GENERATION BY SECTIONS AND k-AMPLENESS

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ABSTRACT. In the article "Submanifold of abelian varieties", A.J. Sommese proved that direct sum and tensor product of two vector bundles E and F over a smooth projective variety are k-ample if E and F are k-ample and are generated by global sections. Here we show that the latter condition is not needed.

1. INTRODUCTION

Let X be a smooth projective variety. The following definition was introduced by Sommese [3].

Definition 1.1. A line bundle L on X is k-ample, if for some $r > 0$, L^r is generated by sections and the fibers of the corresponding morphism

$$\phi : X \to \mathbb{P}H^0(X, L^r)$$

have dimensions less or equal to k.

A vector bundle E is said to be k-ample if $\mathcal{O}_{\mathbb{P}E}(1)$ is k-ample.

Note that 0-ample is the same as ample.

Notation 1.2. Since we often need to use the phrase "generated by sections", we will abbreviate it by "gbs".

Sommese proved in ([3] p. 235, Corollary (1.10)) that direct sums and tensor products of k-ample vector bundles are k-ample, if these vector bundles are generated by sections. The aim of the present article is to remove the latter restriction.

2. GENERATION BY SECTIONS

We start by giving several interpretations of the statement:

$$\mathcal{O}_{\mathbb{P}E}(r) \text{ is gbs.}$$

1) Let $\pi : \mathbb{P}E \to X$ be the projection. The fibre of π over a point $x \in X$ is given by the 1-codimensional subspaces V of E_x, where E_x is the fibre of E over x. Points of $\mathbb{P}E$ will be denoted by pairs (x, V).

Let $V = v^\perp$ for $v \in E_x^*$. A section $s \in H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(r))$ generates the fibre of $\mathcal{O}_{\mathbb{P}E}(r)$ over (x, V) iff

$$\langle s(x, V), v^r \rangle \neq 0,$$

where $\langle\ ,\ \rangle$ denotes the pairing of dual vector spaces.

2) Equivalently, let $\tilde{s}$ be the section in $H^0(X, S^r E)$ which corresponds to s by the natural isomorphism

$$H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(r)) \simeq H^0(X, S^r E).$$

Then s generates the fibre of $\mathcal{O}_{\mathbb{P}E}(r)$ over $(x, v^\perp)$ iff

$$\langle \tilde{s}(x), v^{\otimes r} \rangle \neq 0.$$

3) Given $x \in X$, the value $\tilde{s}(x)$ of each section $\tilde{s} \in H^0(X, S^r E)$ can be regarded as a homogeneous polynomial map of degree r from E_x^* to $\mathbb{C}$. If the fibre of $\mathcal{O}_{\mathbb{P}E}(r)$ over $v^\perp$ is not generated by any global section, then these polynomials have a common zero at v. If $\mathcal{O}_{\mathbb{P}E}(r)$ is gbs, then 0 is the only common zero.

4) Since $v^{\otimes r}$ annihilates the kernel of the natural map $E^{\otimes r} \twoheadrightarrow S^r E$, the fibre of $\mathcal{O}_{\mathbb{P}E}(r)$ over $(x, v^\perp)$ is generated by a section iff there is a section $\tilde{s} \in H^0(X, E^{\otimes r})$ such that

$$\langle \tilde{s}(x), v^{\otimes r} \rangle \neq 0.$$

If $\mathcal{O}_{\mathbb{P}E}(r)$ is gbs, then $\mathcal{O}_{\mathbb{P}E}(kr)$ is gbs for every positive integer k. All these facts will be used in the following without reference.

