On the Algebra of Ghost Fields

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Summary. - We study in detail the algebra of free ghost fields which we realize in a Hilbert-Fock space with positive metric. The investigation of causality clarifies the exact reason for the failure of the spin-statistics theorem and leads to the introduction of the Krein Operator. We study the charge algebra of the ghost fields which gives a representation of $\mathrm{gl}(2, \mathcal{C})$. The symmetries of the S-matrix in ghost space are pointed out.

1. - Introduction

Ghost fields play a vital role in the quantization of non abelian gauge theories. Usually being derived in the framework of path integral quantization [1,2], they have been studied using methods of operator quantization, too. Some authors have proceeded along the canonical road [3], hoping that lack of mathematical precision - caused by the very nature of canonical field quantization - may be overcome by renormalization. There, considerable progress has been made by exploiting BRS symmetry [4]. Non abelian gauge theories have also been studied in the framework of axiomatic quantum field theory [5]. This approach guarantees mathematical consistency by deriving consequences from the fundamental axioms, although the construction of interacting quantum gauge fields (in four dimensions) has not been achieved so far. The operator quantization of gauge theories is usually carried out in the framework of indefinite metric state space [6,7].

Beside these important but difficult matters, fortunately, very simple questions can be asked about ghost fields, too: What is the exact reason for the failure of the spin-statistics theorem? Is it possible to quantize ghost fields in a positive metric Hilbert space? In what sense are the two ghost fields u and $\tilde{u}$ conjugate to each other? One would reasonably expect the answers to these structural questions to be independent of the particular interactions the ghost fields are subjected to. Therefore, this paper gives a detailed investigation of free ghost fields. In this case the above questions can be unambiguously answered, and we will show how they are related to the causality structure of the ghost fields.

The simple case of free ghost fields also deserves consideration for the following reason: Recently non abelian gauge theory has been extensively studied in the framework of causal perturbation theory [8-11]. There [12-15] only free field operators are used to construct the functional S-matrix $S[g]$. Therefore, detailed knowledge of their properties is quite useful. The (extended) functional S-matrix $S[g, j]$ also allows for the definition of interacting quantum fields by functional differentiation [6,12-16] and these share many structural properties with the free ones [16].

The mathematical tools used in this paper are very standard methods from the apparatus of second quantization [15,17-20]. Our main task is to tailor this well known material into a form suitable for the somewhat unconventional ghost fields.

The paper is organized as follows: The next chapter gives the construction of the ghost fields in a Hilbert-Fock space with positive metric. The spin-statistics theorem is dealt with in chapter 3. Chapter 4 studies charge algebra of the ghost fields as a representation of $\mathrm{gl}(2, \mathcal{C})$. The construction of the Krein operator is given in chapter 5. The last chapter discusses the ghost charge conjugation and symmetries in ghost space.

2. - Ghost Fields in Hilbert-Fock Space

In this chapter we give the detailed construction of the free ghost fields, $u_a(x)$ and $\tilde{u}_a(x)$, as used in [8-11]. These are two operator valued distributions acting in the ghost Hilbert-Fock space H_g which satisfy the Klein-Gordon equation:

$$(\partial \cdot \partial + m^2)u_a(x) = (\partial \cdot \partial + m^2)\tilde{u}_a(x) = 0 \tag{2.1}$$

and the following anticommutation relations:

$$\{u_a(x), \tilde{u}_b(y)\}_+ = \frac{1}{i}\delta_{a,b}D(x-y), \quad \{u_a(x), u_b(y)\}_+ = \{\tilde{u}_a(x), \tilde{u}_b(y)\}_+ = 0 \tag{2.2}$$

x and y are points in Minkowski space $\mathcal{R}^4$. $D(x)$ is the Pauli-Jordan commutation function [1]. The above equation has to be understood in the sense of tempered distributions [6], i.e. both sides have to be "integrated" with tempered test functions $F(x, y)$. These free ghost fields are important for the causal construction of non abelian gauge theories [8-11]. As the gauge fields themselves, they are in the adjoint representation of the Lie algebra g of the (global) gauge group G [1]. The index a which runs from 1 to dim G refers to an arbitrary but fixed set $\{T^a\}$ of generators in g. We assume g to be semi-simple and compact, i.e. the Cartan metric of g used for g-covariant summation is the Kronecker tensor. For the free fields considered here the group index a will play a completely trivial role, for it is only the construction of the interaction [8-11] where g enters via the structure constants.

In the following we use covariant notation. The mass m may be positive or zero. Let $\vec{p}$ be a given 3-vector. We assign to it a 4-vector p on the mass-hyperboloid (or cone, resp.) Γ_m by

$$p \stackrel{\text{def}}{=} (p^0, \vec{p}), \quad p^0 \stackrel{\text{def}}{=} p^0(\vec{p}, m) \stackrel{\text{def}}{=} \sqrt{\vec{p}^2 + m^2} \tag{2.3}$$

The invariant volume-measure on Γ_m and the corresponding Dirac distribution are defined by

$$dp \stackrel{\text{def}}{=} \frac{d^3\vec{p}}{2p^0(2\pi)^3}, \qquad \delta(p - p') \stackrel{\text{def}}{=} 2p^0(2\pi)^3\delta(\vec{p} - \vec{p}') \tag{2.4}$$

The n-particle Hilbert space $H_g^{(n)}$ for the ghost fields is defined as follows: Its elements are L^2-functions of n momenta, n group indices, and n ghost indices:

$$H_g^{(n)} \ni \phi^{(n)} = \phi^{(n)}_{a_1,\cdots,a_n;i_1,\cdots,i_n}(p_1, \cdots, p_n) \tag{2.5}$$

which are completely antisymmetric under the simultaneous exchange of arguments and indices: $(p_i, a_i, i_i) \leftrightarrow (p_j, a_j, i_j)$. $p = p(\vec{p}, m)$ is the four-momentum specified above, the G-index a runs from 1 to dim G, and the ghost index i can take the values ± 1, corresponding to a ghost or an antighost particle, respectively. The scalar product in $H_g^{(n)}$ is defined by

$$(\psi^{(n)}, \phi^{(n)}) := \sum_{a_1,\cdots,a_n=1}^{\dim G} \sum_{i_1,\cdots,i_n=\pm 1} \int dp_1 \cdots dp_n \overline{\psi}_{a_1,\cdots,a_n;i_1,\cdots,i_n}(p_1, \cdots, p_n)\phi_{a_1,\cdots,a_n;i_1,\cdots,i_n}(p_1, \cdots, p_n) \tag{2.6}$$

The elements of $H_g^{(n)}$ have to have finite norm: $||\phi^{(n)}|| = (\phi^{(n)}, \phi^{(n)})^{1/2} < \infty$. $H_g^{(0)} = \mathcal{C}$, with scalar product $(\psi^{(0)}, \phi^{(0)}) = \overline{\psi^{(0)}}\phi^{(0)}$. The Hilbert-Fock space H_g for the ghost fields is the Hilbert space sum of all n-particle spaces:

$$H_g = \bigoplus_0^\infty H_g^{(n)} \tag{2.7}$$

That means, its elements are sequences

$$H_g \ni \phi = \{\phi^{(n)}\}_{n=0}^\infty = \{\phi^{(0)}, \phi^{(1)}, \cdots, \phi^{(n)}, \cdots\}, \qquad \phi^{(n)} \in H_g^{(n)} \tag{2.8}$$

with scalar product

$$(\psi, \phi) = \sum_{n=0}^{\infty} (\psi^{(n)}, \phi^{(n)}) \tag{2.9}$$

Again, the norm $||\phi|| = (\phi, \phi)^{1/2}$ has to be finite. The vector

$$\phi_0 \stackrel{\text{def}}{=} \{1, 0, 0, 0, \cdots\} \tag{2.10}$$

is called the vacuum.

