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Phases and Residual Gauge Symmetries of Higgs Models

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Abstract

After elimination of the redundant variables, gauge theories may still exhibit symmetries associated with the gauge fields. The role of these *residual gauge symmetries* is discussed within the Abelian Higgs model and the Georgi-Glashow model. In the different phases of these models, these symmetries are realized differently. The characteristics of emergence and disappearance of the symmetries are studied in detail and the implications for the dynamics in Coulomb, Higgs, and confining phases are discussed.

1 Introduction

An appreciation of the deep and pervasive role of symmetries in quantum mechanics and field theory has been one of the great triumphs of twentieth-century theoretical physics. Indeed, the connection between symmetries and massless particles is so compelling that one's understanding of the symmetries of a system must be fundamentally incomplete if it cannot account for the massless excitations and particles arising in all the phases of that system. By this criterion, there are still gaps in the understanding of symmetries in gauge theories.

The goal of this work is to clarify the role of *residual gauge symmetries* and their relation to the phases of gauge theories, thereby filling some of these gaps. By residual gauge symmetries we mean symmetries associated with gauge fields that remain present after a complete gauge fixing. In particular, these residual symmetries cannot be generated by the Gauss law operator for one of two reasons, either because of some geometrical property or because of some dynamical obstruction. These residual symmetries will be studied in both the Abelian Higgs model [1, 2, 3] and in the non-Abelian Georgi-Glashow model [4]. These models exhibit a rich variety of phases and realizations of residual gauge symmetries, and provide examples of residual symmetries that arise from geometrical properties and from obstructions.

Quite generally, electrodynamics, after elimination of redundant variables, must exhibit a continuous symmetry that by its spontaneous breakdown produces massless particles. Otherwise, the zero value of the photon mass would become accidental after elimination of the redundant variables. Indeed, the photons can be interpreted as Goldstone bosons associated with a shift ("displacement") symmetry (cf. [5, 6, 7, 8, 9]). In our discussion of the Abelian Higgs model, we discuss the peculiar properties of this symmetry and investigate its fate in the Higgs phase. As is well known (for example, see [10]), the Higgs phase, when formulated in terms of the physical degrees of freedom of the unitary gauge, does not exhibit any continuous gauge-like symmetry nor is it a signature of the original global phase symmetry present; on the other hand, the system does not contain massless particles either. In the Abelian Higgs model, the unitary gauge condition is optimal for describing the Higgs phase; its choice, however, is not mandatory. For the discussion of the symmetry we will implement the Coulomb gauge. This is technically more involved and the formalism is less transparent; however, the gauge choice is not restricted to the description of the Higgs phase and is obviously quite appropriate in the Coulomb phase. In this description we shall be able to trace the disappearance of the two continuous symmetries of the Abelian Higgs model and to understand why this disappearance is not accompanied by the appearance of Goldstone bosons. In this discussion it will prove useful to assume one of the space-time coordinates to be compact; in this way the subtle infrared properties of the model associated with the displacements can be controlled and simple topological interpretations of some of our results can be obtained. It will allow us to discuss the realization of the displacement symmetry at finite temperature and to clarify its possible violation by Debye screening. We will demonstrate that the requirement of invariance under displacements imposes severe constraints on the structure of finite-temperature effective Lagrangians. Finally, for our discussion of the non-Abelian center symmetry [11, 12, 13] the introduction of a compact space-time coordinate is indispensable.

The non-Abelian Higgs model, the Georgi-Glashow model [4], exhibits complementary symmetry properties. In the confining phase, the phase with heavy adjoint Higgs scalars, the system does not exhibit any residual symmetry nor does it contain massless particles. We argue it is even not quite correct to count the center symmetry which is realized in the confining phase as a proper symmetry; rather we have to consider it as a discrete leftover of the redundancy expressed by the non-Abelian gauge symmetry and to take it as an indication of an incomplete gauge fixing. In the Higgs phase, on the other hand, the physical degrees of freedom of the unitary gauge contain photons and therefore in the transition from the confining to the Higgs phase a displacement-like symmetry is expected to be present in the Hamiltonian or Lagrangian describing the Higgs phase. Also, the rather mysterious center symmetry has to be broken by the Higgs vacuum. Otherwise static fundamental charges should have a confining interaction, which is certainly not expected in the presence of physical massive and massless vector particles. Within the unitary gauge we will describe the mechanisms by which a symmetry absent in the confining phase can emerge and specify how the center symmetry, a discrete part of the gauge symmetry, is spontaneously broken. As in the Abelian case, a representation of the dynamics in the confining and Higgs phase within one common gauge choice can be attained provided the gauge condition does not involve the Higgs field. Unlike the Coulomb phase of the Abelian Higgs model, the confining phase of Yang-Mills theories, as well as its relation to the Higgs phase, is understood only poorly. We will attempt to shed some light on the connection between the Higgs and confining phases. To this end, we will derive an effective Coulomb-gauge Lagrangian for the description of the Higgs phase and with the help of "gauge-invariant" variables extend this discussion to other gauge conditions. This will allow us to study qualitatively the fading of possible mechanisms of confinement in the transition to the Higgs phase.

2 Abelian Higgs Model

2.1 Residual Gauge Symmetries

In this section we will discuss residual gauge symmetries and their realization in the two phases of the Abelian Higgs model. In standard notation, the Lagrangian is given by

$$\mathcal{L}\left[A,\Phi\right] = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + (\partial_\mu + ieA_\mu)\Phi(\partial^\mu - ieA^\mu)\Phi^\star - V(|\Phi|). \tag{2.1}$$

The Higgs potential $V(|\Phi|)$ and the coupling constant e determine the phases of this model. As emphasized in the Introduction, we also assume the system to be of finite extent L in at least one of the spatial directions (the 3-direction). In conventional treatments, one normally derives the properties of these phases by choosing gauges that simplify the dynamics in the relevant regime. In the weak coupling limit ($e \ll 1$ – see below) of the Abelian Higgs model, any gauge choice that constrains the gauge field such as the Lorentz or Coulomb gauge is appropriate. On the other hand, the description of the strong coupling limit, i.e., of the Higgs phase, simplifies in the "unitary" gauge, which constrains the Higgs field and thereby displays directly the particle content of this phase. In this work, however, we shall perform an analysis within a single gauge choice. Although this necessarily will complicate matters technically, it will enable us to investigate the fate of the symmetries in the two phases and will lead to a unified dynamical description.

In the definition of the generating functional

$$Z[J,k] = \int d[A,\Phi]\,\delta[f(\mathbf{A},\Phi)]e^{iS[A,\Phi]}e^{i\int d^4x\,(J^\mu A_\mu + k\Phi)} \tag{2.2}$$

we assume the gauge condition

$$f(\mathbf{A},\Phi) = 0 \tag{2.3}$$

does not affect the time component of A. In a large part of this section we will choose the Coulomb gauge. We also will use in part of our discussion the polar representation for the Higgs field

$$\Phi = \rho e^{i\varphi}.$$

After implementation of the Coulomb gauge, residual symmetries are still present. Clearly the system described by the above gauge fixed functional is invariant under two global, continuous symmetries, the *global (rigid) phase changes*

$$\Phi(x) \to e^{i\alpha}\Phi(x) \qquad (\varphi(x) \to \varphi(x) + \alpha) \tag{2.4}$$

and *displacements* [9]

$$\mathbf{A}(x) \to \mathbf{A}(x) + \frac{1}{e}\mathbf{d} \qquad \Phi(x) \to e^{i\mathbf{d}\cdot\mathbf{x}}\Phi(x) \qquad (\varphi(x) \to \varphi(x) + \mathbf{d}\cdot\mathbf{x}). \tag{2.5}$$

These transformations do not change the action nor do they change the gauge condition. Strictly speaking, displacements are not continuous transformations, since periodicity in the 3-direction enforces quantization of the 3-component of the displacement vector

$$d_3 = \frac{2\pi n}{L}.$$

It is obvious that a gauge choice that constrains the gauge fields does not affect the global phase symmetry of the theory. The displacement symmetry is a direct consequence of the equations of motion. With the help of current conservation, Maxwell's equations can be written as

$$\partial_\mu(F^{\mu\nu} - ej^\mu x^\nu) = 0. \tag{2.6}$$

In this way, the components of Maxwell's displacement vector

$$\mathbf{D} = \int d^3x\,\left(\mathbf{E} + e\mathbf{x}j^0\right)$$

are identified as the generators of displacements. We note that along a compact direction

$$\int d^3x\,E^i \neq -\int d^3x\,x^i\,\mathrm{div}\,\mathbf{E}$$

and thus displacements cannot be generated by the Gauss law operator $G = \mathrm{div}\,\mathbf{E} - ej^0$. Therefore, after gauge fixing, displacements are present as residual symmetries and the zero modes, $\int d^3x\mathbf{A}(x)$, are gauge invariant. In a gauge-fixed formulation, the interpretation of displacements as residual symmetry transformations becomes manifest. In the Coulomb gauge, with the transverse fields as dynamical degrees of freedom, the symmetry transformation can be softly modulated

$$\mathbf{A}(x) \to \mathbf{A}(x) + \frac{1}{2e}\,\mathrm{rot}\,[\mathbf{d}(\mathbf{x}) \times \mathbf{x}] \qquad \Phi(x) \to e^{i\mathbf{d}(\mathbf{x})\cdot\mathbf{x}}\Phi(x). \tag{2.7}$$

This transformation is clearly not a gauge transformation; the gauge condition is respected by construction and it reduces in the infinite wavelength limit to the symmetry transformation (2.5). Thus for large wavelength modulations the restoring forces generated when applying this transformation to the ground state will be weak and the excited states therefore of low energy. Unlike other symmetry transformations, the displacement symmetry has the peculiar property that in a certain sense, it always gives rise to gapless excitations (or if the 3-space is compact, excitations with energies $\sim 1/L$.) The existence of an operator whose commutator with $\mathbf{D}$ has a nonvanishing expectation value is guaranteed kinematically by the canonical commutation relation

$$\left[D_k, \frac{1}{V}\int d^3x\, A_l\right] = i\delta_{kl}. \tag{2.8}$$

Thus Goldstone bosons associated with this symmetry must exist. The connection to a possibly nontrivial structure of the vacuum is more subtle. In Appendix I the physical content and some general issues concerning spontaneous symmetry breakdown are discussed. We will address the issue of the structure of the vacuum in the context of the Georgi-Glashow model. Related to the kinematical nature of the appearance of the photons as Goldstone bosons is the fact that the Ward identities associated with the displacements and the rigid phase transformations actually imply the Maxwell equations (2.6) (cf. Appendix II).

The presence of two continuous global symmetries has to be expected. The dynamics has not yet been specified and therefore the above gauge-fixed generating functional could, for example, describe decoupled radiation and matter. In this case, the theory would contain massless photons enforced by the displacement symmetry and scalars that as well may be massless as a result of the spontaneous breakdown of the rigid phase symmetry. On the other hand, as is well known, there are no massless excitations in the Higgs phase, nor does the Higgs phase exhibit any symmetries [10, 14]. The disappearance of the residual gauge symmetries in the Higgs phase without emergence of massless particles is the main subject of the following two sections.

