DIAS-STP-96-13

Review of WZW-Toda Reductions*

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Dedication

This article is dedicted to Professor Pratul Bandyopadhyay on the occasion of his retirement. Along with his co-workers I should like to express my best wishes to him on this occasion and to wish him an active and happy retirement. As his interests are broad and have embraced many different aspects of physics I have chosen for my article a review of WZW-Toda reduction. This is a subject which draws together many different strands of recent research and may have some historical interest.

1. Preface

As is well-known, two-dimensional conformal field theories have come to play a central role in present day physics. The reason is that, apart from their intrinsic interest, they consitute a meeting point for three quite different branches of physics, namely the theory of phase-transitions, string theory and statistical mechanics. Furthermore they throw light on ordinary quantum field theory in the sense that many properties such as operator product expansions can be computed reliably and precisely in the two-dimensional context. Among the most important systems which are used in two-dimensional conformal field theory are the Wess-Zumino-Witten system and the Toda field system. The WZW systems are essentially free (linear) systems while the Toda systems are interacting (non-linear) systems. In recent years it has been found, however, that Toda systems may be regarded as constrained WZW systems. What I should like to sketch is how this circumstance allows one to obtain the properties of the Toda systems from those of the free WZW in a relatively simple manner and to show how the WZW-Toda reduction relates to other interesting aspects of physics.

2. Historical Background

Let me begin by summarizing some relevant historical dates, as follows:

1853: Introduction of Liouville Theory.

1966: Discovery of Abelian Toda Systems as integrable systems [2].

1967: Formulation of Kac-Moody (KM) algebras [3].

1978 Construction of Wess-Zumino-Witten systems with KM algebras as symmetry algebras [4].

1979: General solutions of Abelian Toda field equations found [5].

* *Contribution to Retirement Volume for Prof. Pratul Bandyopadyay, Indian Statistical Institute May 1996.*

All these systems were non-linear and hence rather complicated and difficult to quantize. But the situation was simplified when in

1989: Toda systems were discovered to be linearly constrained WZW systems

Since the WZW systems were linear everything then became much more tractable. In fact one now had the following advantages: the Toda action turned out to be nothing but a reduced WZW action, the general solutions of the Toda field equations theory (both abelian and non-abelian) could be obtained quite simply from the (trivial) WZW solutions, the quantization of Toda theories could be effected by applying BRST standard methods to the quantized WZW systems and finally, as might be expected, the W-algebras of Toda theory turned out to be constrained KM-algebras. Symbolically:

$\mathcal{A}_{\text{WZW}}$	$\rightarrow$	$\mathcal{A}_{\text{Toda}}$
WZW Solutions	$\rightarrow$	Toda Solutions (algebraically)
KM Algebra	$\rightarrow$	W-Algebra (algebraically)
KM Quantization	$\rightarrow$	BRST Quantization of Toda

where $\mathcal{A}$ denotes the Action. Furthermore, it turned that the WZW-Toda Reductions and the primariness of the W-Algebras were associated in a one-one manner with the embeddings of the $SL(2,R)$ group in the WZWZ Lie group G [9], the abelian Toda theories being associated with the principal (maximal) embeddings:

Principal	$SL(2,R)$	Embedding	$\leftrightarrow$	Abelian Toda
General	$SL(2,R)$	Embedding	$\leftrightarrow$	General Toda
General	$SL(2,R)$	Embedding	$\leftrightarrow$	Primariness of W-Algebra

Intimate connections were also found between these systems and

Dirac Star Algebras KdV hierarchies R-matrix theory

Finally the relationship between the $WZW \rightarrow$ Toda reductions and their symmetry counterparts $KM \rightarrow W$-Algebra was found to be described by a generalized Miura transformation which had been introduced in the eighties by Drinfeld and Sokolov [10] for a different but related purpose. In the simplest, $SL(N,R)$, case, the DS transformation may be described as $\theta_i(x) \rightarrow W_i(x)$, where

$$(\partial-\theta_1)(\partial-\theta_2)...(\partial-\theta_r)...(\partial-\theta_n) = \partial^n - W_2\partial^{n-2}....W_r\partial^{n-r} + ... + W_n \quad (2.1)$$

with $\sum\theta_i = 0$. In conclusion it should be mentioned that in recent years the supersymmetric ($N=1$ and $N=2$) generalizations of all these relationships have been constructed [11], the $N=2$ case being particularly interesting.

