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The quantum quasi-invariant of the time-dependent nonlinear oscillator and application to betatron dynamics

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Both the classical and quantum approximate invariants are found for the nonlinear time-dependent oscillator of sextupole transverse betatron dynamics. They are represented in terms of the elements of a Lie algebra associated with powers of phase space coordinates. The first order quantum correction to the classical quasi-invariant is found.

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I. INTRODUCTION

Quantum effects in accelerators have been of interest for many years, [1], [2], and [3]; however recently with the development of multi TeV colliders and interest in large linear colliders, they have become the subject of wide spread research [4]. The concept of a quasi-invariant has been introduced in [5], and has been proven useful for representing the properties of nonlinear betatron dynamics. The quantum version of this invariant is developed, and from it quantum corrections to the classical results are found. The similarity between the Lie algebras associated with the classical case and the quantum case are used to obtain the relevant results. At first the classical linear and classical nonlinear cases are studied from the view point of their appropriate Lie algebras. These systems are quantized, and the corresponding Lie algebras are used to determine properties of the quantized systems.

It is well known that the Courant-Snyder [6] invariant is particularly useful for determining the phase space pattern for the transverse dynamics of a particle in a storage ring. Using the Hamiltonian for a time-dependent simple harmonic oscillator, one can obtain the relevant invariant. However, when there are non-linear contributions to the Hamiltonian little success has been achieved in finding invariants. In order to better understand the behavior of a particle beam, it is useful to find an approximate invariant, which is associated with a nonlinear time-dependent Hamiltonian. The method used to find the quasi-invariant for the nonlinear oscillator is first used in the context of classical dynamics, based on using the Lie algebra associated with elements obtained from powers and products of the position and conjugate momentum coordinates. To illustrate the method, an example is given for the linear system, where the invariant is exact, and the relevant algebra is SU(1,1) [7]. The method used for the linear system can be easily generalized to study a nonlinear one dimensional system. The method has the advantage that the time-dependent coefficients of the approximate invariant are found as the solution of a system of linear first order differential equations.

For classical one dimensional transverse dynamics, an approximate invariant associated with a time-dependent Hamiltonian containing a nonlinear sextupole term is found. Both the Hamiltonian and the approximate invariant can be represented as linear sums of the elements of a Lie algebra. The invariant is approximate in the sense that terms of order greater than three, resulting from the Poisson bracket of elements of the algebra, are neglected. This results in a closed Lie algebra. The method is extended to quantum operators, and a similar Lie algebra involving operator elements is found. This is used to obtain the quantum quasi-invariant. The relation between the classical result is established with the aid of coherent states associated with the linear time-dependent oscillator.

The Hamiltonian associated with nonlinear betatron dynamics studied in this paper is a special case of the Hamiltonian for a particle of mass m with the one dimensional conventional form

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + m\omega_0^2 K(t)\frac{\hat{q}^2}{2} + \tilde{S}(t)\hat{q}^3. \tag{1.1}$$

The method used to obtain this equation from a covariant formulation of storage ring dynamic is found in [2] and [8]. This equation is put in dimensionless form

$$H(t) = \frac{\hat{p}^2}{2} + K(t)\frac{\hat{q}^2}{2} + S(t)\hat{q}^3, \tag{1.2}$$

with the transformation to the dimensionless position, momentum, and energy variables q/q_0, p/p_0, and H/E_0. Here

$$\frac{p_0^2}{mE_0} = 1, \quad \frac{E_0}{m\omega_0^2} = q_0^2, \quad p_0 q_0 = \frac{E_0}{\tilde{\epsilon}_0}\hbar, \tag{1.3}$$

with $\tilde{\epsilon}_0 = \omega_0 \hbar$. In addition

$$S(t) = \frac{\tilde{S}(t) q_0^3}{E_0}. \tag{1.4}$$

Introducing the dimensionless parameter

$$\tilde{\epsilon} = \frac{\hbar}{p_0 q_0} = \frac{\tilde{\epsilon}_0}{E_0} \tag{1.5}$$

gives for the quantum bracket of the dimensionless operators $\hat{q}$ and $\hat{p}$

$$[\hat{q}, \hat{p}] = i\frac{\hbar}{p_0 q_0} = i\tilde{\epsilon}. \tag{1.6}$$

This allow the results which depend upon the quantum bracket to be expressed in terms of $\tilde{\epsilon}$. The quantum results, corresponding to various orders of $\hbar$, are found with $\tilde{\epsilon} = 1$, and associated classical results are found with $\tilde{\epsilon} \to 0$, corresponding to the limit $\hbar \to 0$. For applications to betatron dynamics, it is conventional to use $q_0 = 1$ and $p_0 = |\vec{p}|$, which is the magnitude of the three-momentum of a relativistic particle.

II. THE COURANT-SNYDER INVARIANT AND SU(1,1)

The time-dependent Hamiltonian for one dimensional transverse dynamics is written in terms of the position coordinate q and the conjugate momentum p as

$$H(t) = \frac{p^2}{2} + K(t)\frac{q^2}{2}. \tag{2.1}$$

The invariant Courant-Snyder associated with this Hamiltonian is

$$I_0(t) = \frac{\beta(t)p^2 + 2\alpha(t)pq + \gamma(t)q^2}{2}, \tag{2.2}$$

which satisfies the partial differential equation

$$\frac{dI_0(t)}{dt} = \frac{\partial I_0(t)}{\partial t} + \{H(t), I_0(t)\} = 0. \qquad 2.3$$

The Poisson bracket of phase space functions $f(p,q)$ and $g(p,q)$ is defined as

$$\{f(p,q), g(p,q)\} = \frac{\partial f(p,q)}{\partial p}\frac{\partial g(p,q)}{\partial q} - \frac{\partial f(p,q)}{\partial q}\frac{\partial g(p,q)}{\partial p}. \qquad 2.4$$

The functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy the equations

$$\begin{aligned} \frac{d\alpha(t)}{dt} &= K(t)\beta(t) - \gamma(t) \\ \frac{d\beta(t)}{dt} &= -2\alpha(t) \\ \frac{d\gamma(t)}{dt} &= 2K(t)\alpha(t), \end{aligned} \qquad 2.5$$

where

$$\gamma(t) = \frac{1+\alpha^2(t)}{\beta(t)}. \qquad 2.6$$

Both the Hamiltonian Eq. (2.1) and the invariant Eq. (2.2) may be expressed in terms of the elements of the Lie algebra SU(1,1). If one introduces the coordinates

$$\begin{aligned} a &= \frac{q+ip}{\sqrt{2}} \\ a^* &= \frac{q-ip}{\sqrt{2}}, \end{aligned} \qquad 2.7$$

with Poisson bracket

$$\{a, a^*\} = i, \qquad 2.8$$

then the functions

$$A_1 = a^2, \quad A_2 = a^{*2}, \quad A_3 = a^* a, \qquad 2.9$$

satisfy the Lie algebra of SU(1,1). Namely,

$$\begin{aligned} \{A_1, A_2\} &= 4iA_3 \\ \{A_1, A_3\} &= 2iA_1 \\ \{A_2, A_3\} &= -2iA_2. \end{aligned} \qquad 2.10$$