2.2 The Higgs Phase

In this section we derive in Coulomb-like gauges the effective Lagrangian describing the dynamics in the Higgs phase. On the basis of this development we will give a complete account of the symmetries present in the original form of the Lagrangian. To this end we have to integrate out the constrained variables contained in the gauge fixed Lagrangian, the time component of the gauge field. The action is quadratic in the field A_0. The integration can be carried out exactly resulting in an expression for the generating functional which neither contains integration over redundant nor over constrained variables. A straightforward calculation yields for general gauge conditions of the form (2.3) and with vanishing time component of the source J

$$Z[J,k] = \int d[\mathbf{A},\Phi]\,\delta[f(\mathbf{A},\Phi)]e^{iS_{\text{eff}}[\mathbf{A},\Phi]}e^{i\int d^4x(-\mathbf{J}\cdot\mathbf{A}+k\Phi)} \tag{2.9}$$

with

$$S_{\text{eff}} = \int d^4x \, (\mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}}) - \tfrac{1}{2} \operatorname{tr} \{\ln \Gamma\}$$

$$\mathcal{L}_{\text{pot}} = -\tfrac{1}{4} F^{ij} F_{ij} + (\partial_i + ieA_i)\Phi(\partial^i - ieA^i)\Phi^\star - V(|\Phi|) \tag{2.10}$$

$$\mathcal{L}_{\text{kin}} = (e\rho^2 \partial_0 \varphi - \tfrac{1}{2}\partial_i \partial_0 A^i) \frac{2}{\Gamma} (e\rho^2 \partial_0 \varphi - \tfrac{1}{2}\partial_i \partial_0 A^i) + \rho^2 (\partial_0 \varphi)^2 + (\partial_0 \rho)^2 + \tfrac{1}{2}(\partial_0 A^i)^2$$

and

$$\Gamma = \Delta - 2e^2 \Phi \Phi^\star . \tag{2.11}$$

In Coulomb gauge,

$$f(\mathbf{A}, \Phi) = \operatorname{div} \mathbf{A}$$

the expression for $\mathcal{L}_{\text{kin}}$ simplifies

$$\mathcal{L}_{\text{kin}} = \rho^2 \partial_0 \varphi \frac{1}{\Gamma} \Delta \, \partial_0 \varphi + (\partial_0 \rho)^2 + \tfrac{1}{2}(\partial_0 \mathbf{A})^2 . \tag{2.12}$$

Up to this point, we have not yet used any property that is specific to the Higgs phase. As in the standard treatment, we now assume that the Higgs phase emerges if the self-interaction V generates a large expectation value ρ_0 of the Higgs field and that it becomes meaningful to use the polar representation. We decompose the modulus of the Higgs field into this vacuum expectation value and a fluctuating piece

$$\rho(x) = \rho_0 + \frac{1}{\sqrt{2}} \sigma(x)$$

and keep only terms quadratic in the fields σ, φ, and $\mathbf{A}$. In the resulting expression (cf. Eq. (2.11))

$$\mathcal{L}_{\text{pot}} \approx -\tfrac{1}{4} F^{ij} F_{ij} - \tfrac{1}{2}(\nabla \sigma)^2 - \rho_0^2 (\nabla \varphi + e\mathbf{A})^2$$

we single out the zero modes of the fields $\nabla \varphi$ and $\mathbf{A}$

$$\nabla \varphi + e\mathbf{A} = (\nabla \varphi + e\mathbf{A})_0 + (\nabla \varphi + e\mathbf{A})', \quad \text{with} \int d^3x \mathbf{A}' = \int d^3x \nabla \varphi' = 0, \tag{2.13}$$

and note that $\mathbf{A}'$ and φ' are invariant under displacements. We obtain

$$S_{\text{eff}} \approx \int d^4\, x \mathcal{L}_{\text{eff}}$$

with

$$\mathcal{L}_{\text{eff}} = -\tfrac{1}{2}\mathbf{A}'(\Box + m^2)\mathbf{A}' - \tfrac{1}{2}\chi(\Box + m^2)\chi - \tfrac{1}{2}\sigma \Box \sigma - \tfrac{1}{4} V''(\rho_0)\sigma^2 + \mathcal{L}_0 . \tag{2.14}$$

The mass of the vector mesons is

$$m^2 = 2e^2 \rho_0^2$$

and their longitudinal component is given by

$$\chi = \rho_0 \left(\frac{-2\Delta}{-\Delta + m^2} \right)^{\frac{1}{2}} \varphi' . \tag{2.15}$$

The zero-mode dynamics is described by

$$\mathcal{L}_0 = \tfrac{1}{2}(\dot{\mathbf{A}}_0)^2 - \rho_0^2(\nabla\varphi + e\mathbf{A})_0^2. \tag{2.16}$$

No time derivative of $(\nabla\varphi)_0$ appears so it is not a dynamical variable. As far as the 3-component of this vector is concerned, this time independence is a consequence of the periodicity of the scalar field, which obviously requires

$$\int_0^L dz\, \partial_3\varphi = 2n\pi$$

and therefore does not permit continuous changes of the 3-component of $(\nabla\varphi)_0$. Since $(\nabla\varphi)_0$ is essentially an irrelevant constant we may absorb it in $\mathbf{A}_0$ and write

$$\mathbf{A}_0 + \frac{1}{e}(\nabla\varphi)_0 \to \mathbf{A}_0. \tag{2.17}$$

Thus $\mathcal{L}_0$ provides the zero modes for the massive photon field.

It is interesting how, in Coulomb gauge, the standard properties of the Higgs phase emerge. Like in the unitary gauge, the mass of the transverse gauge fields is generated by their coupling to the Higgs condensate. The massive longitudinal degrees of freedom appear due to a dynamically modified kinetic energy of the phase φ of the Higgs field (cf. Eq. (2.12)). Rewriting the kinetic energy in canonical form, Eq. (2.15) makes the transformation of the massless compact variables φ into the massive noncompact fields χ manifest. In the Coulomb gauge, the crucial quantity that distinguishes Coulomb and Higgs phase is the operator Γ of Eq. (2.11). In the Higgs phase, the relevant field configurations must generate a gap in the spectrum of Γ. In this case, the fluctuations of ρ^2 around a nonvanishing expectation value may be neglected. In turn, the transition from the Higgs to the Coulomb phase has to be accompanied by appearance and possible condensation of zero modes of Γ. This is the mechanism of formation of a condensate of vortices [15, 16]in the unitary gauge. In unitary gauge, the appearance of a vortex signifies a failure of the gauge condition – the gauge transformation that eliminates the phase of the Higgs field is ill defined at those points where the Higgs field vanishes. The association of singularities with the zeroes of the Higgs field is thus a gauge artifact and has no physical significance.

We now resume our discussion of the residual symmetries in the Abelian Higgs model. Our construction shows that in the space of redefined fields (cf. Eqs. (2.15), (2.13) and (2.17)) global phase changes (Eq. (2.4)) and displacements (Eq. (2.5))

$$\chi(x) \to \chi(x) \qquad \mathbf{A}'(x) \to \mathbf{A}'(x) \tag{2.18}$$

are reduced to identity transformations. Thus, in the Higgs phase the Lagrangian does not exhibit any symmetries, as one might have expected.

The mechanism that makes the rigid phase symmetry disappear is displayed by Eq. (2.12). In the Higgs phase, it is not φ that is a dynamical variable, rather it is $\sqrt{-\Delta}\varphi$. Thus the global phase of the Higgs field is not dynamical. The physics behind the redefinition of the phase variable is easily understood in the Coulomb gauge. The Coulomb interaction of the charged scalar field in the Higgs phase

$$e^2 \int d^3x\, d^3x' \frac{j^0(x) j^0(x')}{|\mathbf{x}-\mathbf{x}'|} \sim e^2\rho_0^4 \int d^3x\, d^3x' \,\frac{\partial_0\varphi(x)\partial_0\varphi(x')}{|\mathbf{x}-\mathbf{x}'|}$$

gives rise to a nonlocal renormalization of the kinetic energy which is accounted for by the field redefinition in Eq. (2.15) and, by its long-range nature, prevents the emergence of Goldstone bosons. We also observe that in the transition from Coulomb to Higgs phase, the compact φ degree of freedom changes into a noncompact one.

The disappearance of the displacement symmetry in the Higgs phase is most easily understood by enclosing the system in a finite volume and imposing boundary conditions. Otherwise, due to the linear $\mathbf{x}$-dependence of the displacements, certain integrations by parts necessary for deriving the effective Lagrangian are ill defined. With compact space, the absence of displacements in the Higgs phase has a simple topological interpretation. Whenever the gauge field is coupled to matter, the shift in the gauge field is accompanied by a change in the winding number associated with the phase of the matter field. As in the case of the θ-vacuum, in order to form a state that is symmetric under displacements, states with different winding numbers have to be superimposed. Similarly, to generate Goldstone bosons in the broken phase, the possibility for space-dependent long wavelength displacements accompanied by corresponding changes in the winding number must exist. This is not the case in the Higgs phase. Under continuous changes, a field configuration with a certain winding number Φ_n can only be connected to a field configuration with different winding number $\Phi_{n'}$, if these fields vanish somewhere in space. This is effectively ruled out by the assumption of a condensate large enough to make the effect of fluctuations negligible to leading order. Thus in the Higgs phase, the winding number is frozen, i.e., the quantity $(\boldsymbol{\nabla}\varphi)_0$ is not a dynamical variable and therefore the displacement symmetry is not present. In other words, sectors with different winding numbers are separated by barriers of infinite energy, associated with the discontinuous changes in the field configurations. The same topological property that freezes the winding number also prevents the emergence of Goldstone bosons in this symmetry breakdown. The system therefore does not offer in the Higgs phase the possibility for displacements. When the strength of the condensate is continually decreasing, the density of strings of zeroes in 3-space increases and makes it progressively easier for the system to change winding numbers. Eventually, in the perturbative phase, we assume implicitly that the topological constraint is absent as in the free Maxwell theory and quantum fluctuations of the Higgs field around zero field no longer provide any obstacle for change in winding number. More formally, in the perturbative limit the relevant mapping is $x^3 \in S^1 \to \Phi \in C$ rather than the topologically nontrivial $x^3 \in S^1 \to \varphi \in S^1$ relevant for the Higgs limit.

