Resummation of the Two Distinct Large Logarithms in the Broken $O(N)$-symmetric ϕ^4-model

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Abstract

The loop-expansion of the effective potential in the $O(N)$-symmetric ϕ^4-model contains generically two types of large logarithms. To resum those systematically a new minimal two-scale subtraction scheme $\widetilde{\text{2MS}}$ is introduced in an $O(N)$-invariant generalization of $\overline{\text{MS}}$. As the $\overline{\text{2MS}}$ beta functions depend on the renormalization scale-ratio a large logarithms resummation is performed on them. Two partial $\overline{\text{2MS}}$ renormalization group equations are derived to turn the beta functions into $\overline{\text{2MS}}$ running parameters. With the use of standard perturbative boundary conditions, which become applicable in $\overline{\text{2MS}}$, the leading logarithmic $\overline{\text{2MS}}$ effective potential is computed. The calculation indicates that there is no stable vacuum in the broken phase of the theory for $1 < N \le 4$.

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1 Introduction

There are many instances where an ordinary loop-wise perturbative expansion is rendered useless by the occurrence of large logarithms. This is the case eg. in the discussion of scaling violation in deep inelastic scattering (DIS) or in the determination of a reliable approximation to the effective potential (EP) in the standard model (SM). Only after resumming the large logarithms does the violation of Bjorken scaling yield one of the most accurate determinations of the strong coupling constant [1] or may the requirement of vacuum stability be turned into sensible bounds on the Higgs mass [2].

In the case of one type of large logarithms renormalization group (RG) techniques are well established to perform the necessary resummation systematically. However, in certain kinematical regimes in DIS there are two types of large logarithms, in the SM EP for small values of the Higgs field parameter there are five. Although the problem has been recognized by many authors no generally accepted RG techniques have been developed yet to deal with those cases.

Sticking to the $\overline{\text{MS}}$ scheme the decoupling theorem [3] was used in Ref. [4] to obtain some region-wise approximation to leading logarithms (LL) multi-scale summations. Although this is perfectly reasonable, one has to employ "low-energy" parameters, and it is not clear how to obtain sensible approximations for these in terms of the basic parameters of the full theory. Alternatively, one of us [5] argued that one could still apply the standard $\overline{\text{MS}}$ RG equation to multi-scale problems provided "improved" boundary conditions were employed. Although such improved boundary conditions were suggested in some simple cases, no general prescription was given for constructing them, and no improved boundary conditions were apparent for the subleading logarithms summation.

Clearly, one must go beyond the usual mass-independent renormalization schemes if multi-scale problems are to be seriously tackled. In the context of the EP we are aware of two different approaches. In Ref. [6] it was argued that one could employ a mass-dependent scheme in which decoupling of heavy modes is manifest in the perturbative RG functions. Alternatively, in Ref. [7] the usual $\overline{\text{MS}}$ scheme was extended to include several renormalization scales κ_i. While this seems to be an excellent idea, the specific scheme in [7] has two drawbacks. Firstly, the number of renormalization points does not necessarily match the number of generic scales in the problem at hand, as there is a RG scale κ_i associated with each coupling. Secondly, when computing multi-scale RG functions to n loops one encounters contributions proportional to $\log^{n-1}(\kappa_i/\kappa_j)$ (and lower powers). If some of the $\log(\kappa_i/\kappa_j)$ are large then even the perturbative RG functions cannot be trusted and used to sum logarithms. A similar approach to the one of Ref. [7] was outlined in Ref. [8] though no detailed perturbative calculations were performed.

Here we outline a more systematic approach fully developed to include next-to-leading logarithms (NLL) in Ref. [9]. In order to deal with the two-scale problem arising in the analysis of the EP in the $O(N)$-symmetric ϕ^4-theory we introduce a $O(N)$-invariant generalization of $\overline{\text{MS}}$. At each order in a $\overline{\text{MS}}$ loop-expansion we perform a finite renormalization to switch over to a new "minimal two-scale subtraction scheme" $2\overline{\text{MS}}$ which allows for two renormalization scales κ_i corresponding to the two generic scales in the problem. The $\overline{\text{MS}}$ RG functions and $\overline{\text{MS}}$ RGE then split into two $2\overline{\text{MS}}$ "partial" RG functions and two "partial" RGE's. The respective integrability con-

