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A Multi-scale Subtraction Scheme and Partial Renormalization Group Equations in the $O(N)$-symmetric ϕ^4-theory

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Abstract

To resum large logarithms in multi-scale problems a generalization of $\overline{\text{MS}}$ is introduced allowing for as many renormalization scales as there are generic scales in the problem. In the new "minimal multi-scale subtraction scheme" standard perturbative boundary conditions become applicable. However, the multi-loop beta functions depend on the various renormalization scale ratios and a large logarithms resummation has to be performed on them. Using these improved beta functions the "partial" renormalization group equations corresponding to the renormalization point independence of physical quantities allows one to resum the logarithms. As an application the leading and next-to-leading order two-scale analysis of the effective potential in the $O(N)$-symmetric ϕ^4-theory is performed. This calculation indicates that there is no stable vacuum in the broken phase of the theory for $1 < N \leq 4$.

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1. Introduction

The renormalization group (RG) has proved one of the most important tools in refined perturbative analyses. For it has been recognized for a long time that ordinary loop-wise perturbation expansions of important physical quantities are not only restricted to "small" values of the couplings but are often rendered useless by the occurrence of "large" logarithms. RG resummation of these logarithms is then crucial to establish a region of validity for perturbative results.

This is the case in the analysis of vacuum stability (VS) in the Standard Model (SM), where the loop-expansion of the effective potential (EP) contains logarithmic terms. Only after RG summation of these logarithms may the requirement of vacuum stability be turned into bounds on the Higgs mass [1]. Again, the discussion of Bjorken scaling and its violations in deep inelastic scattering (DIS) is reliable only after RG summation of the relevant logarithms yielding in turn high precision tests of QCD and one of the most accurate determinations of the strong coupling constant [2].

To apply the established RG techniques in both cases it is essential that in the region of interest (large absolute values of the scalar field in the discussion of VS, large momentum transfer for fixed Bjorken variable x_B in DIS) there is only one generic scale $\mathcal{M}$. Then, using some mass independent renormalization scheme such as $\overline{\mathrm{MS}}$ $\mathcal{M}$ may be tracked by the corresponding renormalization scale μ, as it occurs in the combination $\hbar\log(\mathcal{M}/\mu^2)$ only. Choosing $\mu^2 = \mathcal{M}$ removes the potentially large logarithms from the perturbation series. Hence, at this scale the perturbative result is trustworthy for "small" values of the couplings and yields the proper boundary condition for the RG evolution to finite values of $\hbar\log(\mathcal{M}/\mu^2)$.

However, there may be many generic scales $\mathcal{M}_i$ in the region of interest. For example, in the computation of finite temperature EP [3] or in supersymmetric extensions of the SM one encounters this problem [4]. But even in the SM there are largely differing effective scales near the tree-level minimum. Although the usual VS analyses of the SM were concerned with large absolute values of the scalar field far away from the tree minimum it is implicitly assumed that the tree minimum is only slightly shifted by quantum corrections. For consistency, one should check this assumption; this is a highly non-trivial multi-scale problem. The breakdown of the ordinary RG analysis of DIS at small and large x_B is again due to the growing importance of generic scales other than the large momentum transfer (for a review see ref. [5]). In both instances different potentially large logarithms $\hbar\log(\mathcal{M}_i/\mu^2)$ occur in the loop-wise perturbative expansion which should be resummed in order to get trustworthy results. But as there is only one renormalization scale one cannot trace the various $\mathcal{M}_i$ at once and remove all the log's from a loop-wise expansion at one particular scale. So, although one still has a perfectly good RG equation there is no longer a proper boundary condition to RG-evolve from. This problem has been recognized by many authors.

Sticking to the $\overline{\mathrm{MS}}$ scheme the decoupling theorem [6] was used in ref. [7] to obtain some regionwise approximation to leading log's (LL) multi-scale summations. Although this is perfectly reasonable, one has to employ "low-energy" parameters, and it is not clear how to obtain sensible approximations for these low-energy parameters in terms of the basic parameters of the full theory. Alternatively, one of us [8] argued that one could still apply the standard $\overline{\text{MS}}$ RG equation to multi-scale problems provided "improved" boundary conditions were employed. Although some improved boundary conditions were suggested in some simple cases, no general prescription was given for constructing these boundary conditions, and no obvious improved boundary conditions were apparent for the subleading log's summation.

Clearly, one must go beyond the usual mass-independent renormalization schemes if multi-scale problems are to be seriously tackled. In the context of the effective potential we are aware of two different approaches. In ref. [9] it was argued that one could employ a mass-dependent scheme in which decoupling of heavy modes is manifest in the perturbative RG functions. Alternatively, in ref. [10] the usual $\overline{\text{MS}}$ scheme was extended to include several renormalization scales κ_i. While this seems to be an excellent idea, the specific scheme in [10] has two drawbacks. Firstly, the number of renormalization points does not necessarily match the number of generic scales in the problem at hand, as there is a RG scale κ_i associated with each coupling. Secondly, when computing multi-scale RG functions to n loops one encounters contributions proportional to $\log^{n-1}(\kappa_i/\kappa_j)$ (and lower powers). If some of the $\log(\kappa_i/\kappa_j)$ are "large" then even the perturbative RG functions cannot be trusted and used to sum logarithms. A similar approach to the one of ref. [10] was outlined in ref. [11] though no detailed perturbative calculations were performed.

In this paper we adopt a more systematic approach. Using the freedom of *finite* renormalizations we introduce a new "minimal multi-scale subtraction scheme" that allows for as many renormalization scales κ_i as there are generic scales in the problem. Hence, removing all large logarithms at scales $\kappa_i^2 = \mathcal{M}_i$ in the new scheme standard perturbative boundary conditions become applicable. As in the approach of ref. [10], the multi-loop RG functions in this scheme *inevitably* depend on the renormalization scale ratios κ_i/κ_j. However, within our minimal multi-scale subtraction scheme we are able to implement a *large logarithms resummation on the RG functions* themselves. Using these improved RG functions the "partial" RGE's corresponding to the renormalization point independence of physical quantities allow us then to resum the logarithms for any other choice of scales.

Much like the SM, the calculation of the effective potential near the tree-level minimum of the broken phase ($m^2 < 0$) in the $O(N)$-symmetric ϕ^4-theory is a two-scale problem for $1 < N < \infty$. In our opinion, this is the simplest non-trivial multi-scale problem in four dimensions, and so we propose to use this model to demonstrate our method. In fact, we are able to *analytically* perform leading order (LO) and next-to-leading order (NLO) multi-scale computations in the $O(N)$-model. Surprisingly, this analysis indicates that the assumption that the tree-level is not significantly shifted by quantum corrections is only valid for $N > 4$. For $1 < N \leq 4$ it appears that there might not even be a stable vacuum in the broken phase.

The outline of the paper is as follows. In section 2 we review the standard $\overline{\text{MS}}$ RG approach to LL summations in the single-scale cases $N = 1$ and $N \to \infty$. In section 3 we motivate the idea of two-scale renormalization and introduce our minimal two-scale subtraction scheme. In section 4 we compute the leading order two-scale RG functions within our minimal prescription. We use these LO beta functions in section 5 to compute the LO running parameters, which are then used in section 6 to compute the two-scale RG improved potential to leading order. In sections 7 and 8 we determine the next-to-leading order contributions to the RG functions and running parameters. In section 9 we obtain the NLO effective potential. Section 10 is devoted to a discussion of the special case $N = 2$. In appendix A we collect the values of various constants and in appendix B we discuss some relevant two-loop integrals.

2. Resumming log's in the effective potential

Let us consider the massive $O(N)$-symmetric field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - \frac{\lambda}{24}\phi^4 - \frac{1}{2}m^2\phi^2 - \Lambda \tag{2.1}$$

where ϕ is an N-component scalar field. Note the inclusion of the cosmological constant Λ [12] which will prove essential in the discussion of the RG later (For a nice discussion of this point in the context of curved spactime calculations we refer to [13]).

We are interested here mainly in the effective potential which arises as the zeroth order term in a derivative expansion of the effective action $\Gamma[\varphi]$

$$\Gamma[\varphi] = \int d^4x \left(-V(\varphi) + \frac{1}{2}Z(\varphi)\partial_\alpha\varphi\partial^\alpha\varphi + \mathcal{O}(\partial^4)\right). \tag{2.2}$$

As usual $\Gamma[\varphi]$ is the Legendre transform of the Schwinger functional $\mathcal{W}[j]$.

A loop-wise perturbation expansion of $V = \sum_n \frac{\hbar^n}{(4\pi)^{2n}} V^{(n\text{-loop})}$ [14, 15] yields in the $\overline{\text{MS}}$-scheme

$$\begin{aligned} V^{(\text{tree})} &= \frac{\lambda}{24}\varphi^4 + \frac{1}{2}m^2\varphi^2 + \Lambda, \\ V^{(\text{1-loop})} &= \frac{{\mathcal{M}_1}^2}{4}\left(\log\frac{\mathcal{M}_1}{\mu^2} - \frac{3}{2}\right) + (N-1)\frac{{\mathcal{M}_2}^2}{4}\left(\log\frac{\mathcal{M}_2}{\mu^2} - \frac{3}{2}\right), \end{aligned} \tag{2.3}$$

where

$$\mathcal{M}_1 = m^2 + \tfrac{1}{2}\lambda\varphi^2, \quad \mathcal{M}_2 = m^2 + \tfrac{1}{6}\lambda\varphi^2, \tag{2.4}$$

and μ is the renormalization scale. The one-loop contribution to the EP thus contains logarithms of the ratios $\mathcal{M}_i/\mu^2$ to the first power and in general the n-loop contribution will be a polynomial of the nth order in these logarithms. (The explicit two-loop result has been obtained in [16].) The EP is independent of the renormalization scale μ which gives rise to a $\overline{\text{MS}}$ RG equation.

In view of these logarithms the loop-wise expansion may be trusted only in a region in field- and coupling-space where simultaneously

$$\frac{\hbar\lambda}{(4\pi)^2} \ll 1, \quad \frac{\hbar\lambda}{(4\pi)^2}\log\frac{\mathcal{M}_i}{\mu^2} \ll 1, \tag{2.5}$$

conditions which may hardly be fulfilled e.g. around the tree-level minimum of the potential, where in the broken phase $\mathcal{M}_2 = 0$, even with a judicious choice of the scale μ. Hence, to obtain a wider range of validity one has to resum the logarithms in the EP [14].

In the two limiting cases $N = 1$ and $N \to \infty$ there is essentially only one relevant scale involved, $\mathcal{M}_1$ for $N = 1$ and $\mathcal{M}_2$ for $N \to \infty$. Setting the renormalization scale μ equal to the relevant scale removes the potentially large logarithms at this scale and we may trust the tree-level EP there. To recover the EP at any other scale we then use the $\overline{\text{MS}}$ RG

$$\mathcal{D}V = 0, \quad \mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_{m^2}\frac{\partial}{\partial m^2} + \beta_\Lambda\frac{\partial}{\partial\Lambda} - \beta_\varphi\varphi\frac{\partial}{\partial\varphi}. \tag{2.6}$$

We next expand the RG functions in powers of $\hbar$. As the expansion coefficient $Z(\varphi)$ in (2.2) does not contain logarithms at the one-loop level no anomalous field-dimension arises and it is an easy task to read off the other one-loop coefficients from the result (2.3). For $N = 1$ we have at *one loop*

$$_1\beta_\lambda = \frac{3\hbar\lambda^2}{(4\pi)^2}, \quad _1\beta_{m^2} = \frac{\hbar\lambda m^2}{(4\pi)^2}, \quad _1\beta_\Lambda = \frac{\hbar m^4}{2(4\pi)^2}, \quad _1\beta_\varphi = 0, \tag{2.7}$$

whereas for $N \to \infty$ we find

$$_2\beta_\lambda = \frac{\hbar N\lambda^2}{3(4\pi)^2}, \quad _2\beta_{m^2} = \frac{\hbar N\lambda m^2}{3(4\pi)^2}, \quad _2\beta_\Lambda = \frac{\hbar N m^4}{2(4\pi)^2}, \quad _2\beta_\varphi = 0 \tag{2.8}$$

which are exact in this limit.