It is clear from the above that many different strands of ideas are brought together in the WZW $\rightarrow$ Toda reduction.

3. WZW Systems

The WZW and Toda systems are different generalization of the single massless free-field system in two dimensions. This system has action

$$\mathcal{A} = \frac{k}{2}\int d^2x\partial_\mu\phi(x)\partial^\mu\phi(x) \qquad k = \text{constant} \tag{3.1}$$

field equations

$$\nabla^2\phi(x) \equiv \partial_+\partial_-\phi(x) = 0 \qquad x_\pm = x_o \pm x_1 \tag{3.2}$$

and solutions

$$\phi(x) = \phi_+(x_+) + \phi_-(x_-) \tag{3.3}$$

where $\phi_\pm$ are arbitrary differentiable functions.

Let us first consider the WZW generalization. This is obtained by noting that in the single free field case the field $g(x) = e^{\phi(x)}$ is an element of a one-parameter Lie group and generalizing to the case when $g(x)$ is an element of any abelian or semi-simple Lie group. For non-abelian groups the generalization is not quite straightforward and is perhaps best described by starting from the solutions (3.3). In analogy with (3.3) one requires that the solution in the general case be of the form

$$g(x) = g_+(x_+)g_-(x_-) \tag{3.4}$$

where $g_\pm$ are arbitrary differentiable group elements. Because of the non-commutative nature of the group one then finds that the field equations must be

$$\partial_+J_- = 0 \quad \text{and} \quad \partial_-\tilde{J}_+ = 0 \quad \text{where} \quad J_- = \big(\partial_-g\big)g^{-1} \qquad \tilde{J}_+ = g^{-1}\partial_+g \tag{3.5}$$

The surprising feature is the appearance of the current $\tilde{J}$ as well as J. Its appearance means that one cannot use the obvious generalization

$$A = \frac{k}{2}\int d^2x\mathrm{tr}\big(J_\mu(x)J^\mu(x)\big) \qquad J_\mu = g^{-1}\partial_\mu g \tag{3.6}$$

of (3.1) as the action, and it took some time before it was discovered by Witten [4] that the correct action was

$$\mathcal{A} = \frac{k}{2}\int d^2x\mathrm{tr}\big(J_\mu(x)J^\mu(x)\big) + \frac{k}{3}\int d^2xdy\epsilon_{\mu\nu\alpha}\mathrm{tr}\big([J_\mu, J_\nu]J_\alpha\big) \tag{3.7}$$

In the second integral on the right-hand side of (3.7) the two-dimensional Minkowski space is regarded as the boundary of a three-dimensional space with extra coordinate y and the point is that although it is a 3-dimensional integral its *variation* is an integral over the boundary. Thus the variation of $\mathcal{A}$ provides a 2-dimensional system of field equations, and this system turns out to be just the system (3.5). Had the sign of the 3-dimensional integral in (3.7) been reversed we would have obtained equations similar to (3.5), but with J and $\tilde{J}$ interchanged. The 3-dimensional integral in (3.7) is a special case of a form that was found earlier by Wess and Zumino [12] in a different context.

Although the WZW action contains a cubic term in the fields, the fact that the field equations are linear in J shows that it is a free system, and the solutions (3.4) to the field equations are simple by construction. As in the abelian case the system is conformally invariant. It is also invariant with respect to the transformations $g(x) \to lg(x)$ and $g(x) \to g(x)r$ where l and r are arbitrary *constant* group elements, and the currents $J_{\pm}(x)$ may be identified as the Noether currents associated with these left and right-handed symmetries. As one might expect from the fact that it is a free system the WZW system is easily quantized. Letting $x = x_+$ and $y = y_-$ denote the usual light-cone coordinates and σ_a the group generators, the commutation relations for the currents are

$$[J_a(x), \tilde{J}_b(y)] = 0 \qquad J_a = \mathrm{tr}(\sigma_a J) \tag{3.8}$$

and the KM algebras

$$[J_a(x), J_b(y)] = f_{ab}^c J_c(y)\delta(x-y) + k g_{ab}\delta'(x-y) \qquad g_{ab} = \mathrm{tr}(\sigma_a \sigma_b) \tag{3.9}$$

for $x = x_+$ and $y = y_+$, and similarly for $\tilde{J}(x_-)$ with $k \to -k$. Writing the group elements in the form (3.4) one finds that their commutation relations are