Lemma 2.1. *Let X be a complex space not necessarily compact and E, F vector bundles on X. Suppose that $\mathcal{O}_{\mathbb{P}E}(r)$ and $\mathcal{O}_{\mathbb{P}F}(r)$ are gbs (see Notation 1.2) for some $r > 0$. Then $\mathcal{O}_{\mathbb{P}(E \oplus F)}(r)$ is gbs.*

Proof: For each $x \in X$ and each non-zero $u \in (E_x^* \oplus F_x^*)$ we have to find a section in $H^0(X, S^r(E \oplus F))$ which is not annihilated by $u^{\otimes r}$. Consider $u = u_E + u_F$, where u_E, u_F are the components in E_x^*, F_x^* resp. We may assume that $u_E \neq 0$. Since $\mathcal{O}_{\mathbb{P}E}(r)$ is gbs, there is a section in $H^0(X, S^r E)$ which is not annihilated by $u_E^{\otimes r}$. The image of this section under the natural injection $H^0(X, S^r E) \hookrightarrow H^0(X, S^r(E \oplus F))$ is not annihilated by $u^{\otimes r}$. $\qquad \square$

Lemma 2.2. *Let X be a complex space, not necessarily compact and E, F vector bundles on X. Suppose that $\mathcal{O}_{\mathbb{P}E}(r)$ and $\mathcal{O}_{\mathbb{P}F}(r)$ are gbs (see Notation 1.2) for some $r > 0$. Let $e = \mathrm{rk}E$. Then there exists a positive integer n depending only on r and e such that $\mathcal{O}_{\mathbb{P}(E \otimes F)}(nr)$ is gbs.*

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The proof of this lemma needs some preparations. We will use the terminology of decorated oriented graphs (see Feynman diagrams in physics). Decoration means that to each vertex and to each arrow another object is associated. The set of arrows of a graph γ will be denoted by $A(\gamma)$. Any arrow a is said to have a head $h(a)$ and a tail $t(a)$. A vertex which is the head of exactly n arrows is said to have indegree n, a vertex which is the tail of exactly n arrows is said to have outdegree n. The bidegree of a vertex is written as (indegree, outdegree).

Definition 2.3.
Let V, W be vector spaces. Let $i = 1, \ldots, n$, $\tilde{v}_i \in S^r V$, $\tilde{w}_i \in S^r W$, $u \in (V \otimes W)^*$. Then $\Gamma_{V,W}(n, r, \tilde{v}_1, \ldots, \tilde{v}_n, \tilde{w}_1, \ldots, \tilde{w}_n, u)$ is the set of all decorated oriented graphs with $2n$ fixed vertices α_i, β_i, such that for each i the vertex α_i has bidegree $(0, r)$ and is decorated by $\tilde{v}(\alpha_i) = \tilde{v}_i$, the vertex β_i has bidegree $(r, 0)$ and is decorated by $\tilde{w}(\beta_i) = \tilde{w}_i$, and such that all rn arrows are decorated by u. We will omit the indices V, W of $\Gamma_{V,W}$, when no ambiguity arises.

Definition 2.4. Let $\gamma \in \Gamma(n, 1, v_1, \ldots, v_n, w_1, \ldots, w_n, u)$. The value $|\gamma|$ of γ is defined by

$$|\gamma| = \prod_{a \in A(\gamma)} u\big(v(h(a)) \otimes w(t(a))\big).$$

Definition 2.5. Let $\gamma \in \Gamma(n, r, \tilde{v}_1, \ldots, \tilde{v}_n, \tilde{w}_1, \ldots, \tilde{w}_n, u)$. Then the value $|\gamma|$ of γ is defined by the following two conditions: Firstly, $|\gamma|$ depends linearly on all $\tilde{v}_i$ and $\tilde{w}_i$. Secondly, if $\tilde{v}_i = v_i^{\otimes r}$, $\tilde{w}_i = w_i^{\otimes r}$ for $i = 1, \ldots, n$, $v_i \in V$, $w_i \in W$ let the expanded graph

$$\gamma_{ex} \in \Gamma\big(rn, 1, \underbrace{v_1, \ldots, v_1}_{r \ times}, \ldots, \underbrace{v_n, \ldots, v_n}_{r \ times}, \underbrace{w_1, \ldots, w_1}_{r \ times}, \ldots, \underbrace{w_n, \ldots, w_n}_{r \ times}, u\big)$$

be a decorated oriented graph with vertices α_i^l, β_i^l, $i = 1, \ldots, n$, $l = 1, \ldots, r$, such that there is a bijection $\xi : A(\gamma_{ex}) \to A(\gamma)$ with $h(a) = \alpha_i$ if $h(\xi(a)) = \alpha_i^l$, $t(a) = \beta_i$ if $t(\xi(a)) = \beta_i^l$. Moreover, let α_i^l be decorated by v_i, β_i^l by w_i, for $i = 1, \ldots, n$, $l = 1, \ldots, r$,. Then

$$|\gamma| = |\gamma_{ex}|.$$

For later use, note that γ_{ex} has a symmetry group of order $(r!)^{2n} s_\gamma^{-1}$, where s_γ is the order of the group of vertex preserving symmetries of γ.