In the Higgs phase, the constant gauge field absorbs part of the phase of the Higgs field (cf. Eq. (2.17)) and thereby turns into the displacement invariant variable $\mathbf{A}_0'$. Thus, emergence of Goldstone bosons is avoided because in the Higgs phase the system contains only dynamical variables that are invariant under displacements. In a description where we introduce from the beginning the phase of the Higgs field as a dynamical variable, this resolution of the symmetry issues is easily seen. In the Higgs phase, the transformation to the *gauge-invariant variables*

$$B_\mu = A_\mu + \frac{1}{e}\partial_\mu \varphi, \quad \rho$$

is well defined; in terms of these variables, the generating functional (2.2) is given by

$$Z = \int d[B] \prod_x \rho(x)\, d\rho(x) \int d[\varphi]\, \delta[f(B - \frac{1}{e}\partial\,\varphi, \rho e^{i\varphi})] e^{i\int d^4x\, \mathcal{L}[B,\rho]} \tag{2.19}$$

with the Lagrangian

$$\mathcal{L}\,[B,\rho] = -\tfrac{1}{4}F^{\mu\nu}[B]F_{\mu\nu}[B] + e^2\rho^2 B_\mu B^\mu + \partial_\mu\rho\,\partial^\mu\rho - V(\rho). \tag{2.20}$$

The integration over φ, which also includes a summation over the winding number, can be used to satisfy the gauge condition. The remaining integration over the zero mode of φ factorizes in Z. Thus the generating functional is completely determined by the fields B and ρ. These variables are invariant under both global phase changes and displacements. Thus, we find once more that in the Higgs phase these two symmetry transformations become the identity; in other words, the corresponding symmetries are implemented trivially. We note that in this way a conflict between the "realization" of the displacement symmetry and its breakdown enforced by the canonical commutation relation (2.8) is avoided.

2.3 Scalar QED at Finite Temperature

In this concluding section on the Abelian Higgs model, we briefly discuss the possible effects of thermal fluctuations on the realization of the displacement symmetry. To this end, we consider the Abelian Higgs model in the Coulomb phase and assume the Higgs potential in Eq.(2.1) to be given by a mass term

$$V(|\Phi|) = m^2\Phi\Phi^\star.$$

At sufficiently high temperatures, the photon is coupled to a finite density of charged particles and may thereby acquire a mass. Such a mechanism is apparently operative leading to Debye screening. However, in perturbation theory no "magnetic" mass is generated. The presence of the Debye mass is not compatible with the displacement symmetry. We study this issue by integrating out the scalar field in the generating functional (2.2), and drop its source term

$$Z[J] = \int d[A]\,\delta[f(A)]e^{-S_{\text{eff}}[A]}e^{i\int d^4x J^\mu A_\mu}. \tag{2.21}$$

The gauge condition has been assumed to be independent of the scalar field. The generating functional is written in Euclidean space and, following the usual convention, the compact direction is denoted in this section as the 0-direction. The effective action is

$$S_{\text{eff}} = \tfrac{1}{4}\int d^4x F_{\mu\nu}F_{\mu\nu} + \operatorname{tr}\ln\left[-D_\mu D_\mu + m^2\right]. \tag{2.22}$$

To lowest order in a perturbative expansion, the determinant is given by tadpole and virtual pair diagrams, which for vanishing photon momentum cancel for "spacelike" gauge fields and yield at high temperatures ($T = 1/L$) the well-known Debye mass ([18],[19])

$$m_D^2 = \tfrac{1}{3}e^2T^2$$

associated with A_0. Inclusion of this lowest order result leads to an effective Lagrangian that is not invariant under displacements. On the other hand, the 'tr log'-contribution to the action is invariant. For an eigenfunction χ

$$\left[-D_\mu D_\mu + m^2\right]\chi = \lambda\chi$$

the transformed eigenfunction

$$\tilde{\chi} = e^{id_\mu x_\mu}\chi$$

satisfies

$$\left[-\tilde{D}_\mu \tilde{D}_\mu + m^2\right]\tilde{\chi} = \lambda\tilde{\chi}$$

with the displacement-transformed covariant derivatives

$$\tilde{D}_\mu = D_\mu - i\, d_\mu.$$

Thus the spectrum $\{\lambda\}$ is invariant under displacements. This invariance indeed excludes, for $\mu = 1, 2, 3$, any dependence of the effective action on the zero modes of the gauge fields

$$a_\mu = \frac{1}{V}\int_V d^4x\, A_\mu(x)$$

and therefore no magnetic mass can be generated. The invariance under the discrete displacements in 0-direction

$$d_0 = \frac{2\pi n}{L}$$

also rules out a purely quadratic mass term for a_0. However, it is compatible with a periodic dependence of S_{eff}

$$S_{\text{eff}}[a_0] = S_{\text{eff}}[a_0 + \frac{2\pi}{eL}]$$

which as we will show below can also be locally quadratic for a_0 near a multiple of $2\pi/eL$. Since displacements contain the inverse of the electric charge, a perturbative evaluation violates the displacement symmetry. Therefore we will evaluate the determinant in the effective potential approximation; i.e., we take into account only zero-momentum photons

$$\operatorname{tr}\ln\left[-D_\mu D_\mu + m^2\right] \approx \operatorname{tr}\ln\left[-(\partial_\mu + iea_\mu)^2 + m^2\right] = V_{\text{eff}}(a).$$

To calculate V_{eff} we add terms that are independent of the gauge-field zero modes and obtain

$$\begin{aligned} V_{\text{eff}}(a) &= \text{const.} + \frac{1}{L}\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\left[\ln\left((\mathbf{k}+e\mathbf{a})^2 + \kappa_n^2(a_0)\right) - \ln\left(\mathbf{k}^2 + \kappa_n^2(0)\right)\right] \\ &= \text{const.} + \frac{1}{L}\sum_{n=-\infty}^{\infty}\int_{\kappa_n^2(0)}^{\kappa_n^2(a_0)} dM^2 \int\frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2 + M^2} = -\frac{1}{6\pi L}\sum_{n=-\infty}^{\infty}|\kappa_n(a_0)|^3 \end{aligned}$$

with

$$\kappa_n^2(a_0) = \left(\frac{2\pi n}{L} + ea_0\right)^2 + m^2.$$

The momentum integral has been evaluated in dimensional regularization. The dependence on the spacelike components of the gauge field has disappeared in the shift of the integration variables. The final sum is calculated with ζ-function regularization (cf. [17])

$$V_{\text{eff}}(a) = -\frac{m^2}{\pi^2 L^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(neLa_0)K_2(nmL).$$

The displacement symmetry is explicitly preserved in the evaluation of the effective potential. Formally we may define a Debye mass by adding and subtracting the quadratic term of the expansion of the cosine

$$m_D^2 = \frac{e^2 m^2}{\pi^2} \sum_{n=1}^{\infty} K_2(nmL).$$

The displacement symmetry makes, however, self-interactions of a_0 beyond the mass term necessary. In the high-temperature limit, the n-sum can be performed and, up to a constant, the effective potential is given by

$$V_{\text{eff}}(a) = \frac{2\pi^2}{3L^4}(\alpha^2 - 2\alpha^3 + \alpha^4) \qquad 0 \leq \alpha \leq 1, \quad mL \ll 1$$

where

$$\alpha = \frac{eLa_0}{2\pi}.$$

We observe both that the effective potential is periodic, preserving displacement symmetry at finite temperature, and that it is locally quadratic for a_0 in the vicinity of zero with curvature corresponding to the standard value of m_0. Thus, while invariance of the finite-temperature effective action under displacements rules out a magnetic mass of the photons, the presence of a mass in the time component of the gauge field as part of a periodic self-interaction is compatible with the displacement symmetry. In addition, if we relabel the axes and reinterpret the system as being at zero temperature with x_3 periodic, in the limit of large L we expect displacement symmetry to manifest itself by excitations with energies $\sim 1/L$.

At sufficiently high temperature we expect the preceding arguments to apply to the Abelian Higgs model. Therefore, in addition to the restoration of the rigid-phase symmetry in the phase transition from the Higgs to the Coulomb phase at finite temperature, we also expect that the displacement symmetry will reappear accompanied by gapless excitations.

3 Georgi-Glashow Model

3.1 Symmetries

In this section we present a general discussion of the symmetries of the Georgi-Glashow model [4] with emphasis on the center symmetry [11, 12, 13]. The Georgi-Glashow Lagrangian describes the $SU(2)$ Yang-Mills theory coupled to an adjoint scalar

$$\mathcal{L}\left[A, \Phi\right] = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} D_\mu \Phi D^\mu \Phi - V(|\Phi|) \tag{3.1}$$

with the covariant derivative of the Higgs field

$$D_\mu \Phi = \partial_\mu \Phi - g A_\mu \times \Phi.$$

We have written the Higgs field and the field strength tensor as vectors in color space. For the following discussion, it is convenient to represent both gauge and Higgs fields in a spherical color basis and we will refer to those objects pointing in the color 3 direction as neutral, those perpendicular as charged. At this point this definition is formal. Only later in the process of gauge fixing will the meaning of the color directions be specified. The charged components of gauge and Higgs fields are defined as

$$A^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(A^1_{\mu} \mp iA^2_{\mu}) \tag{3.2}$$

$$\Phi^{\pm} = \frac{1}{\sqrt{2}}(\Phi^1 \mp i\Phi^2) \tag{3.3}$$

The neutral component of the field strength is

$$F^3_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu - ig(A^-_\mu A^+_\nu - A^-_\nu A^+_\mu) \tag{3.4}$$

and the charged ones are

$$F^{\pm}_{\mu\nu} = (\partial_\mu \pm igA^3_\mu)A^{\pm}_\nu - (\partial_\nu \pm igA^3_\nu)A^{\pm}_\mu. \tag{3.5}$$

The kinetic part of the Higgs field can be similarly rewritten and the following form of the Lagrangian of the Georgi-Glashow model is obtained

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_H$$

with

$$\mathcal{L}_{YM} = -\tfrac{1}{4}F^{3\mu\nu}F^3_{\mu\nu} - \tfrac{1}{2}F^{+\mu\nu}F^-_{\mu\nu} \tag{3.6}$$

$$\begin{aligned}\mathcal{L}_H = \tfrac{1}{2}\left[\partial_\mu\Phi^3 + ig(\Phi^- A^+_\mu - A^-_\mu\Phi^+)\right]\left[\partial^\mu\Phi^3 + ig(\Phi^- A^{+\mu} - A^{-\mu}\Phi^+)\right] \\ + D^{3\dagger}_\mu\Phi^- D^{3\mu}\Phi^+ + ig\Phi^3\left[A^{-\mu}D^3_\mu\Phi^+ - A^{+\mu}D^{3\dagger}_\mu\Phi^-\right] + g^2 A^-_\mu A^{+\mu}(\Phi^3)^2 - V(|\Phi|)\end{aligned} \tag{3.7}$$

where

$$D^3_\mu = \partial_\mu + igA^3_\mu.$$

Center Symmetry As above, we assume that one of the space-time directions, the 3-direction, is compact. In a non-Abelian gauge theory we can associate a loop integral of gauge fields with this compact direction, the Polyakov loop,

$$P_3(x_\perp) = P\exp\left\{ig\int_0^L dz\, A_3(x)\right\}. \tag{3.8}$$

The trace of P_3 can serve as order parameter for the phases of Yang-Mills theories [12, 20, 21]. The expectation value and correlation functions of this variable are related to the self-energy of a single static quark and the interaction energy of two static quarks respectively and therefore distinguish the high-temperature gluon plasma from the low-temperature confining phase. Under gauge transformations $U(x)$, $P_3(x_\perp)$ transforms as

$$P_3(x_\perp) \to U(x_\perp, L)\, P_3(x_\perp) U^\dagger(x_\perp, 0)\ . \tag{3.9}$$