$$[J_a(x_+), g(y_+)] = \sigma_a g(y_+)\delta(x_+ - y_+) \quad \text{and} \quad [g(x_+), g(y_+)] = R g^2(x_+)\delta(x_+ - y_+) \tag{3.10}$$

where in the last equation g denotes the individual elements g_{ik} and $R = R_{ik;st}$ is an R-matrix of the kind used in Yang-Baxter (YB) equations, and similarly for $\tilde{J}$ and $g(x_-)$. Thus the WZW theory provides a dynamical framework for the appearance of KM and YB algebras.

4. Toda Systems

The other type of generalization of the free-field action is to introduce conformally-invariant interactions. For a single scalar field the only way to do this is to use an exponential potential, which leads to the Liouville action

$$\mathcal{A} = \int d^2x \left\{ \frac{1}{2}(\partial\phi)^2 + e^{\phi} \right\} \tag{4.1}$$

The abelian Toda actions are generalizations of the Liouville action to the case of l Lorentz scalar fields $\phi^a(x)$ and the action is

$$\mathcal{A} = \int d^2x \left\{ \frac{1}{2}(\partial\phi^a)(\partial\phi^a) + \sum_{\alpha} \mu_{\alpha} e^{\alpha_a \phi^a} \right\} \tag{4.2}$$

where the sum over a runs from 1 to l where l is the rank of a simple Lie group $\mathcal{G}$ and the α's are the primitive roots of $\mathcal{G}$. The constants α are chosen as primitive roots in order to make the system integrable and if only conformal invariance were required they could be replaced by arbitrary constants. From the properties of the primitive roots one sees that the interaction is actually an exponential interaction between nearest neighbours.

The non-abelian Toda theories are generalizations of the abelian Toda theories in which the single scalar fields are replaced by a set of WZW fields $g^{(a)}(x)$. As we shall see later the systems are associated with the embeddings of $SL(2,R)$ in a simple Lie group G. The embeddings can be integral or half-integral and for the integral case the action takes a form similar to (4.2), namely

$$\mathcal{A} = \sum_a L^{(a)}_{WZW} + \int d^2x g_{(a)} g^{-1}_{(a+1)}(x) \tag{4.3}$$

The abelian Toda theories are the ones associated with the principal $sl(2)$ embeddings.

5. The Magic Constraints

As stated in the Abstract, the general Toda systems (4.3) may be obtained from the WZW systems by placing constraints on the latter. What are these magic constraints? It turns out that they are very simple, namely constraints which are linear in the KM currents and first-class in the sense of Dirac. To specify them more exactly we let G be the Lie algebra of the semi-simple Lie group $\mathcal{G}$ and let $\{M_o, M_\pm\}$ be the standard generators of an $SL(2,R)$ algebra embedded in G. We can then grade the Lie algebra of G with respect to M_o in the usual sense that G can be decomposed as

$$G = \sum G_n \quad \text{where} \quad [M_o, G_n] = nG_n \tag{5.1}$$

An even simpler decomposition of G is

$$G = G_l + G_o + G_r \quad \text{where} \quad G_l = \sum_{n<0} G_n \quad \text{and} \quad G_r = \sum_{n>0} G_n \tag{5.2}$$

The magic constraints are then simply

$$\mathrm{tr}(\sigma_r, j) = 0 \qquad j = J_- - M_- \quad \text{and} \quad \mathrm{tr}(\sigma_l, \tilde{j}) = 0 \qquad \tilde{j} = \tilde{J}_+ - M_+ \tag{5.4}$$

where the $\sigma_{l/r}$ are the generators of $G_{l/r}$. The constraints (5.4) are evidently linear in the currents and they are first class because from (3.10) the KM commutation relations (or Poisson brackets) are of the form

$$[\mathrm{tr}(\sigma_r j(x)), \mathrm{tr}(\sigma_l \tilde{j}(y))] = 0 \qquad [\mathrm{tr}(\sigma_r j(x)), \mathrm{tr}(\sigma_r j(y))] \subset \mathrm{tr}(\sigma_r, j(y))\delta(x-y), \tag{5.5}$$

and a corresponding relation for the $\tilde{j}$'s.