Proposition 2.6. *There is a function $\nu : N \times N \to N$ with the following property. Consider finite dimensional vector spaces V, W with*

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$d = \dim V$ *and subspaces* $A \subset S^r V$, $B \subset S^r W$, *such that the corresponding spaces of polynomial maps have 0 as only common zero. Let* $n = \nu(r, d)$. *Then for any non-trivial* $u \in (V \otimes W)^*$ *there is a positive integer* m *with* $m \leq n$, *elements* $\tilde{v}_i \in A$, $\tilde{w}_i \in B$ *for* $i = 1, \ldots, m$ *and a decorated directed graph* $\gamma \in \Gamma_{V,W}(m, r, \tilde{v}_1, \ldots, \tilde{v}_m, \tilde{w}_1, \ldots, \tilde{w}_m, u)$ *such that* $|\gamma| \neq 0$.

Proof: We will construct a function ν which is far from optimal, but sufficient for our purpose.

We first show that it suffices to prove the proposition for the case that the map $\hat{u} \in Hom(V, W^*)$ corresponding to u is bijective. Otherwise, let $V' = V/ker(\hat{u})$, $W' = W/ker(\hat{u}^*)$, let π_V, π_W be the corresponding projections $\pi_V : S^r V \twoheadrightarrow S^r V'$, $\pi_W : S^r W \twoheadrightarrow S^r W'$ and let u' be the element of $(V' \otimes W')^*$ induced by u. If

$$\gamma \in \Gamma(m, r, V_1, \ldots, V_m, W_1, \ldots, W_m, u), \text{ and}$$

$$\gamma' \in \Gamma(m, r, \pi_V(V_1), \ldots, \pi_V(V_m), \pi_W(W_1), \ldots, \pi_W(W_m), u')$$

have the same underlying undecorated graph, then $|\gamma'| = |\gamma|$.

If u is bijective, we can identify W with V^* and write

$$u(v \otimes w) = \langle v, w \rangle$$

for $v \in V$, $w \in W$. Let SV be the symmetric algebra over V. There are natural multiplication maps

$$m^r : S^r V \to End(SV),$$

and contraction maps

$$i^r : S^r V \to End(SV)$$

which restrict to

$$m_N^r : S^r V \to Hom(S^{N-r} V, S^N V),$$

$$i_N^r : S^r W \to Hom(S^N V, S^{N-r} V),$$

with $S^N V = 0$ for negative N. For $r = 1$ the maps can be characterized by multilinearity and the properties $m^1(v') v^{\otimes N} = \pi_S(v' \otimes v^{\otimes N})$, $i^1(w) v^{\otimes N} = N \langle w, v \rangle v^{\otimes(N-1)}$ for $v', v \in V$, $w \in W$, where π_S is the projection of the tensor algebra of V to the symmetric algebra SV. For general r they are characterized by multilinearity and the properties $m^r(v^{\otimes r}) = m^1(v)^r$, $i^r(w^{\otimes r}) = i^1(w)^r$.

