On the Large Deviations of a Class of Stationary On/Off Sources Which Exhibit Long Range Dependence

Ken Duffy
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Dublin Institute for Advanced Studies.

Abstract

We present a class of stationary two-state sources which exhibit long range dependence. We relate the large deviations of their sojourn times to the large deviations of the sources themselves. We calculate the rate-function, on a non-linear scale, for a two-state source whose sojourn times are distributed by a semi-exponential distribution, and we calculate the rate-function for the multiplex of a finite collection of such sources.

1 Introduction

Long range dependence has been of interest to phenomenologists since Hurst published his 1951 paper [12] on a time-series of water levels of the Nile. His findings showed that statistical tests could imply a complicated, long range, correlation structure in this time-series. In the late 'sixties Mandelbrot and Wallis [20, 21, 22, 23], and Mandelbrot and Van Ness [19], proposed fractional Brownian motion, which is stationary and exhibits long range dependence, as a model for Hurst's time-series. In 1971 O'Connell [26] proposed an ARIMA model as an explanation of Hurst's phenomenon. In 1974 Klemes [15] objected strenuously to long range dependence as an explanation of Hurst's findings and demonstrated that non-stationarities, which seem physically more plausible than infinite memory, could lead to the observed phenomena.

Long range dependence has been of interest to teletraffic engineers since its proposal as an explanation of phenomena, similar to Hurst's, found in data-sets; for example in Leland *et al.* [16], Crovella and Bestavros [5], and Beran

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et al. [2]. Klemes' remarks were reiterated in the teletraffic setting by Duffield *et al.* [7], where again it can be demonstrated that non-stationarities can lead to the observed phenomena. Fractional Brownian motion, which is long range dependent, has been proposed by several authors (see, for example, Norros [25] and Leyland *et al.* [16]) as a model of multiplexed internet data. Long range dependence is also of interest in a number of other fields, for a general reference (and an extensive bibliography) see Beran [1].

From a phenomonology point of view, without addressing Klemes' remarks, fractional Brownian motion has two drawbacks: it is unbounded and it takes negative values. Boundedness arrives naturally in networks from bandwidth restrictions and negative arrivals have a dubious physical interpretation. This motivated the construction by various authors (for example see Boxma [4] and references therein) of a class of stationary two-state sources whose sojourn times are distributed so that the source exhibits long range dependence. Using techniques developed by Russell in [28] we relate the large deviations of the sojourn times of such sources to the large deviations of the sources themselves.

In section 2 we set up our basic notation and introduce Russell's [28] results. In section 3 we construct our class of two-state sources which possess long range dependence and prove two simple lemmas to increase the ease of application of *Russell's random time-change*. In section 4 we present an example where the sojourn times spent in the 'on' and 'off' states are determined by an i.i.d. sequence with semi-exponential distribution. An explicit form is found for the source's rate-function, on a non-linear scale.

2 Notation and Background

We follow a prescription set down by Russell in [28]. Let a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ be given. Let $\{X_t : t \in T\}$ be a stochastic process where T is $\mathbb{R}_+$ or $\mathbb{Z}_+$. For each $t \in T$, define the random function (the sample path) $S_t(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ by

$$S_t(x) := \int_0^{tx} X.\ d\lambda,$$

where λ is Lebesque measure if $T = \mathbb{R}_+$, and $\lambda = \sum_{k=1}^{\infty} \delta_k$, where δ_k is Dirac measure at k, if $T = \mathbb{Z}_+$. We also define the partial sums process $\{S_t : t \in T\}$

by

$$S_t := S_t(1) = \int_0^t X.\ d\lambda.$$

Let $\{T_n : n \in \mathbb{Z}^+\}$ be a sequence of random times and $\{N_t : t \in T\}$ be its adjoint counting process, that is $N_t := \sup\{n : T_n \leq t\}$. For each $n \in \mathbb{Z}_+$ we define the sample path of T_n to be the function, $T_n(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$T_n(x) := T_{\lfloor nx \rfloor}.$$

Similarly for each $t \in T$ we define the sample path of N_t to be the function, $N_t(\cdot) : \mathbb{R}_+ \to \mathbb{Z}_+$, defined by

$$N_t(x) := N_{tx}.$$

Large deviation results relating $\{T_n\}$ and $\{N_t\}$ have been proved by Duffield and Whitt in [8], by Glynn and Whitt in [11], and by Russell in [27].