With the use of the RG functions we next recover the running couplings. Setting $s = \frac{\hbar}{(4\pi)^2}\log(\mu(s)/\mu)$, where μ is the reference scale, we have for $N = 1$

$$\begin{aligned} \lambda(s) &= \lambda(1 - 3\lambda s)^{-1}, \quad m^2(s) = m^2(1 - 3\lambda s)^{-1/3} \\ \Lambda(s) &= \lambda - \frac{m^4}{2\lambda}\left[(1 - 3\lambda s)^{1/3} - 1\right] \end{aligned} \tag{2.9}$$

and for $N \to \infty$

$$\begin{aligned} \lambda(s) &= \lambda(1 - \tfrac{1}{3}N\lambda s)^{-1}, \quad m^2(s) = m^2(1 - \tfrac{1}{3}N\lambda s)^{-1} \\ \Lambda(s) &= \Lambda + \frac{3m^4}{2\lambda}\left[(1 - \tfrac{1}{3}N\lambda s)^{-1} - 1\right]. \end{aligned} \tag{2.10}$$

Imposing the tree-level boundary condition the LL approximation to the effective potential at an arbitrary scale μ becomes

$$V_i^{(0)}(\lambda, m^2, \varphi, \Lambda; \mu) = \frac{\lambda(s_i)}{24}\varphi^4 + \frac{1}{2}m^2(s_i)\varphi^2 + \Lambda(s_i) \tag{2.11}$$

where

$$s_1 = \frac{\hbar}{2(4\pi)^2}\log\frac{\mathcal{M}_1}{\mu^2}, \quad s_2 = \frac{\hbar}{2(4\pi)^2}\log\frac{\mathcal{M}_2}{\mu^2}. \tag{2.12}$$

Higher orders may again be systematically resummed giving rise to the NLL, NNLL, ... approximations to the effective potential [17].

As the usual RG may cope with one scale only this approach does not allow a systematic resummation in the generic case as we have to deal with two relevant scales, at least near the tree-level minimum in the broken phase. Therefore, we have to generalize the usual RG approach allowing for as many renormalization scales as there are relevant scales in the theory, the task we turn to in the next section.

3. Two-scale renormalization

In the previous section we were able to use the renormalization scale μ arising in $\overline{\rm MS}$ to track one relevant scale and to resum the corresponding logarithms with the $\overline{\rm MS}$ RG. This was sufficient to obtain a trustworthy approximation to the EP for $N = 1$ and $N \to \infty$. To deal with the general case we shall introduce a new set of parameters depending on *two* renormalization scales κ_1, κ_2 which allow us to track the two generic scales $\mathcal{M}_i$. That is, we consider a *finite* transformation

$$\begin{aligned}
\lambda_{\overline{\rm MS}} &= F_\lambda(\lambda; \kappa_1, \kappa_2, \mu) \\
m^2_{\overline{\rm MS}} &= m^2 F_{m^2}(\lambda; \kappa_1, \kappa_2, \mu) \\
\Lambda_{\overline{\rm MS}} &= \Lambda + m^4 F_\Lambda(\lambda; \kappa_1, \kappa_2, \mu) \\
\varphi_{\overline{\rm MS}} &= \varphi F_\varphi(\lambda; \kappa_1, \kappa_2, \mu).
\end{aligned} \tag{3.1}$$

Here, the $\overline{\rm MS}$ parameters $\lambda_{\overline{\rm MS}}, m^2_{\overline{\rm MS}}, \Lambda_{\overline{\rm MS}}, \varphi_{\overline{\rm MS}}$ at scale μ may be regarded as "bare" ones as opposed to the new "renormalized" two-scale subtraction scheme parameters $\lambda, m^2, \Lambda, \varphi$.

Our goal is to construct a transformation (3.1) with the following properties:

i) The effective action Γ, when expressed in terms of the new parameters, should be independent of the $\overline{\rm MS}$ scale μ.

ii) When $\kappa_1 = \kappa_2$ the minimal two-scale subtraction scheme should coincide with $\overline{\rm MS}$ at that scale.

iii) When $N = 1$ or $N \to \infty$ one scale should drop and the two-scale scheme should coincide with $\overline{\rm MS}$ at the remaining scale.

iv) When $\kappa_i^2 = \mathcal{M}_i$ the standard loop-expansion should render a reliable approximation to the full EP insofar as $\frac{\hbar}{(4\pi)^2}\lambda(\kappa_1, \kappa_2)$ is "small".

In order to find a suitable transformations (3.1) with the desired properties we first study the associated RG's and RG functions. Having obtained a trusworthy set of RG functions we turn them into running couplings and an improved effective potential.

Our starting point is

$$\Gamma_{\overline{\rm MS}}[\lambda_{\overline{\rm MS}}, m^2_{\overline{\rm MS}}, \Lambda_{\overline{\rm MS}}, \varphi_{\overline{\rm MS}}; \mu] = \Gamma[\lambda, m^2, \Lambda, \varphi; \kappa_1, \kappa_2] \tag{3.2}$$

from which we derive the two RGE's corresponding to variations of scales κ_i, where the other scale κ_j and the $\overline{\rm MS}$ parameters are held fixed, in much the same way as the $\overline{\rm MS}$ RG is usually derived. Specializing to the effective potential we obtain

$$\mathcal{D}_i V = 0, \quad \mathcal{D}_i = \kappa_i \frac{\partial}{\partial \kappa_i} + {}_i\beta_\lambda \frac{\partial}{\partial \lambda} + {}_i\beta_{m^2} \frac{\partial}{\partial m^2} + {}_i\beta_\Lambda \frac{\partial}{\partial \Lambda} - {}_i\beta_\varphi \varphi \frac{\partial}{\partial \varphi}. \tag{3.3}$$

The two sets of RG functions are defined as usual

$${}_i\beta_\lambda = \kappa_i \frac{d\lambda}{d\kappa_i}, \quad {}_i\beta_{m^2} = \kappa_i \frac{dm^2}{d\kappa_i}, \quad {}_i\beta_\Lambda = \kappa_i \frac{d\Lambda}{d\kappa_i}, \quad {}_i\beta_\varphi \varphi = -\kappa_i \frac{d\varphi}{d\kappa_i} \tag{3.4}$$

for $i = 1, 2$. In general they may be functions not only of λ, m^2 as are the $\overline{\rm MS}$ RG functions but also of κ_2/κ_1.

Note that property ii) requires that the sum of the two-scale RG functions at $\kappa_1 = \kappa_2$ coincides with the $\overline{\text{MS}}$ RG function at that scale

$$ {}_1\beta_{.}(\kappa_1 = \kappa_2) +_2 \beta_{.}(\kappa_1 = \kappa_2) = \beta_{.,\overline{\text{MS}}} \tag{3.5} $$

and property iii) fixes the two sets of RG functions in the single-scale limits. For $N = 1$ there are no Goldstone bosons. Hence, we have to choose the usual $N = 1$ $\overline{\text{MS}}$ RG functions as the first set of RG functions, given to $\mathcal{O}(\hbar)$ by eqn. (2.7), and to disregard the second set of RG functions so that $\mathcal{D}_2 = \kappa_2 \partial/\partial\kappa_2$. For $N \to \infty$ there are no Higgs contributions. Accordingly, in this limit we have to disregard the first set of RG functions, so that $\mathcal{D}_1 = \kappa_1 \partial/\partial\kappa_1$, and to choose the second set as the large N $\overline{\text{MS}}$ RG functions, given by eqn. (2.8).

Let us come back to the general case. As we want to vary κ_1 and κ_2 independently we must respect the integrability conditions

$$ [\kappa_1 d/d\kappa_1, \kappa_2 d/d\kappa_2] = [\mathcal{D}_1, \mathcal{D}_2] = 0. \tag{3.6} $$

An essential feature of a mass-independent renormalization scheme such as $\overline{\text{MS}}$ is that the beta functions do not depend on the renormalization scale μ. Unfortunately we cannot generalize this to the multi-scale case and demand that the two sets of beta functions be independent of κ_2/κ_1. The point is that the independence of the RG functions from the scales κ_i, ie. $[\kappa_i \partial/\partial\kappa_i, \mathcal{D}_j] = 0$, is incompatible with the integrability condition eqn. (3.6). However, it is still possible to arrange for one of the two sets of RG functions, or in slight generalization for a linear combination of the two sets, to be κ_i-independent. Hence, we assume that

$$ \begin{aligned} \tilde{\beta}_\lambda &= {}_1\beta_\lambda p_1 + {}_2\beta_\lambda p_2, \quad \tilde{\beta}_{m^2} = {}_1\beta_{m^2} p_1 + {}_2\beta_{m^2} p_2 \\ \tilde{\beta}_\Lambda &= {}_1\beta_\Lambda p_1 + {}_2\beta_\Lambda p_2, \quad \tilde{\beta}_\varphi = {}_1\beta_\varphi p_1 + {}_2\beta_\varphi p_2 \end{aligned} \tag{3.7} $$

depend only on λ, m^2 unlike the RG functions ${}_i\beta_{.}$ in (3.4). Accordingly, their values in a perturbative expansion may be trusted for small λ whatever the value of κ_2/κ_1. p_j are real numbers subject to $p_1 + p_2 = 1$. The corresponding RG operator

$$ \begin{aligned} \tilde{\mathcal{D}} &= p_1 \mathcal{D}_1 + p_2 \mathcal{D}_2 \\ &= p_1 \kappa_1 \frac{\partial}{\partial\kappa_1} + p_2 \kappa_2 \frac{\partial}{\partial\kappa_2} + \tilde{\beta}_\lambda \frac{\partial}{\partial\lambda} + \tilde{\beta}_{m^2} \frac{\partial}{\partial m^2} + \tilde{\beta}_\Lambda \frac{\partial}{\partial\Lambda} - \tilde{\beta}_\varphi \varphi \frac{\partial}{\partial\varphi} \end{aligned} \tag{3.8} $$

commutes with $\kappa_i \partial/\partial\kappa_i$. To recover the κ_2/κ_1-dependence of $\mathcal{D}_i$ we use that eqn. (3.6) implies

$$ [\tilde{\mathcal{D}}, \mathcal{D}_i] = 0, \tag{3.9} $$

yielding RG-type equations for the sought-after ${}_i\beta_{.}$. We remark that the final "improved" potential will have a strong dependence on the p_j-parameters. Each p_j-choice corresponds to a different transformation in eqn. (3.1) which satisfies conditions i), ii) and iii). Accordingly, we should decide for which values of p_j the transformation (3.1) "best" meets condition iv). In section 6 we will argue that the appropriate choice is $p_1 = 1$ and $p_2 = 0$. That is, the first set of beta functions, which track the Higgs scale, are independent of κ_2/κ_1.

4. LO RG functions

To determine the ${}_i\beta_\cdot$ we make a perturbative ansatz

$$ {}_i\beta_\cdot(\lambda, m^2; t) = \sum_{a=0}^{\infty} \frac{\hbar^{a+1}}{(4\pi)^{2a+2}} \, {}_i\beta_\cdot^{(a)}(\lambda, m^2; t), \quad t = \frac{\hbar\lambda}{(4\pi)^2} \log\frac{\kappa_2}{\kappa_1}. \tag{4.1} $$

Note that this is *not* simply a loop-expansion, since although we expand in $\hbar$ we retain all orders in t. Rather, we should view eqn. (4.1) as a LL, NLL, ... expansion of the two-scale RG functions. Hence, we assume the full κ_i-dependence of ${}_i\beta_\cdot$ to enter via t. This immediately allows us to rewrite

$$ p_1\kappa_1\frac{\partial}{\partial\kappa_1} + p_2\kappa_2\frac{\partial}{\partial\kappa_2} = \frac{\hbar}{(4\pi)^2}\lambda(p_2 - p_1)\frac{\partial}{\partial t} \equiv \frac{\hbar}{(4\pi)^2} D. \tag{4.2} $$

The corresponding perturbative decomposition of the RG operators becomes

$$ \begin{aligned} \mathcal{D}_i &= \sum_{a=0}^{\infty} \frac{\hbar^{a+1}}{(4\pi)^{2a+2}} \mathcal{D}_i^{(a)} \\ \mathcal{D}_i^{(a)} &= \frac{(4\pi)^2}{\hbar}\delta^{a0}\kappa_i\frac{\partial}{\partial\kappa_i} + {}_i\beta_\lambda^{(a)}\frac{\partial}{\partial\lambda} + {}_i\beta_{m^2}^{(a)}\frac{\partial}{\partial m^2} + {}_i\beta_\Lambda^{(a)}\frac{\partial}{\partial\Lambda} - {}_i\beta_\varphi^{(a)}\varphi\frac{\partial}{\partial\varphi} \end{aligned} \tag{4.3} $$

with analogous expressions for $\tilde{\mathcal{D}}, \tilde{\mathcal{D}}^{(a)}$. To determine the respective RG-like equation for a given order ${}_i\beta_\cdot^{(a)}$ we need

$$ \begin{aligned} [\tilde{\mathcal{D}}^{(a)}, \mathcal{D}_i^{(b)}] &= \left(\delta^{a0} D\, {}_i\beta_\lambda^{(b)} + \tilde\beta_\lambda^{(a)}\frac{\partial}{\partial\lambda}{}_i\beta_\lambda^{(b)} - {}_i\beta_\lambda^{(b)}\frac{\partial}{\partial\lambda}\tilde\beta_\lambda^{(a)}\right)\frac{\partial}{\partial\lambda} \\ &+ \left(\delta^{a0} D\, {}_i\beta_{m^2}^{(b)} + \tilde\beta_\lambda^{(a)}\frac{\partial}{\partial\lambda}{}_i\beta_{m^2}^{(b)} - {}_i\beta_\lambda^{(b)}\frac{\partial}{\partial\lambda}\tilde\beta_{m^2}^{(a)}\right)\frac{\partial}{\partial m^2} \\ &+ \Bigg(\delta^{a0} D\, {}_i\beta_\Lambda^{(b)} + \tilde\beta_\lambda^{(a)}\frac{\partial}{\partial\lambda}{}_i\beta_\Lambda^{(b)} - {}_i\beta_\lambda^{(b)}\frac{\partial}{\partial\lambda}\tilde\beta_\Lambda^{(a)} \\ &\qquad + \tilde\beta_{m^2}^{(a)}\frac{\partial}{\partial m^2}{}_i\beta_\Lambda^{(b)} - {}_i\beta_{m^2}^{(b)}\frac{\partial}{\partial m^2}\tilde\beta_\Lambda^{(a)}\Bigg)\frac{\partial}{\partial\Lambda} \\ &- \left(\delta^{a0} D\, {}_i\beta_\varphi^{(b)} + \tilde\beta_\lambda^{(a)}\frac{\partial}{\partial\lambda}{}_i\beta_\varphi^{(b)} - {}_i\beta_\lambda^{(b)}\frac{\partial}{\partial\lambda}\tilde\beta_\varphi^{(a)}\right)\varphi\frac{\partial}{\partial\varphi}. \end{aligned} \tag{4.4} $$

