ and $p \in (1,\infty)$ obeying $p < d$ and $s_0 > 1$ be constants. Then there are constants, $\varepsilon = \varepsilon(g,G,p,s_0) \in (0,1]$ and $C = C(g,G,p,s_0) \in [1,\infty)$, with the following significance. If A is a $W^{1,q}$ connection on a smooth principal G-bundle P over X whose curvature obeys* (1.1)*, namely*

$$\|F_A\|_{L^{s_0}(X)} \leq \varepsilon,$$

where $s_0 = d/2$ when $d \geq 3$ or $s_0 > 1$ when $d = 2$, and Γ is the C^∞ flat connection on P produced by Theorem 2.20, then (2.2) *holds for $1 < p < d$, that is,*

$$\|A-\Gamma\|_{W^{1,p}_\Gamma(X)} \leq C\|F_A\|_{L^p(X)}.$$

Theorem 2.24 will in turn follow from extensions of intermediate results established by the author in [30, Section 6] and by Uhlenbeck in [84, Section 3] and which we shall now discuss.

Lemma 2.25 (Sobolev bounds on isomorphisms of principal bundles with sufficiently close transition functions)**.** *(See Feehan* [30, Corollary 6.4] *or Uhlenbeck* [84, Corollary 3.3] *for the case*[9] *$p \in [d/2,q]$.) Let G be a compact Lie group, (X,g) be a compact, smooth Riemannian manifold of dimension $d \geq 2$, and $q > d/2$ and $p \in [1,q]$ be constants. Let $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$ and $\{h_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$ be two sets of $W^{2,q}(U_\alpha\cap U_\beta;G)$ transition functions with respect to a finite open cover, $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$, of X. Then there exist constants, $\varepsilon = \varepsilon(g,G,\mathscr{U}) \in (0,1]$ and $c = c(g,G) \in [1,\infty)$ and $C = C(g,G,p,\mathscr{U}) \in [1,\infty)$, with the following significance. If Inequality* (2.11) *is satisfied, then the maps, $\rho_\alpha : V_\alpha \to G$, constructed in Proposition 2.12 belong to $W^{2,q}(V_\alpha;G)$ and obey the C^0 bounds* (2.12)*, with finite open cover, $\mathscr{V} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$, of X and $V_\alpha \subset U_\alpha$ for all $\alpha\in\mathscr{I}$. Moreover, the following hold.*

(1) (L^r bound for $\nabla\rho_\alpha$.) *If*

$$\max_{\alpha,\beta\in\mathscr{I}} \left\{\|dg_{\alpha\beta}\|_{L^r(U_\alpha\cap U_\beta)},\ \|dh_{\alpha\beta}\|_{L^r(U_\alpha\cap U_\beta)}\right\} \leq \eta_r, \tag{2.24}$$

for $r \in [1,\infty]$ and $\eta_r > 0$, then together with the bounds (2.12)*, the maps, ρ_α, also satisfy*

$$\sup_{\alpha\in\mathscr{I}} \|\nabla\rho_\alpha\|_{L^r(V_\alpha)} \leq c\eta_r. \tag{2.25}$$

[9] In [30, Corollary 6.4], we required the transition functions $g_{\alpha\beta}$ and $h_{\alpha\beta}$ to belong to $W^{2,p}(U_\alpha\cap U_\beta;G)$ for $p > d/2$ for the purpose of regularity but $p = d/2$ is allowed in [30, Inequalities (6.3) and (6.4)].

(2) (L^p bound for $\nabla^2\rho_\alpha$.) *If*

$$\max_{\alpha,\beta\in\mathscr{I}} \left\{\|dg_{\alpha\beta}\|_{L^d(U_\alpha\cap U_\beta)},\, \|dh_{\alpha\beta}\|_{L^d(U_\alpha\cap U_\beta)}\right\} \leq \eta_d, \tag{2.26a}$$

$$\max_{\alpha,\beta\in\mathscr{I}} \left\{\|dg_{\alpha\beta}\|_{W^{1,p}(U_\alpha\cap U_\beta)},\, \|dh_{\alpha\beta}\|_{W^{1,p}(U_\alpha\cap U_\beta)}\right\} \leq \zeta_p, \tag{2.26b}$$

for $\eta_d > 0$ and $\zeta_p > 0$, and $1 \leq p < d$, and the open subsets, $V_\alpha \subset X$, obey an interior cone condition, then

$$\sup_{\alpha\in\mathscr{I}} \|\nabla^2\rho_\alpha\|_{L^p(V_\alpha)} \leq C(1+\eta_d)\zeta_p, \tag{2.27}$$

while if $p \geq d$, then (2.27) *holds with η_d replaced by ζ_p.*

Proof of Lemma 2.25. Consider Item (1). By examining the proof of [30, Corollary 6.4], we see that the bounds,

$$\|\nabla\rho_\alpha\|_{L^r(V_\alpha)} \leq c\eta_r,$$

in (2.25) hold for any $r \in [1,\infty]$, given the bounds in (2.24) for the same r.

Consider Item (2). We now indicate the changes to the remainder of the proof of [30, Corollary 6.4] and first consider the case $1 \leq p < d$. The hypothesis $p \geq d/2$ of [30, Corollary 6.4] is only used in the proof of the bounds on $\|\nabla^2\rho_\alpha\|_{L^p(V_\alpha)}$ in (2.27) when appealing to the Sobolev embedding $W^{1,p}(U) \subset L^{2p}(U)$, for an open subset $U \subset \mathbb{R}^d$ (obeying an interior cone condition). In the second and third lines of the displayed inequality immediately below [30, Inequality (6.8)], rather than apply the continuous Sobolev multiplication,

$$L^{2p}(V_\alpha\cap V_\beta) \times L^{2p}(V_\alpha\cap V_\beta) \to L^p(V_\alpha\cap V_\beta),$$

we instead use $1/p = 1/p^* + 1/d$, with $p^* = dp/(d-p) \in [d/(d-1),\infty)$, and apply the continuous Sobolev multiplication,

$$L^{p^*}(V_\alpha\cap V_\beta) \times L^d(V_\alpha\cap V_\beta) \to L^p(V_\alpha\cap V_\beta), \quad 1 \leq p < d.$$

We then apply the hypothesis (2.26a) to bound the resulting terms,

$$\|dg_{\alpha\beta}\|_{L^d(V_\alpha\cap V_\beta)}, \quad \|dh_{\alpha\beta}\|_{L^d(V_\alpha\cap V_\beta)}, \quad \forall\, \alpha,\beta\in\mathscr{I}.$$

We apply [30, Inequality (6.8)] (valid for any $p \in [1,\infty]$), the continuous Sobolev embedding, $W^{1,p}(V_\alpha\cap V_\beta) \subset L^{p^*}(V_\alpha\cap V_\beta)$ (provided by [2, Theorem 4.12, Part I (C)]), and the hypothesis (2.26b) to bound the resulting terms,

$$\begin{aligned}
\|\nabla\rho_\alpha\|_{L^{p^*}(V_\alpha\cap V_\beta)} &\leq \max_{\alpha,\beta\in\mathscr{I}} C\left\{\|dg_{\alpha\beta}\|_{L^{p^*}(V_\alpha\cap V_\beta)},\, \|dh_{\alpha\beta}\|_{L^{p^*}(V_\alpha\cap V_\beta)}\right\} \\
&\leq \max_{\alpha,\beta\in\mathscr{I}} C\left\{\|dg_{\alpha\beta}\|_{W^{1,p}(V_\alpha\cap V_\beta)},\, \|dh_{\alpha\beta}\|_{W^{1,p}(V_\alpha\cap V_\beta)}\right\} \\
&\leq C\zeta_p, \quad \forall\, \alpha,\beta\in\mathscr{I},
\end{aligned}$$

for $C = C(g,G,p,\mathscr{V}) \in [1,\infty)$. Similarly, in the third line of the displayed inequality immediately below [30, Inequality (6.8)], we apply the hypotheses (2.26a) and (2.26b), respectively, to bound the terms,

$$\begin{aligned}
&\|dh_{\alpha\beta}\|_{L^d(V_\alpha\cap V_\beta)}, \\
&\|dg_{\alpha\beta}\|_{L^{p^*}(V_\alpha\cap V_\beta)} \leq C\|dg_{\alpha\beta}\|_{W^{1,p}(V_\alpha\cap V_\beta)}, \quad \forall\, \alpha,\beta\in\mathscr{I},
\end{aligned}$$

for $C = C(g,G,p,\mathscr{V}) \in [1,\infty)$. We thus find that (2.27) holds for any $p \in [1,d)$.

For the case $p > d$, we apply the continuous Sobolev multiplication,

$$L^\infty(V_\alpha \cap V_\beta) \times L^p(V_\alpha \cap V_\beta) \to L^p(V_\alpha \cap V_\beta), \quad p > d.$$

We then apply the hypothesis (2.26b) to the resulting terms,

$$\|dg_{\alpha\beta}\|_{L^p(V_\alpha \cap V_\beta)}, \quad \|dh_{\alpha\beta}\|_{L^p(V_\alpha \cap V_\beta)}, \quad \forall\, \alpha, \beta \in \mathscr{I}.$$

Similarly to the case $p \in [1, d)$, we apply the [30, Inequality (6.8)], the continuous Sobolev embedding, $W^{1,p}(V_\alpha \cap V_\beta) \subset L^\infty(V_\alpha \cap V_\beta)$ (provided by [2, Theorem 4.12, Part I (A)]), and the hypothesis (2.26b) to bound the resulting terms,

$$\|\nabla \rho_\alpha\|_{L^\infty(V_\alpha \cap V_\beta)}, \quad \forall\, \alpha, \beta \in \mathscr{I}.$$

We thus find that (2.27) holds for $p > d$ when η_d is replaced by ζ_p.

Lastly, for the case $p = d$, we apply the continuous Sobolev multiplication,

$$L^{2d}(V_\alpha \cap V_\beta) \times L^{2d}(V_\alpha \cap V_\beta) \to L^d(V_\alpha \cap V_\beta),$$

the continuous Sobolev embedding, $W^{1,d}(V_\alpha \cap V_\beta) \subset L^{2d}(V_\alpha \cap V_\beta)$ (provided by [2, Theorem 4.12, Part I (B)]), and the hypothesis (2.26b) to bound the terms,

$$\begin{aligned} \|dg_{\alpha\beta}\|_{L^{2d}(V_\alpha \cap V_\beta)} &\leq C\|dg_{\alpha\beta}\|_{W^{1,d}(V_\alpha \cap V_\beta)}, \\ \|dh_{\alpha\beta}\|_{L^{2d}(V_\alpha \cap V_\beta)} &\leq C\|dh_{\alpha\beta}\|_{W^{1,d}(V_\alpha \cap V_\beta)}. \end{aligned}$$

Similarly to the case $p > d$, we bound the resulting terms,

$$\|\nabla \rho_\alpha\|_{L^{2d}(V_\alpha \cap V_\beta)}.$$

We conclude that (2.27) holds for $p = d$ when η_d is replaced by ζ_p. This completes the proof of Lemma 2.25. □

Lemma 2.26 (Sobolev estimates for transition functions of a principal G-bundle with a $W^{1,q}$ connection)**.** *(See Feehan* [30, Lemma 6.5] *for the case*[10] $p \in [d/2, q]$*.) Let G be a compact Lie group, (X, g) be a compact, smooth Riemannian manifold of dimension $d \geq 2$, and $q > d/2$ and $p \in [1, q]$ be constants. Let A be a $W^{1,q}$ connection on a $W^{2,q}$ principal G-bundle, P, over X and $\mathscr{U} = \{U_\alpha\}_{\alpha \in \mathscr{I}}$ be a finite cover of X by open subsets and $\sigma_\alpha : U_\alpha \to P$ be a set of $W^{2,q}$ local sections such that, for i) $r = d/2$ when $p < d$, or ii) $r = p$ when $p \geq d$,*

$$\|\sigma_\alpha^* A\|_{W^{1,r}(U_\alpha)} \leq C_\alpha \|F_A\|_{L^r(U_\alpha)}, \quad \forall\, \alpha \in \mathscr{I}, \tag{2.28}$$

where the $C_\alpha = C_\alpha(g, G, r, U_\alpha) \in [1, \infty)$ are constants. Let $\{g_{\alpha\beta}\}_{\alpha,\beta \in \mathscr{I}}$ be the corresponding set of $W^{2,q}(U_\alpha \cap U_\beta; G)$ transition functions with respect to the set of local sections, $\{\sigma_\alpha\}_{\alpha \in \mathscr{I}}$, so

$$\sigma_\alpha = \sigma_\beta g_{\beta\alpha} \quad \text{on } U_\alpha \cap U_\beta.$$

If the open subsets, $U_\alpha \subset X$, obey an interior cone condition, then there exists a constant, $C = C(g, G, \max_{\alpha \in \mathscr{I}} C_\alpha, p, \mathscr{U}) \in [1, \infty)$, such that

$$\|\nabla g_{\alpha\beta}\|_{L^p(U_\alpha \cap U_\beta)} \leq C\|F_A\|_{L^p(U_\alpha \cup U_\beta)}, \tag{2.29a}$$

$$\|\nabla^2 g_{\alpha\beta}\|_{L^p(U_\alpha \cap U_\beta)} \leq C\left(1 + \|F_A\|_{L^{\bar{p}}(U_\alpha \cup U_\beta)}\right)\|F_A\|_{L^p(U_\alpha \cup U_\beta)}, \quad \forall\, \alpha, \beta \in \mathscr{I}, \tag{2.29b}$$

where $\bar{p} = d/2$ when $1 \leq p < d$ and $\bar{p} = p$ when $p \geq d$.