We consider large deviation principles both for sample paths (SP-LDP) and for partial sums (1D-LDP; one dimensional LDP). See Lewis and Pfister [17] for a review of large deviation theory, and Dembo and Zeitouni [6] for a general reference to large deviation techniques. We follow Russell in considering our SP-LDPs in the topology of pointwise convergence (see Kelly [14] section 3).

We define a scale, $v : \mathbb{R}^+ \to \mathbb{R}^+$, to be a non-decreasing function with $\lim_{t\to\infty} v(t) = \infty$ and define $\mathcal{M}(\mathbb{R}^+, \mathbb{R})$ be the space of right-continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$.

Definition 1 *$\{S_t(\cdot)\}$ satisfies a SP-LDP on the scale $v(t)$ with rate-function $I : \mathbb{R} \to [0, \infty]$ if $I(\zeta)$ is lower semi-continuous, has compact level sets,*

$$\lim_{t\to\infty} \frac{1}{v(t)} \log P\left[\frac{S_t(\cdot)}{t} \in F\right] \leq -\inf_{\zeta\in F} I(\zeta)$$

for all F closed in $\mathcal{M}(\mathbb{R}_+, \mathbb{R})$, and

$$\lim_{t\to\infty} \frac{1}{v(t)} \log P\left[\frac{S_t(\cdot)}{t} \in G\right] \geq -\inf_{\zeta\in G} I(\zeta)$$

for all G open in $\mathcal{M}(\mathbb{R}_+, \mathbb{R})$.

Definition 2 *$\{S_t\}$ satisfies an 1D-LDP on the scale $v(t)$ with rate-function $I^{(1)} : \mathbb{R} \to [0, \infty]$ if $I^{(1)}(x)$ is lower semi-continuous, has compact level sets,*

$$\lim_{t\to\infty} \frac{1}{v(t)} \log \mathbb{P}\left[\frac{S_t}{t} \in F\right] \leq - \inf_{x\in F} I^{(1)}(x)$$

for all F closed in $\mathbb{R}$, and

$$\lim_{t\to\infty} \frac{1}{v(t)} \log \mathbb{P}\left[\frac{S_t}{t} \in G\right] \geq - \inf_{x\in G} I^{(1)}(x)$$

for all G open in $\mathbb{R}$.

In conjunction with Lemma's 5.2 and 5.3, Theorem 5.1 in [28] proves that if $(S_{T_n}(\cdot), T_n(\cdot))$ [the randomly sampled partial sums process and the random sampling] satisfies a joint SP-LDP on the scale $v(t) := t$ (with rate-function $U(x, y)$ which satisfies $U(x, 0) = \infty$ so that, on the scale of large deviations, the probability that the sampling process re-samples the same point indefinitely is zero, with rate ∞), then $(S_t(\cdot), N_t(\cdot))$ [the partial sums process and counting process] satisfies a SP-LDP on the scale $v(t) := t$. Moreover, in Theorem 5.10 he gives a simple relationship between the one dimension rate-functions $U^{(1)}(\cdot,\cdot)$ and $W^{(1)}(\cdot,\cdot)$, for (S_{T_n}, T_n) and (S_t, N_t),

$$W^{(1)}(x, y) = yU^{(1)}\left(\frac{x}{y}, \frac{1}{y}\right).$$

In his work he considers only scaling functions which are linear, that is $v(t) := t$. If we wish to describe a source that exhibits long range dependence we shall require scaling functions which are non-linear so that correlations within the source decay slower than exponentially. As it turns out the condition that we shall require is that $v(t)$ be regularly varying (see Bingham *et al.* [3] for a general reference).