In terms of the elements of the algebra Eq. (2.10), the Hamiltonian and the invariant become

$$\begin{aligned} H(t) &= \alpha_1(t)A_1 + \alpha_2(t)A_2 + \alpha_3(t)A_3 \\ I_0(t) &= \beta_1(t)A_1 + \beta_2(t)A_2 + \beta_3(t)A_3. \end{aligned} \qquad 2.11$$

Requiring $I_0(t)$ to be real gives the relations

$$\begin{aligned} \beta_1(t) &= \beta_2^*(t) \\ \beta_3(t) &= \beta_3^*(t). \end{aligned} \qquad 2.12$$

When these are substituted into Eq. (2.3), one finds, using Eq. (2.10), the set of linear differential equations

$$\begin{pmatrix} \frac{d\beta_1(t)}{dt} \\ \frac{d\beta_2(t)}{dt} \\ \frac{d\beta_3(t)}{dt} \end{pmatrix} = \begin{pmatrix} 2i\alpha_3(t) & 0 & -2i\alpha_1(t) \\ 0 & -2i\alpha_3(t) & 2i\alpha_1(t) \\ 4i\alpha_1(t) & -4i\alpha_1(t) & 0 \end{pmatrix} \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{pmatrix}. \qquad 2.13$$

The functions $\alpha_i(t)$ and $\beta_i(t)$ satisfy the relations

$$\begin{aligned} \alpha_1(t) = \alpha_2(t) &= \frac{K(t)-1}{4} \\ \alpha_3(t) &= \frac{K(t)+1}{2} \end{aligned} \qquad 2.14$$

and

$$\begin{aligned} \beta_1(t) &= \frac{i\alpha(t) - \frac{\gamma(t)-\beta(t)}{2}}{2} \\ \beta_2(t) &= \frac{i\alpha(t) + \frac{\gamma(t)-\beta(t)}{2}}{2} \\ \beta_3(t) &= \frac{\beta(t)+\gamma(t)}{2}. \end{aligned} \qquad 2.15$$

These relations can be used to show that the system of linear differential equations Eq. (2.13) is equivalent to the system Eq. (2.5). With initial values given for $\beta(t)$ and $d\beta(t)/dt$, the system of equations Eq. (2.13) can be integrated numerically, using Eq. (2.14) and Eq. (2.15).

III. THE NONLINEAR SEXTUPOLE SYSTEM

Next the method described above is extended to the classical case when a nonlinear term is added to the linear system Hamiltonian. However, in this case an approximation is made to obtain a finite closed Lie algebra which contains seven elements. As an example, one considers the Hamiltonian Eq. (1.2) where $S(t)$ is the strength of the sextupole term [9]. Now defining functions of a and a^* as

$$\begin{aligned} A_1 &= a^2, & A_2 &= a^{*2} \\ A_3 &= a^* a, & A_4 &= a^3 \\ A_5 &= a^{*3}, & A_6 &= a^2 a^* \\ A_7 &= a^{*2} a, \end{aligned} \qquad 3.1$$

one finds, keeping terms of order less than four in a and a^*, the closed Lie algebra

$$
\begin{array}{ll}
\{A_1, A_2\} = 4iA_3 & \{A_1, A_3\} = 2iA_1, \\
\{A_1, A_4\} = 0, & \{A_1, A_5\} = 6iA_7, \\
\{A_1, A_6\} = 2iA_4, & \{A_1, A_7\} = 4iA_6, \\
\{A_2, A_3\} = -2iA_2, & \{A_2, A_4\} = -6iA_6, \\
\{A_2, A_5\} = 0, & \{A_2, A_6\} = -4iA_7, \\
\{A_2, A_7\} = -2iA_5 & \\
\{A_3, A_4\} = -3iA_4, & \{A_3, A_5\} = 3iA_5, \\
\{A_3, A_6\} = -iA_6, & \{A_3, A_7\} = iA_7, \\
\{A_4, A_5\} = 0, & \\
\{A_4, A_6\} = 0, & \{A_4, A_7\} = 0, \\
\{A_5, A_6\} = 0, & \{A_5, A_7\} = 0, \\
\{A_6, A_7\} = 0. &
\end{array}
\tag{3.2}
$$

The Hamiltonian Eq. (1.2) may be written in the form

$$
H(t) = \sum_{i=1}^{7} \alpha_i(t) A_i, \tag{3.3}
$$

where $\alpha_i(t)$, $i = 1 \to 3$ are given by Eq. (2.14), and

$$
\begin{aligned}
\alpha_4(t) = \alpha_5(t) &= \frac{\sqrt{2}S(t)}{4} \\
\alpha_6(t) = \alpha_7(t) &= 3\alpha_4(t).
\end{aligned}
\tag{3.4}
$$

One can now find an approximate time-invariant associated with the Hamiltonian Eq. (3.3). This is assumed to be of the form

$$
I(t) = \sum_{j=1}^{7} \beta_j(t) A_j, \tag{3.5}
$$

which contains terms up to third order in a and a^*. Since $I(t)$ must be real, one finds

$$
\begin{aligned}
\beta_1(t) = \beta_2^*(t), \quad \beta_3(t) = \beta_3^*(t) \\
\beta_4(t) = \beta_5^*(t), \quad \beta_6(t) = \beta_7^*(t).
\end{aligned}
\tag{3.6}
$$

When this along with the Hamiltonian Eq. (3.3) is substituted into Eq. (2.3), one finds, using the algebra Eq. (3.2), the system of linear first order differential equations

$$
\frac{d\vec{\beta}(t)}{dt} = \mathbf{M}(t)\vec{\beta}(t), \tag{3.7}
$$

where

$$\vec{\beta}(t) = \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \\ \beta_4(t) \\ \beta_5(t) \\ \beta_6(t) \\ \beta_7(t) \end{pmatrix} \qquad 3.8$$

and

$$\mathbf{M}(t) = \begin{pmatrix} 2i\alpha_3(t) & 0 & -2i\alpha_1(t) & 0 & 0 & 0 & 0 \\ 0 & -2i\alpha_3(t) & 2i\alpha_2(t) & 0 & 0 & 0 & 0 \\ -4i\alpha_2(t) & 4i\alpha_1(t) & 0 & 0 & 0 & 0 & 0 \\ 2i\alpha_6(t) & 0 & -3i\alpha_4(t) & 3i\alpha_3(t) & 0 & -2i\alpha_1(t) & 0 \\ 0 & -2i\alpha_7(t) & 3i\alpha_5(t) & 0 & -3i\alpha_3(t) & 0 & 2i\alpha_2(t) \\ 4i\alpha_7(t) & -6i\alpha_4(t) & -i\alpha_6(t) & 6i\alpha_2(t) & 0 & i\alpha_3(t) & -4i\alpha_1(t) \\ 6i\alpha_5(t) & -4i\alpha_6(t) & i\alpha_7(t) & 0 & -6i\alpha_1(t) & 4i\alpha_2(t) & -i\alpha_3(t) \end{pmatrix}. \qquad 3.9$$