The coordinates $x = (x_\perp, 0)$ and $x = (x_\perp, L)$ describe identical points, and we require the periodicity properties imposed on the field strengths not to change under gauge transformation. This is achieved if U satisfies

$$U(x_\perp, L) = c_U \cdot U(x_\perp, 0) \tag{3.10}$$

with c_U being an element of the center of the group. Thus gauge transformations can be classified according to the value of c_U (± 1 in SU(2)). Therefore under gauge transformations

$$\mathrm{tr}\big(P_3(x_\perp)\big) \to \mathrm{tr}\big(c_U P_3(x_\perp)\big) \stackrel{SU(2)}{=} \pm\, \mathrm{tr}\big(P_3(x_\perp)\big). \tag{3.11}$$

A simple example of a class of SU(2) gauge transformations u_n with $c = \pm 1$ is

$$u_n(x) = e^{i\hat{\psi}(x)\cdot\tau n\pi x_3/L}, \qquad c_{u_n} = (-1)^n \tag{3.12}$$

with the arbitrary and, in general, space-time dependent unit vector $\hat{\psi}$ in color space. The transformed gauge and Higgs fields are

$$\begin{aligned} A_\mu &\to u_n\Big(A_\mu - \frac{i}{g}\partial_\mu\Big)u_n^\dagger \qquad \Phi \;\to\; u_n \Phi u_n^\dagger \\ u_n\partial_\mu u_n^\dagger &= i\left[\left(1-\cos\Big(\frac{2\pi n x^3}{L}\Big)\right)\hat{\psi}\times\partial_\mu\hat{\psi} - \sin\Big(\frac{2\pi n x^3}{L}\Big)\partial_\mu\hat{\psi} - \frac{2\pi n}{L}\hat{\psi}\delta_{\mu 3}\right]\frac{\tau}{2} \end{aligned} \tag{3.13}$$

These symmetry transformations are reminiscent of the displacements of the $U(1)$ theory. There are, however, important differences related to the different topological properties of Abelian and non-Abelian theories in the presence of a compact direction. Unlike displacements that cannot be gauged away, since $\Pi_1(U(1)) = Z$, the symmetry transformations in Eq. (3.13) can be continuously connected to the identity for even n, since $\Pi_1(SU(2)) = Z_2$. An example of such a continuous deformation of these transformations into the identity has been given in [22]

$$\begin{aligned} u_{2n}(x,t) &= e^{-(it\pi/2)\hat{\chi}_1\cdot\tau}\, e^{(it\pi/2)\{\hat{\chi}_1\cdot\tau\cos 2n\pi x^3/L + \hat{\chi}_2\cdot\tau \sin 2n\pi x^3/L\}} \\ u_{2n}(x,0) &= 1, \qquad u_{2n}(x,1) = u_{2n}(x), \qquad u_{2n}(0,t) = u_{2n}(L,t) = 1 \end{aligned} \tag{3.14}$$

where the unit-vectors $\hat{\psi}, \hat{\chi}_{1,2}$ form a right-handed, orthogonal basis. In order to single out the topologically nontrivial piece of the above transformation, we define center reflections by

$$Z_k = i e^{i\pi\hat{\psi}_\perp\cdot\tau/2} e^{i(2k+1)\pi\hat{\psi}\cdot\tau x^3/L} \quad \text{with} \quad \hat{\psi}\cdot\hat{\psi}_\perp = 0 \tag{3.15}$$

where k is some fixed integer. These transformations are reflections and change the sign of the Polyakov loop

$$Z_k^2 = 1, \quad c_{Z_k} = -1. \tag{3.16}$$

Center reflections can be used to generate any other gauge transformation changing the sign of $\mathrm{tr}(P_3)$ by multiplication with a strictly periodic ($c = 1$) but otherwise arbitrary gauge transformation. The decomposition of $SU(2)$ gauge transformations into two classes according to $c = \pm 1$ implies a decomposition of each gauge orbit $\mathcal{O}$, into suborbits $\mathcal{O}_\pm$, which are characterized by the sign of the Polyakov loop at some fixed $x_\perp^0$

$$A(x) \in \mathcal{O}_\pm, \qquad \text{if } \pm\,\mathrm{tr}\big(P_3(x_\perp^0)\big) \geq 0. \tag{3.17}$$

Thus, strictly speaking, the trace of the Polyakov loop is not a gauge-invariant quantity. Only $|\,\mathrm{tr}(P_3(x_\perp)|$ is invariant under all of the gauge transformations. Furthermore, the spontaneous breakdown of the center symmetry in Yang-Mills theory as it supposedly happens at small extension or high temperature is a breakdown of the underlying gauge symmetry. It implies that the wave functional describing such a state is different for gauge field configurations that belong to $\mathcal{O}_+$ and $\mathcal{O}_-$ respectively, and that therefore are connected by gauge transformations such as u_{2n+1} in Eq. (3.12). These symmetry considerations apply equally well for the Georgi-Glashow model. Coupling of the Yang-Mills field to matter in the adjoint representation does not affect the center symmetry unlike coupling to fundamental fermions. In the latter case, center symmetry transformations change the boundary condition of fields carrying fundamental charges. Thus realization of the center symmetry should be equally relevant for the phases of the Georgi-Glashow model.

These considerations are also of relevance for understanding the structure of gauge theories after (partially) fixing the gauge. Whenever gauge fixing is carried out exactly and with the help of strictly periodic gauge-fixing transformations ($\Omega, c_\Omega = 1$) the resulting formalism must contain the center symmetry

$$\mathrm{tr}\big(P_3(x_\perp)\big) \to -\,\mathrm{tr}\big(P_3(x_\perp)\big) \tag{3.18}$$

as residual gauge symmetry, in other words, each gauge orbit is represented by two gauge-field configurations.

3.2 Residual Gauge Symmetries in Unitary Gauge

We now consider the role of the residual symmetries of the Georgi-Glashow model in the unitary gauge. The unitary gauge condition is

$$\Phi = \Phi^a(x)\frac{\tau^a}{2} = \rho(x)\frac{\tau^3}{2}. \tag{3.19}$$

The gauge condition does not affect the gauge fields. Therefore the Yang-Mills piece of the Lagrangian remains unchanged and the contribution of the Higgs field simplifies to

$$\mathcal{L}_H = \tfrac{1}{2}\partial_\mu\rho\partial^\mu\rho + g^2\rho^2 A_\mu^- A^{+\,\mu} - V(|\rho|). \tag{3.20}$$

By this gauge fixing, the color 3-direction is identified with the direction of the Higgs field. The symmetry transformations U_n after gauge fixing are obtained using Eq. (3.12) with

$$\hat{\psi} = \hat{\Phi}.$$

With this choice, the unit-vector $\hat{\psi}$ becomes space-time independent after transforming to the unitary gauge and

$$u_n = e^{in\pi x^3\tau^3/L}.$$

We thus consider the following transformations

$$U_n = u_{2n}: \quad A^3_\mu(x) \to A^3_\mu(x) - \frac{4n\pi}{gL}\delta_{\mu,3} \qquad A^+_\mu(x) \to e^{4in\pi x^3/L} A^+_\mu(x)$$
$$F^3_{\mu\nu}(x) \to F^3_{\mu\nu}(x) \qquad F^+_{\mu\nu}(x) \to e^{4in\pi x^3/L} F^+_{\mu\nu}(x)$$
$$\rho(x) \to \rho(x). \tag{3.21}$$

With an appropriate choice of $\hat{\psi}$, center reflections Z_k (Eq. (3.16)) become

$$Z_k = i e^{i\pi\tau^1/2} e^{i(2k+1)\pi x^3 \tau^3/L}$$

and transform the relevant variables as

$$Z_k: \quad A^3_\mu(x) \to -A^3_\mu(x) - \frac{2\pi(2k+1)}{gL}\delta_{\mu,3} \qquad A^+_\mu(x) \to e^{2i(2k+1)\pi x^3/L} A^-_\mu(x)$$
$$F^3_{\mu\nu}(x) \to -F^3_{\mu\nu}(x) \qquad F^+_{\mu\nu}(x) \to e^{2i(2k+1)\pi x^3/L} F^-_{\mu\nu}(x)$$
$$\rho(x) \to -\rho(x). \tag{3.22}$$

With the above choice of the $\hat{\psi}_\perp$-dependent term, we actually have included in Z a charge conjugation[1]

$$Z_k = C u_{2k+1}$$

$$C: \quad A^3_\mu(x) \to -A^3_\mu(x) \qquad A^+_\mu(x) \to A^-_\mu(x)$$
$$F^3_{\mu\nu}(x) \to -F^3_{\mu\nu}(x) \quad F^+_{\mu\nu}(x) \to F^-_{\mu\nu}(x)$$
$$\rho(x) \to -\rho(x). \tag{3.23}$$

We note that these three classes of transformations leave the gauge-fixed Lagrangian invariant

$$U_n,\, Z_k,\, C: \quad \mathcal{L}_{YM} + \mathcal{L}_H \to \mathcal{L}_{YM} + \mathcal{L}_H$$

However, they cannot be simultaneously implemented, since

$$[U_n, Z_k] \neq 0 \qquad [U_n, C] \neq 0 \qquad [Z_k, C] \neq 0.$$

In our analysis of the symmetries we first address the displacements. As argued above, unlike displacements in the Abelian theory, there is no topological obstruction to a change in winding number. This is also true after gauge fixing. However, after gauge fixing, the deformations of the fields in Eq. (3.14) are no longer occurring along equipotential lines. Rather, the necessary rotation of the gauge field in color space introduces terms of the following form:

$$g^2\rho^2 A^-_\mu A^{+\mu} \to g^2\rho^2 A^-_\mu A^{+\mu} + 2g^2\rho^2 A^3_\mu A^{3\mu} \sin^2 t\pi \, \sin^2 \frac{n\pi x^3}{L} + \cdots$$

By construction, along this path the winding of the phase of the charged gauge-field component changes with identical values of the potential energy at initial and final

[1]For a proper definition of C in the Higgs phase one should include a reflection of ρ at the origin, which leaves $\mathcal{L}_H$ invariant; the following discussion will not be affected by such a redefinition.

points $t = 0, 1$. However, although the values of the action corresponding to these initial and final field configurations are the same, they are separated by potential barriers whose height increases with the condensate. Hence, ultimately changes in $A^{3\mu}$, and therefore in the winding number, by such deformations involving all of the color components of the gauge field are prevented dynamically. This mechanism effectively prevents the displacements from being generated by the Gauss law operator and is an example of the second form of residual gauge symmetry mentioned in the Introduction. In the Higgs limit, the system behaves like an Abelian model and like the Abelian model in the weak coupling limit, it possesses the possibility of changing the winding in the phase of the charged gauge fields at their zeroes. Thus, for a sufficiently large condensate and concomitantly large masses of the vector particles, a physical displacement symmetry appears that, as in weakly coupled QED, in turn requires the existence of Goldstone bosons.

We now discuss the discrete symmetries, the center reflection, and the charge conjugation. With the displacement symmetry broken by the mechanism described in the first section the variable

$$2g \int_0^L A_3^3 \, dz = \chi$$

assumes a fixed value, which by a suitable redefinition of the gauge fields can be shifted to the interval

$$0 \leq \chi < 2\pi .$$

In general, both discrete symmetries are broken. The vacuum is invariant under charge conjugation only if $\chi = 0$, and it is center symmetric only if $\chi = \pi$. The classical field configurations of lowest potential energy (cf. Eqs. (3.5,3.20))

$$\rho = \rho_{\min}, \quad A_i^{\pm} = 0, \quad A_i^3 = \text{const.}$$

do not single out a particular value of A_3^3.