$v(t)$ being regularly varying implies that $\lim_{t\to\infty} v(ct)/v(t)$ exists as an extended real number for all $c > 0$. As $v(t)$ is non-decreasing and diverging to $+\infty$ this implies that there exists $G \geq 0$ such that

$$\lim_{t\to\infty} \frac{v(ct)}{v(t)} = c^G,$$

for all $c > 0$.

With this hypothesis in mind the only alteration necessary to Russell's work is a trivial one in Lemma 5.3. The rest of his proofs remain unchanged with the final relationship between the one dimensional contractions changing slightly to be

$$W^{(1)}(x,y) = y^G U^{(1)}\left(\frac{x}{y}, \frac{1}{y}\right). \tag{1}$$

We call this transformation *Russell's random time-change.* We have the following diagram describing the relationship between the sample path large deviation principles and the one dimensional large deviation principles:

$$\begin{array}{rccc} \text{Sample path:} & (S_{T_n}(\cdot), T_n(\cdot)) & \Longleftrightarrow & (S_t(\cdot), N_t(\cdot)) \\ & \Downarrow & & \Downarrow \\ \text{One dimensional:} & (S_{T_n}, T_n) & & (S_t, N_t) \end{array}$$

If we start with a joint large deviations principle for the sample paths then we can deduce a relationship between the one dimensional large deviations of (S_{T_n}, T_n) and (S_t, N_t).

3 A Class of Stationary On/Off Sources

Let $\{\tau_i\}$ and $\{\eta_i\}$ be two stationary sequences of random variables taking values in T. $\{\tau_i\}$ are the sojourn times spent by a source in the 'on' state and $\{\eta_i\}$ are the sojourn times spent by a source in the 'off' state. We define the source's activity X_t at time $t \in T$ to be

$$X_t := \begin{cases} 0 & \text{if} \quad \tau_1 + \eta_1 + \cdots + \tau_k \le t < \tau_1 + \eta_1 + \cdots + \tau_k + \eta_k \\ 1 & \text{if} \quad \tau_1 + \eta_1 + \cdots + \eta_k \le t < \tau_1 + \eta_1 + \cdots + \eta_k + \tau_k, \end{cases}$$

see figure 1 for an illustration.

Define the processes $S_n^\tau := \sum_{i=1}^n \tau_i$, $S_n^\eta := \sum_{i=1}^n \eta_i$, and $T_n := S_n^\tau + S_n^\eta$. S_n^τ is the total time spent in the 'on' state after n 'on' periods, S_n^η is the total time spent in the 'off' state after n 'off' periods and T_n is the time after n 'on' and 'off' periods. We note that $S_{T_n} = S_n^\tau$.

Assumption one: $(S_{T_n}(\cdot), T_n(\cdot))$ satisfies a joint SP-LDP on the regularly varying scale $v(n)$.

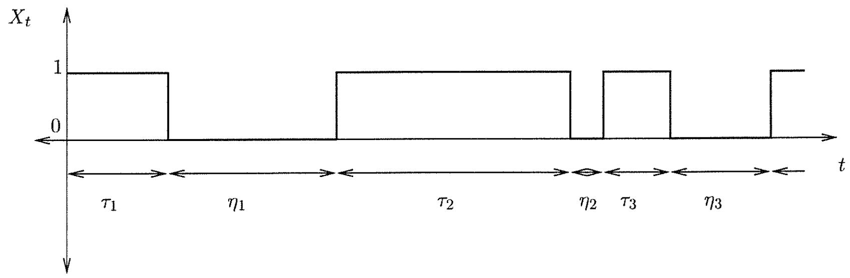

Figure 1: Construction of X_t.