Here, we used the form of eqn. (3.1) implying, in generalization of the single-scale case, that ${}_i\beta_\lambda, {}_i\beta_\varphi$ do not depend on m^2, Λ and ${}_i\beta_{m^2}, {}_i\beta_\Lambda$ not on Λ. We can write

$$ \begin{aligned} {}_i\beta_\lambda^{(a)} &= \lambda^{a+2}\alpha_i^{(a)}(t), \quad {}_i\beta_{m^2}^{(a)} = m^2\lambda^{a+1}\beta_i^{(a)}(t) \\ {}_i\beta_\Lambda^{(a)} &= m^4\lambda^a\gamma_i^{(a)}(t), \quad {}_i\beta_\varphi^{(a)} = \lambda^{a+1}\delta_i^{(a)}(t) \end{aligned} \tag{4.5} $$

with analogous but t-independent expressions for the $\tilde\beta_\cdot^{(a)}$.

At LO we have $a = b = 0$ and eqn. (3.9) reduces to

$$ [\tilde{\mathcal{D}}^{(0)}, \mathcal{D}_i^{(0)}] = 0. \tag{4.6} $$

The corresponding equations for the various ${}_i\beta^{(0)}_{\cdot}$ may be read off from eqn. (4.4). We now solve them in turn.

${}_i\beta^{(0)}_\lambda$ is determined by

$$D\,{}_i\beta^{(0)}_\lambda + \tilde\beta^{(0)}_\lambda \frac{\partial}{\partial\lambda}{}_i\beta^{(0)}_\lambda - {}_i\beta^{(0)}_\lambda \frac{\partial}{\partial\lambda}\tilde\beta^{(0)}_\lambda = 0. \tag{4.7}$$

Inserting the further decomposition (4.5) and taking into account the λ-dependence of t this equation reduces to

$$\left(p_2 - p_1 + \tilde\alpha^{(0)} t\right)\frac{\partial}{\partial t}\alpha^{(0)}_i = 0. \tag{4.8}$$

Hence, $\alpha^{(0)}_i$ is independent of t

$$\begin{aligned} \alpha^{(0)}_i(t) &= a^{(0)}_{1i} = \alpha^{(0)}_i(0), \\ {}_i\beta^{(0)}_\lambda &= \lambda^2\alpha^{(0)}_i. \end{aligned} \tag{4.9}$$

The equation for ${}_i\beta^{(0)}_{m^2}$ is

$$D\,{}_i\beta^{(0)}_{m^2} + \tilde\beta^{(0)}_\lambda \frac{\partial}{\partial\lambda}{}_i\beta^{(0)}_{m^2} - {}_i\beta^{(0)}_\lambda \frac{\partial}{\partial\lambda}\tilde\beta^{(0)}_{m^2} = 0 \tag{4.10}$$

and reduces to

$$\left(p_2 - p_1 + \tilde\alpha^{(0)} t\right)\frac{\partial}{\partial t}\beta^{(0)}_i + \tilde\alpha^{(0)}\beta^{(0)}_i = \beta^{(0)}_i\tilde\alpha^{(0)}. \tag{4.11}$$

Its solution is best expressed in terms of the function f

$$f(t) \equiv \frac{p_2 - p_1 + \tilde\alpha^{(0)} t}{p_2 - p_1} \tag{4.12}$$

and reads

$$\begin{aligned} \beta^{(0)}_i(t) &= b^{(0)}_{1i} + b^{(0)}_{2i} f^{-1}(t), \\ {}_i\beta^{(0)}_{m^2} &= m^2\lambda\beta^{(0)}_i, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} b^{(0)}_{1i} &= \tilde B^{(0)} a^{(0)}_{1i} \quad \text{and} \quad \tilde B^{(0)} = \frac{\tilde\beta^{(0)}}{\tilde\alpha^{(0)}}, \\ b^{(0)}_{2i} &= \frac{1}{\tilde\alpha^{(0)}}\left(\tilde\alpha^{(0)}\beta^{(0)}_i(0) - \alpha^{(0)}_i(0)\tilde\beta^{(0)}\right). \end{aligned} \tag{4.14}$$

The determination of ${}_i\beta^{(0)}_\Lambda$ is a bit more involved

$$D\,{}_i\beta^{(0)}_\Lambda + \tilde\beta^{(0)}_\lambda \frac{\partial}{\partial\lambda}{}_i\beta^{(0)}_\Lambda - {}_i\beta^{(0)}_\lambda \frac{\partial}{\partial\lambda}\tilde\beta^{(0)}_\Lambda + \tilde\beta^{(0)}_{m^2}\frac{\partial}{\partial m^2}{}_i\beta^{(0)}_\Lambda - {}_i\beta^{(0)}_{m^2}\frac{\partial}{\partial m^2}\tilde\beta^{(0)}_\Lambda = 0. \tag{4.15}$$

The corresponding reduced ODE then reads

$$\left(p_2 - p_1 + \tilde\alpha^{(0)} t\right)\frac{\partial}{\partial t}\gamma^{(0)}_i + 2\tilde\beta^{(0)}\gamma^{(0)}_i = 2\beta^{(0)}_i\tilde\gamma^{(0)} \tag{4.16}$$

and is solved by

$$\gamma_i^{(0)}(t) = c_{1i}^{(0)} + c_{2i}^{(0)} f^{-1}(t) + c_{3i}^{(0)} f^{-2\tilde{B}^{(0)}}(t),$$
$$_i\beta_\Lambda^{(0)} = m^4 \gamma_i^{(0)}, \tag{4.17}$$

where

$$c_{1i}^{(0)} = \tilde{C}^{(0)} a_{1i}^{(0)} \quad \text{and} \quad \tilde{C}^{(0)} = \frac{\tilde{\gamma}^{(0)}}{\tilde{\alpha}^{(0)}}, \quad c_{2i}^{(0)} = \frac{2\tilde{C}^{(0)}}{2\tilde{B}^{(0)} - 1} b_{2i}^{(0)},$$
$$c_{3i}^{(0)} = \frac{1}{\tilde{\alpha}^{(0)}} \left(\tilde{\alpha}^{(0)} \gamma_i^{(0)}(0) - \alpha_i^{(0)}(0) \tilde{\gamma}^{(0)} \right) - c_{2i}^{(0)}. \tag{4.18}$$

As for $_i\beta_\varphi^{(0)}$ the trivial boundary condition (see below) implies

$$_i\beta_\varphi^{(0)} = 0. \tag{4.19}$$

In this section we have computed the two-scale LO RG functions for the $O(N)$-model. The results depend on p_j as well as the boundary conditions $\alpha_i^{(0)}(0)$, $\beta_i^{(0)}(0)$, $\gamma_i^{(0)}(0)$, $\delta_i^{(0)}(0)$ which determine the RG functions at $t = 0$ (ie. $\kappa_1 = \kappa_2$). In fact, at LO the boundary conditions are *uniquely* determined by the single-scale limit conditions following from requirements ii) and iii) in section 3

$$\alpha_1^{(0)}(0) = 3, \;\; \beta_1^{(0)}(0) = 1, \;\; \gamma_1^{(0)}(0) = \tfrac{1}{2}, \;\; \delta_1^{(0)}(0) = 0,$$

$$\alpha_2^{(0)}(0) = \tfrac{1}{3}(N-1), \;\; \beta_2^{(0)}(0) = \tfrac{1}{3}(N-1), \;\; \gamma_2^{(0)}(0) = \tfrac{1}{2}(N-1), \;\; \delta_2^{(0)}(0) = 0. \tag{4.20}$$

The LO RG functions for λ and φ are independent of p_j, and are given by (some relevant constants are given in appendix A)

$$_1\beta_\lambda^{(0)} = 3\lambda^2, \quad _2\beta_\lambda^{(0)} = \tfrac{1}{3}(N-1)\lambda^2, \quad _1\beta_\varphi^{(0)} = {}_2\beta_\varphi^{(0)} = 0. \tag{4.21}$$

However, the LO RG functions for m^2 and Λ still have a marked dependence on p_j. As mentioned in the previous section, we are eventually going to adopt the choice $p_1 = 1$, $p_2 = 0$. For this choice eqns. (4.13) and (4.17) reduce to

$$_1\beta_{m^2}^{(0)} = m^2\lambda, \;\; _2\beta_{m^2}^{(0)} = (N-1)\left[\tfrac{1}{9} + \tfrac{2}{9}(1-3t)^{-1}\right] m^2\lambda \tag{4.22}$$

and

$$_1\beta_\Lambda^{(0)} = \tfrac{1}{2}m^4, \;\; _2\beta_\Lambda^{(0)} = (N-1)\left[\tfrac{1}{18} - \tfrac{2}{9}(1-3t)^{-1} + \tfrac{2}{3}(1-3t)^{-\frac{2}{3}}\right] m^4, \tag{4.23}$$

respectively. It is clear that *the beta functions possess Landau poles* at $3t = 1$. Thus, these beta functions are only trustworthy for $1 \gg 3t$. Returning to the general p_j-case, the beta functions have a Landau pole at $p_1 - p_2 = \tilde{\alpha}^{(0)} t$. To avoid this pole we require $p_1 - p_2 \gg \tilde{\alpha}^{(0)} t$ for $p_1 > p_2$ and $p_1 - p_2 \ll \tilde{\alpha}^{(0)} t$ for $p_1 < p_2$. The case $p_1 = p_2 = \frac{1}{2}$ appears to be pathological.

5. LO running two-scale parameters

The running parameters in the minimal two-scale subtraction scheme are functions of the variables

$$s_i = \frac{\hbar}{(4\pi)^2}\log\frac{\kappa_i(s_i)}{\kappa_i}, \quad t = \frac{\hbar\lambda}{(4\pi)^2}\log\frac{\kappa_2}{\kappa_1}, \tag{5.1}$$

where κ_i are the reference scales. Note that $t(s_i)$ as given in eqn. (4.1) is in fact s_i-dependent, $t(s_i) = \frac{\hbar\lambda(s_i)}{(4\pi)^2}\log\frac{\kappa_2(s_2)}{\kappa_1(s_1)}$. The running coupling may be expanded in a series in $\hbar$

$$\lambda(s_i, t) = \sum_{a=0}^{\infty}\frac{\hbar^a}{(4\pi)^{2a}}\lambda^{(a)}(s_i, t) \tag{5.2}$$

with analogous expansions for $m^2(s_i,t), \Lambda(s_i,t), \varphi(s_i,t)$. We now insert these expansions into eqn. (3.4) and solve for the LO parameters.

The equation for the leading order running two-scale coupling is

$$\frac{d\lambda^{(0)}}{ds_i} = \lambda^{(0)^2}\alpha_i^{(0)}. \tag{5.3}$$

As $\alpha_i^{(0)}$ is constant it is easily integrated

$$\lambda^{(0)}(s_i) = \lambda\left(1 - \lambda(\alpha_1^{(0)}s_1 + \alpha_2^{(0)}s_2)\right)^{-1}, \tag{5.4}$$

where the boundary condition is $\lambda(s_i = 0) = \lambda$.