Since classical considerations cannot determine A_3^3, we consider the quantum mechanical ground state energy, which is affected by the presence of "Aharonov-Bohm" fluxes as described by a nonvanishing constant gauge field. To calculate this quantity, we formally eliminate χ by redefining the phase of the charged vector-particles

$$\hat{A}_\mu^{\pm} = e^{\pm i\chi x^3/L} \, A_\mu^{\pm}$$

at the expense of introducing modified boundary conditions

$$\hat{A}_\mu^{\pm}(x^3 = L) = e^{\pm i\chi} \hat{A}_\mu^{\pm}(x^3 = 0).$$

The ground state energy density corresponding to these boundary conditions has been calculated in [23] with the result that the contribution of the charged vector-particles of mass M is given by

$$\varepsilon_{A^{\pm}} = -\frac{3}{\pi^2 L^4} \Bigg[\Big(\frac{1}{2\epsilon} + \tfrac{3}{4} - \tfrac{1}{2}\gamma + \ln(\sqrt{4\pi}\mu/M) \Big) (ML/2)^4 + (ML)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\chi) K_2(nML) \Bigg] \tag{3.24}$$

with the standard parameters ϵ, μ appearing in dimensional regularization. Thus, the vacuum state that is even under charge conjugation ($\chi = 0$) is favored energetically at high temperatures where $K_2(nML)$ is nonnegligible. However, the suppression of nonvanishing values of χ is exponentially small for $M\,L \gg 1$ and therefore irrelevant at temperatures much smaller than the vector particle mass.

To explore the low-temperature regime, we take the approach commonly used in studying symmetry breaking of investigating the response of the system to an arbitrarily weak external perturbation and determining the value of χ that leads to the lowest energy. In the present case, we consider the response to a pair of color charges, which can be expressed in terms of Polyakov loops and will therefore be directly related to center symmetry. For massive charged vector-particles, the correlation function of two Polyakov loops at large distances is dominated by the contribution of the massless neutral fields A_μ^3 and we therefore can write

$$\langle 0|\, T(\mathrm{tr}\, P_3(x_\perp)\, \mathrm{tr}\, P_3(y_\perp)\,|0\rangle \;\approx\; \langle 0|\, T(\mathrm{tr}\, e^{igLa(x_\perp)\tau^3/2}\, \mathrm{tr}\, e^{igLa(y_\perp)\tau^3/2})\,|0\rangle$$

with

$$a(x_\perp) = \frac{1}{L}\int_0^L dz\, A_3^3(x)\,.$$

In the Fock-space decomposition of the ground state we treat the zero-momentum state separately. Here we assume a spontaneous breakdown of the displacement symmetry with the vacuum in a field eigenstate $|\chi\rangle$

$$\Big(\frac{1}{V_\perp}\int_{V_\perp} d^2x\, gLa(x_\perp)\Big)|\chi\rangle = \chi|\chi\rangle.$$

This choice is suggested by the vanishing of the kinetic energy of these zero-modes in the thermodynamic limit (cf. Appendix I).The Fock-states corresponding to finite momentum photons are particle number eigenstates and their contribution can be treated in the standard way. We first calculate the expectation value

$$\begin{aligned}\langle 0_{\chi'}|\, \mathrm{tr}\big(e^{igLa(x_\perp)\tau^3/2}\big)\,|0_\chi\rangle &= 2\delta(\chi'-\chi)\cos\frac{\chi}{2}\langle 0|\, e^{igLa'(x_\perp)/2}\,|0\rangle \\ &= 2\delta(\chi'-\chi)\cos\frac{\chi}{2}e^{-\frac{g^2}{16\pi}L\Lambda/2} \qquad (3.25)\end{aligned}$$

where the matrix-elements of

$$a'(x_\perp) = a(x_\perp) - \frac{\chi}{gL}$$

are those of the corresponding Maxwell theory and have been calculated by replacing the sum over (transverse) momenta by integrals. The divergence in this integral has been regulated with heat kernel regularization (regulator Λ). Obviously, the δ-function in the above expression can be avoided by using a superposition of field eigenstates centered around a specific value. In the evaluation of the correlation function we perform a Euclidean rotation ($x_\perp^2 \to r^2$) and obtain for the interaction energy of two static charges

$$\begin{aligned}e^{-LV(r)} &= \langle 0_{\chi'}|\, \mathrm{tr}\big(e^{igLa(x_\perp)\tau^3/2}\big)\, \mathrm{tr}\big(e^{\pm igLa(y_\perp)\tau^3/2}\big)\,|0_\chi\rangle \\ &= 2\delta(\chi'-\chi)\,\Big[\langle 0|e^{igLa'(x_\perp)/2}e^{-igLa'(y_\perp)/2}|0\rangle + \cos\chi\langle 0|e^{igLa'(x_\perp)/2}e^{igLa'(y_\perp)/2}|0\rangle\Big] \\ &= 2\delta(\chi'-\chi)e^{-\frac{g^2}{16\pi}L\Lambda}\Big[e^{\frac{g^2}{16\pi}\frac{L}{r}} + \cos\chi\, e^{-\frac{g^2}{16\pi}\frac{L}{r}}\Big]. \qquad (3.26)\end{aligned}$$

Thus, for all distances, r, the minimum potential $V(r)$ occurs for the maximum of $\cos\chi$, favoring $\chi = 0$. Although the strength of the "condensate" of Aharonov-Bohm flux, $a(x_\perp)$, is of the order L^{-1}, this constant gauge field has a nonnegligible effect when integrating $j_3 A^3$ over an x^3-independent current. For large separation ($r \gg L$), the interaction energy of static charges behaves as $1/r$ unless χ assumes the center symmetric value π or the charge conjugation symmetric value $\chi = 0$. In the center symmetric case, the interaction energy increases logarithmically with distance, providing a hint of confinement. For the charge-conjugation symmetric case, the potential decreases like r^{-2}. We note that the modification of the ground state energy by the static charges is not suppressed exponentially like the Casimir energy (Eq. (3.24)).

Our calculations thus suggest that there should be no change in the realization of the center symmetry and therefore no discontinuity in the Polyakov loop as a function of temperature throughout the Higgs phase for sufficiently large values of the condensate. With increasing temperature, i.e., decreasing L, the Casimir energy increasingly favors, independent of an external disturbance, the charge symmetric point (cf. Eq. (3.24)) that was already preferred at low temperature.

3.3 Coulomb-gauge Effective Lagrangian

In this section we derive in Coulomb gauge the effective Lagrangian for the Georgi-Glashow model. We proceed as in the Abelian Higgs model and first integrate out the nondynamical A_0 variables. In this first step the gauge condition only has to be assumed to be independent of A_0 but can remain unspecified otherwise. We represent the Higgs field by its modulus ρ and the unit vector $\hat{\Phi}$ defining its orientation. The resulting effective action, including the Faddeev-Popov operator Δ_{FP} is given by

$$S_{\text{eff}} = \int d^4x \, (\mathcal{L}_{\text{kin}} + \mathcal{L}_{sp}) + \text{tr}\{\ln \Delta_{FP}\} - \tfrac{1}{2}\,\text{tr}\{\ln \Gamma\}\,.$$

with the two contributions to the effective Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{pot}} &= -\tfrac{1}{4}F^{ij}F_{ij} + \tfrac{1}{2}\rho\Delta\rho - V(\rho) - \tfrac{1}{2}\rho^2\hat{\Phi}\,\tilde{\Gamma}\,\hat{\Phi} \\
\mathcal{L}_{\text{kin}} &= -\tfrac{1}{2}(D_i\partial_0 A^i + g\rho^2\partial_0\hat{\Phi}\times\hat{\Phi})\frac{1}{\Gamma}(D_i\partial_0 A^i + g\rho^2\partial_0\hat{\Phi}\times\hat{\Phi}) \\
&\quad + \tfrac{1}{2}\rho^2(\partial_0\hat{\Phi})^2 + \tfrac{1}{2}(\partial_0\rho)^2 + \tfrac{1}{2}(\partial_0 A^i)^2
\end{aligned}$$

with

$$\Gamma^{ab} = \tilde{\Gamma}^{ab} + g^2\rho^2(\delta^{ab} - \hat{\Phi}^a\hat{\Phi}^b)$$

and the gauge covariant Laplacian

$$\tilde{\Gamma}^{ab} = -\delta^{ab}\Delta + g\epsilon^{abc}(A^{ic}\partial_i + \partial_i A^{ic}) - g^2(\delta^{ab}A_i^c A^{ic} - A_i^a A^{ib}).$$

We simplify these expressions by assuming that the Higgs field develops a large expectation value. We decompose the modulus of the Higgs field

$$\rho(x) = \rho_0 + \sigma(x),$$

and keep only terms quadratic in the fields $A_i, \partial_\mu\hat{\Phi}$, and σ. Because of the constraint $\hat{\Phi}^2 = 1$, at this point we cannot disregard terms involving higher powers of $\hat{\Phi}$. However, we note that terms involving derivatives of $\hat{\Phi}$ may be truncated at quadratic order, with the result

$$\begin{aligned}\mathcal{L}_{\rm kin} &\approx -\tfrac{1}{2}(\partial_0\partial_i A^i + g\rho_0^2\partial_0\hat{\Phi}\times\hat{\Phi})\,\frac{1}{-\Delta + g^2\rho_0^2(1-\hat{\Phi}\otimes\hat{\Phi})}\,(\partial_0\partial_j A^j + g\rho_0^2\partial_0\hat{\Phi}\times\hat{\Phi})\\ &\quad + \tfrac{1}{2}\rho_0^2(\partial_0\hat{\Phi})^2 + \tfrac{1}{2}(\partial_0\sigma)^2 + \tfrac{1}{2}(\partial_0 A^i)^2\\ \mathcal{L}_{\rm pot} &\approx -\tfrac{1}{4}(\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i) + \tfrac{1}{2}\sigma\Delta\sigma - \tfrac{1}{2}V''(\rho_0)\sigma^2\\ &\quad + \tfrac{1}{2}\rho_0^2\Big[\hat{\Phi}\Delta\hat{\Phi} + 2g(\partial_i\hat{\Phi}\times\hat{\Phi})\,A^i - g^2\left(A^i - \hat{\Phi}(\hat{\Phi}A^i)\right)^2\Big].\end{aligned} \tag{3.27}$$