dition inevitably imposes a dependence of the partial RG functions on the renormalization scale-ratio κ_2/κ_1. Supplementing the integrability with an appropriate subsidiary condition we determine this dependence to all orders in the scale-ratio and obtain a trustworthy set of LL $\overline{\text{2MS}}$ RG functions. With the use of the two "partial" RGE's we then turn those into LL running two-scale parameters exhibiting features similar to the $\overline{\text{MS}}$ couplings such as a Landau pole now in both scaling channels. Using standard perturbative boundary conditions, which become applicable in $\overline{\text{2MS}}$, we calculate the effective potential in this scheme to LL and check it by comparison with two-loop and next-to-large N $\overline{\text{MS}}$ calculations. As a main result we find that for $1 < N \le 4$ there is no stable vacuum in the broken phase. A full analytic determination of the NLL corrections to the results presented here is given in Ref. [9] and shows that the instability is not just an artefact of a LL calculation.

2   The one-loop effective potential in $\overline{\text{MS}}$

Let us consider the $O(N)$-symmetric φ^4-theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\alpha\varphi\,\partial^\alpha\varphi - \frac{\lambda}{24}\varphi^4 - \frac{1}{2}m^2\varphi^2 - \Lambda, \tag{1}$$

where φ is a real N-component scalar field. Note the inclusion of the cosmological constant term which will prove essential in the discussion of the RG and the effective potential later [10].

A loop-wise perturbation expansion of the effective potential [11, 12] yields in the $\overline{\text{MS}}$-scheme to one loop

$$
\begin{aligned}
V^{(\text{tree})} &= \frac{\lambda}{24}\varphi^4 + \frac{1}{2}m^2\varphi^2 + \Lambda, \\
V^{(\text{1-loop})} &= \frac{\hbar}{(4\pi)^2}\frac{\mathcal{M}_1{}^2}{4}\left(\log\frac{\mathcal{M}_1}{\mu^2} - \frac{3}{2}\right) + \frac{\hbar}{(4\pi)^2}(N-1)\frac{\mathcal{M}_2{}^2}{4}\left(\log\frac{\mathcal{M}_2}{\mu^2} - \frac{3}{2}\right),
\end{aligned}
\tag{2}
$$

where

$$\mathcal{M}_1 = m^2 + \tfrac{1}{2}\lambda\varphi^2, \quad \mathcal{M}_2 = m^2 + \tfrac{1}{6}\lambda\varphi^2, \tag{3}$$

and μ is the $\overline{\text{MS}}$-renormalization scale. The one-loop contribution to the EP thus contains logarithms of the ratios $\mathcal{M}_i/\mu^2$ to the first power and in general the n-loop contribution will be a polynomial of the nth order in these logarithms. (The explicit two-loop result has been obtained in [13].)

In view of these logarithms the loop-wise expansion may be trusted only in a region in field- and coupling-space where simultaneously

$$\frac{\hbar\lambda}{(4\pi)^2} \ll 1 \quad \text{and} \quad \frac{\hbar\lambda}{(4\pi)^2}\log\frac{\mathcal{M}_i}{\mu^2} \ll 1. \tag{4}$$

Due to the two largely differing scales $\mathcal{M}_i$ occurring in the logarithms these conditions may hardly be fulfilled eg. around the tree-level minimum of the potential, where $\mathcal{M}_2 = 0$, even with a judicious choice of μ. Hence, to obtain a sensible range of validity one has to resum the logarithms in the EP.

In the one-scale case this would be achieved to LL by solving the one-loop $\overline{\text{MS}}$ RG equation for the effective potential and by employing the corresponding tree-level boundary conditions [14]. Here, we have to deal with two relevant scales. The necessary generalization of the $\overline{\text{MS}}$ scheme and the usual RG approach allowing for as many renormalization scales as there are relevant scales in the theory has been given in [9].

3   The minimal two-scale subtraction scheme $\overline{\text{2MS}}$

To track the two differing logarithms with two corresponding renormalization scales we use the freedom of performing a finite renormalization. Hence, to one loop we add a *finite, $O(N)$-invariant* counterterm to the Lagrangian

$$\Delta\mathcal{L}^{(\text{1-loop})} = \frac{\hbar}{(4\pi)^2}\frac{\mathcal{M}_1^2}{4}\log\frac{\mu^2}{\kappa_1^2} + \frac{\hbar}{(4\pi)^2}(N-1)\frac{\mathcal{M}_2^2}{4}\log\frac{\mu^2}{\kappa_2^2}, \tag{5}$$

where the new renormalization scale κ_1 is tracking the Higgs logarithms and κ_2 is tracking the Goldstone logarithms. Note that $\Delta\mathcal{L}^{(\text{1-loop})}$ is in fact a polynomial of fourth order in φ consistent with renormalizability and the $O(N)$-symmetry.