[10]In [30, Lemma 6.5], we required the transition functions $g_{\alpha\beta}$ to belong to $W^{2,p}(U_\alpha \cap U_\beta; G)$ for $p > d/2$ for the purpose of regularity but $p = d/2$ is allowed in [30, Inequalities (6.10) and (6.11)].

Proof of Lemma 2.26. By examining the proof of [30, Lemma 6.5], we see that the bounds,

$$\|\nabla g_{\alpha\beta}\|_{L^p(U_\alpha\cap U_\beta)} \leq C\|F_A\|_{L^p(U_\alpha\cap U_\beta)},$$

in (2.29a) hold for any $p \in [1,\infty]$ provided the bounds in (2.28) hold with r replaced by p.

We now indicate the changes to the remainder of the proof of [30, Lemma 6.5] and first consider the case $1 \leq p < d$. The hypothesis $p \geq d/2$ of [30, Lemma 6.5] is only used in the proof of the bounds on $\|\nabla^2 g_{\alpha\beta}\|_{L^p(U_\alpha\cap U_\beta)}$ in (2.29b) when appealing to the Sobolev embedding $W^{1,p}(U) \subset L^{2p}(U)$, for an open subset $U \subset \mathbb{R}^d$ (obeying an interior cone condition).

In the proof of [30, Lemma 6.5], rather than apply the continuous Sobolev multiplication,

$$L^{2p}(U_\alpha \cap U_\beta) \times L^{2p}(U_\alpha \cap U_\beta) \to L^p(U_\alpha \cap U_\beta),$$

we instead use $1/p = 1/p^* + 1/d$, with $p^* = dp/(d-p) \in [d/(d-1),\infty)$, and apply the continuous Sobolev multiplication,

$$L^{p^*}(U_\alpha \cap U_\beta) \times L^d(U_\alpha \cap U_\beta) \to L^p(U_\alpha \cap U_\beta), \quad 1 \leq p < d,$$

and the continuous Sobolev embedding, $W^{1,d/2}(U_\alpha\cap U_\beta) \subset L^d(U_\alpha\cap U_\beta)$ (provided by [2, Theorem 4.12, Part I (C)]), to the resulting terms,

$$\|a_\alpha\|_{L^d(U_\alpha\cap U_\beta)},$$

and the embedding $W^{1,p}(U_\alpha \cap U_\beta) \subset L^{p^*}(U_\alpha \cap U_\beta)$ (again provided by [2, Theorem 4.12, Part I (C)]) to the resulting terms,

$$\|a_\alpha\|_{L^{p^*}(U_\alpha\cap U_\beta)}.$$

We thus find that (2.29b) holds for any $p \in [1, d)$. The case $d \leq p < \infty$ (in fact, $d/2 \leq p < \infty$) is covered by [30, Lemma 6.5]. This completes the proof of Lemma 2.26. □

We can now give the

Proof of Theorem 2.24. We shall verify that the inequality (2.2) holds for $1 < p < d$ by adapting our proof in [30, Section 6.3] of that inequality for $p > 1$ obeying $d/2 \leq p \leq q$ when the hypothesis (2.1) in Theorem 2.1 is replaced by the weaker hypothesis (1.1) in Theorem 2.24. Thus, we let

$$\varrho = \frac{1}{2}\operatorname{Inj}(X, g)$$

and let the finite open cover, $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$, in the hypotheses of Lemmas 2.25 and Lemma 2.26 be defined by geodesic open balls, $U_\alpha := B_\varrho(x_\alpha)$, of radius ϱ and center $x_\alpha \in X$.

Corollary 2.22 produces a set, $\{\varsigma_\alpha\}_{\alpha\in\mathscr{I}}$, of C^∞ local sections and a corresponding set, $\{h_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$, of constant local transition functions obeying

$$\varsigma_\beta = \varsigma_\alpha h_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta, \quad \forall\, \alpha, \beta \in \mathscr{I}.$$

Moreover, the local sections, $\varsigma_\alpha : U_\alpha \to P$, identify the flat connection, Γ on $P \restriction U_\alpha$, with the product connection on $U_\alpha \times G$, and the zero local connection one-forms,

$$b_\alpha := \varsigma_\alpha^*\Gamma \equiv 0 \quad \text{on } U_\alpha, \quad \forall\, \alpha \in \mathscr{I}.$$

For small enough $\varepsilon = \varepsilon(g, G, s_0) \in (0, 1]$, the hypothesis (1.1) ensures that

$$\|F_A\|_{L^{s_0}(U_\alpha)} \leq \|F_A\|_{L^{s_0}(X)} \leq \varepsilon, \quad \forall\, \alpha \in \mathscr{I}, \tag{2.30}$$

where $s_0 = d/2$ when $d \geq 3$ or $s_0 > 1$ when $d = 2$. Corollary 2.22 also produces a set, $\{\rho_\alpha^{-1}\}_{\alpha\in\mathscr{I}}$, of $W^{2,q}$ maps, $\rho_\alpha^{-1} : U_\alpha \to G$, taking the set of $W^{2,q}$ local sections, $\{\varsigma_\alpha\}_{\alpha\in\mathscr{I}}$, of P to a set, $\{\sigma_\alpha\}_{\alpha\in\mathscr{I}}$, of $W^{2,q}$ Coulomb-gauge local sections, $\sigma_\alpha : U_\alpha \to P$, obeying

$$\varsigma_\alpha = \sigma_\alpha\rho_\alpha \quad \text{on } U_\alpha, \quad \forall\, \alpha \in \mathscr{I}, \tag{2.31}$$

and the constant transition functions, $\{h_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$, to transition functions, $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$, obeying

$$h_{\alpha\beta} = \rho_\alpha^{-1} g_{\alpha\beta}\rho_\beta \quad \text{on } U_\alpha \cap U_\beta, \quad \forall\, \alpha, \beta \in \mathscr{I}.$$

Let $\mathscr{U}' = \{U'_\alpha\}_{\alpha\in\mathscr{I}}$ be a finite cover of X by open subsets obeying an interior cone condition such that $U'_\alpha \Subset U_\alpha$. Inequality (2.23c) in Corollary 2.22 provides the bound

$$\max_{\alpha,\beta\in\mathscr{I}} \|g_{\alpha\beta} - h_{\alpha\beta}\|_{W^{2,p_0}(U'_\alpha\cap U'_\beta)} \leq C\varepsilon,$$

for any $p_0 \in (1, d)$ with $p_0 \leq q$ and a constant $C = C(g, G, p_0, \mathscr{U}, \mathscr{U}') \in [1, \infty)$. For small enough $\varepsilon = \varepsilon(g, G, p_0, \mathscr{U}, \mathscr{U}') \in (0, 1]$, the preceding $W^{2,p_0}(U'_\alpha \cap U'_\beta; G)$ bound for $g_{\alpha\beta} - h_{\alpha\beta}$ and the continuous Sobolev embeddings, $W^{2,p_0}(U'_\alpha \cap U'_\beta; \mathbb{R}) \subset C(\bar{U}'_\alpha \cap \bar{U}'_\beta; \mathbb{R})$ from [2, Theorem 4.12, Part I (A)], ensure that

$$\max_{\alpha,\beta\in\mathscr{I}} \|g_{\alpha\beta} - h_{\alpha\beta}\|_{C(\bar{U}'_\alpha\cap\bar{U}'_\beta)} \leq C\varepsilon,$$

for a constant $C = C(g, G, p_0, \mathscr{U}, \mathscr{U}') \in [1, \infty)$. Hence, the hypothesis (2.11) (with $\mathscr{U}$ replaced by $\mathscr{U}'$) of Proposition 2.12 and Lemma 2.25 can be satisfied. We now let $\mathscr{V} = \{V_\alpha\}_{\alpha\in\mathscr{I}}$ be the finite cover of X by open subsets produced by Proposition 2.12 such that $V_\alpha \subset U'_\alpha$ and that obey an interior cone condition.

The following inequalities, for a constant $C = C(g, G, r, \mathscr{U}) \in [1, \infty)$,

$$\|a_\alpha\|_{W^{1,r}(U_\alpha)} \leq C\|F_A\|_{L^r(U_\alpha)}, \quad \forall\, \alpha \in \mathscr{I}, \tag{2.32}$$

are provided by (2.23b) in Corollary 2.22 when $r \in (1, d)$ obeys $r \leq q$. In particular, the inequalities (2.32) ensure that the hypothesis (2.28) of Lemma 2.26 holds for $r = d/2$ and $r = p \in (1, d)$ obeying $p \leq q$. Accordingly, Lemma 2.26 provides the bounds (2.29), that is,

$$\begin{aligned}
\|dg_{\alpha\beta}\|_{W^{1,d/2}(U_\alpha\cap U_\beta)} &\leq C\|F_A\|_{L^{d/2}(U_\alpha\cup U_\beta)},\\
\|dg_{\alpha\beta}\|_{W^{1,p}(U_\alpha\cap U_\beta)} &\leq C\|F_A\|_{L^p(U_\alpha\cup U_\beta)}, \quad \forall\, \alpha, \beta \in \mathscr{I},
\end{aligned}$$

noting that $\bar{p} = d/2$ in the conclusion of Lemma 2.26 and $\|F_A\|_{L^{d/2}(U_\alpha\cup U_\beta)} \leq \varepsilon$ by (2.30).

The preceding $W^{1,p}$ bounds for $dg_{\alpha\beta}$ and the fact that $dh_{\alpha\beta} = 0$ on $U_\alpha \cap U_\beta$ imply that the hypotheses (2.24) and (2.26b) of Lemma 2.25 are satisfied with $\eta_p = \zeta_p = C\|F_A\|_{L^p(U_\alpha\cap U_\beta)}$. Similarly, the preceding $W^{1,d/2}$ bounds for $dg_{\alpha\beta}$, the fact that $dh_{\alpha\beta} = 0$ on $U_\alpha \cap U_\beta$, and the continuous Sobolev embedding, $W^{1,d/2}(U_\alpha\cap U_\beta) \subset L^d(U_\alpha\cap U_\beta)$, imply that the hypothesis (2.26a) of Lemma 2.25 is satisfied with $\eta_d = C\|F_A\|_{L^{d/2}(U_\alpha\cap U_\beta)} = C\varepsilon$.