Turning to the running mass we have to solve

$$\frac{dm^{2(0)}}{ds_i} = m^{2(0)}\lambda^{(0)}\beta_i^{(0)}. \tag{5.5}$$

$\beta_i^{(0)}$ is given in eqn. (4.13) in terms of the function $f(t)$. As to leading order

$$t(s_i) = \lambda^{(0)}(s_i)\left(s_2 - s_1 + \frac{t}{\lambda}\right) \tag{5.6}$$

the s_i-dependence of the r.h.s. of eqn. (5.5) is quite involved. Its integration yields

$$m^{2(0)}(s_i) = m^2\left(\frac{\lambda^{(0)}(s_i)}{\lambda}\right)^{B^{(0)}}\left(\frac{f^{(0)}(s_i)}{f}\right)^{\tilde{B}^{(0)}-B^{(0)}}, \tag{5.7}$$

with $B^{(0)} = \frac{\beta_1^{(0)}+\beta_2^{(0)}}{\alpha_1^{(0)}+\alpha_2^{(0)}}$, and with the boundary condition $m^2(s_i = 0) = m^2$. Here, $f^{(0)}(s_i)$ is the function obtained by inserting eqn. (5.6) into eqn. (4.12) defining $f(t)$

$$f^{(0)}(s_i) = \frac{\lambda^{(0)}(s_i)}{\lambda}\left(1 + \frac{(\alpha_1^{(0)} + \alpha_2^{(0)})\lambda(p_1 s_2 - p_2 s_1) + \tilde{\alpha}^{(0)}t}{p_2 - p_1}\right) \tag{5.8}$$

and $f = f^{(0)}(s_i = 0)$. Note that if the two scales coincide we have $t = 0$ and $f^{(0)}(s_1 = s_2) = f = 1$. Requirement ii) provides us then with a *strong check* on the correct algebra for the running LO and NLO parameters.

We finally determine the running cosmological constant from

$$\frac{d\Lambda^{(0)}}{ds_i} = (m^{2(0)})^2 \gamma_i^{(0)}. \tag{5.9}$$

With the use of the results (5.7) for $m^{2(0)}$ and (4.17) for $\gamma_i^{(0)}$ we obtain

$$\begin{aligned} \Lambda^{(0)}(s_i) &= \Lambda + L_1^{(0)} \left[\frac{(m^{2(0)}(s_i))^2}{\lambda^{(0)}(s_i)} - \frac{m^4}{\lambda} \right] \\ &+ L_2^{(0)} \left[\frac{(m^{2(0)}(s_i))^2}{\lambda^{(0)}(s_i)} f^{(0)}(s_i)^{1-2\tilde{B}^{(0)}} - \frac{m^4}{\lambda} f^{1-2\tilde{B}^{(0)}} \right], \end{aligned} \tag{5.10}$$

where

$$L_1^{(0)} = \frac{\tilde{C}^{(0)}}{2\tilde{B}^{(0)} - 1}, \quad L_2^{(0)} = \frac{C^{(0)}}{2B^{(0)} - 1} - L_1^{(0)}, \tag{5.11}$$

with $C^{(0)} = \frac{\gamma_1^{(0)} + \gamma_2^{(0)}}{\alpha_1^{(0)} + \alpha_2^{(0)}}$, and $\Lambda(s_i = 0) = \Lambda$.

To LO there is no anomalous field dimension, and so the field parameter φ does not run.

The LO running coupling $\lambda^{(0)}(s_i)$ has a Landau pole at $\lambda(\alpha_1^{(0)} s_1 + \alpha_2^{(0)} s_2) = 1$ and clearly our approximation will break down before this pole is reached. If we let one of the $s_i \to -\infty$ (ie. the far IR region) while holding the other fixed the coupling will tend to zero as $\lambda^{(0)}(s_i \to -\infty) \propto (-s_i)^{-1}$. Also note that the LO running coupling is independent of p_j which parameterize the class of finite renormalizations under investigation.

The behaviour of the running mass and cosmological constant is more complicated. Consider the combination

$$\left(\frac{f^{(0)}(s_i)}{f} \right) \left(\frac{\lambda^{(0)}(s_i)}{\lambda} \right)^{-1} = 1 + \frac{(\alpha_1^{(0)} + \alpha_2^{(0)}) \lambda (p_1 s_2 - p_2 s_1)}{p_2 - p_1 + \tilde{\alpha}^{(0)} t}. \tag{5.12}$$

In the limit investigated $\frac{f^{(0)}(s_i)}{f}$ is not generally positive unless $p_1 = 0$ or $p_1 = 1$. Of course, we thereby assume that t is chosen such as to avoid the beta function poles in which case $p_2 - p_1 + \tilde{\alpha}^{(0)} t$ has the same sign as $p_2 - p_1$. This is disturbing because in eqns. (5.7) and (5.10) we are required to take non-integer powers of this quantity. Thus, unless $p_1 = 0$ or $p_1 = 1$ we are faced with the disquieting possibility of *complex* running m^2 and Λ in a region where the running coupling is very small. Fortunately, we will see in the next section that a comparison of our p_j-dependent improved potential with standard two-loop and next-to-large N calculations indicates that $p_1 = 1$ is the "natural" choice.

6. LO RG improved potential

It is now an easy task to turn the results for the running two-scale parameters into a RG improved effective potential. Eqn. (3.3) yields the identity

$$V(\lambda, m^2, \varphi, \Lambda; \kappa_1, \kappa_2) = V(\lambda(s_i), m^2(s_i), \varphi(s_i), \Lambda(s_i); \kappa_1(s_1), \kappa_2(s_2)), \tag{6.1}$$

with $\kappa_i(s_i)$ defined in (5.1). Next, we assume the validity of condition iv) in section 3. Hence, if

$$\kappa_i(s_i)^2 = \mathcal{M}_i(s_j) \equiv m^2(s_j) + k_i\,\lambda(s_j)\varphi^2(s_j), \qquad k_1 = \frac{1}{2},\ k_2 = \frac{1}{6} \tag{6.2}$$

the loop-expansion of the EP should render a reliable approximation to the RHS of eqn. (6.1).

To proceed we have to determine the values of s_i fulfilling (6.2). Insertion of the $\kappa_i(s_i)^2$ from (6.2) into (5.1) yields a quite implicit set of equations

$$s_i = \frac{\hbar}{2(4\pi)^2}\log\frac{\mathcal{M}_i(s_j)}{\kappa_i^2}. \tag{6.3}$$

Since we are meant to be summing consistently all logarithms we have to solve (6.3) iteratively

$$s_i = \sum_{a=0}^{\infty}\frac{\hbar^a}{(4\pi)^{2a}}\,s_i^{(a)}(\lambda,...;s_i^{(0)}) \tag{6.4}$$

in terms of the LO log's

$$s_i^{(0)} = \frac{\hbar}{2(4\pi)^2}\log\frac{\mathcal{M}_i}{\kappa_i^2}, \quad \text{where} \quad \mathcal{M}_i = \mathcal{M}_i(s_j = 0). \tag{6.5}$$

This yields contributions to the $s_i^{(a)}$ from both the s_i-dependence of the running two-scale parameters and from their own $\hbar$ expansion (5.2). For later use we also give the NLO term of the result

$$\begin{aligned} s_i^{(1)}(\lambda,...;s_i^{(0)}) &= \tfrac{1}{2}\log\frac{\mathcal{M}_i^{(0)}(s_i^{(0)})}{\mathcal{M}_i}, \quad \text{where} \\ \mathcal{M}_i^{(0)}(s_j) &= m^{2(0)}(s_j) + k_i\lambda^{(0)}(s_j)\varphi^2. \end{aligned} \tag{6.6}$$

To obtain the corresponding series expansion for the RG improved effective potential

$$V(\lambda,...;\kappa_i) = \sum_{a=0}^{\infty}\frac{\hbar^a}{(4\pi)^{2a}}\,V^{(a)}(\lambda,...;\kappa_i) \tag{6.7}$$

we approximate the RHS of eqn. (6.1) with those terms in the minimal two-scale subtraction scheme result for the EP surviving when $\kappa_i(s_i)^2 = \mathcal{M}_i(s_j)$. To $\mathcal{O}(\hbar)$ they are explicitly given by

$$\begin{aligned} V(\lambda(s_i),...;\mathcal{M}_i(s_j)) &= \frac{\lambda(s_i)}{24}\varphi(s_i)^4 + \frac{1}{2}m^2(s_i)\varphi(s_i)^2 + \Lambda(s_i) \\ &- \frac{3\hbar}{2(4\pi)^2}\left(\frac{\mathcal{M}_1(s_i)^2}{4} + (N-1)\frac{\mathcal{M}_2(s_i)^2}{4}\right). \end{aligned} \tag{6.8}$$

We finally insert the running two-scale parameters from (5.2) into the RHS of (6.8) with their arguments s_i coming from (6.4). Accordingly, an expansion in powers of $\hbar$ yields contributions to the $V^{(a)}$ from both the s_i-dependence of the running two-scale parameters and from their own $\hbar$-expansion. Keeping only leading order terms we obtain the LO two-scale RG improved effective potential in the minimal two-scale subtraction scheme

$$V^{(0)}(\lambda, ...; \kappa_i) = \frac{\lambda^{(0)}(s_i^{(0)})}{24}\varphi^4 + \frac{1}{2}m^{2(0)}(s_i^{(0)})\varphi^2 + \Lambda^{(0)}(s_i^{(0)}). \tag{6.9}$$

Let us next examine its properties. In the single-scale limits $N = 1$ and $N \to \infty$ eqn. (6.9) reduces to eqn. (2.11) for $i = 1$ and $i = 2$, respectively. In the general case $1 < N < \infty$ the $m^2\varphi^2$- and Λ-terms in eqn. (6.9) depend on p_j which parameterizes the class of finite renormalizations under consideration. Comparison with two-loop and next-to-large N results will provide us now with a natural value for them.

We have used a *two-scale* RG to track the two scales $\mathcal{M}_1$ and $\mathcal{M}_2$. Once the two log's have been summed up we can set $\kappa_1 = \kappa_2 = \mu$, ie. we may write our improved potential in standard $\overline{\text{MS}}$ parameters. In this way we can compare the improved potential (6.9) with standard perturbation theory. When now inserting the various constants and expanding eqn. (6.9) in $s_i^{(0)}$ up to second order

$$\begin{aligned}
V^{(0)}(\lambda, m^2, \varphi, \Lambda; \mu) \;=\; & \frac{\lambda}{24}\varphi^4\Bigg[1 + 3\lambda s_1^{(0)} + \frac{N-1}{3}\lambda s_2^{(0)} + 9\lambda^2 s_1^{(0)2} \\
& + 2(N-1)\lambda^2 s_1^{(0)} s_2^{(0)} + \frac{(N-1)^2}{9}\lambda^2 s_2^{(0)2}\Bigg] \\
+ \; & \tfrac{1}{2}m^2\varphi^2\Bigg[1 + \lambda s_1^{(0)} + \frac{N-1}{3}\lambda s_2^{(0)} \\
& + \left(2 - \frac{(N-1)p_2}{3(p_2-p_1)}\right)\lambda^2 s_1^{(0)2} + \frac{N-1}{3}\left(2 + \frac{2p_2}{p_2-p_1}\right)\lambda^2 s_1^{(0)} s_2^{(0)} \\
& + \frac{N-1}{9}\left(N + 2 - \frac{3p_2}{p_2-p_1}\right)\lambda^2 s_2^{(0)2}\Bigg] \\
+ \; & \frac{m^4}{\lambda}\Bigg[\tfrac{1}{2}\lambda s_1^{(0)} + \frac{N-1}{2}\lambda s_2^{(0)} + \left(\tfrac{1}{2} - \frac{(N-1)p_2}{3(p_2-p_1)}\right)\lambda^2 s_1^{(0)2} \\
& + \frac{N-1}{3}\left(1 + \frac{2p_2}{p_2-p_1}\right)\lambda^2 s_1^{(0)} s_2^{(0)} \\
& + \frac{N-1}{6}\left(N + 1 - \frac{2p_2}{p_2-p_1}\right)\lambda^2 s_2^{(0)2}\Bigg] + \Lambda
\end{aligned} \tag{6.10}$$

we see that the $\mathcal{O}(s_i^{(0)})$-terms in eqn. (6.10) agree with the logarithmic terms in the one-loop result (2.3). The quadratic, p_j-dependent terms in eqn. (6.10) should be compared with the two-loop $\overline{\text{MS}}$ effective potential [16]

$$\begin{aligned}
V^{(2\text{-loop})} \;=\; & \frac{\lambda\mathcal{M}_1{}^2}{8}\left(1 - \log\frac{\mathcal{M}_1}{\mu^2}\right)^2 + (N^2-1)\frac{\lambda\mathcal{M}_2{}^2}{24}\left(1 - \log\frac{\mathcal{M}_2}{\mu^2}\right)^2 \\
+ \; & (N-1)\frac{\lambda\mathcal{M}_1\mathcal{M}_2}{12}\left(1 - \log\frac{\mathcal{M}_1}{\mu^2} - \log\frac{\mathcal{M}_2}{\mu^2} + \log\frac{\mathcal{M}_1}{\mu^2}\log\frac{\mathcal{M}_2}{\mu^2}\right) \\
- \; & \frac{(\lambda\varphi)^2}{12}I(\mathcal{M}_1, \mathcal{M}_1, \mathcal{M}_1) - (N-1)\frac{(\lambda\varphi)^2}{36}I(\mathcal{M}_2, \mathcal{M}_2, \mathcal{M}_1),
\end{aligned} \tag{6.11}$$

where $I(x,y,z)$ is the general subtracted "sunset" vacuum integral discussed in appendix B. The graphs contributing are given in Fig. 1. [3]

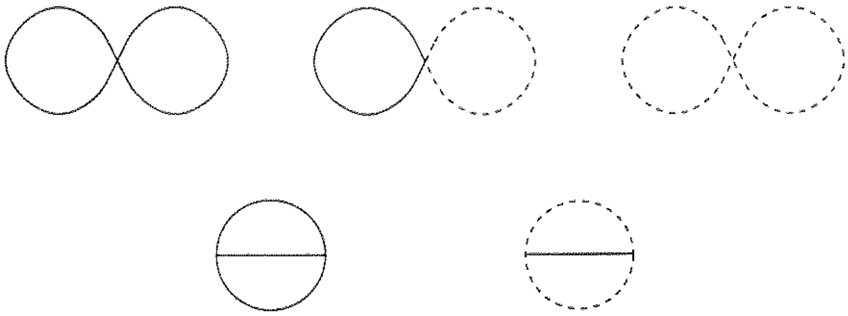

Figure 1: Diagrams contributing to the two-loop EP.