Let $f = \{f(p)\} \in S(\Gamma_m)$ be a tempered test function on Γ_m [6]. Then the ghost annihilation operators $c_{a;i}(f)$ are defined by

$$\{c_{a;i}(f)\phi\}^{(n)}_{a_1,\cdots,a_n;i_1,\cdots,i_n}(p_1, \cdots, p_n) = (n+1)^{1/2} \int dp\overline{f(p)}\phi^{(n+1)}_{a,a_1,\cdots,a_n;i,i_1,\cdots,i_n}(p, p_1, \cdots, p_n) \tag{2.11}$$

Their adjoints are the ghost creation operators $c^+_{a;i}(f)$ with action

$$\{c^+_{a;i}(f)\phi\}^{(n)}_{a_1,\cdots,a_n;i_1,\cdots,i_n}(p_1, \cdots, p_n) = n^{-1/2}[\delta_{a_1,a}\delta_{i_1,i}f(p_1)\phi^{(n-1)}_{a_2,\cdots,a_n;i_2,\cdots,i_n}(p_2, \cdots, p_n) -$$

$$- \sum_{r=2}^{n} \delta_{a_r,a}\delta_{i_r,i}f(p_r)\phi^{(n-1)}_{a_1,\cdots\widehat{a_r}\cdots,a_n;i_1,\cdots\widehat{i_r}\cdots,i_n}(p_1 \cdots \widehat{p_r} \cdots, p_n)] \tag{2.12}$$

A hat over an index or argument means omitting. The creation and annihilation operators are bounded operators defined on the whole of H_g [15]. Their anticommutators are easily calculated:

$$\{c_{a;i}(f), c^+_{b;j}(g)\}_+ = \delta_{a,b}\delta_{i,j} \int dp\overline{f(p)}g(p)dp, \qquad \{c_{a;i}(f), c_{b;j}(g)\}_+ = \{c^+_{a;i}(f), c^+_{b;j}(g)\}_+ = 0 \tag{2.13}$$

We remark that the formulae (2.11)-(2.13) also hold if $f \in L^2(\Gamma_m, dp)$.

In the following we will use the distributional operators $c_{a;i}(p)$ and $c^+_{a;i}(p)$ defined by

$$c_{a;i}(f) = \int dp\overline{f(p)}c_{a;i}(p), \qquad c^+_{a;i}(f) = \int dp\, c^+_{a;i}(p)f(p) \tag{2.14}$$

The "symbols" $c^{(+)}(p)$ are operator valued distributions, i.e. denote the (anti)linear continuous mapping

$$S(\Gamma_m) \ni f \stackrel{c^{(+)}(p)}{\longrightarrow} c^{(+)}(f) \in B(H_g) \tag{2.15}$$

where $B(H_g)$ is the space of the bounded operators over H_g. These distributions are not regular, i.e. there do not exist $B(H_g)$ valued locally integrable functions $c^{(+)}(p)$ such that the above equations hold true in the sense of Lebesgue integrals.

However, it is possible to interpret $c^{(+)}(p)$ as functions in the following sense: We define the dense domain $D_0 \subset H_g$ to consist of all vectors ϕ with only finitely many nonvanishing components $\phi^{(n)}$ which are not merely L^2-functions but tempered test functions of their arguments. Then there are indeed (unbounded) operator valued functions $c_{a;i}(p)$ which map D_0 into itself, defined by

$$[c_{a;i}(p)\phi]^{(n)}_{a_1,\cdots,a_n;i_1,\cdots,i_n}(p_1, \cdots, p_n) = (n+1)^{\frac{1}{2}}\phi^{(n+1)}_{a,a_1,\cdots,a_n;i,i_1,\cdots,i_n}(p, p_1, \cdots, p_n) \tag{2.16}$$

Such the first of eq.(2.14) can be interpreted as a Lebesgue integral in the sense of unbounded operators from D_0 into itself. The adjoint operators $c^+_{a;i}(p)$ are formally defined by

$$[c^+_{a;i}(p)\phi]^{(n)}_{a_1,\cdots,a_n;i_1,\cdots,i_n}(p_1, \cdots, p_n) = n^{\frac{1}{2}}[\delta_{a_1,a}\delta_{i_1,i}\delta(p_1 - p)\phi^{(n-1)}_{a_2,\cdots,a_n;i_2,\cdots,i_n}(p_2, \cdots, p_n) -$$

$$-\sum_{r=2}^{n}\delta_{a_r,a}\delta_{i_r,i}\delta(p_r-p)\phi^{(n-1)}_{a_1,\cdots,\widehat{a_r},\cdots,a_n;i_1,\cdots,\widehat{i_r},\cdots,i_n}(p_1,\cdots,\widehat{p_r},\cdots,p_n) \tag{2.17}$$

However, the vector $c^+_{a;i}(p)\phi$ is, for $\phi \neq 0$, due to the presence of the Dirac distribution an element of D'_0, the topological dual space of D_0, but is not in H_g, i.e. the adjoint operators $c^+_{a;i}(p)$ are defined on the zero vector only. Despite that the functions $c^+_{a;i}(p)$ exist as sesquilinear forms over $D_0 \otimes D_0$ defined by

$$D_0 \otimes D_0 \ni [\phi,\psi] \rightarrow c^+_{a;i}(p)[\phi,\psi] \stackrel{\text{def}}{=} {}_{D_0}(\phi, c^+_{a;i}(p)\psi)_{D'_0} \tag{2.18}$$

where this "scalar product" means the application of the functional in D'_0 to the vector in D_0. By construction, $c^+(p)$ is the sesquilinear form adjoint of $c(p)$:

$$c^+_{a;i}(p)[\phi,\psi] = \overline{c_{a;i}(p)[\psi,\phi]} \stackrel{\text{def}}{=} \overline{(\psi, c_{a;i}(p)\phi)} = (c_{a;i}(p)\phi,\psi) \tag{2.19}$$

Normal products [13,20] of the $c^{(+)}(p)$ are defined as sesquilinear forms over $D_0 \otimes D_0$ in the same way, f.e.

$$\{c^+_{a;i}(p)c_{b;j}(q)\}[\phi,\psi] \stackrel{\text{def}}{=} (c_{a;i}(p)\phi, c_{b;j}(q)\psi) \tag{2.20}$$

The distributional form of eq.(2.13)is

$$\{c_{a;i}(p), c^+_{b;j}(p')\}_+ = \delta_{a,b}\delta_{i,j}\delta(p-p'), \qquad \{c_{a;i}(p), c_{b;j}(p')\}_+ = \{c^+_{a;i}(p), c^+_{b;j}(p')\}_+ = 0 \tag{2.21}$$

Now we will define the distributional ghost field operators in coordinate space, $u_{a;i}(x)$. These are linear combinations of the expressions $\int dp\, c^{(+)}_{a;i}(p)e^{\mp ipx}$. We want the anticommutators of the $u_{a;i}(x)$ to be diagonal in the G-index a. Since the $c^{(+)}_{a;i}(p)$ have already this property we do not discuss G-mixing. Then the general Ansatz is

$$u_{a;i}(x) = \int dp\,[A_{i,j}c_{a;j}(p)e^{-ipx} + B_{i,j}c^+_{a;j}(p)e^{ipx}] \tag{2.22}$$