6. Gauge-Fixing: Cartan and Kostant-Kirillov

Since the magic constraints are first-class they generate a gauge-symmetry and to obtain a reduced system one must gauge-fix. There are two natural ways to gauge-fix for these constraints and each one leads to an interesting form of the reduced system. Both gauge-fixings take the linear form

$$\mathrm{tr}(\theta^a_l, j_r) = 0 \qquad \mathrm{tr}(\theta^a_r, j_l) = 0 \tag{6.1}$$

where the θ's are conjugates of the σ's. The only question is: what kind of conjugates? The first natural choice is let the θ's be the *Cartan* conjugates of the σ's i.e. to let

$$tr(\sigma_l^a, \theta_l^b) = \delta_{ab} \qquad \text{and} \qquad tr(\sigma_r^a, \theta_r^b) = \delta_{ab} \tag{6.2}$$

In the Cartan gauge the currrents take the form

$$J_- = M_- + k_o(j) \quad \text{and} \quad \tilde{J}_+ = M_+ + \tilde{k}_o(\tilde{j}) \tag{6.3}$$

where the k_o and $\tilde{k}_o$ lie in G_o and are (non-local) functionals of the constrained KM currents j and $\tilde{j}$. As we shall see in the next section this gauge leads to the Toda systems. The other natural choice is let to the θ's be the *Kostant-Kirillov* (KK) conjugates of the σ's with respect to the $Sl(2,R)$ generators $M_\pm$ i.e. to let

$$\omega_+(\sigma_l^a, \theta_l^b)) = \delta_{ab} \qquad \text{and} \qquad \omega_-(\sigma_r^a, \theta_r^b)) = \delta_{ab} \tag{6.4}$$

where

$$\omega_\pm(e,f) = \text{tr}(M_\pm[e,f]) \tag{6.5}$$

for any two elements e and f of G. In the KK gauge choice the currents takes the form

$$J_- = M_- + k_+(j) \qquad k_+ \subset \ker M_+ \quad \text{and} \quad \tilde{J}_+ = M_+ + \tilde{k}_-(\tilde{j}) \qquad \tilde{k}_- \subset \ker M_- \tag{6.6}$$

This means that in the KK gauge only the highest weight components of j and the lowest weight components of $\tilde{j}$ with respect to the $SL(2,R)$ subgroup survive. For this reason the KK gauge is sometimes known as the highest weight gauge. It turns out that the k_+ and $\tilde{k}_-$ currents are differential polynomials in the constrained KM currents and as we shall see in section 8 they form W-algebras. In fact the conformal weights s of the W-elements are just $s = m \pm 1$ where the m are the highest/lowest weights.

7. Cartan Fixing and Lagrangian Reduction

There is a standard method of implementing first-class constraints in Lagrangian theory, namely to gauge the theory with respect to the constraint group (without introducing a kinetic term for the gauge fields but rather regarding them as Lagrangian multipliers). This technique is tailor-made for the constraints of section 4 as follows: The group elements of $\mathcal{G}$ may be decomposed into $g_l g_o g_r$ in accordance with the decomposition (5.2) of the lie algebra G and then, using the Polyakov-Wiegmann formula for the decomposition of WZW actions for products, one finds that the WZW action WZW action (3.7) may be written as

$$\mathcal{A} = \mathcal{A}_o + \text{tr}\big(J_-^r g_o J_+ l g_o^{-1}\big) \tag{7.1}$$

where $\mathcal{A}_o$ is the WZW action for $\mathcal{G}_o$ and $J^{l/r}$ denotes the projections of the current onto $G_{l/r}$. Since from (5.4) the gauge-groups in are those generated by $G_{l/r}$ respectively the gauged form of this action is simply

$$\mathcal{A} = \mathcal{A}_o + \int \text{tr}\big((J_-^r - A_r)g_o(J_+^l - A_l)g_o^{-1} + A_r M_- + A_l M_+\big) \tag{7.2}$$

where $A_{l/r}$ are gauge-fields belonging to $G_\pm$ respectively. The action (7.2) is gauge-invariant with respect to the gauge-transformations $X_r \to h_r^{-1}(X_r+\partial_+)h_r$ for $X_r = J_r$ and A_r and $h_r(x) \in G_r$ and similarly for $h_l(x) \in G_l$.