Note that the sunset integrals do not contribute to the m^4-terms. When comparing the m^4-terms in eqns. (6.10) and (6.11) it is easy to see that they only agree for $p_1 = 1$ and $p_2 = 0$. Comparison of the $m^2\varphi^2$- and φ^4-terms is more tricky due to the non-trivial sunset integrals.

We should decompose these integrals into logarithmic and non-logarithmic parts. This is not too difficult for $I(\mathcal{M}_1, \mathcal{M}_1, \mathcal{M}_1)$. Unfortunately, the decomposition of $I(\mathcal{M}_2, \mathcal{M}_2, \mathcal{M}_1)$ is *not unique*. However, as discussed in appendix B it seems natural to adopt the following one

$$\begin{aligned} I(x,y,z) &= -\frac{1}{2}\left[(y+z-x)\log\frac{y}{\mu^2}\log\frac{z}{\mu^2} + (z+x-y)\log\frac{z}{\mu^2}\log\frac{x}{\mu^2}\right. \\ &\quad \left. +(x+y-z)\log\frac{x}{\mu^2}\log\frac{y}{\mu^2}\right] + 2x\log\frac{x}{\mu^2} + 2y\log\frac{y}{\mu^2} \\ &+ \quad 2z\log\frac{z}{\mu^2} + \text{"non-logarithmic" terms.} \end{aligned} \tag{6.12}$$

Inserting eqn. (6.12) into eqn. (6.11) we see that the φ^4-term agrees with the one in eqn. (6.10). The φ^2-terms agree only if $p_1 = 1$ and $p_2 = 0$.

An alternative check on eqn. (6.9) is provided by the large N limit. By construction our improved potential will agree with standard large N results. Examining the next-to-large N result [19] we have found that in the LL approximation the m^4-terms in eqn. (6.9) and in the next-to-large N limit expression only agree if $p_1 = 1$ and $p_2 = 0$. To compare the $m^2\varphi^2$-terms we have again employed a "natural" decomposition of some integrals and once again agreement is achieved for $p_1 = 1$ and $p_2 = 0$. We remark that no other choice of p_j may be obtained by simply adopting a different decomposition of the relevant integrals. We have been unable to check the φ^4-terms since we do not know whether it is possible to perform a "natural" decomposition of some of the contributing integrals.

Thus, a comparison of our improved potential with the standard two-loop and next-to-large N potentials strongly indicates that $p_1 = 1$ and $p_2 = 0$ is the appropriate

[3] The full lines denote the Higgs with (mass)^2 $\mathcal{M}_1$, the dashed ones the Goldstones with (mass)^2 $\mathcal{M}_2$.

choice. This is gratifying, since for this choice one does not encounter the complex running parameters mentioned in the previous section. Let us finally write down explicitly the two-scale improved potential in the two-scale minimal subtraction scheme for this choice of p_j

$$
\begin{aligned}
V^{(0)} &= \frac{\lambda\varphi^4}{24}\left(1-3\lambda s_1^{(0)}-\frac{N-1}{3}\lambda s_2^{(0)}\right)^{-1} \\
&+ \frac{m^2\varphi^2}{2}\left(1-3\lambda s_1^{(0)}-\frac{N-1}{3}\lambda s_2^{(0)}\right)^{-\frac{1}{3}}\left(1-\frac{\frac{N+8}{3}\lambda s_2^{(0)}}{1-3t}\right)^{-\frac{2}{3}\frac{N-1}{N+8}} \\
&- \frac{m^4}{2\lambda}\left[\left(1-3\lambda s_1^{(0)}-\frac{N-1}{3}\lambda s_2^{(0)}\right)^{\frac{1}{3}}\left(1-\frac{\frac{N+8}{3}\lambda s_2^{(0)}}{1-3t}\right)^{-\frac{4}{3}\frac{N-1}{N+8}}-1\right] \\
&+ 2\,\frac{N-1}{N-4}\frac{m^4}{\lambda}(1-3t)^{\frac{1}{3}}\left[\left(1-\frac{\frac{N+8}{3}\lambda s_2^{(0)}}{1-3t}\right)^{-\frac{N-4}{N+8}}-1\right]+\Lambda. \qquad (6.13)
\end{aligned}
$$

For $t=0$ this result has already been obtained in a different way in ref. [8]. In the broken phase ($m^2<0$) the tree-level minimum is at $\mathcal{M}_2=0$ or $s_2^{(0)}\to-\infty$. Hence, as we approach it $\log(\mathcal{M}_2/\mathcal{M}_1)$ will become large. If we are prepared to trust eqn. (6.13) even in the *extreme* case of the tree minimum itself an intriguing property emerges.

As long as $N>4$ the φ^4-and $m^2\varphi^2$-terms vanish and the m^4-term converges to a finite value. As the slope $\frac{dV^{(0)}}{ds_2^{(0)}}(s_2^{(0)}\to-\infty)\searrow 0$ the EP takes its minimum in the broken phase at the tree-level value and becomes complex for even smaller φ^2-values. But for $1<N\leq 4$ the m^4-term, and hence $V^{(0)}$, diverges to minus infinity. This indicates that for these values of N there is *no stable vacuum in the broken phase.* Note especially that for $N=4$, ie. the SM scalar boson content, the divergence is softer but still there, as the penultimate term in eqn. (6.13) becomes a logarithm

$$
V^{(0)}=\,.....-\frac{m^4}{2\lambda}(1-3t)^{\frac{1}{3}}\log\left(1-\frac{4\lambda s_2^{(0)}}{1-3t}\right)+\Lambda. \qquad (6.14)
$$

7. NLO RG functions

The LO results of the last two sections have already been obtained in a less general form in ref. [8] based on the use of the $\overline{\mathrm{MS}}$ RG (2.6) and the conjecture that the correct boundary condition at $\mu^2=\mathcal{M}_2$ are given by the $N=1$ result (2.11). But using those techniques it appeared to be impossible to go beyond LO. The finite renormalization (3.1), introducing the appropriate number of renormalization scales and the corresponding RG equations (3.3), allows us to overcome this problem in a systematic manner. To show the strength of this technique we now determine the NLO RG functions and in the next section the corresponding NLO running parameters.

To NLO eqn. (3.9) yields

$$
[\tilde{\mathcal{D}}^{(1)},\mathcal{D}_i^{(0)}]+[\tilde{\mathcal{D}}^{(0)},\mathcal{D}_i^{(1)}]=0. \qquad (7.1)
$$

The corresponding equations for the various ${}_i\beta^{(1)}$ are obtained with the use of eqn. (4.4). We now solve them in turn.

${}_i\beta_\lambda^{(1)}$ is determined by

$$D\,{}_i\beta_\lambda^{(1)} + \tilde{\beta}_\lambda^{(0)}\frac{\partial}{\partial\lambda}{}_i\beta_\lambda^{(1)} - {}_i\beta_\lambda^{(1)}\frac{\partial}{\partial\lambda}\tilde{\beta}_\lambda^{(0)} + \tilde{\beta}_\lambda^{(1)}\frac{\partial}{\partial\lambda}{}_i\beta_\lambda^{(0)} - {}_i\beta_\lambda^{(0)}\frac{\partial}{\partial\lambda}\tilde{\beta}_\lambda^{(1)} = 0. \tag{7.2}$$

Proceeding in an analogous way as in obtaining the LO RG functions in section 4 we easily obtain the solution

$${}_i\beta_\lambda^{(1)} = \lambda^3\alpha_i^{(1)}, \tag{7.3}$$

where

$$\begin{aligned} \alpha_i^{(1)}(t) &= a_{1i}^{(1)} + a_{2i}^{(1)}f^{-1}(t); \\ a_{1i}^{(1)} &= \tilde{A}^{(1)}a_{1i}^{(0)} \quad \text{and} \quad \tilde{A}^{(1)} = \frac{\tilde{\alpha}^{(1)}}{\tilde{\alpha}^{(0)}}, \\ a_{2i}^{(1)} &= \frac{1}{\tilde{\alpha}^{(0)}}\left(\tilde{\alpha}^{(0)}\alpha_i^{(1)}(0) - \alpha_i^{(0)}(0)\tilde{\alpha}^{(1)}\right). \end{aligned} \tag{7.4}$$

The equation for ${}_i\beta_{m^2}^{(1)}$ is

$$D\,{}_i\beta_{m^2}^{(1)} + \tilde{\beta}_\lambda^{(0)}\frac{\partial}{\partial\lambda}{}_i\beta_{m^2}^{(1)} - {}_i\beta_\lambda^{(1)}\frac{\partial}{\partial\lambda}\tilde{\beta}_{m^2}^{(0)} + \tilde{\beta}_\lambda^{(1)}\frac{\partial}{\partial\lambda}{}_i\beta_{m^2}^{(0)} - {}_i\beta_\lambda^{(0)}\frac{\partial}{\partial\lambda}\tilde{\beta}_{m^2}^{(1)} = 0 \tag{7.5}$$

with the solution

$${}_i\beta_{m^2}^{(1)} = m^2\lambda^2\beta_i^{(1)}, \tag{7.6}$$

where

$$\begin{aligned} \beta_i^{(1)}(t) &= b_{1i}^{(1)} + b_{2i}^{(1)}f^{-1}(t) + f^{-2}(t)[b_{3i}^{(1)} + b_{4i}^{(1)}\log f(t)]; \\ b_{1i}^{(1)} &= \tilde{B}^{(1)}a_{1i}^{(0)} \quad \text{and} \quad \tilde{B}^{(1)} = \frac{\tilde{\beta}^{(1)}}{\tilde{\alpha}^{(0)}}, \quad b_{2i}^{(1)} = \tilde{B}^{(0)}a_{2i}^{(1)}, \\ b_{3i}^{(1)} &= \frac{1}{\tilde{\alpha}^{(0)}}\left(\tilde{\alpha}^{(0)}\beta_i^{(1)}(0) - \alpha_i^{(0)}(0)\tilde{\beta}^{(1)}\right) - b_{2i}^{(1)}, \quad b_{4i}^{(1)} = -\tilde{A}^{(1)}b_{2i}^{(0)}. \end{aligned} \tag{7.7}$$

The equation for ${}_i\beta_\Lambda^{(1)}$ becomes quite involved

$$\begin{aligned} D\,{}_i\beta_\Lambda^{(1)} &+ \tilde{\beta}_\lambda^{(0)}\frac{\partial}{\partial\lambda}{}_i\beta_\Lambda^{(1)} - {}_i\beta_\lambda^{(1)}\frac{\partial}{\partial\lambda}\tilde{\beta}_\Lambda^{(0)} + \tilde{\beta}_{m^2}^{(0)}\frac{\partial}{\partial m^2}{}_i\beta_\Lambda^{(1)} - {}_i\beta_{m^2}^{(1)}\frac{\partial}{\partial m^2}\tilde{\beta}_\Lambda^{(0)} \\ &+ \tilde{\beta}_\lambda^{(1)}\frac{\partial}{\partial\lambda}{}_i\beta_\Lambda^{(0)} - {}_i\beta_\lambda^{(0)}\frac{\partial}{\partial\lambda}\tilde{\beta}_\Lambda^{(1)} + \tilde{\beta}_{m^2}^{(1)}\frac{\partial}{\partial m^2}{}_i\beta_\Lambda^{(0)} - {}_i\beta_{m^2}^{(0)}\frac{\partial}{\partial m^2}\tilde{\beta}_\Lambda^{(1)} = 0. \end{aligned} \tag{7.8}$$