In these expressions the $\alpha(t)^{'s}$ are given in Eq. (2.14) and Eq. (3.4). The first three $\beta(t)^{'s}$ are given in Eq. (2.15). The remaining $\beta(t)^{'s}$ are found as solutions to a system of first order differential equations Eq. (3.7). Using Eq. (3.5), the quasi-invariant may be written in the form

$$I =\sim I_0(t) + 2\Re(\beta_4(t)A_4 + \beta_6(t)A_6). \qquad 3.10$$

The first term $I_0(t)$ is the function Eq. (2.2), which is an invariant for the linear system. The remaining term may be expressed in the form

$$c_1(t)q^3 + c_2(t)q^2p + c_3(t)qp^2 + c_4(t)p^3,$$

with

$$\begin{aligned} \sqrt{2}c_1(t) &= \Re\beta_4(t) + \Re\beta_6(t) \\ \sqrt{2}c_2(t) &= -(3\Im\beta_4(t) + \Im\beta_6(t)) \\ \sqrt{2}c_3(t) &= -(3\Re\beta_4(t) - \Re\beta_6(t)) \\ \sqrt{2}c_4(t) &= \Im\beta_4(t) - \Im\beta_6(t). \end{aligned} \qquad 3.11$$

The functions $c_i(t)$, $i = 1 \rightarrow 4$, satisfy the following system of first order differential equations:

$$\begin{aligned} \dot{c}_1(t) &= K(t)c_2(t) + 3S(t)\alpha(t) \\ \dot{c}_2(t) &= -3c_1(t) + 2K(t)c_3(t) + 3S(t)\beta(t) \\ \dot{c}_3(t) &= -2c_2(t) + 3K(t)c_4(t) \\ \dot{c}_4(t) &= -c_3(t), \end{aligned} \qquad 3.12$$

where dot denotes differentiation with respect to t.

IV. THE QUANTUM LINEAR SYSTEM

As a first approximation for finding the quantum limits associated with the Hamiltonian Eq. (1.2) for transverse betatron oscillations, one neglects the non-linear multipole contributions and considers for each transverse degree of freedom a time-dependent harmonic oscillator with Hamiltonian

$$\hat{H}(t) = \frac{\hat{p}^2 + K(t)\hat{q}^2}{2}, \qquad 4.1$$

where t $(c = 1)$ represents arc length along an ideal storage ring orbit. The dynamical evolution of the conjugate quantum operators $\hat{p} = \dot{\hat{q}}$ and $\hat{q}$ is determined from the Heisenberg equations

$$\begin{aligned} \frac{d\hat{q}}{dt} &= i[\hat{H}, \hat{q}] \\ \frac{d\hat{p}}{dt} &= i[\hat{H}, \hat{p}]. \end{aligned} \qquad 4.2$$

The Courant-Snyder invariant as a function of the quantum operators $\hat{q}$ and $\hat{p}$ takes the form

$$2\hat{I}_0(t) = [(w\hat{p} - \dot{w}\hat{q})^2 + (\hat{q}/w)^2]. \qquad 4.3$$

The invariance follows from

$$\frac{d\hat{I}_0(t)}{dt} = i[\hat{H}(t), \hat{I}_0(t)] + \frac{\partial \hat{I}_0(t)}{\partial t}, \qquad 4.4$$

along with the conditions

$$\begin{aligned} \ddot{w} + K(t)w - \frac{1}{w^3} &= 0 \\ \ddot{\hat{q}} + K(t)\hat{q} &= 0. \end{aligned} \qquad 4.5$$

Expressed in the usual Courant-Synder parameters, one finds for each transverse coordinate $\hat{q}$

$$2\hat{I}_0 = \gamma(t)\hat{q}^2 + a(t)(\hat{q}\hat{p} + \hat{p}\hat{q}) + \beta(t)\hat{p}^2, \qquad 4.6$$

with

$$\begin{aligned} a(t) &= -w\dot{w} \\ \beta(t) &= w^2 \\ \gamma(t) &= \frac{1 + a^2(t)}{\beta(t)}. \end{aligned} \qquad 4.7$$

The quantum states for this system can be constructed with the aid of the squeezing operator [10] and [11] defined as

$$\hat{S} = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})}, \qquad 4.8$$

with complex $\xi = |\xi| exp(i\phi)$ and boson operators $\hat{a}$ and $\hat{a}^\dagger$. The time-independent rationalized Hamiltonian is

$$\hat{H}_o = \frac{\hat{p}^2 + \hat{q}^2}{2} = \hat{a}^\dagger \hat{a} + \frac{1}{2}, \qquad 4.9$$

where the boson operators $\hat{a}$ and $\hat{a}^\dagger$ are found from

$$\hat{q} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \qquad \hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}i}, \qquad 4.10$$

with the commutation relations

$$[\hat{q}, \hat{p}] = i \quad \Longrightarrow \quad [\hat{a}, \hat{a}^\dagger] = 1. \qquad 4.11$$

The Courant-Synder invariant Eq. (4.3) or Eq. (4.6) is found from the time-independent Hamiltonian using the squeezing operator Eq. (4.8) to write

$$\hat{I}_0(t) = \hat{S}\hat{H}_o\hat{S}^\dagger = \left(\hat{b}^\dagger \hat{b} + \frac{1}{2}\right), \qquad 4.12$$

where

$$\hat{b}(t) = \hat{S}e^{i\theta}\hat{a}\hat{S}^\dagger = \frac{1}{2}\left(\frac{1}{w} + w - i\dot{w}\right)\hat{a} + \frac{1}{2}\left(\frac{1}{w} - w - i\dot{w}\right)\hat{a}^\dagger, \qquad 4.13$$

with

$$\begin{aligned} \cosh|\xi| &= \frac{1}{2}\sqrt{(1/w + w)^2 + \dot{w}^2} \\ \tan\theta &= -\frac{w\dot{w}}{1 + w^2} \\ \tan(\theta + \phi) &= -\frac{w\dot{w}}{1 - w^2}. \end{aligned} \qquad 4.14$$