For $r = 1$ let P be a product with the factors $m^1(v_i)$, $i^1(w_i)$, taken in any order, where $v_i \in V$, $w_i \in W$, $i = 1, \ldots, m$. For any non-negative

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integer N, the product P restricts to a map $P_N \in End(S^N V)$. For the trace of P_N one finds

$$tr(P_N) = \sum_{\gamma \in \Gamma_m} c_\gamma |\gamma|,$$

where

$$\Gamma_m = \Gamma_{V,W}(m, 1, v_1, \ldots, v_m, w_1, \ldots, w_m, \langle,\rangle)$$

and

$$c_\gamma = \binom{d + \rho + N - 1}{d + m - 1},$$

where ρ is the cardinality of the set of arrows a of γ such that the factor $m^1(v(h(a)))$ lies to the right of the factor $i^1(w(t(a)))$ in P. For $\rho > 0$ this follows by induction on ρ from $i^1(w)m^1(v) = m^1(v)i^1(w) + \langle w, v \rangle$ and for $\rho = 0$ from cyclic invariance of the trace and induction on N. For $n = 0$ and $N = 0$ the statement is obvious.

For arbitrary r the calculation can be reduced to the case $r = 1$ as in the definition of $|\gamma|$. Let P be a product with the factors $m^r(\tilde{v}_i)$, $i^r(\tilde{w}_i)$, taken in any order, where $\tilde{v}_i \in S^r V$, $\tilde{w}_i \in S^r W$, $i = 1, \ldots, m$. For any N, P restricts to a map $P_N \in End(S^N V)$. To calculate the trace of P_N it is sufficient to consider the case $\tilde{v}_i = v_i^{\otimes r}$, $\tilde{w}_i = w_i^{\otimes r}$ for $i = 1, \ldots, n$, $v_i \in V$, $w_i \in W$. One finds

$$tr(P_N) = \sum_{\gamma \in \Gamma_m^r} c_\gamma |\gamma|,$$

where

$$\Gamma_m^r = \Gamma_{V,W}(m, r, \tilde{v}_1, \ldots, \tilde{v}_m, \tilde{w}_1, \ldots, \tilde{w}_m, \langle,\rangle)$$

and

$$c_\gamma = (r!)^{2m} s_\gamma^{-1} \binom{d + r\rho + N - 1}{d + rm - 1}.$$

Here ρ is the cardinality of the set of arrows a of γ such that the factor $m^r(\tilde{v}(h(a)))$ lies to the right of the factor $i^r(\tilde{w}(t(a)))$ in P, and s_γ is the order of the group of vertex preserving symmetries of γ.

Let $\mathcal{A}_N^r$ be the subalgebra of $End(S^N V)$ generated by the elements $\{m_N^r(\tilde{v})i_N^r(\tilde{w}) | \tilde{v} \in A, \tilde{w} \in B\}$. This algebra is spanned by products $P_N = m_N^r(\tilde{v}_1)i_N^r(\tilde{w}_1) \cdots m_N^r(\tilde{v}_m)i_N^r(\tilde{w}_m)$, where $\tilde{v}_i \in A$, $\tilde{w}_i \in B$ for $i = 1, \ldots, m$. We have seen that the trace of P_N is given by a linear combination of numbers $|\gamma|$, where $\gamma \in \Gamma(m, r, \tilde{v}_1, \ldots, \tilde{v}_m, \tilde{w}_1, \ldots, \tilde{w}_m, \langle,\rangle)$. If those traces all vanish, then $\mathcal{A}^N$ is a nilalgebra. In particular, there is a non-trivial subspace $V_0 \subset S^N V$ such that

$$m_N^r(\tilde{v})i_N^r(\tilde{w})V_0 = 0$$

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for all $\tilde{v} \in A$, $\tilde{w} \in B$. Since the kernel of $m_N^r(\tilde{v})$ vanishes for $\tilde{v} \neq 0$, this means that $i_N^r(\tilde{w})V_0 = 0$ for all $\tilde{w} \in B$. Since the homogeneous polynomial maps in B have 0 as only common zero, one can apply a theorem of Macaulay ([1] Sec. 6, or [2] p. 85 theorem 4.48). According to this theorem the ideal in SW generated by B contains $S^N W$ if $N \geq rd$, which implies $i_N^N(\tilde{w})V_0 = 0$ for all $\tilde{w} \in S^N W$ and yields a contradiction. Thus $\mathcal{A}^N$ is not nil for $N \geq rd$.