We now prove two simple lemmas which allow us to relate the large deviations of (S_{T_n}, T_n) and (S_n^τ, S_n^η).

Lemma 1 *$(S_{T_n}(\cdot), T_n(\cdot))$ satisfies a joint SP-LDP if and only if $(S_n^\tau(\cdot), S_n^\eta(\cdot))$ satisfies a joint SP-LDP.*

PROOF As $(x, y) \to (x, x+y)$ and $(x, y) \to (x, x-y)$ are continuous functions in the topology of pointwise convergence it follows directly from the contraction principle (see theorem 6.4 of [17]) that $(S_{T_n}(\cdot), T_n(\cdot)) = (S_n^\tau(\cdot), S_n^\tau(\cdot) + S_n^\eta(\cdot))$ satisfies a joint SP-LDP if and only if $(S_n^\tau(\cdot), S_n^\eta(\cdot))$ satisfies a joint SP-LDP.

Lemma 2 *(S_{T_n}, T_n) satisfies a joint 1D-LDP with rate-function $U^{(1)}(\cdot,\cdot)$ if and only if (S_n^τ, S_n^η) satisfies a joint 1D-LDP with rate-function, $I^{(1)}(\cdot,\cdot)$, given by*

$$I^{(1)}(x, y) = U^{(1)}(x, x + y).$$

PROOF Define $f : (x, y) \to (x, x - y)$. As f is continuous and (S_{T_n}, T_n) satisfies a joint 1D-LDP it follows directly from the contraction principle that (S_n^τ, S_n^η) satisfies a joint 1D-LDP with rate-function, $I^{(1)}(\cdot,\cdot)$, given by

$$\begin{aligned} I^{(1)}(x, y) &= \inf\{U^{(1)}(a, b) : f(a, b) = (x, y)\} \\ &= \inf\{U^{(1)}(a, b) : (a, a - b) = (x, y)\} \\ &= U^{(1)}(x, x + y). \end{aligned}$$

Simmilarly as $g : (x, y) \to (x, x + y)$ is continuous we have the converse.

Under assumption one we can apply *Russell's random time-change* and see that (S_t, N_t) satisfies a 1D-LDP on the scale $v(t)$ with rate-function

$$W^{(1)}(x,y) = y^G U^{(1)}\left(\frac{x}{y}, \frac{1}{y}\right) = y^G I^{(1)}\left(\frac{x}{y}, \frac{1}{y} - \frac{x}{y}\right).$$

In order to get the large deviations for $\{S_t\}$ all that is left to do is to contract out the effect of $\{N_t\}$. By the contraction principle $\{S_t\}$ satisfies a large deviation principle with rate-function, $K^{(1)}(\cdot)$, given by

$$K^{(1)}(x) = \inf_{y \in \mathbb{R}_+} W^{(1)}(x,y).$$

Note that if $\{\tau_i\}$ and $\{\eta_i\}$ are independent of each other, and if $\{S_n^\tau\}$ and $\{S_n^\eta\}$ satisfy 1D-LDP's on the scale $v(n)$ with rate-functions $I^\tau(\cdot)$ and $I^\eta(\cdot)$ respectively, then (S_n^τ, S_n^η) satisfies a joint large deviation principle with rate-function $I^{(1)}(x,y) = I^\tau(x) + I^\eta(y)$. Hence, by Lemma 2, $U^{(1)}(\cdot,\cdot)$ is given by $U^{(1)}(x,y) = I^\tau(x) + I^\eta(y-x)$. Applying *Russell's random time-change formula* (1) we get

$$W^{(1)}(x,y) = y^G I^\tau\left(\frac{x}{y}\right) + y^G I^\eta\left(\frac{1}{y} - \frac{x}{y}\right).$$