The eigenstates of $\hat{I}_0(t)$ satisfy the eigenvalue equation

$$\begin{aligned} \hat{I}_0(t)|n,t\rangle &= \left(n + \frac{1}{2}\right)|n,t\rangle \\ |n,t\rangle &= \frac{(\hat{b}^\dagger)^n}{\sqrt{n!}}|0\rangle. \end{aligned} \qquad 4.15$$

The states $|n,t\rangle$ are not Schrödinger states, for they are not solutions of the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}|n,t\rangle_s = \hat{H}(t)|n,t\rangle_s. \tag{4.16}$$

However, the Schrödinger states are of the form [12]

$$|n,t\rangle_s = e^{ia_n(t)}|n,t\rangle, \tag{4.17}$$

where the phase, as shown in Appendix A, is

$$a_n(t) = -\left(n+\frac{1}{2}\right)\int^t \frac{dt'}{w^2(t')}. \tag{4.18}$$

To evaluate the quantum correction to $\hat{I}_0(t)$ and to find the uncertainties associated with the operators $\hat{q}(t)$ and $\hat{p}(t)$, one must use the appropriate coherent state associated with $\hat{H}(t)$. This state is the time-dependent generalization of the coherent state [13] obtained from the eigenstates of the time-independent Hamiltonian Eq. (4.9). This is the nearest quantum state to the classical state of the simple harmonic oscillator. The coherent state for a time-dependent simple harmonic oscillator can be generated from the squeezed ground state as

$$|\beta,t\rangle_s = \mathbf{D}(\beta)|0,t\rangle_s, \tag{4.19}$$

where the displacement operator $\mathbf{D}(\beta)$ is defined as

$$\mathbf{D}(\beta) = e^{\beta\hat{b}^\dagger(t)-\beta^*\hat{b}(t)}. \tag{4.20}$$

Here β is a complex parameter, which is the eigenvalue of the operator $\hat{b}(t)$. This parameter is related to the classical value of the invariant $I_0(t)$ since

$${}_s\langle\beta,t|\hat{I}_0(t)|\beta,t\rangle_s = (|\beta|^2+1/2)\hbar/|\overline{p}| = I_0(t)+\hbar/2|\overline{p}|. \tag{4.21}$$

This includes the quantum correction $(1/2)(\hbar/|\overline{p}|)$. The variance of an operator $\hat{q}$ is defined as

$$\sigma^2(q) = {}_s\langle\beta,t|(\hat{q}-\overline{q})^2|\beta,t\rangle_s, \tag{4.22}$$

where the mean value of the operator $\hat{q}$ is

$$\overline{q} = {}_s\langle\beta,t|\hat{q}|\beta,t\rangle_s. \tag{4.23}$$

Time-independent Hamiltonian Eq. (4.9) results are found using the coherent state $|a\rangle$, defined for the complex parameter a as

$$|a\rangle = \mathbf{D}(a)|a\rangle = e^{a\hat{a}^\dagger-a^*\hat{a}}|0\rangle, \tag{4.24}$$

where the parameter a is related to the classical coordinates of position q and momentum p as in Eq. (2.7). They are

$$\begin{aligned}\sigma(q) = \sigma(p) &= \sqrt{\frac{\hbar}{2}} \\ \sigma(p)\sigma(q) &= \frac{\hbar}{2},\end{aligned} \qquad 4.25$$

which yield the minimum value for the uncertainty product. For the Hamiltonian Eq. (4.9), the coherent state $|a\rangle$ represents the quantum state nearest to the classical state, $\hbar \to 0$, for which $\sigma(q)$, $\sigma(p)$, and the uncertainty product are zero.

One can now use the states Eq. (4.19) and the definition of the variance Eq. (4.22) to obtain results appropriate for the a particle collider. For the scaling transformations frequently used in betatron dynamics

$$\begin{aligned}p &\to \frac{p}{|\vec{p}|} \\ H &\to \frac{H}{|\vec{p}|} \\ \hbar &\to \frac{\hbar}{|\vec{p}|},\end{aligned} \qquad 4.26$$

where the three-momentum magnitude $|\vec{p}|$ is

$$|\vec{p}| \approx \frac{\mathcal{E}}{c} \qquad 4.27$$

with relativistic particle energy $\mathcal{E}$, one finds that the uncertainties and the uncertainty product, represented in terms of the Courant-Snyder parameters Eq. (4.7), are

$$\begin{aligned}\sigma(q) &= \sqrt{\frac{\hbar\beta(t)}{2|\vec{p}|}} \\ \sigma\left(\frac{p}{\vec{p}}\right) &= \sqrt{\frac{\hbar\gamma(t)}{2|\vec{p}|}} \\ \frac{p}{|\vec{p}|} &= \frac{dq}{dt} \\ \sigma(q)\sigma\left(\frac{dq}{dt}\right) &= \frac{\hbar\sqrt{\beta(t)\gamma(t)}}{2|\vec{p}|}.\end{aligned} \qquad 4.28$$

Writing the amplitude as $q_{amp} = \sqrt{(\epsilon_0/\pi)\beta(t)}$ with the emittance $\epsilon_0 = 2\pi I_0(t)$, one finds the results

$$\begin{aligned}\frac{\sigma(q_{amp})}{q_{amp}} &= \left(\frac{\epsilon_q}{\epsilon_0}\right)^{1/2} \\ \epsilon_q/\pi &= \frac{\hbar}{2|\vec{p}|} \approx \frac{\hbar c}{2\mathcal{E}} \approx \frac{\lambda_{particle}}{2},\end{aligned} \qquad 4.29$$

where ϵ_q/π, the quantum emittance, represents half the resolution distance of a particle in the beam. With the approximations

$$
\begin{aligned}
\hbar c &\approx 2\times 10^{-19}\,\mathrm{TeV\,m}\\
\mathcal{E} &\approx 2\mathrm{TeV},
\end{aligned}
\qquad 4.30
$$

one finds

$$
\epsilon_q/\pi \approx 5\times 10^{-20}\,\mathrm{m}. \qquad 4.31
$$

For a typical proton collider with $\beta(t)\approx 300\,\mathrm{m}$ and with $q_{amp}\approx 3.5$ mm, one finds

$$
\begin{aligned}
\epsilon_0/\pi &\approx 4\times 10^{-8}\,\mathrm{m}\\
\sigma(q_{amp}) &\approx 3.9\times 10^{-6}\,\mathrm{mm}.
\end{aligned}
\qquad 4.32
$$

Similarly, the angular uncertainty is

$$
\sigma\left(\frac{dq}{dt}\right)\approx 1.3\times 10^{-11}\,\mathrm{rad}. \qquad 4.33
$$