In the minimal two-scale subtraction scheme $\overline{\text{2MS}}$ thence introduced the one-loop contribution to the EP becomes

$$V^{(\text{1-loop})} = \frac{\hbar}{(4\pi)^2}\frac{\mathcal{M}_1^2}{4}\left(\log\frac{\mathcal{M}_1}{\kappa_1^2} - \frac{3}{2}\right) + \frac{\hbar}{(4\pi)^2}(N-1)\frac{\mathcal{M}_2^2}{4}\left(\log\frac{\mathcal{M}_2}{\kappa_2^2} - \frac{3}{2}\right). \tag{6}$$

Hence, in $\overline{\text{2MS}}$ we may again trust the loop-expansion of the EP at $\kappa_1^2 = \mathcal{M}_1$, $\kappa_2^2 = \mathcal{M}_2$ which becomes the boundary condition for the RG evolution in the two-scale case. Note that in this scheme the beta functions inevitably depend on $\log(\kappa_2/\kappa_1)$ and will be trustworthy only after resummation of those logarithms.

As discussed in detail in [9] the general features to be respected by $\overline{\text{2MS}}$ are:

i) The effective action Γ, when expressed in terms of the $\overline{\text{2MS}}$ parameters, should be independent of the $\overline{\text{MS}}$ scale μ.

ii) When $\kappa_1 = \kappa_2$ $\overline{\text{2MS}}$ should coincide with $\overline{\text{MS}}$ at that scale.

iii) When $N = 1$ ($N \to \infty$) the scale κ_2 (κ_1) should drop and $\overline{\text{2MS}}$ should coincide with $\overline{\text{MS}}$ at the remaining scale.

iv) When $\kappa_i^2 = \mathcal{M}_i$ the standard loop-expansion should render a reliable approximation to the full EP insofar as $\frac{\hbar}{(4\pi)^2}\lambda(\kappa_1, \kappa_2)$ is "small".

Starting now from the identity

$$\Gamma_{\overline{\text{MS}}}[\lambda_{\overline{\text{MS}}}, m_{\overline{\text{MS}}}^2, \Lambda_{\overline{\text{MS}}}, \varphi_{\overline{\text{MS}}}; \mu] = \Gamma[\lambda, m^2, \Lambda, \varphi; \kappa_1, \kappa_2] \tag{7}$$

we derive the two partial $\overline{\text{2MS}}$ RGE's corresponding to variations of the scales κ_i, where the other scale κ_j and the $\overline{\text{MS}}$ parameters are held fixed, in much the same way as the $\overline{\text{MS}}$ RG is usually derived. Specializing to the effective potential we obtain

$$\mathcal{D}_i V = 0, \quad \mathcal{D}_i = \kappa_i\frac{\partial}{\partial\kappa_i} + {}_i\beta_\lambda\frac{\partial}{\partial\lambda} + {}_i\beta_{m^2}\frac{\partial}{\partial m^2} + {}_i\beta_\Lambda\frac{\partial}{\partial\Lambda} - {}_i\beta_\varphi\varphi\frac{\partial}{\partial\varphi}. \tag{8}$$

The two sets of RG functions are defined as usual

$$_i\beta_\lambda = \kappa_i\frac{d\lambda}{d\kappa_i}, \quad _i\beta_{m^2} = \kappa_i\frac{dm^2}{d\kappa_i}, \quad _i\beta_\Lambda = \kappa_i\frac{d\Lambda}{d\kappa_i}, \quad _i\beta_\varphi\varphi = -\kappa_i\frac{d\varphi}{d\kappa_i} \tag{9}$$

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for $i = 1, 2$. In general they may be functions not only of λ, m^2 as are the $\overline{\text{MS}}$ RG functions but also of $\log(\kappa_2/\kappa_1)$.