Here $A = (A_{i,j}) = \begin{pmatrix} A_{1,1} & A_{1,-1} \\ A_{-1,1} & A_{-1,-1} \end{pmatrix}$, $B = (B_{i,j})$ are any two 2x2 matrices. Ghost indices $i, j, \cdots$ appearing at least twice in a monomial are understood to be summed over. It follows from the preceding remarks that (2.22) is to be understood in the sense of bilinear forms over $D_0 \otimes D_0$. However, let $F = \{F(x)\} \in S(\mathcal{R}^4)$ be a tempered test function over Minkowski space. Then the smeared bilinear forms

$$u_{a;i}(F) = \int d^4x\, u_{a;i}(x)F(x) \tag{2.23}$$

are generated by bounded operators. This defines the ghost fields as operator valued distributions over $S(\mathcal{R}^4)$. An equivalent, more direct definition of these operators is provided by

$$u_{a;i}(F) = A_{i,j}c_{a;j}(\tilde{F}) + B_{i,j}c^+_{a;j}(\hat{F}),\;\; S(\Gamma_m) \ni \hat{F}(p) \stackrel{\text{def}}{=} \int d^4x F(x)\exp\{ipx\},\;\; \tilde{F}(p) \stackrel{\text{def}}{=} \overline{\hat{F}(-p)} \tag{2.24}$$

Since p is on the mass-hyperboloid(-cone) we certainly have

$$(\partial\cdot\partial + m^2)u_{a;i}(x) = 0 \tag{2.25}$$

The anticommutators are easily calculated:

$$\{u_{a;i}(x), u_{b;j}(y)\}_+ = \frac{1}{i}\delta_{a,b}[A_{i,k}B_{j,k}D^+(x-y) - B_{i,k}A_{j,k}D^-(x-y)] \tag{2.26}$$

Here

$$D^{\pm}(x) = \pm i\int dp e^{\mp ipx} \tag{2.27}$$

are the positive and negative frequency parts of the Pauli-Jordan commutation function [15]

$$D(x) = D^+(x) + D^-(x) \tag{2.28}$$

which is up to a factor the only linear combination of D^+ and D^- with causal support, i.e. vanishes for $x \cdot x < 0$. Since we do not want to violate causality, (2.26) gives the following constraint:

$$A \cdot B^{\mathrm{tr}} = -B \cdot A^{\mathrm{tr}} \tag{2.29}$$

The anticommutators are then given by

$$\{u_{a;i}(x), u_{b;j}(y)\}_+ = \frac{1}{i}\delta_{a,b} C_{i,j} D(x-y) \tag{2.30}$$

where $C := A \cdot B^{\mathrm{tr}}$ is antisymmetrical due to (2.29). This equation has, of course, many solutions. We choose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -C \tag{2.31}$$

This gives the desired ghost fields:

$$u_a(x) := u_{a;1}(x) = \int dp\, c_{a;1}(p) e^{-ipx} + c^+_{a;-1}(p) e^{ipx} \tag{2.32}$$

$$\tilde{u}_a(x) := u_{a;-1}(x) = \int dp\, (-) c_{a;-1}(p) e^{-ipx} + c^+_{a;1}(p) e^{ipx} \tag{2.33}$$

$$\{u_a(x), \tilde{u}_b(y)\}_+ = \frac{1}{i}\delta_{a,b} D(x-y), \qquad \{u_a(x), u_b(y)\}_+ = \{\tilde{u}_a(x), \tilde{u}_b(y)\}_+ = 0 \tag{2.34}$$

which is the standard choice for the free ghost anticommutation rules found in the literature. Moreover, these are exactly the right anticommutators needed for gauge invariance [8-11]. So we have succeeded in realizing the ghost fields in a positive metric Hilbert-Fock space.

We close this chapter by discussing the representation $\{U(a,\Lambda)\}$ of the proper Poincaré group $P^\uparrow_+$ in H_g. This is essentially the same as the representation of $P^\uparrow_+$ for a free hermitian bosonic scalar field [21]: For a vector $\phi \in H_g$ [see(2.8)] one defines

$$[U(a,\Lambda)\phi]^{(0)} = \phi^{(0)}, \quad n \geq 1 \Rightarrow [U(a,\Lambda)\phi]^{(n)}_{a_1,\cdots,a_n;i_1,\cdots,i_n}(p_1,\cdots,p_n) =$$

$$= \exp\{i(p_1 + \cdots + p_n)\cdot a\}\phi^{(n)}_{a_1,\cdots,a_n;i_1,\cdots,i_n}(\Lambda^{-1}p_1,\cdots,\Lambda^{-1}p_n) \tag{2.35}$$

This $U(a,\Lambda)$ is unitary, obeys the spectrum condition [21], and fulfils

$$U(a,\Lambda) u_{a;i}(x) U(a,\Lambda)^{-1} = u_{a;i}(\Lambda x + a) \tag{2.36}$$

i.e. all $2(\dim G)$ ghost fields simply transform independently as scalar fields.

The vacuum vector ϕ_0 is $P^\uparrow_+$ invariant and is determined by this property uniquely up to a phase. We now demand the matrices A and B in (2.22) to be invertible, such that the creation and annihilation operators $c^{(+)}_{a;i}$ can be linearly expressed in terms of the ghost fields $u_{a;i}$ and their adjoints $u^+_{a;i}$. Then the vacuum is also cyclic with respect to these fields, i.e. any vector in H_g can be approximated with arbitrary precision by applying a polynom in the (smeared) ghost fields and their adjoints to the vacuum [21]. The discussion of the discrete symmetries P, T, and C may be found in [22].

It will not have escaped the reader's attention that the construction of the free ghost fields given here is indeed very similar to the definition of free bosonic scalar fields. The differences are: Firstly, to get fermionic ghosts, we have used anticommuting creation and annihilation operators. Secondly, causality forced us to take an unusual mixture of these operators in the definition of the local fields. These seemingly minor differences will entail major consequences, none the less.

3. - The Failure of the Spin-Statistics Theorem

The ghost fields we have constructed in the preceding chapter are spin zero fields and they obey anticommutation relations. This seems to be a contradiction to the famous spin-statistics theorem stating that quantum fields with (half)integer spin obey (anti)commutation relations. To clarify the situation let us shortly remind ourselves of the precise content of the spin-statistics theorem [21]. First, it states that if a single irreducible quantized spinor field Φ_α has causal (anti)commutators with itself and with its adjoint Φ_α^+ and vanishing (anti)commutators with the other fields of the theory, then it has the "right " connection between spin and statistics, that means it has causal (anti)commutators for (half)integer spin. Its second part makes a statement about a theory in which different irreducible spinor fields $\Phi_{\alpha;a}$ and their adjoints $\Phi_{\alpha;a}^+$ have causal (anti)commutators among each other (α is the spin-, a the internal index). Here "abnormal" relations between spin and statistics are possible. Then, however, the theory possesses certain symmetries. This allows for the construction of transformed fields with "normal" (anti)commutation relations.

Since this theorem is derived in the context of axiomatic quantum field theory, the term "quantum field" has to be understood in the precise sense given in [21], i.e. all the Wightman axioms hold true. To check whether the ghost fields fulfil the presumptions of the above theorem we have, among other things, to insure that the ghost fields u and $\tilde{u}$ have causal anticommutators not only with themselves but with their adjoint fields u^+ and $\tilde{u}^+$. This question being of great importance we will not restrict ourselves to the special choice (2.31) here. Instead we go back to the general Ansatz (2.22) only constrained by the causality condition (2.29). Then the adjoint fields are given by

$$u^+_{a;i}(x) = \int dp\, \overline{B_{i,j}} c_{a;j}(p) e^{-ipx} + \overline{A_{i,j}} c^+_{a;j}(p) e^{ipx} \tag{3.1}$$

The causal anticommutators for two adjoint fields are easily obtained by taking the adjoint of (2.22)

$$\{u^+_{a,i}(x), u^+_{b;j}(y)\}_+ = i\delta_{a,b}\overline{C_{i,j}} D(x-y) \tag{3.2}$$

More interesting are the anticommutators of the ghost fields with their adjoints:

$$\{u^+_{a;i}(x), u_{b;j}(y)\}_+ = \frac{1}{i}\delta_{a,b}[\overline{B_{i,k}} B_{j,k} D^+(x-y) - \overline{A_{i,k}} A_{j,k} D^-(x-y)] \tag{3.3}$$

Causality of this expression is equivalent to

$$\overline{B} \cdot B^{tr} = -\overline{A} \cdot A^{tr} \tag{3.4}$$

Transposing this gives

$$B \cdot B^+ = -A \cdot A^+ \tag{3.5}$$

Since for nonvanishing A and B the matrices on the two sides of this equation are of opposite definiteness the only solution is $A = B = 0$. With this choice the ghost fields would be the zero fields. Excluding this trivial case we conclude:

The ghost fields have non causal anticommutation relations with their adjoints. Therefore, they do not fulfil the presumptions of the spin-statistics theorem. Consequently, they are allowed to escape its conclusions.