This representation in terms of the unit vectors $\hat{\Phi}$ is well defined in the Higgs phase with its large expectation value of Φ. Generation of a mass for the components of the transverse vector fields orthogonal to the Higgs field is explicit in the last term of $\mathcal{L}_{\rm pot}$. The appearance of the massive longitudinal components is more subtle. The above expression displays the special roles of the unitary gauge and the Coulomb gauge. In these gauges, the longitudinal components of the gauge fields and the orientation of the Higgs field do not mix to leading order. In unitary gauge the 2 massive longitudinal components are described by the fields

$$\hat{\Phi}^a = \delta_{a,3}: \quad \tilde{\chi}^a = g\rho_0\Big(\frac{1}{-\Delta(-\Delta + g^2\rho_0^2)}\Big)^{\frac{1}{2}}{\rm div}\mathbf{A}^a, \quad a = 1,2$$

and no time derivative of ${\rm div}\mathbf{A}^3$ is present, reflecting the incompleteness of the unitary gauge condition. In Coulomb gauge

$${\rm div}\mathbf{A}^a = 0, \quad a = 1,2,3$$

we have to separate massive from massless excitations. To this end we introduce two vector-fields, an isoscalar (c^i) and a constrained isovector (C^{ia}), by representing the gauge fields as

$$A^i = c^i\hat{\Phi} + C^i\times\hat{\Phi}\,, \quad C^i\cdot\hat{\Phi} = 0. \tag{3.28}$$

In making the harmonic approximation, we again drop terms that are generated when differential operators act on the unit vectors $\hat{\Phi}$. Hence, we write

$$(\partial_0 A^{a,i})^2 \approx (\partial_0 c^{a,i})^2 + (\partial_0 C^{a,i})^2,$$

and correspondingly simplify the Coulomb gauge condition

$$\delta(\partial_i A^i) \approx \delta(\hat{\Phi}\partial_i c^i - \hat{\Phi}\times\partial_i C^i).$$

We replace the functional integral over the 3 constrained vector-fields C by two "Cartesian" vector fields. With interactions neglected, the Faddeev-Popov determinant becomes trivial. Finally to achieve a canonical form for the kinetic terms of the Higgs field unit vectors we define

$$\chi = \rho_0\Big(\frac{-\Delta}{-\Delta + g^2\rho_0^2}\Big)^{\frac{1}{2}}\hat{\Phi} \tag{3.29}$$

and with $g\rho_0 \to \infty$ the constraint on $\hat{\Phi}$ yields for sufficiently small momenta two Cartesian fields

$$D[\Phi_1, \Phi_2, \Phi_3]\, \delta\Big(\sum_{i=1,3} \Phi_i^2\Big) \propto D[\chi_1, \chi_2, \chi_3]\, \delta\Big(\sum_{i=1,3} \chi_i \Delta \chi_i + g^2\rho_0^2\Big) \approx D[\chi_1, \chi_2].$$

This series of small amplitude approximations results in the final expression for the generating functional

$$Z[J] = \int D[c, C^1, C^2, \chi^1, \chi^2, \sigma]\, \delta(\partial_i c^i) \prod_{a=1,2} \delta(\partial_i C^{ia}) e^{i\int d^4x \mathcal{L}_{\text{eff}} + j_c c + ..} \tag{3.30}$$

with the Coulomb-gauge Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{eff}} = & -\tfrac{1}{2} c_i \Box c_i - \tfrac{1}{2} \sum_{a=1,2} \left[C_i^a \left(\Box + g^2\rho_0^2\right) C_i^a + \chi^a \left(\Box + g^2\rho_0^2\right) \chi^a\right] \\ & - \tfrac{1}{2}\sigma \Box \sigma - \tfrac{1}{2} V''(\rho_0)\sigma^2. \end{aligned} \tag{3.31}$$

As in the Abelian case, the small amplitude approximation is essential for the derivation of the Coulomb-gauge fixed Lagrangian and is justified a posteriori by the mass generation. In both cases the massive vector-particles receive their longitudinal components from the compact variables specifying the orientation of the Higgs fields in the internal space ($U(1)$ and $SU(2)$ respectively). This can only happen since in the presence of the mass gap the kinetic energy is modified and thereby the compact $\hat{\Phi}$ are transformed into noncompact variables.

3.4 Gauge-conditions in Higgs and Confining Phases

An efficient description of the Higgs phase can be obtained with the help of gauge invariant variables. The starting point is the partition function

$$Z[J, k] = \int d[A, \Phi] \delta[f(A, \Phi)] \Delta_{FP}[A, \Phi] e^{iS[A,\Phi]} e^{i\int d^4x (J^\mu A_\mu + k\Phi)} \tag{3.32}$$

with a general gauge condition

$$f(A, \Phi) = 0 \tag{3.33}$$

and S is the action of the Georgi-Glashow Lagrangian (3.1). We assume that the modulus of the Higgs field has a large expectation value and parametrize the Higgs field as

$$\Phi = \tfrac{1}{2}\rho\, U \tau^3 U^\dagger. \tag{3.34}$$

The unitary matrix U diagonalizes the Higgs field and can be parametrized, e.g., by

$$U(x) = e^{-i\varphi(x)\tau^3/2} e^{-i\theta(x)\tau^2/2}. \tag{3.35}$$

This diagonalization is determined only up to a rotation around the direction of Φ, i.e a right multiplication of U with $\exp[i\omega(x)\tau^3]$. In analogy with the Abelian case (cf. Eq. (2.20)), we introduce the variables

$$B_\mu = U^\dagger (A_\mu + \frac{1}{ig}\partial_\mu) U. \tag{3.36}$$

Under a gauge transformation Ω, the new set of variables transforms as

$$[B,\rho,U] \stackrel{\Omega}{\to} [B,\rho,\Omega U].$$

In terms of these variables, the Georgi-Glashow Lagrangian is written as

$$\mathcal{L}\,[B,\rho] = -\tfrac{1}{4}F_{\mu\nu}[B]F^{\mu\nu}[B] + \tfrac{1}{2}\left(\partial_\mu\rho\partial^\mu\rho - g^2\rho^2(B^1_\mu B^{1\,\mu} + B^2_\mu B^{2\,\mu})\right) - V(\rho) \qquad (3.37)$$

and the generating functional is given by

$$Z[J,k] = \int d\,[B,\rho,U]\ \delta\left[f({}^UB,\tfrac{1}{2}\rho\,U\,\tau^3U^\dagger)\right]\Delta_{FP}[\,{}^UB,\tfrac{1}{2}\rho\,U\,\tau^3U^\dagger]$$
$$\exp\left\{iS\,[B,\rho] + i\int d^4x\left({}^UJ^\mu B_\mu + {}^Uk\,\rho\tfrac{1}{2}\tau^3 - \frac{i}{g}J^\mu U\partial_\mu U^\dagger\right)\right\} \qquad (3.38)$$

with

$${}^UB_\mu = U(B_\mu + \frac{1}{ig}\partial_\mu)U^\dagger, \quad {}^UJ = U^\dagger JU, \quad {}^Uk = U^\dagger kU.$$

Gauge transformations affect the variable U appearing in the measure of the generating functional; they do not transform the variables appearing in the action. On the other hand, the Lagrangian still exhibits a local $U(1)$ symmetry

$$B_\mu \to e^{i\omega(x)\tau_3}(B_\mu + \frac{1}{ig}\partial_\mu)e^{-i\omega(x)\tau_3} \qquad (3.39)$$

reflecting the ambiguity in the definition of U. However if the gauge condition $f(A,\Phi)$ specifies the gauge completely, the measure will not be invariant.

This formulation also exhibits the particular role of the unitary gauge for describing the physics in the Higgs phase. A complete gauge fixing to the unitary gauge is achieved with the following gauge conditions:

$$\Phi - \tfrac{1}{2}\rho\,V_0\tau^3V_0^\dagger = 0, \qquad \tilde{f}[\mathrm{tr}(\hat{\Phi}A)] = 0.$$

The x-independent unitary matrix V_0 eliminates the functional integration over the angular fields specifying U and $\tilde{f}$ can be chosen to fix the local $U(1)$ symmetry (Eq. (3.39)). Under the change of variables to B (and using the x-independence of V_0)

$$\tilde{f}[\mathrm{tr}(\hat{\Phi}A_\mu)] \to \tilde{f}[B^3_\mu].$$

In generalizing our studies of the previous paragraph we discuss in this section the dynamics in the Higgs phase using a "radiation gauge"

$$f[\,A\,] = 0.$$

Whenever the gauge condition is of this form, i.e., independent of the Higgs field, all the ambiguities ([24, 25]) and obstructions ([26]) which are encountered when eliminating gauge degrees of freedom in pure Yang-Mills theories, and which have been invoked as possible sources for confinement must also be present in the Georgi-Glashow model. Unlike the global gauge fixing of the unitary gauge, radiation gauges cannot be expected to be globally implemented. Horizons associated with the gauge conditions and corresponding restrictions to fundamental domains must be present when implementing the Lorentz or Coulomb gauge and Abelian monopoles must exist also in the Higgs phase when a diagonalization gauge is used [25]. On the other hand, our calculation indicates that such obstructions in implementing the Coulomb gauge should not be relevant for the dynamics in the Higgs phase. In the representation (3.38) of Z in terms of gauge invariant variables, the essential argument can be made quite easily. To this end we treat the gauge condition, for instance the Lorentz-gauge condition

$$f({}^U B, \tfrac{1}{2}\rho\, U\, \tau^3 U^\dagger) = \partial^\mu \Big[U(B_\mu + \frac{1}{ig}\partial_\mu)U^\dagger\Big]$$

approximately and neglect in f the contributions from the charged gauge fields $B_\mu^{1,2}$. In the "gauge invariant" formulation (3.37) these two components are massive and on scales significantly larger than $1/g\rho_0$ fluctuations of these fields should be negligible as compared to the contributions from B_μ^3. The approximate gauge condition

$$\partial^\mu \Big[U(\tfrac{1}{2}B_\mu^3\tau^3 + \frac{1}{ig}\partial_\mu)U^\dagger\Big] = 0$$

is satisfied (cf. Eq. (3.35)) by

$$\partial^\mu B_\mu^3 = 0, \quad \varphi = \text{const.}, \quad \theta = \text{const.},$$

and therefore

$$\int d[U]\delta\Big\{\partial^\mu[U(B_\mu + \frac{1}{ig}\partial_\mu)U^\dagger]\Big\} \approx \delta(\partial^\mu B_\mu^3).$$

The damping of the quantum fluctuations in the Higgs phase by the mass converts the gauge condition effectively into an Abelian one. The Gribov horizon cannot be reached when the charged components acquire a sufficiently large mass. In terms of Gribov's approximate description ([24]), the zero-point fluctuations which reach the Gribov horizon get increasingly damped with increasing vector boson mass and effectively the horizon disappears when

$$g\rho_0 \approx 2\Lambda e^{-3\pi^2/2g^2}$$

with Λ regularizing the gauge field fluctuations. This suppression of particular nonperturbative field configurations in the generating functional as a consequence of the mass term is operative also in other gauges. In the Weyl gauge, for instance, $A_0 = 0$, which can be treated in similar approximate way as the Lorentz gauge, the topologically nontrivial pure gauges

$$A_i = \frac{1}{ig}h^\dagger\partial_i h, \quad h = \exp\Big(in\pi\,\frac{\mathbf{x}\cdot\boldsymbol{\tau}}{\sqrt{\mathbf{x}^2+\xi^2}}\Big) \tag{3.40}$$

are important for the dynamics of Yang-Mills theories. Their degeneracy with $A = 0$ is lifted when coupled to the Higgs field. For a spatially constant Higgs field, for instance, the mass term in the gauge fields gives rise to an increase in the energy

$$\delta E = \frac{n^2\pi^4}{3}\rho_0\,(6 + {}_1F_2(1;2,4;-n^2\pi^2))$$

which in turn prevents the associated nonperturbative phenomenon of vacuum tunneling to occur. Thus in these gauges, the Higgs phase can be reached perturbatively and nonperturbative dynamics possibly related to confinement gets suppressed by coupling gauge and Higgs fields.