After some algebra we find the result

$${}_i\beta_\Lambda^{(1)} = m^4\lambda\gamma_i^{(1)}, \tag{7.9}$$

where

$$\begin{aligned} \gamma_i^{(1)}(t) &= c_{1i}^{(1)} + c_{2i}^{(1)}f^{-1}(t) + f^{-2}(t)[c_{3i}^{(1)} + c_{4i}^{(1)}\log f(t)] \\ &+ f^{-2\tilde{B}^{(0)}}(t)\left[c_{5i}^{(1)} + c_{6i}^{(1)}f^{-1}(t) + c_{7i}^{(1)}f^{-1}(t)\log f(t)\right]; \end{aligned}$$

$$
\begin{aligned}
c_{1i}^{(1)} &= \tilde{C}^{(1)} a_{1i}^{(0)} \quad \text{and} \quad \tilde{C}^{(1)} = \frac{\tilde{\gamma}^{(1)}}{\tilde{\alpha}^{(0)}}, \\
c_{2i}^{(1)} &= \tilde{C}^{(0)} a_{2i}^{(1)} + \left(\frac{\tilde{C}^{(1)}}{\tilde{B}^{(0)}} - \frac{\tilde{C}^{(0)}}{\tilde{B}^{(0)}} \frac{2\tilde{B}^{(1)} - \tilde{A}^{(1)}}{2\tilde{B}^{(0)} - 1} \right) b_{2i}^{(0)}, \\
c_{3i}^{(1)} &= \frac{2\tilde{C}^{(0)}}{2\tilde{B}^{(0)} - 1} b_{3i}^{(1)}, \quad c_{4i}^{(1)} = -\tilde{A}^{(1)} c_{2i}^{(0)}, \\
c_{5i}^{(1)} &= 2\left(\tilde{A}^{(1)} \tilde{B}^{(0)} - \tilde{B}^{(1)} \right) c_{3i}^{(0)}, \\
c_{6i}^{(1)} &= \gamma_i^{(1)}(0) - c_{1i}^{(1)} - c_{2i}^{(1)} - c_{3i}^{(1)} - c_{5i}^{(1)}, \quad c_{7i}^{(1)} = -2\tilde{A}^{(1)} \tilde{B}^{(0)} c_{3i}^{(0)}. \qquad (7.10)
\end{aligned}
$$

To NLO the anomalous dimension is non-trivial and we have to determine ${}_i\beta_\varphi^{(1)}$ from

$$
D\, {}_i\beta_\varphi^{(1)} + \tilde{\beta}_\lambda^{(0)} \frac{\partial}{\partial \lambda} {}_i\beta_\varphi^{(1)} - {}_i\beta_\lambda^{(1)} \frac{\partial}{\partial \lambda} \tilde{\beta}_\varphi^{(0)} + \tilde{\beta}_\lambda^{(1)} \frac{\partial}{\partial \lambda} {}_i\beta_\varphi^{(0)} - {}_i\beta_\lambda^{(0)} \frac{\partial}{\partial \lambda} \tilde{\beta}_\varphi^{(1)} = 0. \qquad (7.11)
$$

The solution is easily obtained

$$
{}_i\beta_\varphi^{(1)} = \lambda^2 \delta_i^{(1)}, \qquad (7.12)
$$

where

$$
\begin{aligned}
\delta_i^{(1)}(t) &= d_{1i}^{(1)} + d_{2i}^{(1)} f^{-2}(t); \\
d_{1i}^{(1)} &= \tilde{D}^{(1)} a_{1i}^{(0)} \quad \text{and} \quad \tilde{D}^{(1)} = \frac{\tilde{\delta}^{(1)}}{\tilde{\alpha}^{(0)}}; \\
d_{2i}^{(1)} &= \frac{1}{\tilde{\alpha}^{(0)}} \left(\tilde{\alpha}^{(0)} \delta_i^{(1)}(0) - \alpha_i^{(0)}(0) \tilde{\delta}^{(1)} \right). \qquad (7.13)
\end{aligned}
$$

So far we have not specified the values of the NLO boundary constants $\alpha_i^{(1)}(0)$, $\beta_i^{(1)}(0)$, $\gamma_i^{(1)}(0)$ and $\delta_i^{(1)}(0)$. At LO the relevant constants were completely determined by the single-scale limit conditions following from requirements ii) and iii). Unfortunately they do not anymore *uniquely* fix the NLO constants. For suppose we expand $\alpha_i^{(1)}(0)$, $\beta_i^{(1)}(0)$, $\gamma_i^{(1)}(0)$ and $\delta_i^{(1)}(0)$ in powers of $(N-1)$. Then the large N limit condition forbids any terms proportional to $(N-1)^2$ and higher powers of $(N-1)$ [18], and the $N=1$ limit condition fixes the contributions proportional to $(N-1)^0$. However, these limits tell us nothing about NLO terms proportional to $(N-1)$. Of course, we still have the condition that the sums of the two sets of RG functions at $t=0$ are just the usual $\overline{\text{MS}}$ RG functions, ie. ${}_1\beta_\cdot^{(1)}(t=0) + {}_2\beta_\cdot^{(1)}(t=0) = \beta_{\cdot,\overline{\text{MS}}}^{(\text{2-loop})}$. In $\overline{\text{MS}}$ $\beta_{\Lambda,\overline{\text{MS}}}^{(\text{2-loop})} = 0$ and the other two-loop beta functions can be found eg. in ref. [16]. Putting all this together we have

$$
\begin{aligned}
\alpha_1^{(1)}(0) &= -\tfrac{17}{3} - [1 + q_1](N-1), \quad \alpha_2^{(1)}(0) = q_1(N-1), \\
\beta_1^{(1)}(0) &= -\tfrac{5}{6} - [\tfrac{5}{18} + q_2](N-1), \quad \beta_2^{(1)}(0) = q_2(N-1), \\
\gamma_1^{(1)}(0) &= q_3(N-1), \quad \gamma_2^{(1)}(0) = -q_3(N-1), \\
\delta_1^{(1)}(0) &= \tfrac{1}{12} + [\tfrac{1}{36} + q_4](N-1), \quad \delta_2^{(1)}(0) = -q_4(N-1), \qquad (7.14)
\end{aligned}
$$

where q_j are real numbers which are independent of N. We shall comment further on sensible choices for q_j in the discussion of the NLO effective potential in section 9.

8. NLO running two-scale parameters

Using the LO results and the set of RG functions obtained in the last section we now calculate the NLO running two-scale parameters, which will be used to construct the NLO effective potential.

The equation for the next-to-leading order running two-scale coupling is

$$\frac{d\lambda^{(1)}}{ds_i} = 2\lambda^{(0)}\alpha_i^{(0)}\lambda^{(1)} + {\lambda^{(0)}}^3\alpha_i^{(1)}. \tag{8.1}$$

With the use of the results (5.4) for $\lambda^{(0)}$ and (7.3) for $\alpha_i^{(1)}$ we may integrate this equation and find

$$\lambda^{(1)}(s_i) = \lambda^{(0)}(s_i)^2 \log\left(\left(\frac{\lambda^{(0)}(s_i)}{\lambda}\right)^{A^{(1)}} \left(\frac{f^{(0)}(s_i)}{f}\right)^{\tilde{A}^{(1)}-A^{(1)}}\right). \tag{8.2}$$

Above $A^{(1)} = \frac{\alpha_1^{(1)}+\alpha_2^{(1)}}{\alpha_1^{(0)}+\alpha_2^{(0)}}$.

Turning to the NLO running mass we have to solve

$$\frac{dm^{2(1)}}{ds_i} = \lambda^{(0)}\beta_i^{(0)}m^{2(1)} + m^{2(0)}\left(\beta_i^{(0)}\lambda^{(1)} + \lambda^{(0)}\frac{\partial\beta_i^{(0)}}{\partial\lambda}\lambda^{(1)} + {\lambda^{(0)}}^2\beta_i^{(1)}\right). \tag{8.3}$$

The integration of this equation is quite involved and yields

$$\begin{aligned} m^{2(1)}(s_i) &= m^{2(0)}(s_i)\Bigg[M_1^{(1)}\left[\lambda^{(0)}(s_i)-\lambda\right] + M_2^{(1)}\left[\frac{\lambda^{(0)}(s_i)}{f^{(0)}(s_i)} - \frac{\lambda}{f}\right] \\ &\quad +\lambda^{(0)}(s_i)\left[M_3^{(1)}\log\left(\frac{f^{(0)}(s_i)}{f}\right) + M_4^{(1)}\log\left(\frac{\lambda^{(0)}(s_i)}{\lambda}\right)\right] \\ &\quad +M_5^{(1)}\frac{\lambda^{(0)}(s_i)}{f^{(0)}(s_i)}\log\left(\left(\frac{f^{(0)}(s_i)}{f}\right)\left(\frac{\lambda^{(0)}(s_i)}{\lambda}\right)^{-1}\right)\Bigg], \end{aligned} \tag{8.4}$$

where

$$\begin{aligned} M_1^{(1)} &= \tilde{B}^{(1)} - \tilde{B}^{(0)}\tilde{A}^{(1)}, \\ M_2^{(1)} &= B^{(1)} - B^{(0)}A^{(1)} - M_1^{(1)} + \tilde{A}^{(1)}(\tilde{B}^{(0)} - B^{(0)})\log f, \\ M_3^{(1)} &= \tilde{B}^{(0)}(\tilde{A}^{(1)} - A^{(1)}), \quad M_4^{(1)} = \tilde{B}^{(0)}A^{(1)}, \\ M_5^{(1)} &= M_4^{(1)} - B^{(0)}A^{(1)}. \end{aligned} \tag{8.5}$$

Above $B^{(1)} = \frac{\beta_1^{(1)}+\beta_2^{(1)}}{\alpha_1^{(0)}+\alpha_2^{(0)}}$.

The NLO running cosmological constant is determined by

$$\frac{d\Lambda^{(1)}}{ds_i} = 2m^{2(0)}\gamma_i^{(0)}m^{2(1)} + (m^{2(0)})^2\Big(\frac{\partial\gamma_i^{(0)}}{\partial\lambda}\lambda^{(1)} + \lambda^{(0)}\gamma_i^{(1)}\Big). \tag{8.6}$$

With the use of the various results above we obtain after a tedious computation

$$\Lambda^{(1)}(s_i) = \lambda\, L_1^{(1)}\left[\frac{(m^{2(0)}(s_i))^2}{\lambda^{(0)}(s_i)} - \frac{m^4}{\lambda}\right]$$

$$
\begin{aligned}
&+ \; \lambda\, L_2^{(1)} \left[\frac{(m^{2(0)}(s_i))^2}{\lambda^{(0)}(s_i)} f^{(0)}(s_i)^{1-2\tilde{B}^{(0)}} - \frac{m^4}{\lambda} f^{1-2\tilde{B}^{(0)}} \right] \\
&+ \; L_3^{(1)} \left[(m^{2(0)}(s_i))^2 - m^4 \right] + L_4^{(1)} \left[\frac{(m^{2(0)}(s_i))^2}{f^{(0)}(s_i)} - \frac{m^4}{f} \right] \\
&+ \; L_5^{(1)} \left[\frac{(m^{2(0)}(s_i))^2}{f^{(0)}(s_i)^{2\tilde{B}^{(0)}}} - \frac{m^4}{f^{2\tilde{B}^{(0)}}} \right] \\
&+ \; (m^{2(0)}(s_i))^2 \left[L_6^{(1)} \log\left(\frac{f^{(0)}(s_i)}{f} \right) + L_7^{(1)} \log\left(\frac{\lambda^{(0)}(s_i)}{\lambda} \right) \right] \\
&+ \; L_8^{(1)} \frac{(m^{2(0)}(s_i))^2}{f^{(0)}(s_i)} \log\left(\frac{f^{(0)}(s_i)}{f} \right) \left(\frac{\lambda^{(0)}(s_i)}{\lambda} \right)^{-1} \\
&+ \; L_9^{(1)} \frac{(m^{2(0)}(s_i))^2}{f^{(0)}(s_i)^{2\tilde{B}^{(0)}}} \log\left(\frac{f^{(0)}(s_i)}{f} \right) \left(\frac{\lambda^{(0)}(s_i)}{\lambda} \right)^{-1}, \qquad (8.7)
\end{aligned}
$$

where

$$
\begin{aligned}
L_1^{(1)} &= -2\left(M_1^{(1)} + \frac{1}{f} M_2^{(1)} \right) L_1^{(0)}, \quad L_2^{(1)} = -2\left(M_1^{(1)} + \frac{1}{f} M_2^{(1)} \right) L_2^{(0)}, \\
L_3^{(1)} &= \frac{\tilde{C}^{(0)}}{2\tilde{B}^{(0)}} \left(2M_1^{(1)} + \frac{\tilde{C}^{(1)}}{\tilde{C}^{(0)}} - \tilde{A}^{(1)} \right), \quad L_4^{(1)} = \frac{2\tilde{C}^{(0)}}{2\tilde{B}^{(0)} - 1} M_2^{(1)} \\
L_5^{(1)} &= \frac{1}{2B^{(0)}} \left(C^{(1)} - C^{(0)} A^{(1)} \right) + \frac{C^{(0)}}{B^{(0)}} \left(B^{(1)} - B^{(0)} A^{(1)} \right) - \frac{\tilde{C}^{(1)}}{2\tilde{B}^{(0)}} \\
&- \frac{\tilde{C}^{(0)}}{2\tilde{B}^{(0)} - 1} \left(2B^{(1)} - 2B^{(0)} A^{(1)} + \frac{\tilde{A}^{(1)}}{2\tilde{B}^{(0)}} - \frac{\tilde{B}^{(1)}}{\tilde{B}^{(0)}} \right) \\
&+ \tilde{A}^{(1)} \left(\tilde{C}^{(0)} - C^{(0)} - \frac{2\tilde{C}^{(0)}}{2\tilde{B}^{(0)} - 1} (\tilde{B}^{(0)} - B^{(0)}) \right) \log f, \\
L_6^{(1)} &= \tilde{C}^{(0)} \tilde{A}^{(1)} - L_7^{(1)}, \quad L_7^{(1)} = \tilde{C}^{(0)} A^{(1)}, \\
L_8^{(1)} &= \frac{2\tilde{C}^{(0)}}{2\tilde{B}^{(0)} - 1} M_5^{(1)}, \quad L_9^{(1)} = -C^{(0)} A^{(1)} + L_7^{(1)} - L_8^{(1)}. \qquad (8.8)
\end{aligned}
$$

Above $C^{(1)} = \frac{\gamma_1^{(1)} + \gamma_2^{(1)}}{\alpha_1^{(0)} + \alpha_2^{(0)}}$. We remark that most individual integrals occurring in the computation of not only $\Lambda^{(1)}$ but also $\Lambda^{(0)}$ and $m^{2(1)}$ yield hypergeometric functions and that only the respective sums of those are again expressible in terms of elementary functions as given above.