It is easy to check that this non-causal anticommutators is the only "deficiency" of the ghost fields, i.e. all other Wightman axioms hold true [21]. We should remark that the demand for causal (anti)commutators of quantum fields with each other and with their adjoints does not only constitute one presumption of the spin-statistics theorem, it is a far more general assumption entering the definition of an axiomatic quantum field theory [21] from the very beginning. This means that all the theorems of (standard) axiomatic quantum field theory can not be applied to the ghost fields without further check. One would naturally expect inconsistencies for fields with acausal behaviour, especially for the construction of the causal S-matrix [12-15]. The resolution of this problem will be given in chapter 5. There it will prove useful to have some general formulae for transformations of the ghost fields at hand. These are systematically derived in the next chapter.

4. - The Charge Algebra of the Ghost Fields

In this chapter we are going to study the charges of the ghost fields, their algebra, and the transformations they induce in H_g. The charges are expressions bilinear in the fundamental fields. We consider again an arbitrary complex matrix $a = (a_{i,j})$ acting in a two dimensional complex formal ghost-antighost vector space V_2. We assign to it the following charge operator $Q = Q(a)$ acting in the Hilbert-Fock space H_g:

$$Q(a) := \sum_{r;a;i} c^+_{a;i}(f_r) a_{i,j} c_{a,j}(f_r) \tag{4.1}$$

Here f_r is any complete orthonormal basis in $L^2(\Gamma_m, dp)$. These charges are unbounded operators defined on the common dense domain $D \subset H_g$ which consists of all vectors ϕ with only finitely many nonvanishing components $\phi^{(n)}$. We obviously have $D_0 \subset D \subset H_g$. The above sums converge strongly on D and the charge operators map this domain into itself. So their products are well defined, too. The (smeared) creation and annihilation operators $c^{(+)}(f)$ and the smeared fields $u^{(+)}(F), \tilde{u}^{(+)}(F)$ also map D into itself, thereby giving a meaning to products of themselves with the charges Q. The charges can also be expressed as

$$Q(a) = \int dp\, c^+_i(p) a_{i,j} c_j(p) \tag{4.2}$$

where this integrals have again to be understood in the sense of sesquilinear forms over $D_0 \otimes D_0$, or in the sense of integrable distributions [6]. Here, and in the following we have suppressed the G-index a.

The set $\{a\}$ of all complex two by two matrices forms the complex Lie algebra $\mathrm{gl}(2, \mathcal{C})$. A short calculation shows

$$Q([a, b]_-) = [Q(a), Q(b)]_- \tag{4.3}$$

That means the linear map $a \to Q(a)$ gives a representation of $\mathrm{gl}(2, \mathcal{C})$. In $\{a\}$ we have the additional antilinear structure of passing from a matrix a to its adjoint a^+. Also this structure is preserved by the "Quantization" (4.1):

$$Q(a^+) = Q(a)^+ \tag{4.4}$$

Here we have to add the technical remark that the Hilbert space adjoint of $Q(a)$: $Q(a)^+_H$, may generally be defined on a domain somewhat larger than D. Then we mean by $Q(a)^+$ its restriction to D.

A hermitian basis (over $\mathcal{C}$) of $\mathrm{gl}(2, \mathcal{C})$ is given by the four Pauli matrices $\{\sigma_s\}_{s=0}^3$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.5}$$

The corresponding essentially selfadjoint basis of the charge algebra $\{Q\}$ is given by: The ghost number

$$N = Q(\sigma_0) = \int dp\, [c^+_1(p) c_1(p) + c^+_{-1}(p) c_{-1}(p)] \tag{4.6}$$

the ghost charge

$$Q_g = Q(\sigma_3) = \int dp\, [c^+_1(p) c_1(p) - c^+_{-1}(p) c_{-1}(p)] \tag{4.7}$$

and two operators

$$\Gamma = Q(\sigma_1) = \int dp\, [c^+_1(p) c_{-1}(p) + c^+_{-1}(p) c_1(p)] \tag{4.8}$$

$$\Omega = Q(\sigma_2) = \frac{1}{i} \int dp\, [c^+_1(p) c_{-1}(p) - c^+_{-1}(p) c_1(p)] \tag{4.9}$$

which replace one antighost by a ghost and (with a different relative sign) vice versa.

We now want to calculate the commutators of these charges with the ghost creation and annihilation operators. The following general formulae are easily established:

$$[Q(a), c^+_i(p)]_- = c^+_j(p) a_{j,i}, \quad [Q(a), c_i(p)]_- = -a_{i,j} c_j(p), \quad Q(a)\phi_0 = 0 \tag{4.10}$$

This implies the action of these charges on a general state $\phi \in D$ (We again suppress all G-indices):

$$\phi = \{\phi^{(0)}, \phi^{(1)}_{i_1}(p_1), \cdots, \phi^{(n)}_{i_1,\cdots,i_n}(p_1, \cdots, p_n), \cdots\},$$

$$Q(a)\phi = \tilde{\phi} = \{\tilde{\phi}^{(0)}, \tilde{\phi}^{(1)}_{i_1}(p_1), \cdots, \tilde{\phi}^{(n)}_{i_1,\cdots,i_n}(p_1, \cdots, p_n) \cdots\},$$

$$\tilde{\phi}^{(0)} = 0, \quad n \geq 1 \Rightarrow \tilde{\phi}^{(n)}_{i_1,\cdots,i_n}(p_1, \cdots, p_n) = \sum_{r=1}^{n} \left\{ \sum_{j_r=\pm 1} a_{i_r,j_r} \phi^{(n)}_{i_1,\cdots,j_r,\cdots,i_n}(p_1, \cdots, p_n) \right\} \tag{4.11}$$

This means that the representation of gl$(2, \mathcal{C})$ in the n particle space $H_g^{(n)}$ is, for fixed p, the n-fold direct sum of the self-representation. It follows that the charge algebra $\{Q\}$ is a faithful representation of gl$(2, \mathcal{C})$.