V. THE QUANTUM NONLINEAR SYSTEM

The method described can be extended to the case when a nonlinear term is added to the quantum Hamiltonian. As an example, one considers the quantum operator Hamiltonian Eq. (1.2). Defining operator elements of $\hat{a}$ and $\hat{a}^\dagger$, with $[\hat{a},\hat{a}^\dagger]=\tilde{\epsilon}$, as

$$
\begin{aligned}
&\hat{A}_1=\hat{a}^2 && \hat{A}_2=\hat{a}^{\dagger 2}\\
&\hat{A}_3=(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)/2 && \hat{A}_4=\hat{a}^3\\
&\hat{A}_5=a^{\dagger 3} && \hat{A}_6=(\hat{a}^2\hat{a}^\dagger+\hat{a}\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}^2)/3\\
&\hat{A}_7=(\hat{a}^{\dagger 2}\hat{a}+\hat{a}^\dagger\hat{a}\hat{a}^\dagger+\hat{a}\hat{a}^{\dagger 2})/3,
\end{aligned}
\qquad 5.1
$$

one finds, keeping terms of order less than four in $\hat{a}$ and $\hat{a}^\dagger$ and first order in the where ${}_s\langle\beta, t|\hat{I}_0(t)|\beta, t\rangle_s$ is given in Eq. (4.21). The correction to the linear invariant is

$$ {}_s\langle\beta, t|\hat{I}_1(t)|\beta, t\rangle_s = I_1 + I_{1qc}. \qquad 6.12 $$

The classical correction to the linear invariant is

$$ I_1 = c_1(t)\bar{q}^3 + c_2(t)\bar{q}^2\bar{p} + c_3(t)\bar{p}^2\bar{q} + c_4(t)\bar{p}^3, \qquad 6.13 $$

and the quantum correction is

$$ \begin{aligned} I_{1qc} = & ((\bar{q}/2)[\beta(t)c_1(t) + \gamma(t)c_3(t) - \alpha(t)c_2(t)] \\ & + (\bar{p}/2)[\beta(t)c_2(t) + 3\gamma(t)c_4(t) - \alpha(t)c_3(t)])(\hbar/|\bar{p}|). \end{aligned} \qquad 6.14 $$

VII. RESULTS AND CONCLUSIONS

The nonlinear time-dependent Hamiltonian for one dimensional transverse classical dynamics is written in terms of the position coordinate q and the conjugate momentum p in Eq. (1.2). For this Hamiltonian, the equation of motion is found from Hamilton's equations

$$ \begin{aligned} \dot{q} &= \frac{\partial H(t)}{\partial p} \\ \dot{p} &= -\frac{\partial H(t)}{\partial q} \end{aligned} \qquad 7.1 $$

to be

$$ \ddot{q} + K(t)q + 3S(t)q^2 = 0. \qquad 7.2 $$

The classical approximate invariant associated with this Hamiltonian is

$$ \begin{aligned} I(t) = & \frac{\beta(t)p^2 + 2\alpha(t)pq + \gamma(t)q^2}{2} \\ & + c_1(t)q^3 + c_2(t)q^2p + c_3(t)qp^2 + c_4(t)p^3. \end{aligned} \qquad 7.3 $$

The time-dependent functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are found from the Eq. (2.5) or Eq. (2.13), and the functions $c_i(t)$ can be found from the differential equations Eq. (3.12). These system of equations are equivalent to the system of linear equations Eq. (3.7).

Numerical results are given which confirm the analytical development in the previous sections. Periodic solutions for the functions $c_i(t)$ allow the determination of these functions at a fixed point in a lattice with a sextupole nonlinearity. The values of the functions q and p are determined from nonlinear tracking for the first five circuits of the lattice. After the j^{th} turn, the quasi-invariant becomes

$$ I(j) = I_0(j) + c_1(j)g(1,j) + c_2(j)g(2,j) + c_3(j)g(3,j) + c_4(4,j), \qquad 7.4 $$

with

$$\begin{aligned} g(1,j) &= q(j)^3 \\ g(2,j) &= q(j)^2 p(j) \\ g(3,j) &= q(j) p(j)^2 \\ g(4,j) &= p(j)^3 . \end{aligned} \qquad 7.5$$

From the requirement that

$$I(k) - I(1) = 0, \qquad 7.6$$

for $k = 2 \to 5$, one finds the system of linear equations

$$\Delta(k) = \sum_{i=1}^{4} c_i(j) \Delta g(i,k), \qquad 7.7$$

with

$$\begin{aligned} \Delta(k) &= -(I_0(k) - I_0(1)) \\ \Delta g(i,k) &= g(i,k) - g(i,1) \end{aligned} \qquad 7.8$$

For the numerical results, the FODO approximation is used to find the lattice function $\beta(t)$. This function is derived in Appendix B. The system Eq. (7.7) and Eq. (7.8) is solved numerically to find the coefficients $c_i(t)$. The thin lens approximation is used where the lattice is made up of a single thin sextupole element and identical cells of length L. Each cell consists of a focusing and defocusing magnet separated by a bending drift magnet. The focal length of the focusing and defocusing magnets is f, and the phase advance per cell μ is found from

$$\sin(\mu/2) = \frac{L}{2f}. \qquad 7.9$$

The tune ν is obtained from

$$\nu = \frac{\mu N_c}{2\pi}, \qquad 7.10$$

where N_c is the number of cells. The maximum value of $\beta(t)$ occurs when $\alpha(t) = 0$. and $\beta(t) = 1/\gamma(t)$, and it is found from

$$\beta(t)_{max} = 2f \left(\frac{1 + \sin(\mu/2)}{1 - \sin(\mu/2)} \right). \qquad 7.11$$

The phase space plot of $\beta_{max} p$ cm and q cm for the classical quasi-invariant I is shown in Figure. 1. The classical results for both the invariant for the linear system I_0 cm and the quasi-invariant I cm for the nonlinear system are plotted in Figure 2. as a function of turn number. It is clearly seen here that the methods leading to the quasi-invariant produce a more stable quantity than I_0. For the example considered, the values $N_c = 4$, $\mu = \pi/2$, $L/2 = 8875$ mm.