A different class of gauges which has been proposed ([25, 27]) to be of relevance for the description of confinement are the diagonalization gauges in which the gauge condition is of the form

$$f = E(x) - \tfrac{1}{2}e(x)\tau^3$$

where $E(x)$ is the adjoint quantity to be diagonalized with $e(x)$ its modulus. The gauge condition becomes ill defined whenever E vanishes resulting in a monopole singularity of the gauge field. Formally this gauge condition can be explicitly implemented in the representation (3.38) of the generating functional

$$Z[J,k] = \int d[e] \int d\,[B,\rho,U]\;\delta\left[U\,E\,U^\dagger - \tfrac{1}{2}\,e\,\tau^3\right] \prod_x e^2(x)$$
$$\exp\Big\{iS[B,\rho] + i\int d^4x\left({}^{U}\!J^\mu B_\mu + {}^{U}\!k\,\rho\tfrac{1}{2}\tau^3 - \frac{i}{g}J^\mu U\partial_\mu U^\dagger\right)\Big\}. \qquad (3.41)$$

Unless E vanishes the two angular fields in the U integration can be identified with the angular fields characterizing the orientation of $E(x)$. Examples which have been discussed are

$$E(x) = F_{12}(x)$$

or the Polyakov gauge (cf. [27, 28, 29]) with (cf. Eq. (3.8))

$$E(x) = \frac{1}{2i}\left(P_3(x_\perp) - P_3^\dagger(x_\perp)\right).$$

Both of these choices have the property that the gauge condition fails in the perturbative regime. Both gauges are singular for $A = 0$ and therefore gauge fields fluctuating with small amplitude around zero generate a large number of monopoles. In such a situation, a small amplitude expansion of the action will in general not be justified. While also after gauge fixing the field strength for such configurations remains finite and actually small, this small value is obtained by a cancellation of the singular Abelian and the singular commutator term in the field strength. In these gauges, dropping the commutator term (and possibly neglecting all but the diagonal gauge field components in a subsequent "Abelian projection" [30]) explicitly suppresses these weak field configurations exponentially beyond the geometric suppression by the Faddeev-Popov determinant (the $e^2(x)$ in the integrand (3.41)) by assigning large values of the action to these field configurations. If such a description is indeed appropriate for the confining phase of the Yang-Mills theory, the Higgs phase appears in such gauges to be separated by nonperturbative dynamics such as the annihilation of the "Abelian projected" monopoles.

4 Conclusion

When characterizing phases of systems that contain gauge degrees of freedom, one is invariably confronted with the difficulty of disentangling symmetry properties of physical degrees of freedom from consequences of the redundancy in the set of variables in a locally gauge invariant description. At first glance, the standard procedure of representing each gauge orbit by exactly one representative gauge field appears to resolve the issue. However, this resolution is incomplete and in some instances incorrect. It is incomplete, since this procedure does not address the existence of symmetries associated with the gauge degrees of freedom after elimination of the redundancy. It is incorrect for systems like pure Yang-Mills theories with their ability to spontaneously break a discrete part of the gauge symmetry, the center symmetry.

We have addressed these issues in the framework of the Abelian and non-Abelian Higgs models and it is useful to briefly compare and contrast the two cases. In the Abelian Higgs model, a state that is symmetric under displacements requires superposition of states of different matter field winding number. In the Coulomb phase, the copious zeros in Φ allow the winding number to change so that the displacement symmetry is present and gives rise to a massless photon. In the Higgs phase, however, the large magnitude of Φ freezes the winding number, prevents displacement symmetry, and thus prevents massless particles. Formally, the degrees of freedom relevant to displacement symmetry disappear in the Higgs phase, and when expressed in terms of gauge invariant variables, the action displays no residual symmetries. In the absence of residual symmetries, there was no need to further consider the structure of the ground state.

Unlike the Abelian case, in the Georgi-Glashow model the apparent displacement symmetry can be continuously connected to unity. In the confining phase, the copious zeros in Φ allow would-be displacements to be connected to unity by gauge fields which do not accumulate large contributions to the action. Thus, unlike in the Coulomb phase of the Abelian model, where the zeros in Φ facilitated displacement symmetry by allowing winding number change, in the non-Abelian case, the zeros prevent displacement symmetry by allowing the displacements to be generated by true gauge transformations. In the Higgs phase, however, the finite value of Φ prevents displacements from being connected to unity by gauge fields with low actions, producing a residual displacement symmetry and thus a massless particle. When one transforms to gauge invariant variables in the Higgs phase, the residual symmetries are explicit. Given the presence of residual symmetries, we also determined the structure of the ground states. In the confined phase, confinement requires center symmetry so that χ is symmetrically distributed around π and charge conjugation symmetry is broken. In the Higgs phase, we determined that the ground state is charge conjugation symmetric and breaks center symmetry. We have not studied the ground state of the deconfined phase where center symmetry must be broken, and thus do not know its behavior under charge conjugation.

Ultimately, one would like to understand what happens to aspects of confinement and other nonperturbative behavior as one goes from the confined phase to the Higgs phase in the non-Abelian Higgs model. As a starting point, our formal developments have highlighted a number of similarities between the Abelian transition from the Coulomb to the Higgs phase and the non-Abelian transition from the confining to the Higgs phase. In the Abelian Higgs model, the transition from the Higgs to the Coulomb phase is accompanied by a "condensation" of 2-dimensional manifolds of zeroes of the Higgs field which in the unitary gauge can be interpreted as a condensation of vortices. We similarly expect the corresponding Higgs to confining phase transition to be accompanied by condensation of the world-lines of the zeroes of the Higgs field. In both the Abelian and non-Abelian cases, a differential operator appears whose inverse has to develop a gap in the Higgs phase and the transition to the Higgs phase is accompanied by a transition from compact to noncompact variables. However, in the transition to the confining phase additional mechanisms must be also operative, and we have several hints from this work. An effective field theory description based on "radiation" type gauges indicates that when decreasing the mass of the vector-particles, Gribov horizons appear and the density of vacuum-like states increases near the ground state energy. Thus, below some sufficiently low value of the Higgs field, one expects both confinement and nonperturbative phenomena like instantons.

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Appendix I: Spontaneous Symmetry Breakdown

In this appendix, we collect some facts about spontaneous symmetry breaking in non-gauged Higgs models which are useful for better understanding the bulk of this paper. Our main focus will be on the global phases of the scalar fields, their canonical treatment, the role of boundary conditions, and a comparison with the standard interpretation of spontaneous symmetry breaking.

For simplicity let us suppose that the Higgs field $\phi_a(x)$ is real and in the fundamental (vector) representation of SO(N), so that there is only one fundamental group-invariant, namely $\rho = \sqrt{\phi_a \phi_a} \geq 0$. The generalization to other cases is straightforward. The Higgs Lagrangian is

$$L = \int \tfrac{1}{2}(\partial\phi)^2 - V(\rho) \ . \qquad \text{(I.1)}$$

Since, apart from ρ, the potential contains only constant parameters, the point ρ_0 where it reaches its minimum must be a constant. In general $\rho_0 = 0$ and $\rho_0 \neq 0$ correspond to different phases. Let us suppose that we are in the phase $\rho_0 \neq 0$. Then we are permitted to use the polar decomposition $\phi = \rho\hat{\phi}(\theta)$ with $\hat{\phi}^2 = 1$, where the θ's are polar angles, and the Lagrangian density may be written as

$$L = L_\rho + L_{\theta,\rho} \ , \qquad L_\rho = \int \tfrac{1}{2}(\partial\rho)^2 - V(\rho) \ , \qquad L_{\theta,\rho} = \int \tfrac{1}{2}\rho^2 g_{ab}(\theta)\partial\theta^a\partial\theta^b \qquad \text{(I.2)}$$

where g_{ab} is the appropriate positive metric. The minimal configuration for the Lagrangian is then evidently

$$\rho = \rho_0 \neq 0 \ , \qquad \partial\theta_a = 0 \ .$$

This shows that the fields ρ and θ are on a very different footing. The field ρ has its minimal value, which is sharp, dictated by the potential and is massive, with mass $V''(\rho_0) > 0$. But, although the minimal values of the θ's are constant, their actual values are left completely undetermined and need not even be sharp, and they are evidently massless Goldstone fields. We see also that since for ρ the situation would be the same even if there were no θ variables, ρ is related to phase-transitions rather than symmetry breaking. Indeed we shall see that even for $\rho_0 \neq 0$, with the concomitant existence of massless Goldstone modes, the rigid symmetry is not necessarily broken.

The symmetry is carried by the θ's and to investigate the situation with respect to these we note that the θ-momenta, and the θ phase-space are

$$\pi_a = \rho^2 g_{ab}\dot{\theta}^b \ , \qquad [\pi_a(x), \theta^b(y)] = i g_a^b \delta(x-y)$$

while the θ-Hamiltonian and the group generators are

$$H_{\theta,\rho} = \tfrac{1}{2}\int g_{ab}\big(\rho^{-2}\pi^a\pi^b + \rho^2\nabla\theta^a\cdot\nabla\theta^b\big) \quad \text{and} \quad Q = \int_\Omega \pi_a^\theta \delta\theta^a \ , \tag{I.3}$$

respectively. We now extract the zero-modes of θ and π_θ with respect to the spacial coordinates according to

$$\theta^a(x,t) = \theta_0^a(t) + \tilde{\theta}^a(x,t) \ , \qquad \int_\Omega \tilde{\theta}^a = 0 \ , \qquad \theta_0^a = \frac{1}{\Omega}\int_\Omega \theta^a \tag{I.4}$$

and

$$\pi_a(x,t) = \frac{1}{\Omega}\pi_a^0(t) + \tilde{\pi}_a(x,t) \ , \qquad \int_\Omega \tilde{\pi}_a = 0 \ , \qquad \pi_a^0 = \int_\Omega \pi_a(x,t) \ . \tag{I.5}$$

Then the phase-space decomposes into

$$[\tilde{\pi}_a(x), \tilde{\theta}_b(y)] = i\delta_{ab}\Big(\delta(x-y) - \Omega^{-1}\Big) \ , \qquad [\pi_a^0, \tilde{\theta}_b] = [\tilde{\pi}^a, \theta_0^b] = 0 \ , \qquad [\pi_a^0, \theta_0^b] = i \ . \tag{I.6}$$

The normalization has been chosen so that θ_0 and π^0 satisfy the conventional quantum mechanical relation with no factor Ω.