By inserting the Pauli matrices for a in (4.10) we get the following commutators of our four basis operators in $\{Q\}$ with the ghost creation and annihilation operators:

$$[N, c_j^+(p)]_- = c_j^+(p), \quad [N, c_j(p)]_- = -c_j(p) \tag{4.12}$$

$$[Q_g, c_j^+(p)]_- = jc_j^+(p), \quad [Q_g, c_j(p)]_- = -jc_j(p) \tag{4.13}$$

$$[\Gamma, c_j^+(p)]_- = c_{-j}^+(p), \quad [\Gamma, c_j(p)]_- = -c_{-j}(p) \tag{4.14}$$

$$[\Omega, c_j^+(p)]_- = ijc_{-j}^+(p), \quad [\Omega, c_j(p)]_- = ijc_{-j}(p) \tag{4.15}$$

and their commutators with the local fields are

$$[N, u(x)]_- = -\tilde{u}^+(x), \quad [N, \tilde{u}(x)]_- = u^+(x) \tag{4.16}$$

$$[Q_g, u(x)]_- = -u(x), \quad [Q_g, \tilde{u}(x)]_- = \tilde{u}(x) \tag{4.17}$$

$$[\Gamma, u(x)]_- = \tilde{u}(x), \quad [\Gamma, \tilde{u}(x)]_- = u(x) \tag{4.18}$$

$$[\Omega, u(x)]_- = -i\tilde{u}(x), \quad [\Omega, \tilde{u}(x)]_- = iu(x) \tag{4.19}$$

It is remarkable that the operator N is distinguished from the other three charges Q_g, Γ, Ω for two very different reasons. Firstly, from the algebraic point of view: Its linear span $\lambda N, \lambda \in \mathcal{C}$, is the center of the charge algebra $\{Q\}$ (as the linear span of σ_0 is the center of gl$(2, \mathcal{C})$). Secondly, form the standpoint of causal field theory: Its commutators with the local fields $u(x), \tilde{u}(x)$ give the not relatively local fields [21] $u^+(x), \tilde{u}^+(x)$, while the commutators of the other charges with the local fields give back local fields.

This is further illustrated by writing down the following conserved currents:

$$j_N^\mu(x) = i : u^+(x) \overleftrightarrow{\partial^\mu} u(x) :, \quad j_g^\mu(x) = i : \tilde{u}(x) \overleftrightarrow{\partial^\mu} u(x) :$$

$$j_u^\mu(x) = i : u(x) \overleftrightarrow{\partial^\mu} u(x) :, \quad j_{\tilde{u}}^\mu(x) = i : \tilde{u}(x) \overleftrightarrow{\partial^\mu} \tilde{u}(x) : \tag{4.20}$$

Here the double dot means normal ordering [2,6,15,20]. These Wick powers [6,20] of the free ghost fields are again operator valued distributions over $S(\mathcal{R}^4)$, i.e. they have to be smeared with tempered test functions $F(x)$ over Minkowski space. They are (after smearing) unbounded operators defined on the dense domain D_0 (see chapter 2) and map this domain into itself. D_0 is actually a dense, common, and invariant domain for all operators over H_g appearing in the paper at hand. The currents can, again, likewise be interpreted as functions with values in the sesquilinear forms over $D_0 \otimes D_0$.

The charges are related to these currents by

$$N = \int_{t=\text{const.}} d^3\vec{x} j_N^0(x), \quad Q_g = \int_{t=\text{const.}} d^3\vec{x} j_g^0(x)$$

$$\Gamma = \frac{1}{2}[Q_u - Q_{\tilde{u}}] \stackrel{\text{def}}{=} \frac{1}{2}\left[\int\limits_{t=\text{const.}} d^3\vec{x} j^0_u(x) - \int\limits_{t=\text{const.}} d^3\vec{x} j^0_{\tilde{u}}(x)\right]$$

$$\Omega = \frac{i}{2}[Q_u + Q_{\tilde{u}}] \stackrel{\text{def}}{=} \frac{i}{2}\left[\int\limits_{t=\text{const.}} d^3\vec{x} j^0_u(x) + \int\limits_{t=\text{const.}} d^3\vec{x} j^0_{\tilde{u}}(x)\right] \tag{4.21}$$

These formulae have to be understood in the sense of bilinear forms over $D_0 \otimes D_0$, or in the sense of integrable distributions [6].

The current j_N uses u^+, so it is not a relatively local quantum field. No wonder then that the derivation implemented by its charge N acts nonlocally on the field algebra. The three other currents, however, are relatively local to the basic fields $u, \tilde{u}$. The existence of three bilinear, conserved, local currents for a scalar field is a unique feature of the ghost field (The more conventional bosonic charged scalar field has only one such current).

We now pass from the Lie algebra gl$(2, \mathcal{C})$ to the corresponding Lie group GL$(2, \mathcal{C})$, i.e. the set $\{A\}$ of all complex two by two matrices A with $\det A \neq 0$, the group product being the usual matrix multiplication. Again, we look at it as a 4-dimensional complex Lie group. Each group element can be written in the form

$$A = \exp\{ia\} = \exp\{iz_s\sigma_s\}, \quad z_s = x_s + iy_s \in [-\pi, \pi) + i\mathcal{R} \tag{4.22}$$

In $\{A\}$ we also have the anti-isomorphism of passing from a matrix A to its adjoint A^+. Real z's represent unitary matrices, obviously.

We now assign to each A the following transformation operator $T{=}T(A)$ in H_g:

$$T(A) = T(\exp\{ia\}) \stackrel{\text{def}}{=} \exp\{iQ(a)\} \tag{4.23}$$

Like the charges $\{Q\}$ the transformations $\{T\}$ are defined on the domain D and map it into itself. The exponential on the right side of the last equation is defined by its power series which converges strongly on D. Using the Baker-Campbell-Hausdorff formula and (4.3) gives

$$T(AB) = T(A)T(B) \tag{4.24}$$

So the transformations $\{T\}$ give a representation of GL$(2, \mathcal{C})$. Inserting (4.4) into (4.24) gives in addition

$$T(A^+) = T(A)^+ \tag{4.25}$$

i.e. the "quantization" preserves the anti-isomorphism. For the definition of T^+ the technical remark after (4.4) applies. The quantization of unitary matrices are isometries on D. These can, of course, be extended to unitary operators on H_g, and this process is always understood to have been carried out.

We now study the adjoint action of the transformations $\{T\}$ in H_g. The following formulae hold true on D:

$$\text{Ad}T(A)[c_i^+(f)] \stackrel{\text{def}}{=} T(A)c_i^+(f)T^{-1}(A) = c_j^+(f)A_{j,i} \tag{4.26}$$

$$\text{Ad}T(A)[c_i(f)] \stackrel{\text{def}}{=} T(A)c_i(f)T^{-1}(A) = A^{-1}_{i,j}c_j(f) \tag{4.27}$$

This implies the following action of the transformations $\{T\}$ on the states $\phi \in D$:

$$\phi = \{\phi^{(0)}, \phi^{(1)}_{i_1}(p_1), \cdots, \phi^{(n)}_{i_1,\cdots,i_n}(p_1, \cdots, p_n), \cdots\}$$

$$T(A)\phi = \phi' = \{\phi'^{(0)}, \phi'^{(1)}_{i_1}(p_1), \cdots, \phi'^{(n)}_{i_1,\cdots,i_n}(p_1, \cdots, p_n), \cdots\}$$

$$\phi'(0) = \phi^{(0)}, \quad n \geq 1 \Rightarrow \phi'^{(n)}_{i_1,\cdots,i_n}(p_1, \cdots, p_n) = \sum_{j_1,\cdots,j_n=\pm 1} \left\{ \left(\prod_{r=1}^{n} a_{i_r,j_r} \right) \phi^{(n)}_{j_1,\cdots,j_n}(p, \cdots, p_n) \right\} \tag{4.28}$$

This means that the representation of $GL(2,C)$ in the n particle space $H_g^{(n)}$ is, for fixed p, the n-fold direct product of the self-representation. It follows that the transformation group $\{T\}$ is a faithful representation of $GL(2,C)$.

The adjoint action of the transformation group $\{T\}$ on the charge algebra $\{Q\}$ is easily calculated by means of (4.26),(4.27):

$$\mathrm{Ad}T(A)[Q(b)] \stackrel{\mathrm{def}}{=} T(A)Q(b)T^{-1}(A) = Q(AbA^{-1}) \stackrel{\mathrm{def}}{=} Q\left(\mathrm{Ad}A[b]\right) \tag{4.29}$$

i.e. the adjoint action commutes with the "quantization". This will turn out to be quite useful in the next chapter since it essentially reduces operator calculations in H_g to matrix algebra in V_2.