To see how (7.2) leads to a Toda system we simply choose a gauge so that $J_-^r = J_+^l = 0$ to obtain

$$\mathcal{A} = \mathcal{A}_o + \int \mathrm{tr}\big(A_r g_o A_l g_o^{-1} + A_r M_- + A_l M_+\big) \tag{7.3}$$

and then either integrate out the A-fields (or eliminate them by means of the field equations in classical theory) In this way we obtain by inspection

$$\mathcal{A} = \mathcal{A}_o + \int \mathrm{tr}\big(M_- g_o M_+ g_o^{-1}\big) \tag{7.4}$$

which is nothing but the Toda action! In the quantum case the gauge-fixing is accompanied by some Faddeev-Popov ghosts but these are easily handled [9] using the BRST mechanism.

8. KK-Fixing and W-Algebras

To see how the KK choice of gauge-fixing leads to W-algebras we note that currents can be transformed to the form in which only the highest/lowest weight components survive by means of a gauge-transformation of the form

$$e^{\alpha.\sigma_+}\Big(M_- + j + \partial\Big)e^{-\alpha.\sigma_+} = M_- + k_+(j) \qquad k_+(j) \subset \mathrm{ker} M_+ \tag{8.1}$$

where $\alpha.\sigma$ denotes a sum over all gauge-parameters α and the corresponding gauge-group generators σ, and similarly for $\tilde{j}$. The interesting point is that the parameters α in (8.1) can be determined by iteration. Hence (8.1) is a complete gauge-fixing and the parameters are differential polynomials of the KM current components. It follows that the final current components k_+ are differential polynomials of the KM current components. But they are also gauge-invariant and since the commutators and PB's of both differential polynomials and gauge-invariants close we have

$$\{k_+(j), k_+(j)\} = \mathcal{DP}(k_+(j)) \tag{8.2}$$

where $\mathcal{DP}$ denotes differential polynomial. Thus we have an algebra of gauge-invariant DP's. Furthermore since the gauge-fixing is complete they form a basis for all gauge-invariants. Thus (8.2) actually defines the algebra of all gauge-invariants. So the algebra of all gauge-invariants is a differential polynomial algebra. To show that it is a W-algebra it suffices to show that it contains a Virasoro subalgebra and has a primary basis. This is not difficult. Indeed the Virasoro operator is just $\mathrm{tr}(M_-, k_+(j))$ i.e. is the element of k_+ that corresponds to the highest root of the embedded $SL(2,R)$ and the components corresponding to the other highest weights of $SL(2,R)$ are already primary. As might be expected, the W-algebra generated in this way is just the symmetry-algebra of Toda theory found in [8].

A bonus of using the gauge-fixing procedure (8.1) is that solving (8.1) for α is the equivalent of carrying out for arbitrary simple groups the DS transformation indicated in (2.1).

9. Gauge-Fixing and the Dirac Star Algebra

As we have just seen, the W-algebra is the algebra of invariants with respect to the gauge group generated by the first class constraints. On the other hand, for any set of first-class constraints $F_a = 0$ for $a = 1..n$ the addition of a set $H_a = 0$ for $a = 1...n$ of complete gauge-fixing constraints produces a set of second-class constraints in the sense of Dirac i.e. a set of constraints C_i $i = 1...2n$ such that

$$\Delta_{ik} \neq 0 \quad \text{where} \quad \Delta_{ik} = \{C_i, C_k\}_{C=0} \tag{9.1}$$

Then for any operators A we can then define the reduced operators

$$A^* = A - \{A, C_i\}_{C=0}\left(\Delta^{-1}\right)^{ik}_{C=0} C_k \quad \text{so that} \quad \{A^*, C_k\}_{C=0} = 0 \tag{9.2}$$

From (9.2) it follows (i) that the PB algebra $\{A^*, B^*\}$ of the reduced operators closes and (ii) that the reduced operators A^* are gauge-invariant on the constrained gauge-fixed surface. From these results it follows that the reduced PB algebra and the algebra of gauge-invariants are isomorphic.

The isomorphism of reduced and gauge-invariant algebras is a completely general result but applying it to the present case it shows that the W-algebra is just the Dirac star algebra of the reduced system and a bonus is that the differential polynomiality of the W-algebra implies the differential-polynomial invertibility of the Dirac constraint matrix Δ. This can be verified directly and is quite a remarkable result because the polynomial invertibility of Δ is very unusual. Indeed I do not know of any other non-trivial example, and in the literature direct use of the Dirac star algebra is usually avoided precisely because of the difficulty of inverting Δ.