Note that property ii) requires the sum of the $\overline{\text{2MS}}$ RG functions at $\kappa_1 = \kappa_2$ to coincide with the $\overline{\text{MS}}$ RG function at that scale

$$_1\beta.(\kappa_1 = \kappa_2) +_2 \beta.(\kappa_1 = \kappa_2) = \beta_{.,\overline{\text{MS}}}, \tag{10}$$

where the set of $\overline{\text{MS}}$ beta functions is given to one loop by

$$\beta_{\lambda,\overline{\text{MS}}}^{(\text{1-loop})} = \frac{\hbar}{(4\pi)^2}\left(3 + \frac{N-1}{3}\right)\lambda^2, \quad \beta_{m^2,\overline{\text{MS}}}^{(\text{1-loop})} = \frac{\hbar}{(4\pi)^2}\left(1 + \frac{N-1}{3}\right)\lambda m^2,$$

$$\beta_{\Lambda,\overline{\text{MS}}}^{(\text{1-loop})} = \frac{\hbar}{(4\pi)^2}\left(\frac{1}{2} + \frac{N-1}{2}\right)m^4, \quad \beta_{\varphi,\overline{\text{MS}}}^{(\text{1-loop})} = 0. \tag{11}$$

In the $N = 1$ limit property iii) fixes the $_1\beta.$ to be the usual $N = 1$ $\overline{\text{MS}}$ RG functions, given to $\mathcal{O}(\hbar)$ by eqns. (11) with $N = 1$, and requires to disregard the second set of RG functions so that $\mathcal{D}_2 = \kappa_2\partial/\partial\kappa_2$. For $N \to \infty$ there are no Higgs contributions and the $_2\beta.$ are the $N \to \infty$ $\overline{\text{MS}}$ RG functions, again given to $\mathcal{O}(\hbar)$ by eqns. (11) in the large N limit. The first set of RG functions is then trivial, hence $\mathcal{D}_1 = \kappa_1\partial/\partial\kappa_1$.

4 The LL resummed $\overline{\text{2MS}}$ RG functions

As we want to vary κ_1 and κ_2 independently we must respect the integrability condition

$$[\kappa_1 d/d\kappa_1, \kappa_2 d/d\kappa_2] = [\mathcal{D}_1, \mathcal{D}_2] = 0, \tag{12}$$

which allows us now to determine the $\overline{\text{2MS}}$ beta functions. An essential feature of a mass-independent renormalization scheme such as $\overline{\text{MS}}$ is that the beta functions do not depend on the renormalization scale μ. Unfortunately we cannot generalize this to the multi-scale case and demand that the two sets of beta functions be independent of $\log(\kappa_2/\kappa_1)$. In fact, the independence of the RG functions from the scales κ_i, ie. $[\kappa_i\partial/\partial\kappa_i, \mathcal{D}_j] = 0$, is incompatible with the integrability condition (12). However, as we have one subsidiary condition at our disposal it is possible to arrange eg. for the first set of RG functions to be κ_i-independent

$$[\kappa_i\partial/\partial\kappa_i, \mathcal{D}_1] = 0. \tag{13}$$

Hence, at LL we have the first set of RG functions fixed to be the $N = 1$ values from eqns. (11)

$$_1\beta_\lambda^{(\text{LL})} = \frac{\hbar}{(4\pi)^2}3\lambda^2, \quad _1\beta_{m^2}^{(\text{LL})} = \frac{\hbar}{(4\pi)^2}\lambda m^2,$$

$$_1\beta_\Lambda^{(\text{LL})} = \frac{\hbar}{(4\pi)^2}\frac{1}{2}m^4, \quad _1\beta_\varphi^{(\text{LL})} = 0. \tag{14}$$

In general, we could assume a linear combination $\tilde{\beta}. = p \cdot {_1\beta.} + (1-p) \cdot {_2\beta.}$ of the two sets of beta functions to be κ_i-independent. As analyzed in detail in [9] the results for the beta functions, the running parameters and the EP are then p-dependent. p

has to be fixed eg. by comparison with the 2-loop and the next-to-large N EP and in our case it turns out that $p = 1$ is the appropriate choice [9].

As $\mathcal{D}_1$ is now fixed eqn. (12) yields RG-type equations for the $_2\beta$ which we solve next. Setting

$$t = \frac{\hbar\lambda}{(4\pi)^2} \log \frac{\kappa_2}{\kappa_1} \tag{15}$$

the equation for $_2\beta_\lambda$ becomes to leading order

$$-\frac{\hbar\lambda}{(4\pi)^2} \frac{\partial}{\partial t} {}_2\beta_\lambda^{(\mathrm{LL})} + {}_1\beta_\lambda^{(\mathrm{LL})} \frac{\partial}{\partial\lambda} {}_2\beta_\lambda^{(\mathrm{LL})} - {}_2\beta_\lambda^{(\mathrm{LL})} \frac{\partial}{\partial\lambda} {}_1\beta_\lambda^{(\mathrm{LL})} = 0. \tag{16}$$

The solution does not exlicitly depend on t

$$_2\beta_\lambda^{(\mathrm{LL})}(t) = \frac{\hbar}{(4\pi)^2} \frac{N-1}{3} \lambda^2. \tag{17}$$

Note that to fix the boundary conditions above and in what follows we use property ii) leading to the relevant condition (10).