We close this algebraic chapter by giving explicitly the adjoint action of the one parameter groups generated by the charges $Q(\sigma_s)$ on the creation and annihilation operators and on the local fields:

$$\mathrm{Ad}\exp\{izN\}[c_j^+(p)] = \exp\{iz\}c_j^+(p) \tag{4.30}$$
$$\mathrm{Ad}\exp\{izN\}[c_j(p)] = \exp\{-iz\}c_j(p) \tag{4.31}$$
$$\mathrm{Ad}\exp\{izN\}[u(x)] = (\cos z)u(x) + (\sin z)(i\tilde{u})^+(x) \tag{4.32}$$
$$\mathrm{Ad}\exp\{izN\}[i\tilde{u}(x)] = -(\sin z)u^+(x) + (\cos z)i\tilde{u}(x) \tag{4.33}$$
$$\mathrm{Ad}\exp\{izQ_g\}[c_j^+(p)] = \exp\{jiz\}c_j^+(p) \tag{4.34}$$
$$\mathrm{Ad}\exp\{izQ_g\}[c_j(p)] = \exp\{-jiz\}c_j(p) \tag{4.35}$$
$$\mathrm{Ad}\exp\{izQ_g\}[u(x)] = \exp\{-iz\}u(x) \tag{4.36}$$
$$\mathrm{Ad}\exp\{izQ_g\}[\tilde{u}(x)] = \exp\{iz\}\tilde{u}(x) \tag{4.37}$$
$$\mathrm{Ad}\exp\{iz\Gamma\}[c_j^+(p)] = (\cos z)c_j^+(p) + i(\sin z)c_{-j}^+(p) \tag{4.38}$$
$$\mathrm{Ad}\exp\{iz\Gamma\}[c_j(p)] = (\cos z)c_j(p) - i(\sin z)c_{-j}(p) \tag{4.39}$$
$$\mathrm{Ad}\exp\{iz\Gamma\}[u(x)] = (\cos z)u(x) + (\sin z)i\tilde{u}(x) \tag{4.40}$$
$$\mathrm{Ad}\exp\{iz\Gamma\}[i\tilde{u}(x)] = -(\sin z)u(x) + (\cos z)i\tilde{u}(x) \tag{4.41}$$
$$\mathrm{Ad}\exp\{iz\Omega\}[c_j^+(p)] = (\cos z)c_j^+(p) - j(\sin z)c_{-j}^+(p) \tag{4.42}$$
$$\mathrm{Ad}\exp\{iz\Omega\}[c_j(p)] = (\cos z)c_j(p) - j(\sin z)c_{-j}(p) \tag{4.43}$$
$$\mathrm{Ad}\exp\{iz\Omega\}[u(x)] = (\cos z)u(x) + (\sin z)\tilde{u}(x) \tag{4.44}$$
$$\mathrm{Ad}\exp\{iz\Omega\}[\tilde{u}(x)] = -(\sin z)u(x) + (\cos z)\tilde{u}(x) \tag{4.45}$$

5.- The Construction of the Krein operator

We have seen in chapter 3 that the adjoint ghost fields $u^+, \tilde{u}^+$ have non causal anticommutators with the ghost fields $u, \tilde{u}$. Since causality is a cornerstone of relativistic quantum field theory we expect trouble. This is avoided in the most simple way conceivable: The theory has to be constructed by using only the local fields $u, \tilde{u}$ while the adjoint fields $u^+, \tilde{u}^+$ will not appear at all. In canonical field theory, for example, the Lagrangian or Hamiltonian has to be (and is!) constructed by solely using u and$\tilde{u}$ (beside other non-ghost fields). An axiomatic framework, too, would have to exclude u^+ and $\tilde{u}^+$ from its field content from the very beginning.

The situation is quite transparent in causal perturbation theory [12-15] which aims at the construction of the functional S-matrix in the form of the following power series

$$S[g] = 1 + \sum_{n=1}^{\infty} e^n \int d^4x_1 \cdots d^4x_n \, T^{(n)}(x_1, \cdots, x_n)g(x_1)\cdots g(x_n) \tag{5.1}$$

Indeed, if only we are given $T^{(1)}(x)$, all higher $T^{(n)}(x)$ can be derived by merely using causality and Poincaré invariance of $S[g]$ [12-15]! Roughly speaking, $T^{(n)}$ is just the n-fold time ordered product of n times $T^{(1)}$. $T^{(1)}(x)$ is a certain combination of Wick powers of free quantum fields and its field content and special form determine the framework of the theory and its interaction. Only those fields will appear in $T^{(n)}$ which are already present in $T^{(1)}$. Therefore, all we have to do is to avoid using the non causal fields $u^+, \tilde{u}^+$ in the construction of $T^{(1)}$ and we will forever have got rid of them.

In massless Yang-Mills theories [8-11], for example, one considers the interaction

$$T^{(1)}(x) = -\frac{i}{2} f_{a,b,c}\{: A_{\mu a} A_{\nu b} F_c^{\mu\nu} : (x) + : A_{\mu a} u_b \partial^\mu \tilde{u}_c : (x)\} \tag{5.2}$$

$f_{a,b,c}$ are the structure constants of G, A_a^μ are the free quantized gauge potentials, $F_a^{\mu\nu}$ are the corresponding free field strengths, and u_a and $\tilde{u}_a$ are our now familiar ghost fields. This $T^{(1)}(x)$ is invariant under gauge transformations [8] generated by the differential operator [23]

$$Q = \int_{t=\mathrm{const.}} d^3\vec{x}[(\partial_\nu A_a^\nu) \overleftrightarrow{\partial^0} u_a](x) \tag{5.3}$$

Since the noncausal fields $u^+, \tilde{u}^+$ do indeed not appear causality is preserved.

The story, however, goes on. For, by suspending the use of the adjoint fields we run into the the next problem: The S-matrix (5.1) should be unitary. Then $T^{(1)}(x)$ must be antihermitian. This is, however, not the case, since the construction of an antihermitian quantity certainly requires using both the ghost fields and their adjoints. In canonical quantization it is the Lagrangian or Hamiltonian which has to be hermitian, and in the axiomatic approach [21] the adjoint fields always appear on equal footing with the field themselves.

The solution of this problem is typical to gauge theories. The S-matrix, or $iT^{(1}(x)$ [the Lagrangian, Hamiltonian], is no longer supposed to be unitary but pseudo-unitary, or pseudo-hermitian, respectively. This means the following: In the Hilbert space H of the underlying gauge theory there exists a distinguished bounded linear operator $J : H \to H$ satisfying

$$J^+ = J, \quad J^2 = 1 \tag{5.4}$$

i.e. J is hermitian and unitary. It is, however, not a positive but an indefinite operator. The pseudo-adjoint O^K of an operator O over H is then defined by

$$O^K \stackrel{\mathrm{def}}{=} JO^+J \tag{5.5}$$

The rigorous discussion of domain questions can be found in [7]. This pseudo-adjugation shares all the algebraic properties of the usual adjugation, i.e

$$(O_1 + O_2)^K = O_1^K + O_2^K,\ (zO)^K = \bar{z}O,\ (O_1O_2)^K = O_2^K O_1^K,\ \left(O^K\right)^K = O \tag{5.6}$$

Positivity, however, is lost:

$$\exists O : \ O^K O \not\geq 0 \tag{5.7}$$

An operator H which satisfies $H^K = H$ is called pseudo-hermitian, an operator U obeying $UU^K = 1$ is called pseudo-unitary. For referring explicitly to the operator J the terms J-hermitian and J-unitary are used, too. The operator J is called the Krein operator and the pair $\{H, K\}$ is called a Krein space. Let (ϕ, ψ) be the (positive definite) scalar product in H. The operator J defines a second, indefinite scalar product by

$$\langle \phi, \psi \rangle \stackrel{\mathrm{def}}{=} (\phi, J\psi) \tag{5.8}$$

Krein spaces are well studied in the mathematical literature [7] and they are the appropriate spaces to study quantized gauge theories [3,5,6,24]. The J-unitarity of the S-matrix in quantized gauge theories is important because, together with gauge invariance, it implies the unitary of the S-matrix on the physical subspace [3,11,15].