Finally we determine the non-trivial NLO running of $\varphi(s_i)$

$$
\frac{d\varphi^{(1)}}{ds_i} = -\varphi^{(0)} \lambda^{(0)2} \delta_i^{(1)}, \quad \varphi(s_i = 0) = \varphi. \qquad (8.9)
$$

The integration of this equation is straightforward and yields

$$
\begin{aligned}
\varphi^{(1)}(s_i) &= -\varphi \tilde{D}^{(1)} \left[\lambda^{(0)}(s_i) - \lambda \right] \\
&+ \varphi(\tilde{D}^{(1)} - D^{(1)}) \left[\frac{\lambda^{(0)}(s_i)}{f^{(0)}(s_i)} - \frac{\lambda}{f} \right], \qquad (8.10)
\end{aligned}
$$

where $D^{(1)} = \frac{\delta_1^{(0)}+\delta_2^{(0)}}{\alpha_1^{(0)}+\alpha_2^{(0)}}$.

It is easy to see that $\lambda^{(1)}$, $m^{2(1)}$ and $\varphi^{(1)}$ vanish for $N > 1$ in the limit of one $s_i \to -\infty$ while holding the other fixed. $\Lambda^{(1)}$ will tend to a finite value in this limit only for $N > 4$. However, it will diverge for $1 < N \leq 4$ if $p_1 = 1$ and $s_2 \to -\infty$ with the same rate as $\Lambda^{(0)}$ due to the first two terms in (8.7).

9. NLO RG improved potential

It is straightforward to extract the two-scale NLO potential from the standard perturbative boundary condition eqn. (6.8)

$$\begin{aligned} V^{(1)}(\lambda,...;\kappa_i) &= \frac{\lambda^{(1)}(s_i^{(0)})}{24}\varphi^4 + \frac{\lambda^{(0)}(s_i^{(0)})}{6}\varphi^3\,\varphi^{(1)}(s_i^{(0)}) \\ &+ \frac{1}{2}m^{2(1)}(s_i^{(0)})\varphi^2 + m^{2(0)}(s_i^{(0)})\,\varphi\,\varphi^{(1)}(s_i^{(0)}) + \Lambda^{(1)}(s_i^{(0)}) \\ &+ \sum_{i=1}^{2}\left[{}_i\beta_\lambda^{(0)}(s_j^{(0)})\frac{\varphi^4}{24} + {}_i\beta_{m^2}^{(0)}(s_j^{(0)})\frac{\varphi^2}{2} + {}_i\beta_\Lambda^{(0)}(s_j^{(0)})\right]s_i^{(1)}(\lambda,...;s_i^{(0)}) \\ &+ \frac{3}{2}\left(\frac{\mathcal{M}_1^{(0)}(s_i^{(0)})^2}{4} + (N-1)\frac{\mathcal{M}_2^{(0)}(s_i^{(0)})^2}{4}\right). \end{aligned} \tag{9.1}$$

The different contributions come from the expansion of the running two-scale parameters, from the expansion of their s_i-dependence and from the explicit one-loop term in (6.8). In practice, we immediately set $p_1 = 1$ and $p_2 = 0$ as has been done in the LO result.

Next, we fix the values of q_j used to parameterize the NLO boundary functions in eqn. (7.14) by comparing the q_j-dependent NLO potential *and the NLO* $Z(\varphi)^{(1)}$*-function* with the corresponding standard two-loop results. This immediately fixes $q_3 = 0$ and hence $\gamma_i^{(1)}(0) = 0$. The value of q_4 depends on how we decompose the two-loop integral J for $Z(\varphi)^{(2\text{-loop})}$ given in Fig. 2 [4] into its logarithmic and non-logarithmic pieces.

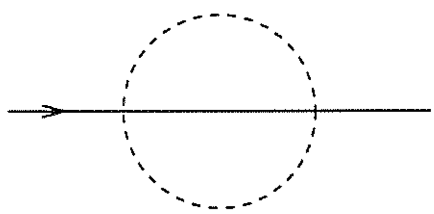

Figure 2: Diagram corresponding to J

In order to determine the "natural" decomposition of this integral it is helpful to consider the *general* integral $J(x,y,z)$ as given in appendix B. It is symmetric in x, y, z. Accordingly, a natural decomposition should respect this property. In fact, there is only

[4] In fact, a derivative wrt. p^2 at $p^2=0$ has to be taken as indicated in eqn. (B.5) from appendix B.

one decomposition which does this

$$J(x,y,z) \propto \log\frac{x}{\mu^2} + \log\frac{y}{\mu^2} + \log\frac{z}{\mu^2} + \text{“non-logarithmic” terms.} \qquad (9.2)$$

We are interested in the case $J(\mathcal{M}_2, \mathcal{M}_2, \mathcal{M}_1)$ and so we choose the coefficient of the $\log(\mathcal{M}_2/\mu^2)$-term in $J(\mathcal{M}_2, \mathcal{M}_2, \mathcal{M}_1)$ to be twice that of the $\log(\mathcal{M}_1/\mu^2)$-term. This implies that the coefficient of $(N-1)$ in $\delta_2^{(1)}(0)$ must be twice the coefficient of $(N-1)$ in $\delta_1^{(1)}(0)$ or $q_4 = -\frac{1}{54}$.

To determine q_1 and q_2 we need the subleading logarithms in $I(\mathcal{M}_2, \mathcal{M}_2, \mathcal{M}_1)$. Using the decomposition (6.11) yields $q_1 = -\frac{10}{27}$ and $q_2 = -\frac{5}{27}$. Putting this all together, the complete set of boundary functions are

$$\begin{aligned}
\alpha_1^{(1)}(0) &= -\tfrac{17}{3} - \tfrac{17}{27}(N-1), \quad \alpha_2^{(1)}(0) = -\tfrac{10}{27}(N-1),\\
\beta_1^{(1)}(0) &= -\tfrac{5}{6} - \tfrac{5}{54}(N-1), \quad \beta_2^{(1)}(0) = -\tfrac{5}{27}(N-1),\\
\gamma_1^{(1)}(0) &= 0, \quad \gamma_2^{(1)}(0) = 0,\\
\delta_1^{(1)}(0) &= \tfrac{1}{12} + \tfrac{1}{108}(N-1), \quad \delta_2^{(1)}(0) = \tfrac{1}{54}(N-1).
\end{aligned} \qquad (9.3)$$

The behaviour of the NLO contribution is of most interest around the broken phase tree-level minimum, where $\mathcal{M}_2 = 0$ or $s_2^{(0)} \to -\infty$. As in the LO case all the terms in eqn. (9.1) will vanish or converge to a finite limit if $N > 4$. But for $1 < N \leq 4$ $\Lambda^{(1)}$ and ${}_2\beta_\Lambda^{(0)} \cdot s_2^{(1)}$ will diverge. It is easy to check that they diverge at the same rate as $\Lambda^{(0)}$ in the LO analysis. However, as the NLO divergence is suppressed by a factor $\frac{\hbar\lambda}{(4\pi)^2} \ll 1$ qualitatively nothing will change.

10. The relevance of $N = 2$

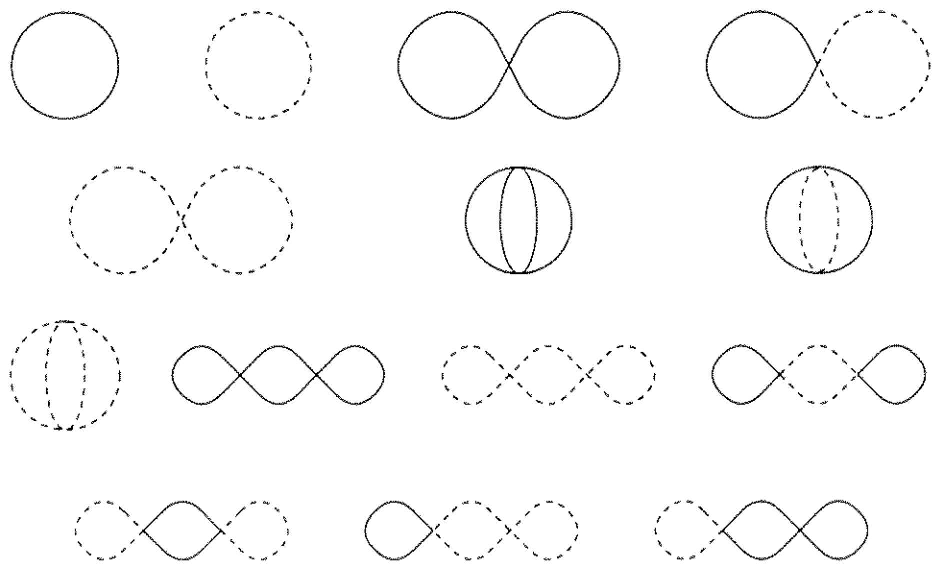

Figure 3: Diagrams contributing to Λ to three loops

From diagrammatic considerations we would expect the m^4-term in the RG improved potential to have a certain exchange symmetry in the $N = 2$ case. Note that these graphs will also contribute to the $m^2\varphi^2$- and φ^4-term. Now, for the case $N = 2$ these contributions are invariant under the exchange of Higgs- and Goldstone-lines. We would therefore expect that for $\kappa_1 = \kappa_2 = \mu$ the m^4-terms in eqn. (6.13) should be symmetric in $s_1^{(0)}$ and $s_2^{(0)}$. A glance at eqn. (6.13) in this case,

$$\begin{aligned} V^{(0)} &= -\frac{m^4}{2\lambda}\left[(1 - 3\lambda s_1^{(0)} - \tfrac{1}{3}\lambda s_2^{(0)})^{\frac{1}{3}}(1 - \tfrac{10}{3}\lambda s_2^{(0)})^{-\frac{2}{15}} + 2(1 - \tfrac{10}{3}\lambda s_2^{(0)})^{\frac{1}{5}} - 3\right] \\ &+ \text{ other terms,} \end{aligned} \qquad (10.1)$$

clearly shows that the m^4-term is *not* symmetric in $s_1^{(0)}$ and $s_2^{(0)}$. We find it somewhat disturbing that our approximation scheme does not respect this symmetry.

We know from section 6 that eqn. (10.1) matches standard perturbation theory through to two loops. Therefore, this $s_1^{(0)} \leftrightarrow s_2^{(0)}$ symmetry must go down *beyond* the two-loop level. Expanding eqn. (10.1) in powers of $s_1^{(0)}$ and $s_2^{(0)}$ up to $\mathcal{O}(\lambda^5)$

$$\begin{aligned} V^{(0)} &= -\frac{m^4}{2}\Big[s_1^{(0)} + s_2^{(0)} + \lambda\left(s_1^{(0)2} + \tfrac{2}{3}s_1^{(0)}s_2^{(0)} + s_2^{(0)2}\right) \\ &+\lambda^2\left(\tfrac{5}{3}s_1^{(0)3} + s_1^{(0)2}s_2^{(0)} + s_1^{(0)}s_2^{(0)2} + \tfrac{5}{3}s_2^{(0)3}\right) \\ &+\lambda^3\left(\tfrac{10}{3}s_1^{(0)4} + \tfrac{20}{9}s_1^{(0)3}s_2^{(0)} + \tfrac{4}{3}s_1^{(0)2}s_2^{(0)2} + \tfrac{20}{9}s_1^{(0)}s_2^{(0)3} + \tfrac{10}{3}s_2^{(0)4}\right) \\ &+\lambda^4\Big(\tfrac{22}{3}s_1^{(0)5} + \tfrac{50}{9}s_1^{(0)4}s_2^{(0)} + \tfrac{80}{27}s_1^{(0)3}s_2^{(0)2} \\ &\quad + \tfrac{20}{9}s_1^{(0)2}s_2^{(0)3} + \tfrac{178}{27}s_1^{(0)}s_2^{(0)4} + \tfrac{986}{135}s_2^{(0)5}\Big)\Big] \end{aligned} \qquad (10.2)$$

we see that the $s_1^{(0)} \leftrightarrow s_2^{(0)}$ symmetry *survives* at three and four loops, but breaks down at *five* loops. So we see that the failure of our approximation to observe it only appears at quite a high order in perturbation theory. We are unable to explain this phenomenon further.