We now have the following diagram describing the relationship between the SP-LDPS's and 1D-LDPS's:

$$\begin{array}{llllll}
\text{SP-LDP:} & (S_n^\tau(\cdot), S_n^\eta(\cdot)) & \Longleftrightarrow & (S_{T_n}(\cdot), T_n(\cdot)) & \Longleftrightarrow & (S_t(\cdot), N_t(\cdot)) & \\
 & & & \Downarrow & & \Downarrow & \\
\text{1D-LDP:} & (S_n^\tau, S_n^\eta) & \Longleftrightarrow & (S_{T_n}, T_n) & & (S_t, N_t) & \Rightarrow (S_t)
\end{array}$$

If we start with a joint large deviation principle for the sample paths $(S_n^\tau(\cdot), S_n^\eta(\cdot))$, then we can deduce a relationship between the joint one dimensional large deviations of (S_n^τ, S_n^η) and the one dimensional large deviations of $\{S_t\}$.

In order to use *Russell's random time-change* we must prove that $(S_n^\tau(\cdot), S_n^\eta(\cdot))$ satisfies a joint SP-LDP. In section two of [28], Russell proves a SP-LDP on the scale $v(t) := t$ under the assumption of a mixing condition adapted from Lewis *et al.* [18]. This condition does not move to the case of more general scalings as it relies on the use of the sub-additivity lemma.

The most common example which is of interest is where the sojourn times spent in the 'on' and 'off' states were described by a power-tail distribution. That is, $\{\tau_i\}$ and $\{\eta_i\}$ are independent i.i.d sequences, each τ_i and η_i being equal in distribution to τ, where

$$\mathbb{P}[\tau \geq x] := a(x)x^{-\alpha},$$

$\alpha > 1$ and $a(x)$ is slowly varying (see Bingham *et al.* [3]), that is

$$\lim_{x\to\infty} \frac{a(cx)}{a(x)} = 1,$$

for all $c > 0$. However large deviation theory is inappropriate for the partial sums of these random variables; in [9] Gantert proves that $\{S_n^\tau\}$ satisfies a 1D-LDP on the scale $v(n) := \log(n)$ with rate-function, $I^\tau(x)$, given by

$$I^\tau(x) = \begin{cases} \alpha - 1 & \text{if } \ x > M \\ 0 & \text{if } \ x = M \\ \infty & \text{if } \ x < M, \end{cases}$$

where $M := \mathbb{E}[\tau]$. This rate-function does not contain any detailed information and as it does not have compact level sets one can not use the contraction prinicple directly. Moreover, it is dificult to prove a SP-LDP for such a sequence as the lack of compact level sets means it is not sufficient just to check that the upper and lower deviation functions coincide. Instead we turn our attention to the case where the sojourn times are described by a random variable who's distribution has a semi-exponential tail.

4 Example: Semi-Exponential Tails

Let $\{\tau_i\}$ and $\{\eta_i\}$ be independent sequences of i.i.d. random variables with

$$\mathbb{P}[\tau_1 \geq x] := \mathbb{P}[\eta_1 \geq x] := a(x)\exp(-b(x)x^r),$$

where $a(x)$ and $b(x)$ are slowly varying and $r \in (0,1)$. Note that all the moments of τ_1 are finite but the cumulant generating function does not exist in a neighbourhood of the origin. Define $M := \mathbb{E}[\tau_1]$.

In [10] Gantert proves a SP-LDP for $\{S_n^\tau(\cdot)\}$ on the scale $v(n) := b(n)n^r$ with a rate-function that has compact level sets. By the characterisation theorem,

Theorem 1.4.1 of Bingham *et al.* [3], $v(x) := b(x)x^r$ is a regularly varying function. Hence, by Lemma 1, $(S_{T_n}(\cdot), T_n(\cdot))$ satisfies a joint SP-LDP on the scale $v(n)$. Thus the LDP and scaling hypotheses are satisfied and we may use *Russell's random time change.*