Let us go back to the Yang-Mills theory (5.2). Its Hilbert space is the direct product of the gauge field Fock space and the ghost Fock space, and the Krein operator factorizes accordingly:

$$H = H_A \otimes H_g, \;\; J = J_A \otimes J_g \tag{5.9}$$

J_A is given by

$$J_A \stackrel{\text{def}}{=} \prod_{a=1}^{\dim G} (-1)^{N_a^0} \tag{5.10}$$

where N_a^0 is the number operator for SO$(3, \mathcal{R})$-scalar gauge particles of G-colour a. The gauge potentials are pseudo-hermitian with respect to J_A:

$$(A_a^\mu)^K \stackrel{\text{def}}{=} J_A \, (A_a^\mu)^+ \, J_A = A_a^\mu \tag{5.11}$$

The reader can find the details of this construction in refs.[11,15].

We, instead, proceed in our discussion of the ghost field algebra. Our aim is the construction of the Krein operator J_g for the ghost space H_g. This will then define the pseudo-adjoint ghost fields

$$u^K(x) \stackrel{\text{def}}{=} J_g u^+(x) J_g, \;\; \tilde{u}^K(x) \stackrel{\text{def}}{=} J_g \tilde{u}^+(x) J_g \tag{5.12}$$

The key to the construction of J_g is causality. For, we know that the J_g-adjoint fields $u^K, \tilde{u}^K$ have to appear in the construction of J_g-hermitian quantities like $T^{(1)}(x)$. Since we want to preserve causality these fields have to be relatively local to the other fields in $T^{(1)}$, especially to the ghost fields themselves. We know already that the adjoint fields $u^+, \tilde{u}^+$ do not have this property. Therefore, the last equation tells us that the operator J_g has to restore causality.

This story can be compared to the construction of the covariant derivative in differential geometry. There the partial derivative fails to be covariant. Covariance is restored by adding to it a connection term which takes the parallel transport into account. Obviously, this can only work because the connection term is not covariant itself.

By analogy we find that the operator J_g cannot be (quasi)local itself, i.e. it has to act nonlocally on the ghost fields. We will construct J_g using the charges and transformations discussed in the preceding chapter. It follows that we would not succeed if solely using the quasilocal charges Q_g, Γ, Ω. Instead, we will certainly need the non quasilocal ghost number N.

Since $J_g^2 = 1$ the first guess would be to take for J_g the operator

$$E \stackrel{\text{def}}{=} (-1)^N = \exp\{i\pi N\} \tag{5.13}$$

However, N is not the only charge operator with integer spectrum. The mutually commuting operators

$$N_j \stackrel{\text{def}}{=} \frac{1}{2}[N + jQ_g], \; j \in \{1, -1\} \tag{5.14}$$

which separately measure the number of ghost and antighost particles have integer spectrum, too. This implies

$$E = (-1)^N = (-1)^{N_1 + N_{-1}} = (-1)^{N_1}(-1)^{N_{-1}} = (-1)^{N_1}(-1)^{-N_{-1}} = (-1)^{N_1 - N_{-1}} = (-1)^{Q_g} \tag{5.15}$$

showing that this particular function of the not quasilocal operator N is actually equal to a transformation generated by the quasilocal operator Q_g. Thus it cannot be the right choice for J_g. Instead, E is the grading operator for the $\mathcal{Z}_2$-graded operator algebra [23] $\{O\}$. For, if O_b and O_f denote Bose and Fermi operators, respectively [i.e. even resp. odd polynomials in the operators $c^{(+)}(f)$], we have

$$E O_b E = +O_b, \;\; E_b O_f E = -O_f \tag{5.16}$$

We now use one of the operators N_j for the construction of the Krein operator since these operators still have an admixture of the nonlocal operator N. So we define

$$I \stackrel{\text{def}}{=} (-1)^{N_{-1}} = \exp\{i\frac{\pi}{2}[N - Q]\} = T(\Sigma_3) \tag{5.17}$$

We have used a capital letter for the Pauli matrix Σ_3 because we here interpret it as an element of the Lie group GL$(2,\mathcal{C})$ while we use small letters if we interpret it as an element of the Lie algebra gl$(2,\mathcal{C})$. I is indeed hermitian and unitary. Its action on operators follows from

$$Ic_j^{(+)}(p)I = jc_j^{(+)}(p) \tag{5.18}$$

Let us denote the pseudo-adjoint with respect to I by a star:

$$O^* \stackrel{\text{def}}{=} I0^+I \tag{5.19}$$

Then we have

$$u^*(x) = \tilde{u}(x), \qquad \tilde{u}^*(x) = u(x) \tag{5.20}$$

So the I-adjoint of the local fields are local fields again. This is exactly what we wanted. Moreover, u and $\tilde{u}$ are indeed I-adjoint to each other, which answers the last of the three questions in the introduction. So one might well use the symbol $\sim$ instead of $*$ for the I-adjugation. I would be a perfectly right choice for the Krein operator. The reason why it is not used, however, is that neither $iT^{(1)}(x)$ in (5.2) nor Q in (5.3) would be I-hermitian. While this would be easily corrected by giving different though equivalent definitions in (5.2, 5.3), expressions (5.2, 5.3) have historically preceded the discussion of the Krein structure of this theory.

The Krein operator J_g is obtained from I by the following unitary transformation S in H_g:

$$S = T(U),\ U \stackrel{\text{def}}{=} \frac{i}{\sqrt{2}}(\Sigma_1 + \Sigma_3) \tag{5.21}$$

Since

$$U = \exp\left\{i\frac{\pi}{2\sqrt{2}}(\sigma_1 + \sigma_3)\right\} \tag{5.22}$$

we explicitly have

$$S = \exp\left\{i\frac{\pi}{2\sqrt{2}}(\Gamma + Q_g)\right\} \tag{5.23}$$

The Krein operator is finally defined by

$$J_g \stackrel{\text{def}}{=} SIS^{-1} \tag{5.24}$$

By using the equations obtained in the preceding chapter we find

$$J_g = T(\Sigma_1) = \exp\left\{i\frac{\pi}{2}(N - \Gamma)\right\} = i^{N-\Gamma} \tag{5.25}$$

Its action on operators follows from

$$J_g c_j^{(+)}(p) J_g = c_{-j}^{(+)}(p) \tag{5.26}$$

This implies that the J_g-adjoint fields in (5.12) are given by

$$u^K(x) = u(x), \qquad \tilde{u}^K(x) = -\tilde{u}(x) \tag{5.27}$$

It is then easy to check that the desired pseudo-hermiticity properties hold true:

$$\left[T^{(1)}(x)\right]^K = -T^{(1)}(x), \quad Q^K = Q \tag{5.28}$$

To obtain these important properties, equations (5.27) have often been taken as the definition of the pseudo-adjugation K [1,3,11]. Our aim here was to show how this fits naturally in the framework of Krein spaces, to construct J_g explicitly, and to show its relation to the causality structure of the ghost fields.