quantum parameter $\tilde{\epsilon}$, the closed approximate Lie algebra

$$
\begin{array}{ll}
[\hat{A}_1, \hat{A}_2] = 4\tilde{\epsilon}\hat{A}_3 & [\hat{A}_1, \hat{A}_3] = 2\tilde{\epsilon}\hat{A}_1, \\
[\hat{A}_1, \hat{A}_4] = 0, & [\hat{A}_1, \hat{A}_5] = 6\tilde{\epsilon}\hat{A}_7, \\
[\hat{A}_1, \hat{A}_6] = 2\tilde{\epsilon}\hat{A}_4, & [\hat{A}_1, \hat{A}_7] = 4\tilde{\epsilon}\hat{A}_6, \\
[\hat{A}_2, \hat{A}_3] = -2\tilde{\epsilon}\hat{A}_2, & [\hat{A}_2, \hat{A}_4] = -6\tilde{\epsilon}\hat{A}_6, \\
[\hat{A}_2, \hat{A}_5] = 0, & [\hat{A}_2, \hat{A}_6] = -4\tilde{\epsilon}\hat{A}_7, \\
[\hat{A}_2, \hat{A}_7] = -2\tilde{\epsilon}\hat{A}_5 & \\
[\hat{A}_3, \hat{A}_4] = -3\tilde{\epsilon}\hat{A}_4, & [\hat{A}_3, \hat{A}_5] = 3\tilde{\epsilon}\hat{A}_5, \\
[\hat{A}_3, \hat{A}_6] = -\tilde{\epsilon}\hat{A}_6, & [\hat{A}_3, \hat{A}_7] = \tilde{\epsilon}\hat{A}_7, \\
[\hat{A}_4, \hat{A}_5] = 0, & \\
[\hat{A}_4, \hat{A}_6] = 0, & [\hat{A}_4, \hat{A}_7] = 0, \\
[\hat{A}_5, \hat{A}_6] = 0, & [\hat{A}_5, \hat{A}_7] = 0, \\
[\hat{A}_6, \hat{A}_7] = 0. &
\end{array}
\tag{5.2}
$$

These algebraic relations are the same as those associated with the Poisson bracket relations. The hermitian Hamiltonian operator and the hermitian quasi-invariant operator are

$$\hat{H}(t) = \sum_{i=1}^{7} \alpha_i(t)\hat{A}_i, \tag{5.3}$$

and

$$\hat{I}(t) = \sum_{j=1}^{7} \beta_j(t)\hat{A}_j, \tag{5.4}$$

where the $\alpha_i(t)$ and $\beta_j(t)$ are defined as before. The equation which must be satisfied by the hermitian quasi-invariant operator is

$$\frac{d\hat{I}(t)}{dt} = \frac{\partial \hat{I}(t)}{\partial t} + i[\hat{H}(t), \hat{I}(t)] = 0. \tag{5.5}$$

This equation along with the quantum algebra leads to the same set of differential equations Eq. (3.7) that appear in the classical case.

VI. QUANTUM CORRECTIONS

The quantum corrections to the quasi-invariant are obtained using first order perturbation theory. The Boson operators which occur in the linear invariant Eq. (4.12) can be written as

$$\begin{pmatrix} \hat{b} \\ \hat{b}^\dagger \end{pmatrix} = \begin{pmatrix} f_1(w) & f_2(w) \\ f_2^*(w) & f_1^*(w) \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}. \tag{6.1}$$

The inverse transformation is

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \begin{pmatrix} f_1^*(w) & -f_2(w) \\ -f_2^*(w) & f_1(w) \end{pmatrix} \begin{pmatrix} \hat{b} \\ \hat{b}^\dagger \end{pmatrix}, \tag{6.2}$$

where

$$\begin{aligned} f_1(w) &= \frac{1}{2}\left(\frac{1}{w} + w - i\dot{w}\right) \\ f_2(w) &= \frac{1}{2}\left(\frac{1}{w} - w - i\dot{w}\right), \end{aligned} \tag{6.3}$$

and

$$|f_1(w)|^2 - |f_2(w)|^2 = 1. \tag{6.4}$$

These scaled Boson operators satisfy the commutation relations

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= 1 \\ [\hat{b}, \hat{b}^\dagger] &= 1. \end{aligned} \tag{6.5}$$

The operators $\hat{q}$ and $\hat{p}$ become

$$\begin{aligned} \hat{q} &= (w/\sqrt{2})(\hat{b} + \hat{b}^\dagger) \\ \hat{p} &= (1/iw\sqrt{2})(\hat{b} - \hat{b}^\dagger) + (\dot{w}/\sqrt{2})(\hat{b} + \hat{b}^\dagger). \end{aligned} \tag{6.6}$$

The quantum corrections to the quasi-invariant are found from the operator

$$\hat{I}(t) = \hat{I}_0(t) + \hat{I}_1(t) \tag{6.7}$$

where from Eq. (4.12)

$$\hat{I}_0(t) = (1/2)[(w\hat{p} - \dot{w}\hat{q})^2 + (\hat{q}/w)^2], \tag{6.8}$$

and

$$\begin{aligned} \hat{I}_1(t) =& c_1(t)\hat{q}^3 + c_2(t)\frac{1}{3}(\hat{q}^2\hat{p} + \hat{q}\hat{p}\hat{q} + \hat{p}\hat{q}^2) + \\ & c_3(t)\frac{1}{3}(\hat{p}^2\hat{q} + \hat{p}\hat{q}\hat{p} + \hat{q}\hat{p}^2) + c_4(t)\hat{p}^3. \end{aligned} \tag{6.9}$$

The classical values of the operators $\hat{q}$ and $\hat{p}$ are

$$\begin{aligned} \bar{q} &=_s \langle \beta, t|\hat{q}|\beta, t\rangle_s \\ \bar{p} &=_s \langle \beta, t|\hat{p}|\beta, t\rangle_s. \end{aligned} \tag{6.10}$$

The expectation value of the quasi-invariant operator is

$${}_s\langle \beta, t|\hat{I}(t)|\beta, t\rangle_s =_s \langle \beta, t|\hat{I}_0(t)|\beta, t\rangle_s +_s \langle \beta, t|\hat{I}_1(t)|\beta, t\rangle_s, \tag{6.11}$$

and $\nu = 0.33666667 + N_c\mu/(2\pi)$, with near resonance fractional tune contribution, have been used. The initial values $q = 0.3$ cm, $\beta_{max}p = 0$ along with the sextupole strength $3s_e = 0.1 \times 10^{-5}\,\mathrm{cm}^{-2}$ have been used. For integer j, the sextupole function is approximated by $S(t) = (s_e/3)\delta(t - jT_0)$, where T_0 is the orbital period. For the present case, the values of the periodic functions $c_i(0)$ are

$$\begin{aligned}
c_1 &= -3.41219 \times 10^{-6}\ \ \mathrm{cm}^{-2} \\
c_2/\beta_{max} &= -0.91910 \times 10^{-7}\ \ \mathrm{cm}^{-2} \\
c_3/\beta^2_{max} &= +0.99563 \times 10^{-5}\ \ \mathrm{cm}^{-2} \\
c_4/\beta^3_{max} &= -1.11468 \times 10^{-7}\ \ \mathrm{cm}^{-2},
\end{aligned} \tag{7.12}$$

with $\beta_{max} = 38389.279$ cm.