The local connection one-forms,

$$a^0_\alpha := \varsigma^*_\alpha A = \varsigma^*_\alpha(A - \Gamma) \quad \text{and} \quad a_\alpha := \sigma^*_\alpha A \quad \text{on } U_\alpha,$$

are related through (2.31) by

$$a^0_\alpha = \rho_\alpha^{-1} a_\alpha \rho_\alpha + \rho_\alpha^{-1} d\rho_\alpha \quad \text{a.e. on } U_\alpha. \tag{2.33}$$

The estimate (2.25) (with $r = p$) in Lemma 2.25, the inequalities (2.32) (with $r = p$), and the pointwise identity (2.33) imply that

$$\|a^0_\alpha\|_{L^p(V_\alpha)} \leq C\|F_A\|_{L^p(U_\alpha)}, \quad \forall\, \alpha \in \mathscr{I}.$$

Taking the covariant derivative of the pointwise identity (2.33) yields

$$\nabla a_\alpha^0 = -\rho_\alpha^{-1}(\nabla\rho_\alpha)\rho_\alpha^{-1}\otimes a_\alpha\rho_\alpha + \rho_\alpha^{-1}(\nabla a_\alpha)\rho_\alpha + \rho_\alpha^{-1}a_\alpha\otimes\nabla\rho_\alpha \\ - \rho_\alpha^{-1}(\nabla\rho_\alpha)\rho_\alpha^{-1}\otimes\nabla\rho_\alpha + \rho_\alpha^{-1}\nabla^2\rho_\alpha \quad \text{a.e. on } U_\alpha.$$

The estimate (2.25) (with $r = p$ and d) for $\|\nabla\rho\|_{L^p(V_\alpha)}$ and estimate (2.27) for $\|\nabla^2\rho\|_{L^p(V_\alpha)}$ provided by Lemma 2.25, the estimates (2.32) (with $r = p$ and $r = d/2$) for $\|a_\alpha\|_{W^{1,p}(U_\alpha)}$, the continuous Sobolev multiplication, $L^{p^*}(U_\alpha)\times L^d(U_\alpha)\to L^p(U_\alpha)$ for $p^* = dp/(d-p)$, the continuous Sobolev embeddings, $W^{1,p}(U_\alpha)\subset L^{p^*}(U_\alpha)$ and $W^{1,d/2}(U_\alpha)\subset L^d(U_\alpha)$ from [2, Theorem 4.12, Part I (C)], and the preceding pointwise identity imply that

$$\|\nabla a_\alpha^0\|_{L^p(V_\alpha)} \leq C\|F_A\|_{L^p(U_\alpha)}, \quad \forall\,\alpha\in\mathscr{I},$$

for a constant $C = C(g,G,p,\mathscr{U})\in[1,\infty)$.

Combining the preceding $L^p(V_\alpha)$ estimates for $a_\alpha^0 = \varsigma_\alpha^*(A-\Gamma)$ and ∇a_α^0 yields

$$\|A-\Gamma\|_{W_\Gamma^{1,p}(V_\alpha)} \leq C\|F_A\|_{L^p(U_\alpha)}, \quad \forall\,\alpha\in\mathscr{I},$$

for a constant $C = C(g,G,p,\mathscr{U})\in[1,\infty)$. Combining the preceding $L^p(V_\alpha)$ estimates for $a_\alpha^0 = \varsigma_\alpha^*(A-\Gamma)$ and ∇a_α^0 yields the global $W_\Gamma^{1,p}(X)$ estimate,

$$\|A-\Gamma\|_{W_\Gamma^{1,p}(X)} \leq C\|F_A\|_{L^p(X)},$$

and this is the estimate (2.2). This completes the proof of Theorem 2.24. □

Remark 2.27 (An extension of Theorem 2.24 to the cases $p = d$ and $p > d$). We first note that when $r \geq d$, the inequalities (2.32) are implied by Corollary 2.11, provided F_A now obeys the stronger bound (2.8).

For $p = d$, we proceed exactly as for the case $1 < p < d$ in the proof of Theorem 2.24 but use the continuous Sobolev multiplication, $L^{2p}(U_\alpha)\times L^{2p}(U_\alpha)\to L^p(U_\alpha)$ and the continuous Sobolev embedding, $W^{1,p}(U_\alpha)\subset L^t(U_\alpha)$ from [2, Theorem 4.12, Part I (B)] with $1\leq t<\infty$ for $p = d$.

For $p > d$, we proceed as for the case $1 < p < d$ but use the continuous Sobolev multiplication, $L^p(U_\alpha)\times L^\infty(U_\alpha)\to L^p(U_\alpha)$ and the continuous Sobolev embedding, $W^{1,p}(U_\alpha)\subset L^\infty(U_\alpha)$ from [2, Theorem 4.12, Part I (A)].

2.7. **Existence of $W^{2,q}$ Coulomb gauge transformations for $W^{1,q}$ connections that are L^d close to a reference connection.** In order to prove the forthcoming Proposition 2.29, we shall need a stronger version of the slice theorem for the action of the group of gauge transformations, going beyond the usual statements found in standard references such as Donaldson and Kronheimer [23] or Freed and Uhlenbeck [33] and proved by applying the Implicit Function Theorem. We thus recall a slight enhancement of [32, Theorem 9].

Theorem 2.28 (Existence of $W^{2,q}$ Coulomb gauge transformations for $W^{1,q}$ connections that are L^d close to a reference connection)**.** *(See Feehan and Maridakis* [32, Theorem 9]*.) Let (X,g) be a closed, smooth Riemannian manifold of dimension $d\geq 2$, and G be a compact Lie group, and P be a smooth principal G-bundle over X and $r_0 > 2$ be a constant. If A_1 is a C^∞ connection on P, and A_0 is a $W^{1,q}$ Sobolev connection on P, with $d/2 < q < \infty$ and $p\in(1,\infty)$ a constant obeying $p\leq q$, then there exists a constant $\zeta = \zeta(A_0,A_1,g,G,p,q,r_0)\in(0,1]$ with the following* *significance. If A is a $W^{1,q}$ connection on P that obeys*[11]

$$\|A - A_0\|_{L^{r_0}(X)} < \zeta, \tag{2.34a}$$

$$\|A - A_0\|_{W^{1,p}_{A_1}(X)} \leq M, \tag{2.34b}$$

for some constant $M \in [1, \infty)$ and $r_0 = d$ when $d \geq 3$ or $r_0 > 2$ when $d = 2$, then there exists a $W^{2,q}$ gauge transformation $u \in \operatorname{Aut}(P)$ such that

$$d^*_{A_0}(u(A) - A_0) = 0,$$

and

$$\|u(A) - A_0\|_{W^{1,p}_{A_1}(X)} < N\|A - A_0\|_{W^{1,p}_{A_1}(X)},$$

where $N = N(A_0, A_1, g, G, M, p, q, r_0) \in [1, \infty)$ is a constant.

The essential point in Theorem 2.28 is that the result holds for the critical exponent, $p = d/2$ with $d \geq 3$, when the Sobolev space $W^{2,p}(X)$ fails to embed in $C(X)$ (see [2, Theorem 4.12]) and a proof of Theorem 2.28 by the Implicit Function Theorem in the case $p > d/2$ fails when $p = d/2$. In this situation, a $W^{2,\frac{d}{2}}$ gauge transformation u of P is not continuous, the set $\operatorname{Aut}^{2,\frac{d}{2}}(P)$ of $W^{2,\frac{d}{2}}$ gauge transformations is not a manifold, and $\operatorname{Aut}^{2,\frac{d}{2}}(P)$ cannot act smoothly on the affine space $\mathscr{A}^{1,\frac{d}{2}}(P)$ of $W^{1,\frac{d}{2}}$ connections on P. When $d = 4$ and $p \geq 2$, this phenomenon is discussed by Freed and Uhlenbeck in [33, Appendix A].

The version of Theorem 2.28 that we state here enhances [32, Theorem 9] by relaxing its hypothesis [32, Equation (1.26)], namely,

$$\|A - A_0\|_{W^{1,p}_{A_1}(X)} < \zeta,$$

to that of (2.34), and allowing $p < d/2$ when $d \geq 3$. Fortunately, the required modifications to the proof are small and we indicate them below.

Proof of Theorem 2.28. First, the restriction $p \geq d/2$ in the hypothesis [32, Proposition 2.3] was included only for the sake of consistency with the remainder of that article and can be omitted.

Second, the restriction $p \geq d/2$ in the hypothesis of [32, Proposition 2.1] was used in the proof of that result but can be omitted by making use of more refined Sobolev multiplication and embedding results when $d \geq 3$ and $1 < p < d/2$. We begin by replacing the final inequality in [32, p. 21] by

$$\begin{aligned}\|(\Delta_A - \Delta_{A_s})\xi\|_{L^p(X)} \leq z\|\nabla_{A_1} a\|_{L^{d/2}(X)}\|\xi\|_{L^{p^{**}}(X)} + \|a \times \nabla_{A_1}\xi\|_{L^p(X)} \\ + z\|a_1\|_{C(X)}\|a\|_{L^d(X)}\|\xi\|_{L^{p^*}(X)} + z\||a|^2\|_{L^{d/2}(X)}\|\xi\|_{L^{p^{**}}(X)},\end{aligned}$$

where $p^* = dp/(d-p) \in (d/(d-1), d)$ and $1/p = 1/d + 1/p^*$, giving a continuous Sobolev multiplication map, $L^d(X;\mathbb{R}) \times L^{p^*}(X;\mathbb{R}) \to L^p(X;\mathbb{R})$, and continuous embedding, $W^{1,p}(X;\mathbb{R}) \subset L^{p^*}(X;\mathbb{R})$; also $1/p = 1/(d/2) + 1/p^{**}$, where $p^{**} = dp/(d-2p) \in (d/(d-2), \infty)$, giving a continuous Sobolev multiplication map, $L^{d/2}(X;\mathbb{R}) \times L^{p^{**}}(X;\mathbb{R}) \to L^p(X;\mathbb{R})$, and continuous embedding, $W^{2,p}(X;\mathbb{R}) \subset L^{p^{**}}(X;\mathbb{R})$. (We appeal here to [2, Theorem 4.12, Part I (C)].) Noting

[11]The hypothesis (2.34) is milder than that of the original statement of [32, Theorem 9].

that $W^{1,q}(X;\mathbb{R}) \subset L^d(X;\mathbb{R})$ for $q > d/2$ yields the first inequality in [32, p. 21],

$$\|(\Delta_A - \Delta_{A_s})\xi\|_{L^p(X)} \leq z\left(\|\nabla_{A_1} a\|_{L^q(X)} + \|a\|^2_{W^{1,q}_{A_1}(X)}\right)\|\xi\|_{W^{2,p}_{A_1}(X)} \\ + z\|a_1\|_{C(X)}\|a\|_{W^{1,q}_{A_1}(X)}\|\xi\|_{W^{1,p}_{A_1}(X)} + \|a \times \nabla_{A_1}\xi\|_{L^p(X)}.$$

Similarly, the last term in the preceding inequality can be estimated by

$$\|a \times \nabla_{A_1}\xi\|_{L^p(X)} \leq z\|a\|_{L^d(X)}\|\nabla_{A_1}\xi\|_{L^{p^*}(X)} \leq z\|a\|_{W^{1,d/2}(X)}\|\nabla_{A_1}\xi\|_{W^{1,p}(X)} \\ \leq z\|a\|_{W^{1,q}(X)}\|\xi\|_{W^{2,p}_{A_1}(X)},$$

replacing [32, Equation (2.9)]. The remainder of the proof of [32, Proposition 2.1] is unchanged.

Third, the restriction $p \geq d/2$ in the hypothesis of [32, Corollary 2.5] was also used in the proof of that result but again can be omitted by making use of different argument. We instead observe that one can use a standard method, exactly analogous to the proof of [35, Lemma 9.17], to eliminate the usual term $\|a\|_{L^p(X)}$ from the right-hand side of the *a priori* estimate below,

$$\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C\|\Delta_A\xi\|_{L^p(X)}, \quad \forall\, \xi \in (\operatorname{Ker}\Delta_A)^\perp \cap W^{2,p}_{A_1}(X;\Lambda^l \otimes \mathrm{ad}P), \tag{2.35}$$

and this is [32, Inequality (2.5)], the conclusion of [32, Corollary 2.5].

Fourth, we observe that the hypothesis [32, Equation (2.14)] in [32, Lemma 2.8] for $a \in W^{1,q}(X;\Lambda^1 \otimes \mathrm{ad}P)$ may be relaxed, in the case $d = 2$, to

$$\|a\|_{L^{r_0}(X)} < \delta,$$

for $r_0 \in (2,\infty)$ (typically close to 2). We then replace the continuous Sobolev multiplication, $L^4(X;\mathbb{R}) \times L^4(X;\mathbb{R}) \to L^2(X;\mathbb{R})$, in the final displayed series of inequalities in [32, p. 28] by $L^{r_0}(X;\mathbb{R}) \times L^t(X;\mathbb{R}) \to L^2(X;\mathbb{R})$, where $t \in (2,\infty)$ (typically large) is defined by $1/2 = 1/r_0 + 1/t$ and use the continuous Sobolev embedding, $W^{1,2}(X;\mathbb{R}) \times L^t(X;\mathbb{R})$, provided by [2, Theorem 4.12, Part I (B)] when $d = 2$.