In [9] Gantert proves that $\{S_n^\tau\}$ satisfies a 1D-LDP on the scale $v(n)$ with rate-function, $I^\tau(x)$, given by

$$I^\tau(x) = I^\eta(x) = \begin{cases} (x-M)^r & \text{if } x \geq M \\ \infty & \text{if } x < M. \end{cases}$$

Clearly $I(x)$ has compact level sets. It is possible to get more precise expansions for the tail probabilities of sums of these random variables, see Nagaev [24], when one is not just interested in logarithmic asymptotics. The exponential rate suffices for our needs. This is quite an unusual 1D-LDP: the rate-function, $I(x)$, is not convex. This is as large deviations are caused by the tail of individual random variables rather than aggregate behaviour of sums.

By Lemma 2, (S_{T_n}, T_n) satisfies a joint 1D-LDP on the scale $v(n)$ with rate-function, $U^{(1)}(x, y)$, given by

$$U^{(1)}(x,y) = I^\tau(x) + I^\eta(y-x),$$

that is

$$U^{(1)}(x,y) = \begin{cases} \infty & \text{if } x < M \text{ or } y - x < M \\ (x-M)^r + (y-x-M)^r & \text{otherwise.} \end{cases}$$

By *Russell's random time-change* (1), (S_t, N_t) satisfies a 1D-LDP on the scale $v(t)$ with rate-function

$$W^{(1)}(x,y) = y^r U^{(1)}\left(\frac{x}{y}, \frac{1}{y}\right),$$

that is

$$W^{(1)}(x,y) = \begin{cases} \infty & \text{if } \frac{x}{y} < M \text{ or } \frac{1}{y} - \frac{x}{y} < M \\ (x-My)^r + (1-x-My)^r & \text{otherwise.} \end{cases}$$

For a graph of $W^{(1)}(x,y)$, with $a(x) = b(x) = 1$ and $r = 1/2$, see figure 2.

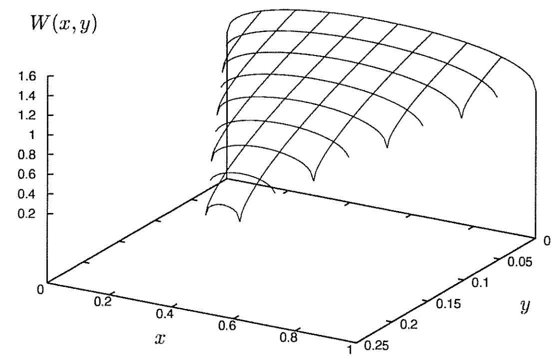

Figure 2: Rate function for (S_t, N_t) on the scale $v(t) := \sqrt{t}$.

We now wish to contract down to remove the y dependence in order to evaluate $K^{(1)}(x)$, the rate-function for $\{S_t\}$. As $U^{(1)}(x, y)$ is concave in both its arguments, is increasing at its left boundry, and is decreasing at its right boundary, its minimum is attained at one or other of its boundaries. Hence

$$K^{(1)}(x) = \begin{cases} (1-2x)^r & \text{if } 0 \leq x \leq 1/2 \\ (2x-1)^r & \text{if } 1/2 \leq x \leq 1 \\ \infty & \text{otherwise.} \end{cases}$$

Note that $K^{(1)}(x)$ is zero at $x = 1/2$. This is as the mean sojourn times spent in both the on, and off, states are finite and equal. $K^{(1)}(x)$ is graphed, with $a(x) = b(x) = 1$ and $r = 1/2$, in figure 3. We note that $K^{(1)}(x)$ has non-convex structure, and that this is quite unusual for a large deviation rate-function.