It is clear from the Figure 2. that the quasi-invariant is nearly stable. It remains this way for increasingly larger number of turns. It oscillates with small amplitude and with period of 100 turns. The amplitude of the oscillation depends upon the strength of the sextupole nonlinearity, and the period results from the nearness of the fractional tune to the third integer resonance. Although, the present quasi-invariant, which includes terms in q and p through third order, becomes increasingly unstable for large values of the sextupole strength or large initial values of the amplitude q, it is clear that the method can be extended to include arbitrarily higher order corrections which will improve the stability of the quasi-invariant. The quantum correction associated with the quasi-invariant can be found from Eq. (4.21) and Eq. (6.14), and for the numerical example being considered it takes the value

$$I_{qc} \approx (1/2 + \bar{q}/2(\beta_{max}c_1 + c_3/\beta_{max}))\hbar/|\vec{p}| \approx 0.538\,\hbar/|\vec{p}|. \qquad 7.13$$

Although very small for a hadron collider, it would be more significant for a low energy nonlinear time-dependent oscillator of the type described by the Hamiltonian Eq. (1.2).

In conclusion, it is seen that the Lie algebra methods used for both the classical and quantum quasi-invariants provide a useful approximation for the invariant associated with the time-dependent nonlinear oscillator. For applications to betatron dynamics, this method provides a complimentary method to the usual nonlinear-map tracking methods. In addition, the quantum states $|\beta, t\rangle_s$ of Eq. (4.19) provides the connection between the quantum operator for the quasi-invariant and the classical result when these states are used to form matrix elements of the type used to obtain the quantum uncertainties Eq. (4.28) and the quantum correction Eq. (6.14).

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APPENDIX A: THE SCHRÖDINGER STATE PHASE

The phase Eq. (4.18) can be found by first differentiating Eq. (4.17) with respect to t, and then using Eq. (4.16) to write the matrix element

$$i\frac{\partial a_n(t)}{\partial t} + \langle n,t|\frac{\partial}{\partial t}|n,t\rangle = -i\langle n,t|\mathbf{H}(t)|n,t\rangle$$

$$= \frac{1}{2}(\dot{w}^2 + K(t)w^2 + 1/w^2)(n+1/2), \qquad A.1$$

where Eq. (4.13) is used to express the Hamiltonian Eq. (4.1) as a function of $\mathbf{b}(t)$ and its adjoint. The matrix elements of this operator can be found from

$$\mathbf{b}^\dagger|n,t\rangle = \sqrt{n+1}\,|n+1,t\rangle \qquad A.2$$

and

$$\mathbf{b}\,|n,t\rangle = \sqrt{n}\,|n-1,t\rangle \quad \Longrightarrow \quad \langle n,t|\mathbf{b}^\dagger = \sqrt{n}\langle n-1,t|. \qquad A.3$$

Making the replacement $n \to n-1$ in Eq. (A.2), one can derive the identity

$$\langle n,t|\frac{\partial \mathbf{b}^\dagger}{\partial t}|n-1,t\rangle + \langle n,t|\mathbf{b}^\dagger\frac{\partial}{\partial t}|n-1,t\rangle$$

$$= \sqrt{n}\langle n,t|\frac{\partial}{\partial t}|n,t\rangle, \qquad A.4$$

where

$$\frac{\partial \mathbf{b}^\dagger}{\partial t} = \frac{1}{2}(i(w\ddot{w} - \dot{w}^2) - 2\dot{w}/w)\mathbf{b} + i(w\ddot{w} - \dot{w}^2)\mathbf{b}^\dagger. \qquad A.5$$

Therefore

$$\langle n,t|\frac{\partial}{\partial t}|n,t\rangle = \langle 0,t|\frac{\partial}{\partial t}|0,t\rangle + i\,\frac{t}{2}(w\ddot{w} - \dot{w}^2). \qquad A.6$$

Choosing

$$\langle 0,t|\frac{\partial}{\partial t}|0,t\rangle = \frac{i(w\ddot{w} - \dot{w}^2)}{4}, \qquad A.7$$

one finds from Eq. (4.5), Eq. (A.1), Eq. (A.6), and Eq. (A.7) the differential equation

$$\frac{da_n(t)}{dt} = -(n+\frac{1}{2})\,\frac{1}{w^2(t)}, \qquad A.8$$

which has the solution Eq. (4.18).

APPENDIX B: THE BETA LATTICE FUNCTION

In this appendix, following the methods of [6], the lattice function $\beta(t)$ used in the numerical calculations is derived. It is found for a lattice made up of similar cells of the FODO (focusing, drift, defocusing, drift) form. Focusing and defocusing are achieved with thin lens quadrupole magnets, and the drifts occur through bending dipole magnets of length L and strength B_0. The function $\beta(t)$ has period $2L$, and the function on the interval $L < t < 2L$ is found from that on the interval $0 < t < L$ using

$$f(t)_{L<t<2L} = f(2L - t)_{0<t<L}. \qquad B.1$$

The beta functions for a lattice with phase advance μ per cell are found from the $(1,2)$ component of the transfer matrix. The function $\beta(t)$ is found from

$$\beta(t)\sin\mu = (\mathbf{O}(t)\mathbf{F}\mathbf{O}(L)\mathbf{D}\mathbf{O}(L-t))_{12}, \qquad B.2$$

where the focusing and defocusing matrices for lenses of focal length f are, respectively,

$$\mathbf{F} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}. \qquad B.3$$

The matrix for a drift of distance t is

$$\mathbf{O(t)} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \qquad B.4$$

The resulting beta function for $0 < t < L$ is

$$\beta(t) = \frac{2L}{\sin\mu}\left(1 + \frac{L}{2f} - (\frac{1}{f} + \frac{L}{2f^2})t + \frac{t^2}{2f^2}\right), \qquad B.5$$

where $L = 2f\sin(\mu/2)$.

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FIGURE CAPTIONS

FIG. 1. The quasi-invariant in phase space using $\beta_{max}\, p$ cm and q cm.

FIG. 2. The Courant-Snyder invariant, $\epsilon_0/2\pi = I_0$ cm, and the quasi-invariant $\epsilon/2\pi = I$ cm, as a function of turn-number.