Fifth, we observe that the statement of [32, Proposition 2.11] extends without change to include the case $1 < p < d/2$ (when $d \geq 3$) with only minor changes to its proof. Indeed, recall from [2, Theorem 4.12, Part I] that the stated Sobolev embedding [32, Equation (2.30)] also holds with p replaced by q for the indicated three cases. Writing $1/p = 1/p^* + 1/d$, for $p^* = dp/(d-p)$, there is a continuous Sobolev multiplication map,

$$L^{p^*}(X;\mathbb{R}) \times L^d(X;\mathbb{R}) \to L^p(X;\mathbb{R}),$$

and hence, the final term in the displayed series of inequalities in [32, Equation p. 31] can be estimated instead by

$$\begin{aligned}\|a_0 \times \nabla_{A_1} u_0\|_{L^p(X)} &\leq z\|a_0\|_{L^d(X)}\|\nabla_{A_1} u_0\|_{L^{p^*}(X)} \\ &\leq zC_0^2\|a_0\|_{W^{1,d/2}_{A_1}(X)}\|\nabla_{A_1} u_0\|_{W^{1,p}_{A_1}(X)} \\ &\leq zC_0^2\|a_0\|_{W^{1,q}_{A_1}(X)}\|u_0\|_{W^{2,p}_{A_1}(X)}.\end{aligned}$$

The remaining restrictions to $p \geq d/2$ in the proof of [32, Proposition 2.11] were made only for the sake of consistency and can be omitted.

Sixth, we indicate the changes to the proof of [32, Theorem 9] required to include the case $1 \geq p < d/2$ (when $d \geq 3$). Recall that S in the beginning of the proof of [32, Theorem 9] was defined to be the set of $t \in [0,1]$ such that there exists a $W^{2,q}$ gauge transformation $u_t \in \operatorname{Aut}(P)$ with the property that

$$d_{A_0}^*(u_t(A_t) - A_0) = 0 \quad \text{and} \quad \|u_t(A_t) - A_0\|_{W_{A_1}^{1,p}(X)} < 2N\|A_t - A_0\|_{W_{A_1}^{1,p}(X)}.$$

In the beginning of Step 1 in the proof of [32, Theorem 9], we can apply [32, Proposition 2.11] to ensure that

$$\|A_{t_0} - A_0\|_{W_{A_1}^{1,p}(X)} < \zeta,$$

and so hypothesis [32, Equation (1.26)] of [32, Theorem 9] holds for A replaced by A_{t_0}. In the second last paragraph of [32, p. 38], the need to appeal to the Sobolev Embedding Theorem for $W^{1,p}(X;\mathbb{R})$ when $d \geq 3$ or $d = 2$ is eliminated because we now use the hypothesis (2.34) in Theorem 2.28 (rather than the hypothesis [32, Equation (1.26)] of [32, Theorem 9]) to directly ensure that [32, Lemma 2.8] may be applied. (And in that application, we require only that $\|a\|_{L^r(X)} < 2C_1N\zeta$ rather than $\|a\|_{L^4(X)} < 2C_1N\zeta$.) The remainder of the proof of [32, Theorem 9] is unchanged. This concludes the proof of Theorem 2.28. □

2.8. **Existence of a global $W^{2,q}$ Coulomb gauge transformation for the critical exponent.** In this subsection, we will establish the existence of a $W^{2,q}$ gauge transformation $u \in \operatorname{Aut}(P)$ bringing A into Coulomb gauge relative to Γ when A has $L^{d/2}(X)$-small curvature. Specifically, we will prove the following extension of Item (3) in Theorem 2.1 and thus establish Item (3) in Theorem 1 by choosing $K = 1$ in Proposition 2.29.

Proposition 2.29 (Existence of a global $W^{2,q}$ gauge transformation bringing a connection A with $L^{d/2}(X)$-small curvature into Coulomb gauge relative to Γ)**.** *Assume the hypotheses of Theorem 2.24 and, in addition, that $\|F_A\|_{L^p(X)} \leq K$ for a constant $K \in [1,\infty)$. Then there is a $W^{2,q}$ gauge transformation, $u \in \operatorname{Aut}(P)$, such that $u(A)$ is in Coulomb gauge with respect to Γ, so* (2.3) *holds, and obeys the bound* (2.4) *with constant $C_1 = C_1(g,G,K,p,q,s_0)$.*

Proof. Theorem 2.24 provides the bound, with $C_p = C_p(g,G,p,s_0) \in [1,\infty)$,

$$\|A - \Gamma\|_{W_\Gamma^{1,p}(X)} \leq C_p\|F_A\|_{L^p(X)},$$

for $p \in (1,q]$ obeying $p < d$ and, in particular when $d \geq 3$ and $p = d/2$,

$$\|A - \Gamma\|_{W_\Gamma^{1,d/2}(X)} \leq C_d\|F_A\|_{L^{d/2}(X)},$$

and when $d = 2$, we recall that we restrict $p > 1$. Moreover, the Sobolev Embedding Theorem [2, Theorem 4.12, Part I (C)] implies that

$$\|A - \Gamma\|_{L^d(X)} \leq \kappa_d\|A - \Gamma\|_{W_\Gamma^{1,d/2}(X)},$$

which we apply when $d \geq 3$ only, and

$$\|A - \Gamma\|_{L^{p^*}(X)} \leq \kappa_p\|A - \Gamma\|_{W_\Gamma^{1,p}(X)},$$

where $p^* = 2p/(2-p) > 2$ and $1 < p < 2$, which we apply when $d = 2$ only, for a constant $\kappa_p = \kappa_p(g,G,p) \in [1,\infty)$. By hypothesis (1.1), we have

$$\|F_A\|_{L^{d/2}(X)} \leq \varepsilon \quad \text{for } d \geq 3 \quad \text{and} \quad \|F_A\|_{L^{s_0}(X)} \leq \varepsilon \quad \text{for } d = 2.$$

For small enough $\varepsilon = \varepsilon(g,G,p,q,s_0) \in (0,1]$, we can therefore apply Theorem 2.28 (with $A_1 = A_0 = \Gamma$ and $\zeta = \kappa_p C_p\varepsilon$ and $M = C_pK$) to find the desired $W^{2,q}$ gauge transformation, $u \in \operatorname{Aut}(P)$, such that $d_\Gamma^*(u(A) - \Gamma) = 0$ and

$$\|u(A) - \Gamma\|_{W_\Gamma^{1,p}(X)} \leq C\|A - \Gamma\|_{W_\Gamma^{1,p}(X)}.$$

Combining the preceding two inequalities yields (2.4) with the indicated constant. □

3. Optimal Łojasiewicz–Simon inequalities for the Yang–Mills energy function near the critical variety of flat connections

Our goal in this section is to complete the

Proof of Theorem 2. According to Theorem 1, there is a C^∞ flat connection, Γ on P, and, for $p \in (1,\infty)$ obeying $p \le q$, a constant, $C_p = C_p(g, G, p, s_0) \in [1,\infty)$, such that

$$\|A - \Gamma\|_{W^{1,p}_\Gamma(X)} \le C_p \|F_A\|_{L^p(X)}. \tag{3.1}$$

Choosing $p = 2$ in (3.1) gives the optimal Łojasiewicz–Simon distance inequality (1.12).

We now prove the optimal Łojasiewicz–Simon gradient inequality (1.13). We may assume that $A \neq \Gamma$ without loss of generality. Write $a := A - \Gamma \in W^{1,q}_\Gamma(X; \Lambda^1 \otimes \mathrm{ad}P)$ and note that

$$\left(W^{1,2}_\Gamma(X; \Lambda^1 \otimes \mathrm{ad}P)\right)^* = W^{-1,2}_\Gamma(X; \Lambda^1 \otimes \mathrm{ad}P)$$

is the continuous dual space of the Hilbert space, $W^{1,2}_\Gamma(X; \Lambda^1 \otimes \mathrm{ad}P)$. We have

$$d_A a = d_\Gamma a + 2a \wedge a = F_A + a \wedge a,$$

using (1.8), since $F_A = F_{\Gamma + a} = F_\Gamma + d_\Gamma a + a \wedge a = d_\Gamma a + a \wedge a$ by (1.6). Thus, we obtain

$$\begin{aligned}
\|d_A^* F_A\|_{W^{-1,2}_\Gamma(X)} &= \sup_{b \in W^{1,2}_\Gamma(X;\Lambda^1 \otimes \mathrm{ad}P)\setminus\{0\}} \frac{(d_A^* F_A, b)_{L^2(X)}}{\|b\|_{W^{1,2}_\Gamma(X)}} \\
&\ge \frac{(d_A^* F_A, a)_{L^2(X)}}{\|a\|_{W^{1,2}_\Gamma(X)}} = \frac{(F_A, d_A a)_{L^2(X)}}{\|a\|_{W^{1,2}_\Gamma(X)}} = \frac{(F_A, F_A + a \wedge a)_{L^2(X)}}{\|a\|_{W^{1,2}_\Gamma(X)}},
\end{aligned}$$

and therefore,

$$\|d_A^* F_A\|_{W^{-1,2}_\Gamma(X)} \ge \frac{\|F_A\|^2_{L^2(X)}}{\|a\|_{W^{1,2}_\Gamma(X)}} + \frac{(F_A, a \wedge a)_{L^2(X)}}{\|a\|_{W^{1,2}_\Gamma(X)}}. \tag{3.2}$$

We recall that [2, Theorem 4.12, Part I (B) and (C)] provides a continuous embedding of Sobolev spaces with norm $\kappa_r = \kappa_r(g) \in [1,\infty)$,

$$W^{1,2}(X) \subset L^r(X) \quad \text{for } \begin{cases} 1 \le r < \infty, & \text{if } d = 2, \\ 1 \le r \le 2^* = 2d/(d-2), & \text{if } d > 2, \end{cases}$$

and a continuous embedding, $W^{1,d/2}(X) \subset L^d(X)$, for all $d \ge 2$. When $d > 2$, we claim that

$$\|a \wedge a\|_{L^2(X)} \le c\kappa_r \kappa_d \|a\|_{W^{1,2}_\Gamma(X)} \|a\|_{W^{1,d/2}_\Gamma(X)}, \tag{3.3}$$

for a constant $c = c(d, G) \in [1,\infty)$.

The proof of (3.3) is straightforward. Indeed, writing $1/2 = 1/r + 1/d$ (for $d > 2$ and $r = 2^* = 2d/(d-2) \in (2,\infty)$), we have

$$\|a \wedge a\|_{L^2(X)} \le c\|a\|_{L^r(X)} \|a\|_{L^d(X)},$$

for a constant $c = c(d, G) \in [1,\infty)$. Combining the preceding inequality with the continuous embeddings, $W^{1,2}(X) \subset L^r(X)$, when $r = 2^* = 2d/(d-2)$, and $W^{1,d/2}(X) \subset L^d(X)$ yields (3.3).

The gradient inequality (1.13) now follows for all $d \geq 2$. Indeed, for $d > 2$,

$$\begin{aligned}
\|d_A^* F_A\|_{W_\Gamma^{-1,2}(X)} &\geq \frac{\|F_A\|_{L^2(X)}^2}{\|a\|_{W_\Gamma^{1,2}(X)}} - \frac{\|F_A\|_{L^2(X)}\|a \wedge a\|_{L^2(X)}}{\|a\|_{W_\Gamma^{1,2}(X)}} \quad \text{(by (3.2))} \\
&\geq \frac{\|F_A\|_{L^2(X)}^2}{\|a\|_{W_\Gamma^{1,2}(X)}} - c\kappa_r\kappa_d \frac{\|F_A\|_{L^2(X)}\|a\|_{W_\Gamma^{1,2}(X)}\|a\|_{W_\Gamma^{1,d/2}(X)}}{\|a\|_{W_\Gamma^{1,2}(X)}} \quad \text{(by (3.3))} \\
&= \frac{\|F_A\|_{L^2(X)}^2}{\|a\|_{W_\Gamma^{1,2}(X)}} - c\kappa_r\kappa_d \|F_A\|_{L^2(X)}\|a\|_{W_\Gamma^{1,d/2}(X)} \\
&\geq C_2^{-1}\|F_A\|_{L^2(X)} - c\kappa_r\kappa_d C_{d/2}\|F_A\|_{L^2(X)}\|F_A\|_{L^{d/2}(X)} \quad \text{(by (3.1))} \\
&\geq (C_2^{-1} - c\kappa_r\kappa_d C_{d/2}\varepsilon)\|F_A\|_{L^2(X)} \quad \text{(by (1.1))}.
\end{aligned}$$

Now choose $\varepsilon = \frac{1}{2}C_2^{-1}/(c\kappa_r\kappa_d C_{d/2})$ to give (1.13) for $d > 2$, noting that because $M_0(P)$ is compact, the explicit dependence of the $W^{-1,2}$ norm on Γ may be dropped.