Multiplex's of heavy tailed sojourn sources are often considered (see, for example, Jelenkovic and Lazar [13], and references therein) as models for long range dependent data passing through large switches. Consider $L \in \mathbb{N}$ independent copies $\{X_t^i : i \in \{1, \dots, L\}\}$ of the two-state source with semi-exponential sojourn times. Define $\{S_t^i\}$ to be the arrivals from source i up to time t and let $S_t := \sum_{i=1}^{L} S_t^i$ be the total arrivals up to time t. As each $\{S_t^i\}$ satisfies a 1D-LDP on the scale $v(t) := b(t)t^r$ with rate-function $K^{(1)}(x)$, and as $\phi : \mathbb{R}^L \to \mathbb{R}$ defined by $\phi(x_1, \dots, x_L) = x_1 \cdots + x_L$ is continuous, we can use the contraction principle to see that $\{S_t\}$ satisfies a 1D-LDP with rate-function, $S^{(1)}(x)$, given by

$$S^{(1)}(x) := \inf\{K^{(1)}(x_1) + \cdots + K^{(1)}(x_L) : x_1 + \cdots + x_L = x\}.$$

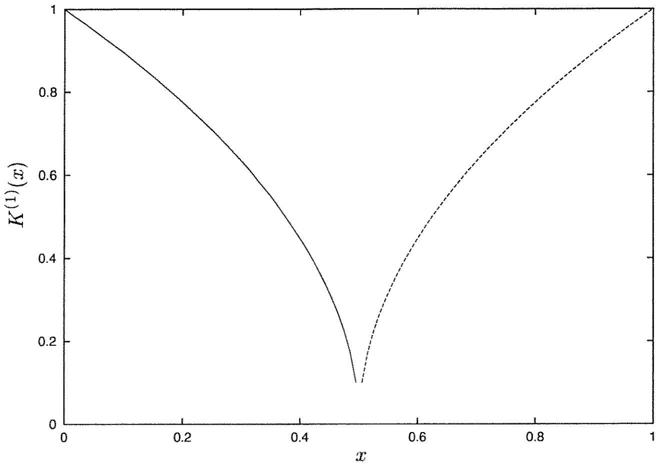

Figure 3: Rate function for $\{S_t\}$ on the scale $v(t) := \sqrt{t}$.

Due to the concave nature of $K^{(1)}(x)$, we have that

$$S^{(1)}(x) := \begin{cases} \lfloor 2x-1 \rfloor + K^{(1)}\left(x + \frac{1}{2} - \frac{\lfloor 2x-1 \rfloor}{2}\right) & \text{if } 1/2 \leq x \leq (L+1)/2 \\ -\lceil 2x-1 \rceil + K^{(1)}\left(-x - \frac{1}{2} - \frac{\lceil 2x-1 \rceil}{2}\right) & \text{if } -(L-1)/2 \leq x \leq 1/2 \\ +\infty & \text{otherwise.} \end{cases}$$

$S^{(1)}(x)$ is graphed, with $a(x) = b(x) = 1$, $r = 1/2$ and $L = 5$, in figure 4.

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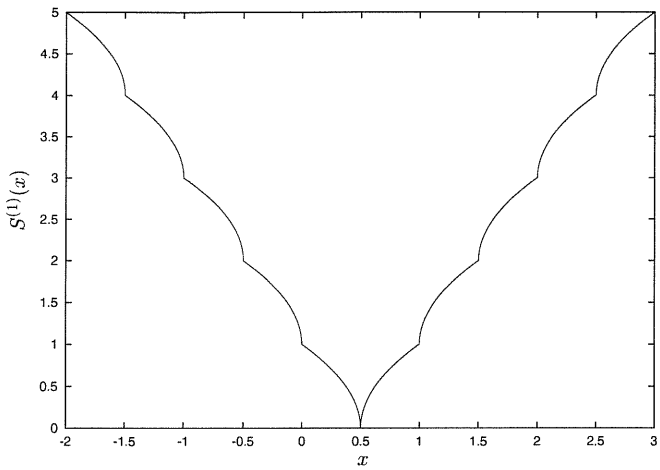

Figure 4: Rate function for multiplex of five on/off sources on the scale $v(t) := \sqrt{t}$.

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