Let $\mathcal{W}_m^N$ be the subspace of $\mathcal{A}^N$ generated by products of length $\leq m$ of elements of the form $m_N^r(\tilde{v})i_N^r(\tilde{w})$. For each m one has either $\dim(\mathcal{W}_{m+1}^N) > \dim(\mathcal{W}_m^N)$ or $\mathcal{W}_m^N = \mathcal{A}^N$.

Let $D(N) = (\dim S^N V)^2$. Since $D(N) \geq \dim\mathcal{A}^N$, one has $\mathcal{W}_{D(N)}^N = \mathcal{A}^N$. In particular, $\mathcal{W}_{D(rd)}^{rd}$ contains elements of non-vanishing trace. Thus it suffices to take for $\nu(r, d)$ the least common multiple of all integers less or equal to $D(rd)$.

$\square$

Proof of Lemma 2.2:

For each $x \in X$ and each non-zero $u \in (E_x \times F_x)^*$ we have to find a section in $H^0(X, (E \otimes F)^{\otimes nr})$ which is not annihilated by $u^{\otimes nr}$. Let

$$I_E : H^0(X, S^r E)^{\otimes n} \hookrightarrow H^0(X, E^{\otimes rn}),$$

$$I_F : H^0(X, S^r F)^{\otimes n} \hookrightarrow H^0(X, F^{\otimes rn}),$$

$$J : H^0(X, E^{\otimes rn}) \otimes H^0(X, F^{\otimes rn}) \hookrightarrow H^0(X, (E \otimes F)^{\otimes rn})$$

be the canonical injections. The permutation group $S(rn)$ acts on $E^{\otimes rn}$ in the standard way, which yields a map

$$\Sigma : S(rn) \times H^0(X, E^{\otimes rn}) \to H^0(X, E^{\otimes rn}).$$

Let

$$\Phi : S(rn) \times H^0(X, S^r E)^{\otimes n} \otimes H^0(X, S^r F)^{\otimes n} \to H^0(X, (E \otimes F)^{\otimes nr})$$

be defined by $\Phi = J \circ (\Sigma \otimes Id) \circ (Id \times I_E \otimes I_F)$. Though Φ factors through the coset map $S(rn) \to S(rn)/S(r)^{\times n}$, it yields sufficiently many sections for our purpose. For $x \in X$, $\sigma \in S(rn)$, $\tilde{s}_i \in H^0(X, S^r E)$, $\tilde{t}_i \in H^0(X, S^r F)$, there is an element

$$\gamma \in \Gamma_{E_x, F_x}(n, r, \tilde{s}_1(x), \ldots, \tilde{s}_n(x), \tilde{t}_1(x), \ldots, \tilde{t}_n(x), u),$$

such that

(2.1) $\quad \langle \Phi(\sigma, \tilde{s}_1 \otimes \cdots \otimes \tilde{s}_n \otimes \tilde{t}_1 \otimes \cdots \otimes \tilde{t}_n)_x, u^{\otimes nr} \rangle = |\gamma|.$

Conversely, for each $\gamma \in \Gamma(n, r, \tilde{s}_1(x), \ldots, \tilde{s}_N(x), \tilde{t}_1(x), \ldots, \tilde{t}_n(x), u)$ one can find a permutation $\sigma \in S(rn)$ satisfying the equation(2.1).

By Proposition 2.6 there exist sections $\tilde{s}_i$, $\tilde{t}_i$ and

$$\gamma \in \Gamma(n, r, \tilde{s}_1(x), \ldots, \tilde{s}_N(x), \tilde{t}_1(x), \ldots, \tilde{t}_n(x), u)$$

such that $|\gamma| \neq 0$. The image of the corresponding section $\Phi(\sigma, \tilde{s}_1 \otimes \cdots \otimes \tilde{s}_n \otimes \tilde{t}_1 \otimes \cdots \otimes \tilde{t}_n)$ in $H^0(X, S^{\otimes nr}(E \otimes F))$ yields a section in $H^0(\mathbb{P}(E \otimes F), \mathcal{O}_{\mathbb{P}(E \otimes F)}(nr))$ which generates the fibre over $(x, u^\perp)$. $\square$