For $d = 2$ and $s_0 \in (1,2)$ (we may assume $s_0 < 2$ without loss of generality), we can instead use $1/2 = 1/r + 1/s_0^*$, where $s_0^* := 2s_0/(2-s_0) \in (2,\infty)$ and $r \in (2,\infty)$, to give $\|a \wedge a\|_{L^2(X)} \leq c\|a\|_{L^r(X)}\|a\|_{L^{s_0^*}(X)}$ and continuous Sobolev embeddings, $W^{1,s_0}(X;\mathbb{R}) \subset L^{s_0^*}(X;\mathbb{R})$ and $W^{1,2}(X;\mathbb{R}) \subset L^r(X;\mathbb{R})$. Now arguing exactly as in the calculation for $d > 2$ gives (1.13) for $d = 2$. This finishes the proof of Theorem 2. □

4. Łojasiewicz–Simon gradient inequalities for Morse–Bott functions

Our goal in this section is to give the

Proof of Theorem 3. We begin with four reductions that simplify the proof. First, observe that if $\mathscr{E}_0 : \mathscr{U} \to \mathbb{R}$ is defined by $\mathscr{E}_0(x) := \mathscr{E}(x + x_\infty)$, then $\mathscr{E}_0'(0) = 0$, so we may assume without loss of generality that $x_\infty = 0$ and relabel $\mathscr{E}_0$ as $\mathscr{E}$.

Second, recall that by hypothesis, $\mathscr{X} = \mathscr{X}_0 \oplus K$ (a direct sum of Banach spaces), where $\mathscr{X}_0 \subset \mathscr{X}$ is a closed subspace (a Banach space) complementing $K = \operatorname{Ker}\mathscr{E}''(0) = \operatorname{Ker}\mathscr{M}'(0)$. Hence, by applying a C^2 diffeomorphism to a neighborhood of the origin in $\mathscr{X}$ and possibly shrinking $\mathscr{U}$, we may assume without loss of generality that $\mathscr{U} \cap \operatorname{Crit}\mathscr{E} = \mathscr{U} \cap K$, recalling that $K = T_{x_\infty}\operatorname{Crit}\mathscr{E}$ by hypothesis that $\mathscr{E}$ is Morse–Bott at x_∞.

Third, if $x \in \mathscr{U} \cap K$, then $\mathscr{E}(x) = \mathscr{E}(0)$, and we may restrict our attention to $x \in \mathscr{U} \cap \mathscr{X}_0$ without loss of generality in the remainder of the proof. To see this, observe that if $x = x_0 + k \in \mathscr{U} \cap (\mathscr{X}_0 \oplus K)$ then $\mathscr{E}(k) = \mathscr{E}(0)$ (because $\mathscr{E}$ is constant along $\mathscr{U} \cap K$) and

$$\|\mathscr{M}(x)\|_{\mathscr{H}} \geq Z|\mathscr{E}(x) - \mathscr{E}(0)|^{1/2} \iff \|\mathscr{M}(x)\|_{\mathscr{H}} \geq Z|\mathscr{E}(x) - \mathscr{E}(k)|^{1/2}.$$

If we define $\bar{\mathscr{E}}(x_0) := \mathscr{E}(x_0 + k)$ and thus $\bar{\mathscr{M}}(x_0) = \mathscr{M}(x_0 + k)$ for $x = x_0 + k \in \mathscr{U}$, then

$$\|\mathscr{M}(x)\|_{\mathscr{H}} \geq Z|\mathscr{E}(x) - \mathscr{E}(k)|^{1/2} \iff \|\bar{\mathscr{M}}(x_0)\|_{\mathscr{H}} \geq Z|\bar{\mathscr{E}}(x_0) - \bar{\mathscr{E}}(0)|^{1/2}.$$

We now relabel $\bar{\mathscr{E}}$ and $\bar{\mathscr{M}}$ as $\mathscr{E}$ and $\mathscr{M}$ respectively.

Fourth, observe that if $\mathscr{E}_0 : \mathscr{U} \to \mathbb{R}$ is defined by $\mathscr{E}_0(x) := \mathscr{E}(x) - \mathscr{E}(0)$, then $\mathscr{E}_0(0) = 0$, so we may once again relabel $\mathscr{E}_0$ as $\mathscr{E}$ and assume without loss of generality that $\mathscr{E}(0) = 0$.

By hypothesis, $\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{K}$ (again a direct sum of Banach spaces), where $\mathscr{K} = \operatorname{Ker}\mathscr{M}_1(0)$ has closed complement $\mathscr{G}_0$ (a Banach space), and $\mathscr{H}_0 = \operatorname{Ran}\mathscr{M}_1(0) \subset \mathscr{H}$ is a closed subspace (a Banach space). Hence, the bounded operator, $\mathscr{M}_1(0) : \mathscr{G}_0 \to \mathscr{H}_0$, is bijective and thus invertible by the Open Mapping Theorem. Note that $K \subset \mathscr{K}$ by definition of $\mathscr{M}_1(0)$ and $\mathscr{X}_0 \subset \mathscr{G}_0$ by hypothesis.

By the Mean Value Theorem and the hypothesis that $\mathscr{M} : \mathscr{U} \to \mathscr{Y}$ is C^1, we have

$$\begin{aligned}\mathscr{M}(x) &= \int_0^1 \mathscr{M}'(tx)x\,dt \\ &= \mathscr{M}'(0)x + \int_0^1 (\mathscr{M}'(tx) - \mathscr{M}'(0))x\,dt \\ &= \mathscr{M}'(0)x + \int_0^1 (\mathscr{M}_1(tx) - \mathscr{M}_1(0))x\,dt.\end{aligned}$$

Thus,

$$\|\mathscr{M}(x)\|_{\mathscr{H}} \geq \|\mathscr{M}'(0)x\|_{\mathscr{H}} - \max_{t\in[0,1]} \|(\mathscr{M}_1(tx) - \mathscr{M}_1(0))x\|_{\mathscr{H}}.$$

Because $x \in \mathscr{X}_0 \subset \mathscr{G}_0$ and $\mathscr{M}_1(0) : \mathscr{G}_0 \to \mathscr{H}_0$ is invertible, we have

$$\|x\|_{\mathscr{G}} = \|x\|_{\mathscr{G}_0} = \|\mathscr{M}_1(0)^{-1}\mathscr{M}_1(0)x\|_{\mathscr{G}_0} \leq \|\mathscr{M}_1(0)^{-1}\|_{\mathscr{L}(\mathscr{H}_0,\mathscr{G}_0)}\|\mathscr{M}_1(0)x\|_{\mathscr{H}_0}.$$

Therefore,

$$\|\mathscr{M}_1(0)x\|_{\mathscr{H}} = \|\mathscr{M}_1(0)x\|_{\mathscr{H}_0} \geq \frac{\|x\|_{\mathscr{G}}}{\|\mathscr{M}_1(0)^{-1}\|_{\mathscr{L}(\mathscr{H}_0,\mathscr{G}_0)}} =: 2C_0\|x\|_{\mathscr{G}}.$$

On the other hand, given $\varepsilon \in (0,1]$,

$$\begin{aligned}\max_{t\in[0,1]} \|(\mathscr{M}_1(tx) - \mathscr{M}_1(0))x\|_{\mathscr{H}} &\leq \max_{t\in[0,1]} \|\mathscr{M}_1(tx) - \mathscr{M}_1(0)\|_{\mathscr{L}(\mathscr{G},\mathscr{H})}\|x\|_{\mathscr{G}} \\ &\leq \varepsilon\|x\|_{\mathscr{G}},\end{aligned}$$

for $\|x\|_{\mathscr{X}} \leq \delta = \delta(\varepsilon) \in (0,1]$, where the final inequality follows by the hypothesis of continuity of $\mathscr{M}_1(x) \in \mathscr{L}(\mathscr{G},\mathscr{H})$ with respect to $x \in \mathscr{U}$. Consequently, choosing $\varepsilon \leq C_0$ yields

$$\|\mathscr{M}(x)\|_{\mathscr{H}} \geq C_0\|x\|_{\mathscr{G}}, \quad \forall\, x \in \mathscr{X}_0 \text{ such that } \|x\|_{\mathscr{X}} \leq \delta. \tag{4.1}$$

In the other direction,

$$\mathscr{E}(x) = \int_0^1 \mathscr{E}''(tx)x^2\,dt = \mathscr{E}''(0)x^2 + \int_0^1 (\mathscr{E}''(tx) - \mathscr{E}''(0))x^2\,dt.$$

Now, $\mathscr{E}''(0)x^2 = \langle x, \mathscr{M}'(0)x\rangle_{\mathscr{X}\times\mathscr{X}^*} = \langle x, \mathscr{M}_1(0)x\rangle_{\mathscr{G}\times\mathscr{G}^*}$ (using the continuous embeddings, $\mathscr{X} \subset \mathscr{G}$ and $\mathscr{H} \subset \mathscr{G}^*$, the latter with norm $\kappa \in [1,\infty)$). Therefore,

$$\begin{aligned}|\mathscr{E}''(0)x^2| &= |\langle x, \mathscr{M}_1(0)x\rangle_{\mathscr{G}\times\mathscr{G}^*}| \\ &\leq \|x\|_{\mathscr{G}}\|\mathscr{M}_1(0)x\|_{\mathscr{G}^*} \leq \kappa\|x\|_{\mathscr{G}}\|\mathscr{M}_1(0)x\|_{\mathscr{H}} \\ &\leq \kappa\|\mathscr{M}_1(0)\|_{\mathscr{L}(\mathscr{G},\mathscr{H})}\|x\|_{\mathscr{G}}^2 =: \frac{1}{2}C_1\|x\|_{\mathscr{G}}^2.\end{aligned}$$

Similarly, $\mathscr{E}''(tx)x^2 = \langle x, \mathscr{M}'(tx)x\rangle_{\mathscr{X}\times\mathscr{X}^*} = \langle x, \mathscr{M}_1(tx)x\rangle_{\mathscr{G}\times\mathscr{G}^*}$ and

$$\begin{aligned}\left|\int_0^1 (\mathscr{E}''(tx) - \mathscr{E}''(0))x^2\,dt\right| &= \left|\int_0^1 \langle x, (\mathscr{M}_1(tx) - \mathscr{M}_1(0))x\rangle_{\mathscr{G}\times\mathscr{G}^*}\,dt\right| \\ &\leq \|x\|_{\mathscr{G}} \max_{t\in[0,1]} \|(\mathscr{M}_1(tx) - \mathscr{M}_1(0))x\|_{\mathscr{G}^*} \\ &\leq \kappa\|x\|_{\mathscr{G}} \max_{t\in[0,1]} \|(\mathscr{M}_1(tx) - \mathscr{M}_1(0))x\|_{\mathscr{H}} \\ &\leq \kappa\|x\|_{\mathscr{G}}^2 \max_{t\in[0,1]} \|\mathscr{M}_1(tx) - \mathscr{M}_1(0)\|_{\mathscr{L}(\mathscr{G},\mathscr{H})} \\ &\leq \kappa\varepsilon\|x\|_{\mathscr{G}}^2, \quad \text{for } \|x\|_{\mathscr{X}} \leq \delta.\end{aligned}$$

Consequently, choosing $\varepsilon \in (0,1]$ so that $\kappa\varepsilon \leq \frac{1}{2}C_1$, we obtain

$$|\mathscr{E}(x)| \leq C_1\|x\|_{\mathscr{G}}^2, \quad \forall\, x \in \mathscr{X}_0 \text{ such that } \|x\|_{\mathscr{X}} \leq \delta. \tag{4.2}$$

Combining (4.1) and (4.2) yields,

$$\|\mathscr{M}(x)\|_{\mathscr{H}} \geq Z|\mathscr{E}(x)|^{1/2}, \quad \forall\, x \in \mathscr{X}_0 \text{ such that } \|x\|_{\mathscr{X}} \leq \delta,$$

for $Z := C_0/\sqrt{C_1}$. This completes the proof of Theorem 3. □

5. Morse–Bott property of Yang–Mills energy functions

In our articles [31, 32] with Maridakis we only gave a few examples where the energy functions $\mathscr{E}$ were known to be Morse–Bott. In this section, we provide two criteria for when Yang–Mills energy functions are Morse–Bott. Those criteria are simplest in the case of the self-dual Yang–Mills energy function near anti-self-dual connections over four-dimensional manifolds, which we discuss in Section 5.1 (and where we prove Theorem 6), and in the case of the Yang–Mills energy function near flat connections over manifolds of dimension $d \geq 2$, which we discuss in Section 5.2 (and where we prove Theorem 7). Finally, in Section 5.3 we give the short proof of Theorem 8.