10. General Solution

In this section we show how the general solution of the Toda systems may be obtained algebraically from the (trivial) general solution of the corresponding WZW case. The idea is to use the Gauss decomposition $g = g_l g_o g_r$ of the group elements with respect to the M_o grading. Then taking the WZW solution one writes each component in Gauss form to obtain

$$g(x_-)\gamma(x_+) = g_l(x_-)g_o(x_-)g_r(x_-)\gamma_l(x_+)\gamma_o(x_+)\gamma_r(x_+) \tag{10.1}$$

where the elements belong to $\mathcal{G}_{l/r}$ and $\mathcal{G}_o$ in the manner indicated. The components $g(x_-)$ and $\gamma(x_+)$ on the extreme left and right may be eliminated by a gauge transformation and the constraints on the currents imply that the off-diagonal components $g_l(x_+)$ and $g_r(x_-)$ are determined by the diagonal components $g_l(x_+)$ and $g_r(x_-)$ respectively through

$$\partial g_r(x_-) = g_0(x_-)g_r(x_-) \quad \text{and} \quad \partial\gamma_l(x_+) = \gamma_l(x_+)\gamma_o(x_+) \tag{10.2}$$

On converting the gauge-modified (10.1) to the overall Gauss form

$$g_o(x_-)g_r(x_-)\gamma_l(x_+)\gamma_o(x_+) = \alpha_l(x)h_o(x)\beta_r(x) \tag{10.3}$$

and removing the matrices on the left and right by a further gauge transformation one sees that the Toda solution is simply $h_o(x)$. Thus all one has to do is determine the matrix $h_o(x)$, and, apart from solving the relatively trivial differential equations (10.2), this is a purely algebraic problem.

For example in the Liouville case, after gauging away the matrices on the extreme laft and right in (10.1) and using (10.2) we obtain

$$\begin{pmatrix} e^{\partial a(x_-)} & 0 \\ 0 & e^{-\partial a(x_-)} \end{pmatrix} \begin{pmatrix} 1 & a(x_-) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b(x_+) & 1 \end{pmatrix} \begin{pmatrix} e^{\partial a(x_+)} & 0 \\ 0 & e^{-\partial a(x_+)} \end{pmatrix} \tag{10.4}$$

and after Gauss conversion this becomes

$$\begin{pmatrix} 1 & a(x_-) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\theta(x)} & 0 \\ 0 & e^{-\theta(x)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b(x_+) & 1 \end{pmatrix} \tag{10.5}$$

where

$$\theta(x) = \ln\Big(\frac{\partial a(x_-)\partial b(x_+)}{1 + a(x_-)b(x_+)}\Big) \tag{10.6}$$

This the well-known general solution in the Liouville case.

11. Quantization

For quantization one may proceed in a number of ways. One may quantize the reduced system canonically [8][13][14], one may quantize it using the functional integral formalism [15] or one may apply the BRST method to the gauged Lagrangian [9][16]. The latter procedure is simply a special case of the usual BRST procedure. I have not space to describe these three procedures here but it is perhaps worth mentioning that all three procedures agree and lead to a value of the Virasoro central charge of the reduced system of the form

$$c = \dim(G_o) + 12\pi(2k + \hbar g)\mathrm{tr}\Big[M_o + \frac{\hbar m_o}{4\pi(2k + \hbar g)}\Big]^2 \tag{11.1}$$

where M_o and m_o are the grading operators for the actual and principal embeddings respectively and g is the Coxeter number for the Lie group $\mathcal{G}$. This formula may also [14] be written as

$$c = \Big[\hbar \mathrm{dim} C + \sum_\alpha \frac{\hbar k_\alpha \mathrm{dim} G_\alpha}{2k_\alpha + \hbar g_\alpha}\Big] + 12\pi(2k + \hbar g)\mathrm{tr}\Big[M_o + \frac{\hbar \bar{M}}{4\pi(2k + \hbar g)}\Big]^2 \tag{11.2}$$

where C is the centre G_o and $\bar{M} = m_o - \sum_a m_o^{(a)}$ where the m_o^α are the principal grading operators for the simple subalgebras G^α of G_o. The advantage of the form (11.2) is that it seperates the part of the centre due to the free WZW theory for G_o (first square bracket) from the part due to the interaction.

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