11. Conclusions

In order to deal systematically with the two-scale problem arising in the analysis of the effective potential in the $O(N)$-symmetric ϕ^4-theory we have introduced a generalization of $\overline{\text{MS}}$. At each order in a $\overline{\text{MS}}$ loop-expansion we have performed a finite renormalization to switch over to a new "minimal two-scale subtraction scheme" allowing for two renormalization scales κ_i corresponding to the two generic scales in the problem. The $\overline{\text{MS}}$ RG functions and $\overline{\text{MS}}$ RGE then split into two minimal two-scale subtraction scheme "partial" RG functions and two "partial" RGE's. The respective integrability condition inevitably imposes a dependence of the partial RG functions on the renormalization scale ratio κ_2/κ_1. Supplementing the integrability with an appropriate subsidiary condition we have been able to determine this dependence to all orders in the scale ratio and have obtained a trustworthy set of LO and NLO two-scale subtraction scheme RG functions. With the use of the two "partial" RGE's we have then turned those into LO and NLO running two-scale parameters exhibiting features similar to the $\overline{\text{MS}}$ couplings such as a Landau pole now in both scaling channels. Using standard perturbative boundary conditions, which become applicable in the minimal two-scale subtraction scheme, we have calculated the effective potential in this scheme to LO and NLO. To fix the remaining renormalization freedom we have compared our results with two-loop and next-to-large N limit $\overline{\text{MS}}$ calculations. As a main result we have found in both LO and NLO that for $1 < N \leq 4$ there is no stable vacuum in the broken phase.

The vacuum instability in the broken phase of the $O(N)$-model raises immediately the possibility of a similar outcome in a multi-scale analysis of the SM effective potential. As the method outlined generalizes naturally to problems with more than two scales we are in a position to investigate systematically the different possible scenarios. Before turning to the SM itself it proves useful thereby to study the effects of adding either fermions as in a Yukawa-type model or gauging the simplest case of $N = 2$ as in the Abelian-Higgs model. The Yukawa case will either be a two- or three-scale problem, depending on whether one includes Goldstone bosons or not. The Abelian-Higgs model in the Landau gauge will be a three-scale problem to which the methods in this paper are easily extended. Now one has *three* integrability conditions $[\mathcal{D}_i, \mathcal{D}_j] = 0$ and one must impose three independent subsidiary conditions analogous to $[\kappa_1 \partial/\partial\kappa_1, \mathcal{D}_1] = 0$ which we used in our $O(N)$-model analysis. Note that for the general n-scale problem one would have $\frac{1}{2}n(n-1)$ integrability conditions which should be supplemented by $\frac{1}{2}n(n-1)$ subsidiary conditions. The question of whether fermions or gauge fields may stabilize the effective potential for small N in a full multi-scale analysis is under investigation.

We do not see any fundamental problem in applying the framework presented here to multi-scale computations at finite temperature, to the analysis of the multi-scale EP in supersymmetric extensions of the SM or to a full multi-scale treatment of DIS problems in the regions of very large or small x_B. The necessary adaptions are under investigation.

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Appendix A. Values of various constants

Here, we give the values of various constants appearing in the paper. We quote them for the choice $p_1 = 1$ and $p_2 = 0$.

$$B^{(0)} = \frac{N+2}{N+8}, \quad C^{(0)} = \frac{3N}{2(N+8)} \tag{A.1}$$

$$\tilde{B}^{(0)} = \frac{1}{3}, \quad \tilde{C}^{(0)} = \frac{1}{6} \tag{A.2}$$

$$
\begin{aligned}
A^{(1)} &= -\frac{3N+14}{N+8}, \quad B^{(1)} = -\frac{5(N+2)}{6(N+8)}, \\
C^{(1)} &= 0, \quad D^{(1)} = \frac{N+2}{12(N+8)}
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\tilde{A}^{(1)} &= -\frac{17(N+8)}{81}, \quad \tilde{B}^{(1)} = -\frac{5(N+8)}{162}, \\
\tilde{C}^{(1)} &= 0, \quad \tilde{D}^{(1)} = \frac{N+8}{324}
\end{aligned}
\tag{A.4}
$$

$$
L_1^{(0)} = -\frac{1}{2}, \quad L_2^{(0)} = \frac{2(N-1)}{N-4}
\tag{A.5}
$$

$$
\begin{aligned}
M_1^{(1)} &= \frac{19(N+8)}{486}, \\
M_2^{(1)} &= -\frac{(N-1)(19N^2-578N-2600)}{486(N+8)^2} + \frac{(N-1)(34N^2+544N+2178)}{243(N+8)^2}\log f, \\
M_3^{(1)} &= -\frac{(N-1)(17N+46)}{243(N+8)}, \quad M_4^{(1)} = -\frac{3N+14}{3(N+8)}, \\
M_5^{(1)} &= \frac{2(N-1)(3N+14)}{3(N+8)^2}
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
L_1^{(1)} &= \frac{19(N+8)}{486} - \frac{(N-1)(19N^2-578N-2600)}{486(N+8)^2 f} \\
&+ \frac{(N-1)(34N^2+544N+2178)}{243(N+8)^2 f}\log f, \\
L_2^{(1)} &= -\frac{38(N-1)(N+8)}{243(N-4)} + \frac{2(N-1)^2(19N^2-578N-2600)}{243(N-4)(N+8)^2 f} \\
&- \frac{4(N-1)^2(34N^2+544N+2178)}{243(N-4)(N+8)^2 f}\log f, \quad L_3^{(1)} = \frac{35(N+8)}{486}, \\
L_4^{(1)} &= \frac{(N-1)(19N^2-578N-2600)}{486(N+8)^2} - \frac{(N-1)(34N^2+544N+2178)}{243(N+8)^2}\log f, \\
L_5^{(1)} &= -\frac{(N-1)(N^3-42N^2-360N-760)}{9(N+2)(N+8)^2} + \frac{34(N-1)}{81}\log f, \\
L_6^{(1)} &= -\frac{(N-1)(17N+46)}{486(N+8)}, \quad L_7^{(1)} = -\frac{3N+14}{6(N+8)}, \\
L_8^{(1)} &= -\frac{2(N-1)(3N+14)}{3(N+8)^2}, \quad L_9^{(1)} = \frac{2(N-1)(3N+14)}{(N+8)^2}
\end{aligned}
\tag{A.7}
$$

Appendix B. The integrals I and J

Here, we list some useful formulae regarding the two-loop integrals I and J. The general unsubtracted scalar sunset integral in D dimensions is defined as

$$I_D(x,y,z) = \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{(k^2+x)(l^2+z)((k+l)^2+z)}. \qquad \text{(B.1)}$$

A full calculation of this integral is rather involved [20]. However, there is a formula in ref. [21] which nicely splits the integral into a very simple, for $D = 4$ divergent expression plus a finite term which is proportional to $I_{D-2}(x,y,z)$, ie. the same integral in two lower dimensions.

$$\begin{aligned} I_D(x,y,z) &= (4\pi)^{-D}\frac{\Gamma^2(2-\frac{1}{2}D)}{(D-2)(D-3)}\Big[(x-y-z)(yz)^{\frac{1}{2}D-2} \\ &\quad +(y-z-x)(zx)^{\frac{1}{2}D-2}+(z-x-y)(xy)^{\frac{1}{2}D-2}\Big] \\ &\quad - (4\pi)^{-2}(x^2+y^2+z^2-2xy-2yz-2zx)\, I_{D-2}(x,y,z). \end{aligned} \qquad \text{(B.2)}$$

Since the last term is finite we regard it as a "non-logarithmic" term and ascribe the logarithmic terms purely to the simple, divergent piece. The renormalized $I(x,y,z)$ referred to in the text is then given as

$$\begin{aligned} I(x,y,z) &= \text{FP}\Big[(4\pi e^{-\gamma}\mu^2)^{2\epsilon}\Big(I_{4-2\epsilon}(x,y,z) \\ &\quad -\frac{1}{(4\pi)^2\epsilon}(K_{4-2\epsilon}(x)+K_{4-2\epsilon}(y)+K_{4-2\epsilon}(z))\Big)\Big] \end{aligned} \qquad \text{(B.3)}$$

where FP denotes the finite part, γ is Euler's constant and

$$K_D(x) = \int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+x}. \qquad \text{(B.4)}$$

The K_D-terms in eqn. (B.3) are due to the subtraction of one-loop sub-divergences.

The unsubtracted $J_D(x,y,z)$ is defined as

$$J_D(x,y,z) = \frac{\partial}{\partial p^2}\int \frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{(k^2+x)(l^2+y)((k+l+p)^2+z)}\Bigg|_{p^2=0}. \qquad \text{(B.5)}$$

The renormalized $J(x,y,z)$ which enters into $Z(\varphi)^{(\text{2-loop})}$ is simply

$$J(x,y,z) = \text{FP}\left[(4\pi e^{-\gamma}\mu^2)^{2\epsilon}J_{4-2\epsilon}(x,y,z)\right]. \qquad \text{(B.6)}$$

Above, the x, y, z are the (masses)^2 on the three internal lines.

References

[1] M.J. Duncan, R. Phillippe and M. Sher, Phys. Lett. B 153 (1985) 165;
M. Sher and H.W. Zaglauer, Phys. Lett. B 206 (1988) 527;

M. Lindner, M. Sher and H.W. Zaglauer, Phys. Lett. B 228 (1989) 139;
M. Sher, Phys. Rep. 179 (1989) 274;
M. Sher, Phys. Lett. B 317 (1993) 159;
J.A. Casas, J.R. Espinosa and M. Quiros, hep-ph/9603227 (1996).

[2] V.N. Gribov and I.N. Lipatov, Sov. J. Nucl. Phys. 15 (1972) 438;
A.J. Buras, Rev. Mod. Phys. 52 (1980) 199;
G. Altarelli, Phys. Rep. 81 (1982) 1;
L.V. Gribov, E.M. Levin and M.G. Ryshkin, Phys. Rep. 100 (1983) 1;
M. Virchaux and A. Milsztajn, Phys. Lett. B 274 (1992) 221.

[3] J. Kapusta, Finite Temperature Field Theory (Cambridge University Press, Cambridge, 1989).

[4] M. Carena, M. Quiros and C.E.M. Wagner, Nucl. Phys. B 461 (1996) 407.

[5] R.G. Roberts, The structure of the proton (Cambridge University Press, Cambridge, 1993);
S. Forte and R.D. Ball, hep-ph/9512208 (1995).

[6] T. Appelquist and J. Carazzone, Phys. Rev. D 11 (1975) 2856;
K. Symanzik, Comm. Math. Phys. 34 (1973) 7.

[7] M. Bando, T. Kugo, N. Maekawa and H. Nakano, Prog. Theor. Phys. 90 (1993) 405.

[8] C. Ford, Phys. Rev. D 50 (1994) 7531.

[9] H. Nakano and Y. Yoshida, Phys. Rev. D 49 (1994) 5393.

[10] M.B. Einhorn and D.R.T. Jones, Nucl. Phys. B 230 [FS10] (1984) 261.

[11] K. Nishijima, Prog. Theor. Phys. 88 (1992) 993; Prog. Theor. Phys. 89 (1993) 917.

[12] M. Bando, T. Kugo, N. Maewaka and H. Nakano, Phys. Lett B 301 (1992) 83.

[13] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, Effective Action in Quantum Gravity (Institute of Physics Publishing, Bristol, 1992).

[14] S. Coleman and E. Weinberg, Phys. Rev D 10 (1973) 1888.

[15] R. Jackiw, Phys. Rev. D 9 (1974) 1686.

[16] C. Ford and D.R.T. Jones, Phys. Lett. B 274 (1992) 409; Phys. Lett. B 285 (1992) 399(E).

[17] B. Kastening, Phys. Lett. B 283 (1992) 287.

[18] H.J. Schnitzer, Phys. Rev D 10 (1974) 1800.

[19] R.G. Root, Phys. Rev. D 10 (1974) 3322.

[20] A.B. Lahanas, K. Tamvakis and C.E. Vayonakis, Nucl. Phys. B 196 (1982) 11;
C. Ford, I. Jack and D.R.T. Jones, Nucl. Phys. B 387 (1992) 373;
A.I. Davydychev and J.B. Tausk, Nucl. Phys. B 397 (1993) 123.

[21] A.I. Davydychev and J.B. Tausk, hep-ph/9504431 (1995).