5.1. **Self-dual Yang–Mills energy function near anti-self-dual connections.** In this subsection, we assume that (X, g) is a closed, *four*-dimensional, smooth Riemannian manifold and that, as usual, G is a compact Lie group and P is a smooth principal G-bundle over X. The self-dual Yang–Mills energy function, $\mathscr{E}_+ : \mathscr{A}(P) \to \mathbb{R}$ in (1.25), has Hessian operator, $\mathscr{E}_+'' : \mathscr{A}(P) \to T^*\mathscr{A}(P) \times T^*\mathscr{A}(P)$, given by

$$\mathscr{E}_+''(A)(a,b) = (d_A^+ a, d_A^+ b)_{L^2(X)} + (F_A^+, a\wedge b)_{L^2(X)}, \tag{5.1}$$

for all $a, b \in T_A\mathscr{A}(P) = W^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P)$.

Lemma 5.1 (Morse–Bott property of the self-dual Yang–Mills energy function at regular anti–self-dual connections)**.** *Let (X, g) be a closed, four-dimensional, smooth Riemannian manifold, G be a compact Lie group, P be a smooth principal G-bundle over X, and $q > 2$ be a constant. If A is a $W^{1,q}$ anti-self-dual Yang–Mills connection on P such that* $\operatorname{Coker} d_A^+ = 0$*, then $\mathscr{E}_+ : \mathscr{A}(P) \to \mathbb{R}$ is a Morse–Bott function at A in the sense of Definition 1.5. Moreover, if in addition the isotropy group of A in* $\operatorname{Aut}(P)$ *is the center of G, then $\mathscr{E} : \mathscr{B}^*(P) \to \mathbb{R}$ is a Morse–Bott function at $[A]$.*

Proof. We first consider $\mathscr{E}_+ : \mathscr{A}(P) \to \mathbb{R}$. From Donaldson and Kronheimer [23, Section 4.2.5], the intersection of the subvariety, $\tilde{M}_+(P, g) = \{B \in \mathscr{A}(P) : F_B^+ = 0\}$, with an open ball, $\tilde{U}_A(\varepsilon) \subset \mathscr{A}(P)$, with center A and small enough radius $\varepsilon = \varepsilon(A, g) \in (0, 1]$, is a smooth manifold if $\operatorname{Coker} d_A^+ = 0$, since the latter property means that $0 \in L^q(X; \Lambda^+ \otimes \mathrm{ad}P)$ is a regular value of the map $\mathscr{A}(P) \ni A \mapsto F_A^+ \in L^q(X; \Lambda^+ \otimes \mathrm{ad}P)$. Moreover, $\operatorname{Coker} d_B^+ = 0$ for small enough ε and all $B \in \tilde{U}_A(\varepsilon)$ since the property of d_A^+ being surjective is open. Hence, from the discussion in Section 1.4,

$$\tilde{M}_+(P,g) \cap \tilde{U}_A(\varepsilon) = \widetilde{\mathrm{Crit}}(\mathscr{E}_+) \cap \tilde{U}_A(\varepsilon),$$

and $\widetilde{\mathrm{Crit}}(\mathscr{E}_+) \cap \tilde{U}_A(\varepsilon)$ is a smooth manifold. Because $F_A^+ = 0$, we have by (5.1) that

$$\mathscr{E}_+''(A)(a,b) = (d_A^+ a, d_A^+ b)_{L^2(X)} = (d_A^{+,*} d_A^+ a, b)_{L^2(X)}.$$

On the other hand, the tangent space to $\widetilde{\mathrm{Crit}}(\mathscr{E}_+) \subset \mathscr{A}(P)$ at A is given by

$$T_A \widetilde{\mathrm{Crit}}(\mathscr{E}_+) = \mathrm{Ker}\left(d_A^+ : W^{1,q}(X;\Lambda^1 \otimes \mathrm{ad}P) \to L^q(X;\Lambda^+ \otimes \mathrm{ad}P)\right).$$

But then

$$\begin{aligned}\mathrm{Ker}\,\mathscr{E}_+''(A) &= \mathrm{Ker}\left(d_A^{+,*} d_A^+ : W^{1,q}(X;\Lambda^1 \otimes \mathrm{ad}P) \to W^{-1,q}(X;\Lambda^1 \otimes \mathrm{ad}P)\right) \\ &= \mathrm{Ker}\left(d_A^+ : W^{1,q}(X;\Lambda^1 \otimes \mathrm{ad}P) \to L^q(X;\Lambda^+ \otimes \mathrm{ad}P)\right) = T_A \widetilde{\mathrm{Crit}}(\mathscr{E}_+),\end{aligned}$$

and thus $\mathscr{E}_+ : \mathscr{A}(P) \to \mathbb{R}$ is a Morse–Bott function at A by Definition 1.5.

We now consider $\mathscr{E}_+ : \mathscr{B}^*(P) \to \mathbb{R}$. The argument here is very similar and again relies on [23, Section 4.2] for a description of the manifold structures of $M_+(P,g)$ and $\mathscr{B}(P)$. We let $U_{[A]}(\varepsilon) \subset \mathscr{B}^*(P)$ denote the open ball with center $[A]$ and radius ε and now find that

$$M_+^*(P,g) \cap U_{[A]}(\varepsilon) = \mathrm{Crit}(\mathscr{E}_+) \cap U_{[A]}(\varepsilon),$$

and $\mathrm{Crit}(\mathscr{E}_+) \cap U_{[A]}(\varepsilon)$ is a smooth manifold. The tangent space to $\mathrm{Crit}(\mathscr{E}_+) \subset \mathscr{B}^*(P)$ at $[A]$ is thus given by

$$T_A \,\mathrm{Crit}(\mathscr{E}_+) = \mathrm{Ker}\left(d_A^+ : \mathrm{Ker}\, d_A^* \cap W^{1,q}(X;\Lambda^1 \otimes \mathrm{ad}P) \to L^q(X;\Lambda^+ \otimes \mathrm{ad}P)\right).$$

But then

$$\begin{aligned}\mathrm{Ker}\,\mathscr{E}_+''(A) &= \mathrm{Ker}\left(d_A^{+,*} d_A^+ : \mathrm{Ker}\, d_A^* \cap W^{1,q}(X;\Lambda^1 \otimes \mathrm{ad}P) \to \mathrm{Ker}\, d_A^* \cap W^{-1,q}(X;\Lambda^1 \otimes \mathrm{ad}P)\right) \\ &= \mathrm{Ker}\left(d_A^+ : \mathrm{Ker}\, d_A^* \cap W^{1,q}(X;\Lambda^1 \otimes \mathrm{ad}P) \to L^q(X;\Lambda^+ \otimes \mathrm{ad}P)\right) = T_{[A]} \,\mathrm{Crit}(\mathscr{E}_+),\end{aligned}$$

and thus $\mathscr{E}_+ : \mathscr{B}^*(P) \to \mathbb{R}$ is a Morse–Bott function at $[A]$ by Definition 1.5. □

The Łojasiewicz–Simon gradient inequality (1.31) in Theorem 6 may be proved as a consequence of the Morse–Bott property of $\mathscr{E}_+$ and Theorem 3 or directly using standard arguments in Yang–Mills gauge theory. We shall provide both arguments.

Proof of Inequality (1.31) *using the Morse–Bott property of* $\mathscr{E}_+$. We first observe that our proof of [32, Theorem 3], giving analyticity (we just need C^1 here) of the gradient map for the boson coupled Yang–Mills energy function and its Łojasiewicz–Simon gradient inequality, for some exponent $\theta \in [1/2, 1)$, carries over *mutatis mutandis* for the self-dual Yang–Mills energy function and, indeed, is easier since X is restricted to have dimension $d = 4$ and the structure of the energy function is much simpler. Moreover, when $\mathrm{Coker}\, d_{A_\infty}^+ = 0$, we verified that $\mathscr{E}_+$ has the Morse–Bott property (in the sense of Definition 1.5) at A_∞ in Lemma 5.1. Inequality (1.31) now follows from Theorem 3 and [32, Theorem 3] (with $p = 2$ and $\theta = 1/2$). In order to apply [32, Theorem 3], we must strengthen the hypothesis (1.33) to

$$\|A - \Gamma\|_{W^{1,2}(X)} < \sigma,$$

corresponding to the hypothesis [32, Inequality (1.16)]. □

Proof of Theorem 6, including direct proof of Inequality (1.31). The first and final assertions regarding the Morse–Bott properties of $\mathscr{E}_+ : \mathscr{A}(P) \to \mathbb{R}$ and $\mathscr{E}_+ : \mathscr{B}^*(P) \to \mathbb{R}$ both follow from Lemma 5.1.

In the remainder of the proof, we may assume without loss of generality that A_∞ is a C^∞ connection by choosing a $W^{2,q}$ gauge transformation $u \in \operatorname{Aut}(P)$ such that $u(A_\infty)$ is a C^∞ anti-self-dual connection. To see this, we observe that the hypothesis (1.29) is equivalent to

$$\|u(A) - u(A_\infty)\|_{W^{1,2}_{u(A_\infty)}(X)} < \sigma$$

and the inequalities (1.30), (1.31) are equivalent to their analogues with $u(A)$ and $u(A_\infty)$. The existence of a $W^{2,q}$ gauge transformation follows from standard arguments; see Donaldson and Kronheimer [23, Section 4.4], Uhlenbeck [85, 84], and Feehan [25, Section 3.1].

Because $F^+_{A_\infty} = 0$, we have an elliptic complex [23, Equation (4.2.26)],

$$\Omega^0(X; \operatorname{ad}P) \xrightarrow{d_{A_\infty}} \Omega^1(X; \operatorname{ad}P) \xrightarrow{d^+_{A_\infty}} \Omega^{2,+}(X; \operatorname{ad}P)$$

and an L^2-orthogonal Hodge decomposition [36, Theorem 1.5.2]

$$W^{1,q}_{A_\infty}(X; \Lambda^1 \otimes \operatorname{ad}P) = \operatorname{Ker}\left(d^+_{A_\infty} + d^*_{A_\infty}\right) \oplus \operatorname{Ran} d_{A_\infty} \oplus \operatorname{Ran} d^{+,*}_{A_\infty}.$$

Note that $\operatorname{Ran} d_{A_\infty} \subset \operatorname{Ker} d^+_{A_\infty}$. We now write $A = A_\infty + a$ for $a \in W^{1,q}_{A_\infty}(X; \Lambda^1 \otimes \operatorname{ad}P)$ and split $a = a_\perp + a_\parallel$, where $a_\perp, a_\parallel \in W^{1,q}_{A_\infty}(X; \Lambda^1 \otimes \operatorname{ad}P)$ and $a_\perp$ is L^2-orthogonal to $\operatorname{Ker} d^+_{A_\infty}$ while $a_\parallel \in \operatorname{Ker} d^+_{A_\infty}$.