3. Direct sums and tensor products of k-ample vector bundles

Theorem 3.1. *If E, F are k-ample, then $E \oplus F$ is k-ample.*

Proof: We use criterion (1.7.3) of proposition (1.7) of Sommese [3]. First note that $\mathcal{O}_{\mathbb{P}E}(r_1)$ gbs (see Notation 1.2) and $\mathcal{O}_{\mathbb{P}F}(r_2)$ gbs imply that $\mathcal{O}_{\mathbb{P}E}(r)$ and $\mathcal{O}_{\mathbb{P}F}(r)$ are gbs whenever r is a common multiple of r_1, r_2. By Lemma 2.1 this implies that $\mathcal{O}_{\mathbb{P}(E \oplus F)}(r)$ is gbs. Now assume that there is a holomorphic finite to one map $\phi: Z \to X$ of a compact analytic space Z to X, such that $\dim Z = k + 1$ and that there is a surjective map $\phi^*(E \oplus F) \twoheadrightarrow Q$ with a trivial bundle Q. In particular there is a non-trivial section of $\phi^*(E^* \oplus F^*)$. We may assume that the component $s: Z \to \phi^* E^*$ of this section is non-trivial. Since $\mathcal{O}_{\mathbb{P}E}(r)$ is gbs, for any $z \in Z$ such that $s(z) \neq 0$ there is a section $\sigma: X \to S^r E$ such that $\langle \phi^* \sigma(z), s(z)^{\otimes r} \rangle \neq 0$. In particular, $\langle \phi^* \sigma, s^r \rangle$ yields a non-trivial section of the trivial line bundle over Z. Such a section must be constant, which implies that s cannot vanish anywhere. Consequently it yields a trivial quotient bundle of $\phi^* E$, contrary to the assumption that E is k-ample. $\square$

Theorem 3.2. *If E, F are k-ample, then $E \otimes F$ is k-ample.*

Proof: We use criterion (1.7.4), proposition (1.7) of Sommese [3]. We first show that $S^2 E$ is k ample. By Lemma 1 the bundle $E^{\oplus e}$ is k-ample, where e is the rank of E. Since each irreducible summand of $S^n(S^2 E)$ is isomorphic to an irreducible summand of $S^{2n}(E^{\oplus e})$, we have $H^j(X, F \otimes S^n(S^2 E)) = 0$ for $j > k$, for any coherent sheaf F on X and $n >> 0$. Since $\mathcal{O}_{\mathbb{P}(S^2 E)}(1)$ is a restriction of $\mathcal{O}_{\mathbb{P}(E^{\otimes 2})}(1)$, by Lemma 2.2 $\mathcal{O}_{\mathbb{P}(S^2 E)}(N)$ is gbs for some N. Thus $S^2 E$ is k ample. Since $E \otimes F$ is a quotient of the k-ample bundle $S^2(E \oplus F)$, the theorem follows. $\square$

Remark 3.3. *Theorem 3.2 implies that there is no need for the gbs conditions in Lemma (1.11.4) and Proposition (1.13) of [3]. One obtains the following.*

Lemma 3.4. *Let E be a vector bundle on a compact analytic space X, $Gr(s,E)$ be the bundle of s-codimensional subspaces of the fibres of E and $\xi(s,E)$ the tautological line bundle of $Gr(s,E)$. Then $\xi(s,E)$ is k-ample if E is k-ample.*

Proposition 3.5. *For $i = 1,\ldots,n$ let E_i be a vector bundle of rank r_i on a projective manifold X, If for all i E_i is k_i-ample. Then*

$$H^p(X, \wedge^q T_X^* \otimes \wedge^{s_1} E_1 \otimes \cdots \otimes \wedge^{s_n} E_n) = 0$$

if $p + q > \dim X + \sum_i s_i(r_i - s_i) + \min_i\{k_i\}$.

References

[1] P. Macaulay, *The algebraic geometry of modular systems*, Cambridge Univ. Press, 1916.

[2] B. Shiffmann, A.J. Sommese, *Vanishing theorems on complex manifolds*, Progress in mathematics Vol. **56**(1985).

[3] A.J. Sommese, *Submanifold of Abelian Varieties,*, Math.Ann.**233**(1978), 229-256.

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