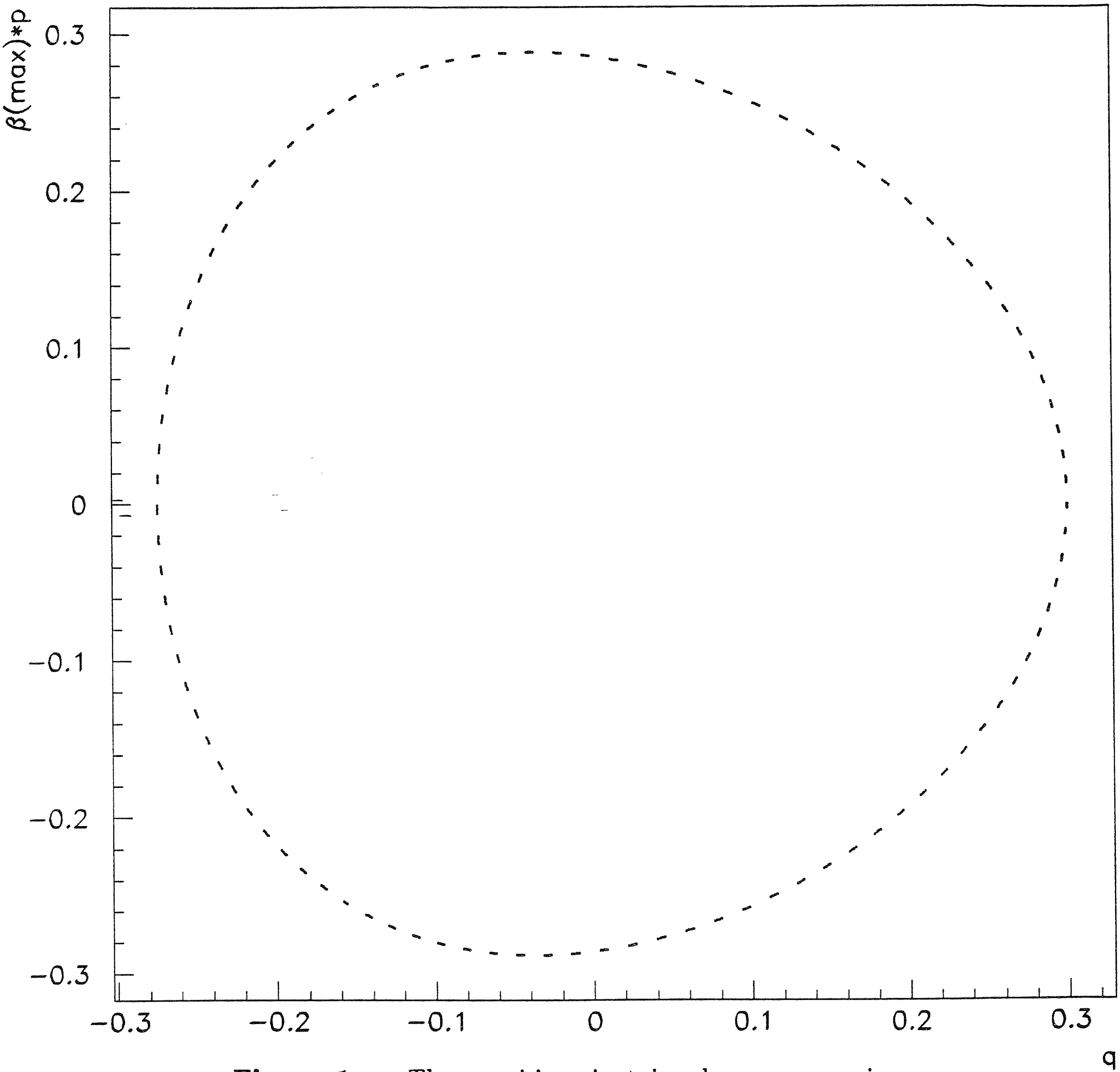
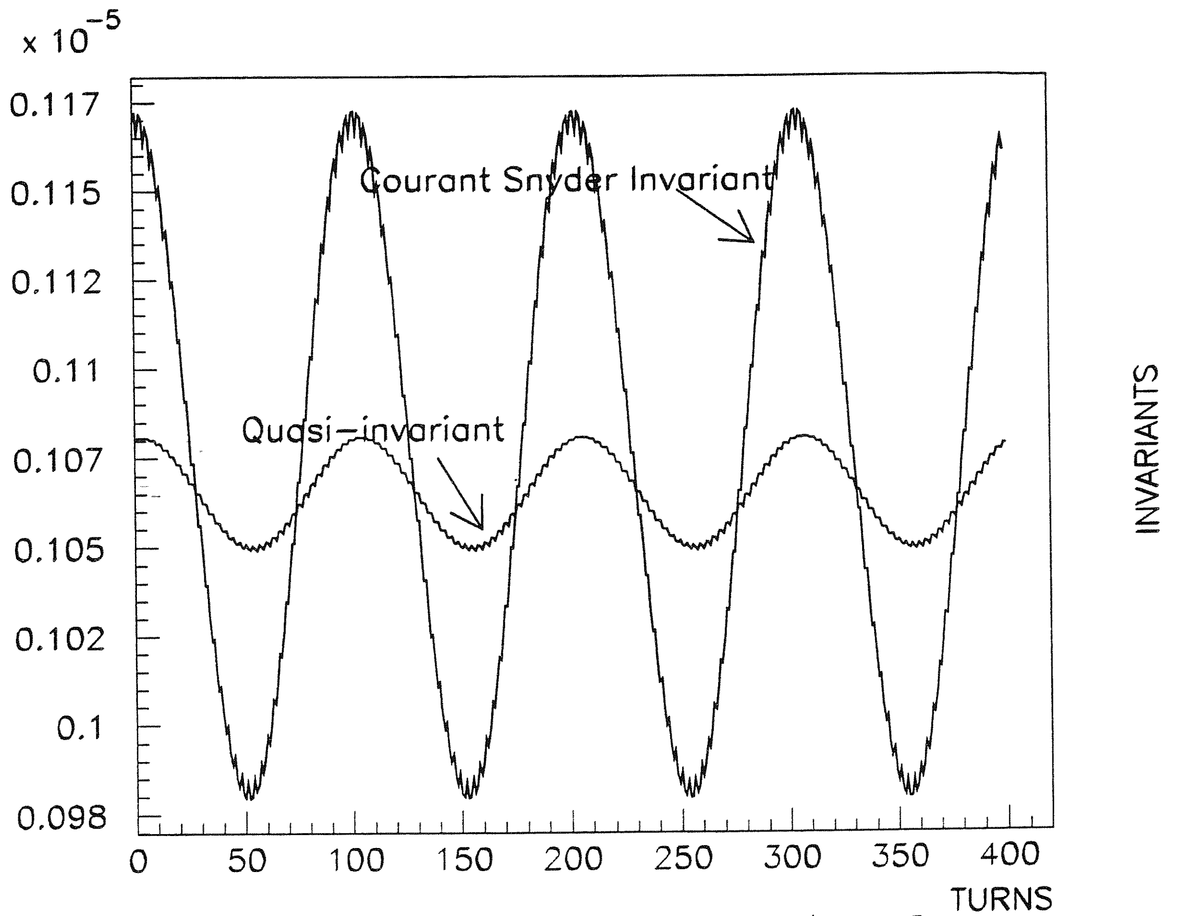


Figure 1: The quasi-invariant in phase space using $\beta_{max}\, p$ cm and q cm.



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