We first consider the case where $a_\parallel = 0$ and observe that $a = a_\perp = d^{+,*}_{A_\infty} v$ for $v \in W^{2,q}_{A_\infty}(X; \Lambda^2 \otimes \operatorname{ad}P)$ by the Hodge decomposition. Because $F^+_{A_\infty} = 0$, we have

$$F^+_A = F^+_{A_\infty + a} = d^+_{A_\infty} a + (a \wedge a)^+. \tag{5.2}$$

We claim that a obeys the following *a priori* estimate, with $p \in (1, \infty)$ obeying $p \leq q$ and a constant $C = C(A_\infty, g, G, p) \in [1, \infty)$:

$$\|a\|_{W^{1,p}_{A_\infty}(X)} \leq C \|d^+_{A_\infty} a\|_{L^p(X)}. \tag{5.3}$$

To see this, we observe that

$$\|d^{+,*}_{A_\infty} v\|_{W^{1,p}_{A_\infty}(X)} \leq c \|v\|_{W^{2,p}_{A_\infty}(X)} \leq C \|d^+_{A_\infty} d^{+,*}_{A_\infty} v\|_{L^p(X)}$$

for constants $c = c(g, G)$ and $C = C(A_\infty, g, G, p)$ in $[1, \infty)$. Ellipticity of the second-order operator $d^+_{A_\infty} d^{+,*}_{A_\infty}$ follows from its Bochner–Weitzenböck formula [33, Equation (6.26)], as that implies that its principal symbol coincides with that of the covariant Laplace operator $\nabla^*_{A_\infty} \nabla_{A_\infty}$ and thus a scalar multiple (the Riemannian metric on T^*X) of the identity. The *a priori* $W^{2,p}$ elliptic estimate for v follows from [35, Theorem 9.14] or [26, Theorem 14.60] for $d^+_{A_\infty} d^{+,*}_{A_\infty}$ and an argument exactly analogous to the proof of [35, Lemma 9.17] to eliminate the term $\|v\|_{L^p(X)}$ from the right-hand side. Hence, the claim (5.3) follows.

Because $1/p = 1/p^* + 1/4$ with $p^* = 4p/(4-p) \in (4, \infty)$, we have

$$\|(a \wedge a)^+\|_{L^p(X)} \leq c \|a\|_{L^{p^*}(X)} \|a\|_{L^4(X)} \leq C \|a\|_{W^{1,p}_{A_\infty}(X)} \|a\|_{L^4(X)},$$

for a constant $c = c(g, G) \in [1, \infty)$ and $C = C(g, G, p) \in [1, \infty)$. Consequently,

$$\begin{aligned}
\|a\|_{W^{1,p}_{A_\infty}(X)} &\leq C\|d^+_{A_\infty} a\|_{L^p(X)} \quad \text{(by (5.3))}\\
&\leq C\|F^+_A\|_{L^p(X)} + C\|(a \wedge a)^+\|_{L^p(X)} \quad \text{(by (5.2))}\\
&\leq C\|F^+_A\|_{L^p(X)} + C\|a\|_{W^{1,p}_{A_\infty}(X)}\|a\|_{L^4(X)}.
\end{aligned}$$

Since $\|a\|_{L^4(X)} < \sigma$ by (1.29), then rearrangement, for small enough $\sigma = \sigma(A_\infty, g, G, p) \in (0, 1]$, yields

$$\|a\|_{W^{1,p}_{A_\infty}(X)} \leq C\|F^+_A\|_{L^p(X)}, \tag{5.4}$$

and thus for $p = 2$ we obtain (1.30).

To prove (1.31), write $d^+_A a = d^+_{A_\infty} a + 2(a \wedge a)^+ = F^+_A + (a \wedge a)^+$ and note that

$$\begin{aligned}
\|d^{+,*}_A F^+_A\|_{W^{-1,2}_{A_\infty}(X)} &= \sup_{b \in W^{1,2}_{A_\infty}(X;\Lambda^1\otimes \mathrm{ad}P)\setminus\{0\}} \frac{(d^{+,*}_A F^+_A, b)_{L^2(X)}}{\|b\|_{W^{1,2}_{A_\infty}(X)}}\\
&\geq \frac{(d^{+,*}_A F^+_A, a)_{L^2(X)}}{\|a\|_{W^{1,2}_{A_\infty}(X)}} = \frac{(F^+_A, d^+_A a)_{L^2(X)}}{\|a\|_{W^{1,2}_{A_\infty}(X)}} = \frac{(F^+_A, F^+_A + (a \wedge a)^+)_{L^2(X)}}{\|a\|_{W^{1,2}_{A_\infty}(X)}}.
\end{aligned}$$

Therefore,

$$\|d^{+,*}_A F^+_A\|_{W^{-1,2}_{A_\infty}(X)} \geq \frac{\|F^+_A\|^2_{L^2(X)}}{\|a\|_{W^{1,2}_{A_\infty}(X)}} + \frac{(F^+_A, (a \wedge a)^+)_{L^2(X)}}{\|a\|_{W^{1,2}_{A_\infty}(X)}}. \tag{5.5}$$

The gradient inequality (1.31) now follows. Indeed,

$$\|(a \wedge a)^+\|_{L^2(X)} \leq c\|a\|^2_{L^4(X)} \leq C\|a\|_{L^4(X)}\|a\|_{W^{1,2}_{A_\infty}(X)},$$

for constants c and C with the same dependencies as above, and

$$\begin{aligned}
\|d^{+,*}_A F^+_A\|_{W^{-1,2}_{A_\infty}(X)} &\geq \frac{\|F^+_A\|^2_{L^2(X)}}{\|a\|_{W^{1,2}_{A_\infty}(X)}} - \frac{\|F^+_A\|_{L^2(X)}\|(a \wedge a)^+\|_{L^2(X)}}{\|a\|_{W^{1,2}_{A_\infty}(X)}} \quad \text{(by (5.5))}\\
&\geq \frac{\|F^+_A\|^2_{L^2(X)}}{\|a\|_{W^{1,2}_{A_\infty}(X)}} - C\frac{\|F^+_A\|_{L^2(X)}\|a\|_{L^4(X)}\|a\|_{W^{1,2}_{A_\infty}(X)}}{\|a\|_{W^{1,2}_{A_\infty}(X)}}\\
&\geq C^{-1}\|F^+_A\|_{L^2(X)} - C\sigma\|F^+_A\|_{L^2(X)} \quad \text{(by (1.29) and (1.30))}.
\end{aligned}$$

Now choose σ small enough that $\sigma \leq 1/(2C^2)$ to give (1.31). This completes the proof of the optimal Łojasiewicz–Simon inequalities when $a_\| = 0$.

When $a_\| \neq 0$, we instead choose a $W^{1,q}$ anti-self-dual connection $\tilde{A}_\infty$ on P such that $A = \tilde{A}_\infty + \tilde{a}$, where $\tilde{a} \in W^{1,q}_{A_\infty}(X; \Lambda^1 \otimes \mathrm{ad}P)$ is L^2-orthogonal to $\mathrm{Ker}\, d^+_{\tilde{A}_\infty}$ and obeys $\|\tilde{a}\|_{L^4(X)} < 2\sigma$. The existence of $\tilde{a}$ follows because an open neighborhood of A_∞ in $\tilde{M}_+(P, g) \subset \mathscr{A}(P)$ is a smooth submanifold by our hypothesis that $\mathrm{Coker}\, d^+_{A_\infty} = 0$ and so has an L^2-normal tubular neighborhood in $\mathscr{A}(P)$ (compare [39, Theorem 4.5.2] in the case of finite-dimensional manifolds). To see this explicitly, we note that by [23, Section 4.2.5] for small enough $\sigma = \sigma(A_\infty, g, G) \in (0, 1]$,

$$\mathscr{U} := \{b \in W^{1,q}_{A_\infty}(X; \Lambda^1 \otimes \mathrm{ad}P) : F^+_{A_\infty+b} = 0 \text{ and } \|b\|_{L^4(X)} < \sigma\}$$

is an open, smooth submanifold of $W^{1,q}_{A_\infty}(X;\Lambda^1\otimes\mathrm{ad}P)$, with

$$T_b := \operatorname{Ker} d^+_{A_\infty+b} \cap W^{1,q}_{A_\infty}(X;\Lambda^1\otimes\mathrm{ad}P),$$

as tangent space at b and smooth normal bundle, $\mathscr{N}$, with fiber over b,

$$N_b := \left(\operatorname{Ker} d^+_{A_\infty+b}\right)^\perp \cap W^{1,q}_{A_\infty}(X;\Lambda^1\otimes\mathrm{ad}P),$$

where $(\operatorname{Ker} d^+_{A_\infty+b})^\perp$ is the L^2-orthogonal complement of $\operatorname{Ker} d^+_{A_\infty+b} \cap W^{1,q}_{A_\infty}(X;\Lambda^1\otimes\mathrm{ad}P)$. The differential of the smooth map,

$$\mathscr{N} \ni (b,\eta) \mapsto b+\eta \in W^{1,q}_{A_\infty}(X;\Lambda^1\otimes\mathrm{ad}P) = T_0\oplus N_0,$$

is the identity at the origin $(0,0)$ and so the existence of an L^2-normal tubular neighborhood now follows from the Implicit Function Theorem for smooth maps on Banach spaces. Because $F^+_{\tilde{A}_\infty}=0$, we have

$$F^+_A = F^+_{\tilde{A}_\infty+\tilde{a}} = d^+_{\tilde{A}_\infty}\tilde{a} + (\tilde{a}\wedge\tilde{a})^+,$$

and so the inequalities (1.30) and (1.31) now follow almost exactly as before, noting that $\|\tilde{A}_\infty - A_\infty\|_{L^4(X)} < \sigma$. This completes the proof of Theorem 6. □

5.2. **Yang–Mills energy function near flat connections.** We shall proceed by analogy with our development in Section 5.1 but return to the general case where X may have any dimension $d\geq 2$. If $F_A=0$, then $\mathscr{E}'(A)\equiv 0$ by (1.11) and A is a critical point of $\mathscr{E}:\mathscr{A}(P)\to\mathbb{R}$, so that

$$\tilde{M}_0(P,g) \subset \widetilde{\operatorname{Crit}}(\mathscr{E}) \cap \mathscr{A}(P),$$

where $\widetilde{\operatorname{Crit}}(\mathscr{E}_+)$ denotes the critical set of $\mathscr{E}:\mathscr{A}(P)\to\mathbb{R}$. Conversely, suppose $A\in\widetilde{\operatorname{Crit}}(\mathscr{E})$. The Bianchi Identity [23, Equation (2.1.21)] implies that $d_AF_A=0$, so $F_A\in\operatorname{Ker} d_A\cap L^q(X;\Lambda^2\otimes\mathrm{ad}P)$ and if A is a *regular point* of the map $\mathscr{A}(P)\ni A\mapsto F_A\in L^q(X;\Lambda^2\otimes\mathrm{ad}P)$ in the sense that

$$\operatorname{Ker} d_A\cap L^q(X;\Lambda^2\otimes\mathrm{ad}P) = \operatorname{Ran} d_A\cap L^q(X;\Lambda^2\otimes\mathrm{ad}P),$$

then (1.11) implies that $F_A=0$ and $A\in\tilde{M}_0(P,g)$. Of course, in the absence of an assumption that A is regular in the preceding sense, then A is (by definition) a *Yang–Mills connection* as in (1.36),

$$d_A^*F_A = 0,$$

and of course need not be flat. However, if we require in addition to (1.36) that

$$\|F_A\|_{L^{d/2}(X)} \leq \varepsilon, \tag{5.6}$$

for $\varepsilon=\varepsilon(g,G)\in(0,1]$, then A is necessarily flat by Feehan [30, Theorem 1] and thus we obtain the reverse inclusion,

$$\widetilde{\operatorname{Crit}}(\mathscr{E})\cap\mathscr{A}_\varepsilon(P) \subset \tilde{M}_0(P,g),$$

where $\mathscr{A}_\varepsilon(P) := \{A\in\mathscr{A}(P) : A \text{ obeys (5.6)}\}$. Our proof of [30, Theorem 1] employed the Łojasiewicz–Simon gradient inequality for the Yang–Mills energy function at one step [30, p. 578], but an elementary argument which avoids the Łojasiewicz–Simon gradient inequality has recently been provided by Huang [43].