We close this chapter by discussing the J_g adjugation on the charge algebra $\{Q\}$. This is given by

$$N^K = N, \quad Q_g^K = -Q_g, \quad \Gamma^K = \Gamma, \quad \Omega^K = -\Omega \tag{5.29}$$

The set $\{H_J\} \subset \{Q\}$ of all J_g-hermitian charges H_J, i.e. the charges satisfying $H_J^K = H_J$, is consequently given by the real linear span of $\{N, iQ_g, \Gamma, i\Omega\}$ and forms a real four-dimensional Lie subalgebra of the complex four dimensional Lie algebra $\{Q\}$. It follows from the representation properties discussed in the last chapter that its elements can be written as

$$H_J = Q(m), \quad \text{with } m = m^k \stackrel{\text{def}}{=} \Sigma_1 m^+ \Sigma_1 \tag{5.30}$$

The set of the 2x2 matrices $\{m\}$ fulfilling the last equation forms the real Lie algebra $u(1,1)$. $\{H_J\}$ is therefore a faithful representation of this Lie algebra.

The J_g-unitary transformations $S_J \in \{T\}$, i.e. the transformations satisfying $S_J^K S_J = 1$, form a real four-dimensional Lie subgroup of the complex four-dimensional Lie group $\{T\}$. They can be written as

$$S_J = T(M), \quad \text{with } MM^k \stackrel{\text{def}}{=} M\Sigma_1 M^+ \Sigma_1 = 1 \tag{5.31}$$

The set of the 2x2 matrices M fulfilling the last equation forms the real Lie group $U(1,1)$. $\{S_J\}$ is therefore a faithful representation of this Lie group. Its identity component is given by the exponentiation of $\{H_J\}$.

6. - Ghost Charge Conjugation and Symmetries in Ghost Space

In this chapter we will discuss the ghost charge conjugation C_g. As the name suggests this is an operator reflecting the ghost charge, i.e. satisfying

$$C_g Q_g C_g^{-1} = -C_g \tag{6.1}$$

It is easy to check that the Krein operator $J_g = i^{\Gamma - N}$ would do this job. To be a symmetry of the theory (of the S-matrix, f.e.) C_g should be a quasilocal operator, however. Such it is easily constructed by taking the "quasilocal part" of J_g ,i.e. we define

$$C_g \stackrel{\text{def}}{=} i^N J_g = i^\Gamma = T(i\Sigma_1) \tag{6.2}$$

Indeed, since N acts trivially on the charge algebra we have

$$C_g Q(a) C_g^{-1} = J_g Q(a) J_g, \quad \forall a \tag{6.3}$$

This gives the action of C on $\{Q\}$ as

$$C_g N C_g^{-1} = N, \quad C_g Q_g C_g^{-1} = -Q_g, \quad C_g \Gamma C_g^{-1} = \Gamma, \quad C_g \Omega C_g^{-1} = -\Omega \tag{6.4}$$

The square of C_g is given by the grading operator (5.15):

$$C_g^2 = (i^N J_g)^2 = (-1)^N J_g^2 = (-1)^N = (-1)^Q = E; \quad C_g^4 = 1 \tag{6.5}$$

The adjoint of C_g is given by

$$C_g^+ = \left[i^N J_g\right]^+ = J_g(-i)^N = J_g^{-1}\left[i^N\right]^{-1} = C_g^{-1} \tag{6.6}$$

i.e. C_g is unitary. Since it commutes with J_g:

$$[C_g, J_g]_- = 0 \tag{6.7}$$

it is J_g-unitary as well:

$$C_g C_g^K = 1 \tag{6.8}$$

The action of the ghost charge conjugation on the particle operators follows from (4.38,4.39):

$$C_g c_j^+(p) C_g^{-1} = i c_{-j}^+(p), \quad C_g c_j(p) C_g^{-1} = -i c_{-j}(p) \tag{6.9}$$

while (4.40,4.41) imply its action on the local fields:

$$C_g u(x) C_g^{-1} = i\tilde{u}(x), \quad C_g \tilde{u}(x) C_g^{-1} = i u(x) \tag{6.10}$$

This transformation of the ghost fields has also been discussed in [3].

We remind the reader not to confuse the ghost charge conjugation C_g with the charge conjugation C which is discussed in [22]. The latter one is the baryonic charge conjugation: The Yang-Mills fields (gluons) have vanishing baryonic charge, of course. However, due to their coupling to the baryons (quarks) they have to transform nontrivially under the conjugation of the baryonic charge. The ghost fields, in turn, couple to the Yang-Mills fields and consequently transform nontrivially under C themselves.

Let us go back to the interaction $T^{(1)}(x)$ in eq. (5.2). Beside being gauge invariant it is invariant under the transformations generated by the ghost charge:

$$[Q_g, T^{(1)}(x)]_- = 0 \tag{6.11}$$

However, neither the other generators of the transformation group $\{T\}$: N, Γ, and Ω, nor the ghost charge conjugation C_g commute with $T^{(1)}(x)$.

This lack of symmetry can be easily cured by a slight modification of the interaction in the ghost sector: Consider the interaction described by

$$\Theta^{(1)}(x) = -\frac{i}{2} f_{a,b,c}\{: A_{\mu a} A_{\nu b} F_c^{\mu\nu} : (x) + \frac{1}{2} A_{\mu a} u_b \overleftrightarrow{\partial^\mu} \tilde{u}_c : (x)\} \tag{6.12}$$

This operator differs from $T^{(1)}(x)$ only by a pure divergence term (i.e. a term of the form $\partial_\mu H^\mu$ [8]) and a Q-boundary (i.e. a term of the form $\{Q, K\}_+$ [3,23]). It therefore remains gauge invariant [3,8]. It additionally commutes with 3 basis charges in $\{Q\}$:

$$[Q_g, \Theta^{(1)}(x)]_- = [\Gamma, \Theta^{(1)}(x)]_- = [\Omega, \Theta^{(1)}(x)]_- = 0 \tag{6.13}$$

and is, therefore, C_g invariant,too. Only the central charge N does not commute with $\Theta^{(1)}(x)$. The transformations T_0 generated by the 3 charges in the last eq. can be written as

$$T_0 = T(A_0), \quad \text{with } \det A_0 = 0 \tag{6.14}$$

i.e. $\{T_0\}$ is a faithful representation of $\mathrm{SL}(2, \mathcal{C})$.

A symmetry of a gauge theory living in a Krein space should also preserve the indefinite form (5.8), i.e. should be implemented by pseudo-unitary transformation. This condition selects those transformation S_0 in $\{T_0\}$ which are simultaneously in $\{S_J\}$. They can be written as

$$S_0 = T(M_0), \quad \text{with } \det M_0 = 0 \wedge M M^k = 1 \tag{6.15}$$

The matrices M_0 form the real three-dimensional Lie group $\mathrm{SU}(1,1)$ and $\{S_0\}$ is a faithful representation of this group.

$\Theta^{(1)}(x)$ is, due to its symmetry in ghost space, pseudo-antihermitian with respect to both Krein operators I and J_g discussed in the last chapter:

$$\left[\Theta^{(1)}(x)\right]^K = \left[\Theta^{(1)}(x)\right]^* = -\Theta^{(1)}(x) \tag{6.16}$$

In fact, it would be pseudo-antihermitian with respect to any Krein operator which originates from I by an arbitrary unitary transformation in $\{T\}$.

Let us also state that the representation of $P_+^\uparrow$ discussed in the end of chapter 2 is pseudo-unitary: Fore, it is unitary and it commutes with all charges $Q(a)$ (these are $P_+^\uparrow$-scalars) and transformations $T(A)$. It is, therefore, pseudo-unitary with respect to any Krein operator chosen in $\{T\}$, too.

We close by remarking that the reasoning which leads to the introduction of the Krein operator J_A (5.9,5.10) in the Yang-Mills sector of the theory is very different: The adjoint Yang-Mills fields $[A_a^\mu(x)]^+$ do have causal commutators there. The representation of $P_+^\uparrow$, however, is not unitary, and J_A has to be introduced for the reason of covariance. This already happens in QED [15].

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