The Hamiltonian. The group generators decompose into

$$Q = Q_0 + \tilde{Q} \ , \qquad Q_0 = \pi_a^0 \delta\theta_a^0 \ , \qquad \tilde{Q} = \int_\Omega \tilde{\pi}^a \delta\tilde{\theta}_a$$

and the Hamiltonian becomes

$$H_\theta = \Big(\frac{\Omega_\rho}{\Omega^2}\Big)\frac{g_{ab}\pi_0^a\pi_b^0}{2} + \frac{\pi_0^a}{\Omega}\int \rho^{-2} g_{ab}\tilde{\pi}_0^b + \tilde{H} \tag{I.7}$$

where $\Omega_\rho = \int \rho^{-2}$ and

$$\tilde{H} = \int_\Omega : \rho^{-2} g_{ab}\tilde{\pi}^a\tilde{\pi}_b + \rho^2\nabla\tilde{\theta}^a\cdot\nabla\tilde{\theta}^b := \int (\rho^{-2}\tilde{\pi}^a + i\sqrt{\nabla}\tilde{\theta}_a)\rho^2 g_{ab}(\rho^{-2}\tilde{\pi}^b - i\sqrt{\nabla}\tilde{\theta}^b) \ . \tag{I.8}$$

The tilde-vacuum is defined by

$$(\rho^{-2}\tilde{\pi}^b - i\sqrt{\nabla}\tilde{\theta}^b)|\rangle = 0 \ .$$

Both of the tilded variables $\tilde{\theta}$ and $\tilde{\pi}$ appear quadratically in the Hamiltonian and hence we can normal order and define a vacuum in the usual way. Taking the tilded vacuum expectation value of the Hamiltonian and using $\langle\rho\rangle = \rho_0$ we obtain

$$\langle H_\theta \rangle \equiv H_0 = \rho_0^2 \frac{g_{ab}\pi_0^a \pi_0^b}{2\Omega} , \qquad \langle G \rangle = G_0 = \pi_0 \delta\theta_0 . \tag{I.9}$$

Since the number of zero-mode variables is finite, H_0 is just an ordinary quantum Hamiltonian. However, since the θ_0 do not appear, it is not possible to normal order or define a normalized ground state (H_0 has an infimum but no minimum). If we allow infinite norm states then for finite volume the minimization of H_0 requires that $\pi_0^\theta = 0$, in which case $Q_0 = 0$ and the system is in the broken but symmetric phase. However, in the infinite volume limit H_0 becomes zero and if the system were isolated it would make no statement about the ground state θ_0 configuration. Any state compatible with the group relation $[Q, \theta_0^a] = i\delta\theta_0^b$ would be permissible, in particular the symmetric and nonsymmetric extremes $Q_0 = 0$ and θ_0 sharp. What determines the choice of configuration is not the system itself but the *boundary conditions*. For example, with an interaction of the form $\epsilon(\theta \cdot \Phi)^2$, where Φ represents an external heath-bath and ϵ is the coupling constant (trigger), the result depends on the order in which the limits $\Omega \to \infty$ and $\epsilon \to 0$ are taken. If $\epsilon \to 0$ first, then we reach the broken but symmetric phase $Q_0 = 0$ but if $\Omega \to \infty$ first then we reach the broken unsymmetric phase, θ_0 sharp, where the value is dictated by Φ. This dependence on the boundary conditions corresponds to the usual situation for ferromagnets and is supposed to correspond to what happens in the early universe.

Although the choice θ_0 sharp looks fairly innocuous from the mathematical, or even quantum mechanical, point of view, it makes a profound physical difference in field theory because of the factor Ω. To see this consider either the quantum mechanical amplitude A or the average energy ΔE needed for a change $\theta_0 \to \theta_0 + \Delta\theta_0$ in time T, namely

$$A = \sqrt{\frac{T}{\Omega}} e^{-i\frac{\Omega(\Delta\theta_0)^2}{2T}} \quad \text{and} \quad \Delta E = \frac{\Omega}{2T}(\Delta\theta_0)^2 ,$$

respectively. The critical parameter in both cases is Ω/T. It is clear that when Ω/T becomes infinite the probability of changing to another value of θ_0, or of changing to a symmetric configuration, is zero, and that the reason for this is that it would require an infinite energy-input. Thus, once the system has been forced into a configuration where θ_0 is sharp by the boundary conditions, it is forced to stay in that configuration. The only question is when the limit $\Omega/T \to \infty$ occurs. It is usually assumed to occur in two and three space dimensions on the dimensional grounds that $\Omega = T^d$ where d is the space dimension. For one space dimension (two-dimensional space-time), on the other hand, $\Omega = T$ so the ratio is not automatically infinite for infinite Ω. In this case, whether the symmetry is spontaneously broken or not depends on the model.

For the path integral, the nonsymmetric case corresponds to inserting a delta function $\delta(\theta_0 - \Theta_0)$ in the measure $d\theta_0$ for the variable θ_0. This corresponds to taking a single point in the group-orbit of θ_0^a but this is natural because the θ_0 themselves and, in the infinite volume limit, their conjugates π_0, do not appear in the Hamiltonian. Furthermore, the different sectors correspond to the same physics and are separated by infinite energy barriers. In particular, the group transformations that link the different sectors are forbidden.

Comparison with Standard Group Statements. Let us digress for a moment to consider the conventional spontaneous symmetry breaking statements about the generators in the spontaneously broken case, namely

$$[Q, \phi^a(x)] = i\delta\phi^a(x) \qquad \Rightarrow \qquad \langle \phi_0^a \rangle \neq 0 \quad \rightarrow \quad \langle Q^2 \rangle \neq 0 \tag{I.10}$$

which links spontaneous breakdown with noninvariance of the vacuum, and

$$\langle Q^2 \rangle = \int dx\, dy\, \langle j(x) j(y) \rangle = \Omega \int dx\, \langle j(x) j(0) \rangle \tag{I.11}$$

where $j(x)$ is the time-component of a local current, which purports to show that Q becomes infinite when $\Omega \to \infty$.

The generator for the fluctuating part $\tilde{Q}$ of Q is quadratic and can be normal-ordered in the usual way. It is therefore zero on the vacuum, and we then see that these conventional statements are actually statements about the zero mode parts, namely

$$\rho_0 \neq 0 \quad \rightarrow \quad [Q_0, \theta_0^a] = i\delta\theta_0^a \neq 0 \quad \rightarrow \quad Q_0 \neq 0 \tag{I.12}$$

and

$$\langle Q^2 \rangle = Q_0^2 = (\pi_a^0 \delta\theta_0^a)^2 \tag{I.13}$$

From these equations we see that the conventional conclusion that the generators become infinite is not correct. What happens in (I.11) is that the j's are of order Ω^{-1} and thus the whole expression is of order 1. The correct statement seems to be that the generators operate in the usual manner on the phase-space $\{\pi^0, \theta_0\}$, but, as we have seen, because of the factor Ω the dynamics of this phase-space becomes trivial in the infinite Ω limit.

Appendix II: Ward Identities of the Displacement Symmetry

Quantum mechanically one identifies the massless photons as Goldstone bosons associated with the symmetry breakdown of the displacement symmetry with the help of the associated Ward-Identities. For their derivation we consider soft modulations of the rigid displacements (2.5)

$$\Phi(x) \to e^{-id_\mu(x)x^\mu}\Phi(x), \quad A_\mu(x) \to A_\mu(x) + \frac{1}{e} d_\mu(x). \tag{II.1}$$

This is not a gauge transformation; we can restrict the arbitrary $d_\mu(x)$ to not change at all the chosen gauge. In Coulomb gauge, the following choice

$$d_0(x) = 0\,, \quad \operatorname{div} \mathbf{d}(x) = 0 \tag{II.2}$$

guarantees that only physical variables – the transverse gauge fields -are affected by the transformation. These transformations are transverse to the gauge orbits. They change the value of the action and for infinitesimal transformations

$$\delta S = \int d^4x d_\nu(x)\, \partial_\mu \left[\frac{1}{e} F^{\mu\nu} - x^\nu j^\mu \right] \tag{II.3}$$

where the current of the charged matter field is given by

$$j_\mu = i\,\Phi^\star \overleftrightarrow{\partial}_\mu \Phi - 2eA_\mu\Phi^\star\Phi.$$

Accounting also for the corresponding changes in the external source terms we obtain the fundamental functional identity

$$0 = \int d[A,\Phi]\,\delta[\mathrm{div}\,\mathbf{A}]e^{iS[A,\Phi]}e^{i\int d^4x\,(J^\mu A_\mu + k^\star\Phi + k\Phi^\star)} \int d^4x\, d_\nu(x) \\ \cdot\left(\partial_\mu\left[\frac{1}{e}F^{\mu\nu} - x^\nu j^\mu\right] + \frac{1}{e}J^\nu - ix^\nu(k^\star\Phi - k\Phi^\star)\right) \qquad \text{(II.4)}$$

which is the quantum mechanical version of Eq. (2.6). This identifies the four conserved Noether currents associated with the displacement symmetry

$$C^{\mu\nu} = F^{\mu\nu} - ex^\nu j^\mu$$

or, in the case of Coulomb gauge the two components after transverse projection.

Conservation of the Noether currents C has been derived independently of the conservation of the charge current j. In order to account for the latter we proceed as usual. Under the transformation

$$\Phi(x) \to e^{i\alpha(x)}\Phi(x), \quad A_\mu(x) \to A_\mu(x)$$

the action changes for infinitesimal α

$$\delta S = \int d^4x\,\alpha(x)\partial_\mu j^\mu \qquad \text{(II.5)}$$

and yields

$$0 = \int d[A,\Phi]\,\delta[\mathrm{div}\,\mathbf{A}]e^{iS[A,\Phi]}e^{i\int d^4x\,(J^\mu A_\mu + k^\star\Phi + k\Phi^\star)} \int d^4x\,\alpha(x)\left(\partial_\mu j^\mu + ik^\star\Phi - k\Phi^\star\right).$$

This equation can be used to eliminate the explicit x-dependence in Eq. (II.4) by choosing $\alpha = d\,x$, with the transverse but otherwise arbitrary vector field $\mathbf{d}$ (cf. Eq. (II.2)) with the result

$$0 = \int d[A,\Phi]\,\delta[\mathrm{div}\,\mathbf{A}]e^{iS[A,\Phi]}e^{i\int d^4x\,(J^\mu A_\mu + k^\star\Phi + k\Phi^\star)}\left(\Box\mathbf{A}(x) - e\mathbf{j}_{tr}(x) + \mathbf{J}_{tr}(x)\right) \qquad \text{(II.6)}$$

where tr denotes the transverse component of the corresponding vectors. This is the quantum mechanical equation of motion from which the standard Schwinger-Dyson equation may be obtained by functional differentiation with respect to the source $J(y)$ evaluated at $J = 0$

$$\Box_x\langle 0|T(A^i_{tr}(x)A^j_{tr}(y))|0\rangle + e\langle 0|T(j^i_{tr}(x)A^j_{tr}(y))|0\rangle + i\left(\delta_{i,j} - \frac{\nabla_i\nabla_j}{\Delta}\right)\delta(x-y) = 0.$$

Ward identities always follow from properties of the equations of motion; they are obtained by applying a particular subset of variable transformations which yield the equations of motion. In the case of displacements, this particular subset happens to coincide with the most general transformations.

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