If Γ is a flat connection on P, then its exterior covariant derivative defines an elliptic complex,

$$\cdots\Omega^i(X;\mathrm{ad}P) \xrightarrow{d_\Gamma} \Omega^{i+1}(X;\mathrm{ad}P) \xrightarrow{d_\Gamma} \Omega^{i+2}(X;\mathrm{ad}P)\cdots$$

for $i \geq 0$, since $d_\Gamma^2 = F_\Gamma = 0$. By analogy with their definitions based on the deformation complex for an anti-self-dual connection [23, Section 4.2.5] on a principal G-bundle P over a four-dimensional Riemannian manifold, we define

$$H_\Gamma^i := \operatorname{Ker} d_\Gamma \cap \Omega^i(X; \operatorname{ad}P)/\operatorname{Ran} d_\Gamma, \quad i \geq 0.$$

By analogy with the construction in [23, Section 4.2.5] of a local Kuranishi model for an open neighborhood of a point $[A] \in M_+(P, g) \subset \mathscr{B}(P)$ when X has dimension four, we observe that if $H_\Gamma^2 = 0$, then there is an open neighborhood $\tilde{\mathscr{U}}_\Gamma \subset \mathscr{A}(P)$ of a flat connection, Γ, on P such that

$$\tilde{\mathscr{U}}_\Gamma \cap \tilde{M}_0(P) \subset \mathscr{A}(P)$$

is an open, smooth submanifold. Moreover, if the isotropy group of Γ in $\operatorname{Aut}(P)$ is the center of G, then the quotient,

$$\mathscr{U}_\Gamma \cap M_0(P) \subset \mathscr{B}^*(P),$$

is an open, smooth submanifold. In general, the moduli space $M_0(P)$ will not be a smooth submanifold but rather a finite-dimensional, real analytic subvariety (compare [23, p. 139]).

By gauge invariance, the Yang–Mills energy function is well-defined on the quotient, $\mathscr{E} : \mathscr{B}^*(P) \to \mathbb{R}$ (with $q > 2$), and we have the equality,

$$M_0^*(P) = \operatorname{Crit}(\mathscr{E}) \cap \mathscr{B}_\varepsilon^*(P),$$

where $\operatorname{Crit}(\mathscr{E})$ denotes the critical set of $\mathscr{E} : \mathscr{B}^*(P) \to \mathbb{R}$, and $\mathscr{B}_\varepsilon(P) := \{[A] \in \mathscr{B}(P) : A \text{ obeys } (5.6)\}$, and $\mathscr{B}_\varepsilon^*(P) := \mathscr{B}_\varepsilon(P) \cap \mathscr{B}^*(P)$, and $M_0^*(P) := M_0(P) \cap \mathscr{B}^*(P)$.

Given the preceding remarks, the proof of Lemma 5.1 adapts *mutatis mutandis* to give the

Lemma 5.2 (Morse–Bott property of the Yang–Mills energy function at regular flat connections)**.** *Let (X, g) be a closed, smooth Riemannian manifold of dimension $d \geq 2$, and G be a compact Lie group, P be a smooth principal G-bundle over X, and $q > 2$. If Γ is a $W^{1,q}$ flat connection on P such that $H_\Gamma^2 = 0$, then $\mathscr{E} : \mathscr{A}(P) \to \mathbb{R}$ is a Morse–Bott function at Γ in the sense of Definition 1.5. Moreover, if in addition the isotropy group of Γ in* $\operatorname{Aut}(P)$ *is the center of G, then $\mathscr{E} : \mathscr{B}^*(P) \to \mathbb{R}$ is a Morse–Bott function at* $[\Gamma]$.

When $H_\Gamma^2 = 0$, we shall give a second proof of a simpler version of the Łojasiewicz–Simon gradient inequality (1.13), namely (1.35) in Theorem 7, using the Morse–Bott property of $\mathscr{E}$ at Γ from Lemma 5.2 and Theorem 3. We shall also give a direct proof of (1.35) using arguments in Yang–Mills gauge theory that are considerably simpler than those used to prove Theorem 1. To verify the preceding results, we outline the modifications required to the corresponding proofs in Section 5.1 for $\mathscr{E}_+$ when X has dimension four.

Proof of Inequality (1.35) *using the Morse–Bott property of $\mathscr{E}$ when* $2 \leq d \leq 4$. Our [32, Theorem 3], giving analyticity (we just need C^1 here) of the gradient map for the boson coupled Yang–Mills energy function and its Łojasiewicz–Simon gradient inequality, for some exponent $\theta \in [1/2, 1)$, specializes to the Yang–Mills energy function. When $H_\Gamma^2 = 0$, we verified that $\mathscr{E}$ has the Morse–Bott property (in the sense of Definition 1.5) at Γ in Lemma 5.2. Inequality (1.35) with $\theta = 1/2$ thus follows from Theorem 3 and [32, Theorem 3] (for $\theta = 1/2$ and with $p = 2$, a valid choice for $q \in [2, \infty)$ obeying $q > d/2$ and $p \in [2, \infty)$ obeying $d/2 \leq p \leq q$ when $2 \leq d \leq 4$). In order to apply [32, Theorem 3], we must strengthen the hypothesis (1.33) to

$$\|A - \Gamma\|_{W^{1,2}(X)} < \sigma,$$

corresponding to the hypothesis [32, Inequality (1.16)]. □

Proof of Theorem 7, including direct proof of Inequality (1.35). The first and final assertions regarding the Morse–Bott properties of $\mathscr{E} : \mathscr{A}(P) \to \mathbb{R}$ and $\mathscr{E} : \mathscr{B}^*(P) \to \mathbb{R}$ both follow from Lemma 5.2. For the remainder of the proof, we highlight the modifications required to the proof of Theorem 6.

As before, we may assume without loss of generality that Γ is a C^∞ connection by choosing a $W^{2,q}$ gauge transformation $u \in \operatorname{Aut}(P)$ such that $u(\Gamma)$ is a C^∞ flat connection. Similarly, we have an L^2-orthogonal Hodge decomposition [36, Theorem 1.5.2],

$$W^{1,q}_\Gamma(X;\Lambda^1 \otimes \operatorname{ad}P) = \operatorname{Ker}(d_\Gamma + d_\Gamma^*) \oplus \operatorname{Ran} d_\Gamma \oplus \operatorname{Ran} d_\Gamma^*.$$

Note that $\operatorname{Ran} d_\Gamma \subset \operatorname{Ker} d_\Gamma$ and write $A = \Gamma + a$ for $a \in W^{1,q}_\Gamma(X;\Lambda^1 \otimes \operatorname{ad}P)$ and split $a = a_\perp + a_\|$, where $a_\perp, a_\| \in W^{1,q}_\Gamma(X;\Lambda^1 \otimes \operatorname{ad}P)$ and $a_\perp$ is L^2-orthogonal to $\operatorname{Ker} d_\Gamma$ while $a_\| \in \operatorname{Ker} d_\Gamma$.

We first consider the case where $a_\| = 0$ and observe that $a = a_\perp = d_\Gamma^* v$ for $v \in W^{2,q}_\Gamma(X;\Lambda^2 \otimes \operatorname{ad}P)$ by the Hodge decomposition. Because $F_\Gamma = 0$, we have

$$F_A = F_{\Gamma+a} = d_\Gamma a + a \wedge a. \tag{5.7}$$

The proof of (5.3) carries over without change to show that a obeys the following *a priori* estimate, with $p \in (1,\infty)$ obeying $p \le q$ and a constant $C = C(A_\infty, g, G, p) \in [1,\infty)$:

$$\|a\|_{W^{1,p}_\Gamma(X)} \le C\|d_\Gamma a\|_{L^p(X)}. \tag{5.8}$$

Moreover, the proof of (5.4) adapts to show that, for small enough $\sigma = \sigma(g, G, p, \Gamma) \in (0, 1]$,

$$\|a\|_{W^{1,p}_\Gamma(X)} \le C\|F_A\|_{L^p(X)}, \tag{5.9}$$

where $C = G(g, G, p, \Gamma) \in [1,\infty)$ and for $p \in (1, d)$ or $p = 2$ when $d = 2$ obeying $p \le q$.

The only change in the proof of (5.4) is that we now use the continuous Sobolev multiplication, $L^d(X) \times L^{p^*}(X) \to L^p(X)$, and continuous Sobolev embedding, $W^{1,p}(X) \subset L^{p^*}(X)$ for $p \in (1, d)$ and $p^* = dp/(d-p) \in (d,\infty)$ to estimate,

$$\|a \wedge a\|_{L^p(X)} \le C\|a\|_{L^d(X)}\|a\|_{W^{1,p}_\Gamma(X)}.$$

For $d = 2$ and $p = 2$, which is excluded by the preceding requirement that $p \in (1, d)$, we recall that $r_0 > 2$ and choose $t_0 \in (2,\infty)$ by writing $1/2 = 1/r_0 + 1/t_0$ and use the continuous Sobolev multiplication, $L^{r_0}(X) \times L^{t_0}(X) \to L^2(X)$, and continuous Sobolev embedding, $W^{1,p}(X) \subset L^{t_0}(X)$, to estimate

$$\|a \wedge a\|_{L^2(X)} \le C\|a\|_{L^{r_0}(X)}\|a\|_{W^{1,2}_\Gamma(X)}.$$

For all $d \ge 2$, we thus obtain (5.8), now using the condition (1.33) in place of the condition (1.29) used to obtain (5.4).

By choosing $p = 2$ in (5.9) we obtain (1.34). To establish (1.35), we write $d_A a = d_\Gamma a + 2a \wedge a = F_A + a \wedge a$ and adapt the argument in the proof of Theorem 6 used to prove (1.31). The only significant change is that, for $d \ge 3$, we now use the continuous Sobolev multiplication, $L^d(X) \times L^{2^*}(X) \to L^2(X)$, and continuous Sobolev embedding, $W^{1,2}(X) \subset L^{2^*}(X)$ for $2^* = 2d/(d-2) \in (d,\infty)$. For $d = 2$ and $r_0 > 2$, we use the continuous Sobolev multiplication, $L^{r_0}(X) \times L^{t_0}(X) \to L^2(X)$, and continuous Sobolev embedding, $W^{1,2}(X) \subset L^{t_0}(X)$, as discussed above. This completes the proof of the optimal Łojasiewicz–Simon inequalities when $a_\| = 0$.

When $a_\| \ne 0$, we instead choose a $W^{1,q}$ flat connection $\tilde\Gamma$ on P such that $A = \tilde\Gamma + \tilde a$, where $\tilde a \in W^{1,q}_\Gamma(X;\Lambda^1 \otimes \operatorname{ad}P)$ is L^2-orthogonal to $\operatorname{Ker} d_{\tilde\Gamma}$ and obeys $\|\tilde a\|_{L^{r_0}(X)} < 2\sigma$. The existence of $\tilde a$ follows because an open neighborhood of Γ in $\tilde M_0(P, g) \subset \mathscr{A}(P)$ is a smooth submanifold by our hypothesis that $H^2_\Gamma = 0$ and so has an L^2-normal tubular neighborhood in $\mathscr{A}(P)$, by the same argument as used in the proof of Theorem 6. In the present context, we recall that

$$T_\Gamma := \operatorname{Ker} d_\Gamma \cap W^{1,q}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P)$$

is the tangent space at Γ to $\{A \in \mathscr{A}(P) : F_A = 0\}$ and

$$N_\Gamma := (\operatorname{Ker} d_\Gamma)^\perp \cap W^{1,q}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P)$$

is the corresponding normal space. Because $F_{\tilde\Gamma} = 0$, we have

$$F_A = F^+_{\tilde\Gamma+\tilde a} = d_{\tilde\Gamma}\tilde a + \tilde a \wedge \tilde a,$$

and so the inequalities (1.34) and (1.35) now follow almost exactly as before, noting that $\|\tilde\Gamma - \Gamma\|_{L^{r_0}(X)} < \sigma$. This completes the proof of Theorem 7. □

5.3. **Yang–Mills energy function near arbitrary critical points when $2 \le d \le 4$.** It remains to give the short

Proof of Theorem 8. Our [32, Theorem 3], giving analyticity (again, we just need C^1 here) of the gradient map for the boson coupled Yang–Mills energy function and its Łojasiewicz–Simon gradient inequality, for some exponent $\theta \in [1/2, 1)$, specializes to the Yang–Mills energy function. By hypothesis, $\mathscr{E}$ has the Morse–Bott property (in the sense of Definition 1.5) at A_∞. Inequality (1.40) now follows from Theorem 3 and [32, Theorem 3] (for $\theta = 1/2$ and with $p = 2$, a valid choice for $q \in [2,\infty)$ obeying $q > d/2$ and $p \in [2,\infty)$ obeying $d/2 \le p \le q$ when $2 \le d \le 4$). □

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