We turn to the equation for $_2\beta_{m^2}$

$$-\frac{\hbar\lambda}{(4\pi)^2} \frac{\partial}{\partial t} {}_2\beta_{m^2}^{(\mathrm{LL})} + {}_1\beta_\lambda^{(\mathrm{LL})} \frac{\partial}{\partial\lambda} {}_2\beta_{m^2}^{(\mathrm{LL})} - {}_2\beta_\lambda^{(\mathrm{LL})} \frac{\partial}{\partial\lambda} {}_1\beta_{m^2}^{(\mathrm{LL})} = 0. \tag{18}$$

This is easily solved by

$$_2\beta_{m^2}^{(\mathrm{LL})}(t) = \frac{\hbar}{(4\pi)^2} \frac{N-1}{9} \left(1 + 2\left(1 - 3t\right)^{-1}\right) \lambda m^2. \tag{19}$$

Next we determine $_2\beta_\Lambda$ from

$$-\frac{\hbar\lambda}{(4\pi)^2} \frac{\partial}{\partial t} {}_2\beta_\Lambda^{(\mathrm{LL})} + {}_1\beta_\lambda^{(\mathrm{LL})} \frac{\partial}{\partial\lambda} {}_2\beta_\Lambda^{(\mathrm{LL})} - {}_2\beta_\lambda^{(\mathrm{LL})} \frac{\partial}{\partial\lambda} {}_1\beta_\Lambda^{(\mathrm{LL})}$$
$$+ {}_1\beta_{m^2}^{(\mathrm{LL})} \frac{\partial}{\partial m^2} {}_2\beta_\Lambda^{(\mathrm{LL})} - {}_2\beta_{m^2}^{(\mathrm{LL})} \frac{\partial}{\partial m^2} {}_1\beta_\Lambda^{(\mathrm{LL})} = 0. \tag{20}$$

For later convenience we give the result partly in terms of $_2\beta_\lambda^{(\mathrm{LL})}(t)$ and $_2\beta_{m^2}^{(\mathrm{LL})}(t)$

$$_2\beta_\Lambda^{(\mathrm{LL})}(t) = \frac{\hbar}{(4\pi)^2} \frac{2(N-1)}{3} (1-3t)^{-\frac{2}{3}} m^4 + \frac{1}{2} {}_2\beta_\lambda^{(\mathrm{LL})}(t) \left(\frac{m^2}{\lambda}\right)^2 - {}_2\beta_{m^2}^{(\mathrm{LL})}(t) \frac{m^2}{\lambda}. \tag{21}$$

Finally $_2\beta_\varphi$ remains trivial

$$_2\beta_\varphi^{(\mathrm{LL})}(t) = 0. \tag{22}$$

It is obvious that *the beta functions possess Landau poles at* $1 - 3t = 0$. Hence, they are trustworthy only for $1 \gg 3t$. On the other hand, the limit $t \to -\infty$ exists for the whole set of $_2\beta^{(\mathrm{LL})}(t)$. This will allow us later to discuss the non-trivial behaviour of the two-scale EP around the tree-level minimum.

5 The LL $\overline{\text{2MS}}$ running two-scale parameters

The running parameters in $\overline{\text{2MS}}$ are functions of the variables

$$s_i = \frac{\hbar}{(4\pi)^2} \log \frac{\kappa_i(s_i)}{\kappa_i}, \quad t = \frac{\hbar\lambda}{(4\pi)^2} \log \frac{\kappa_2}{\kappa_1}, \tag{23}$$

where κ_i are the reference scales. Note that $t(s_i)$ as given in eqn. (15) is in fact s_i-dependent, $t = \frac{\hbar\lambda(s_i)}{(4\pi)^2} \log \frac{\kappa_2(s_i)}{\kappa_1(s_i)}$. The above variables may be expanded in series in $\hbar$ the LL terms of which we determine now from eqn. (9).

The equations for the leading order running two-scale coupling are

$$\frac{d\lambda^{(\text{LL})}}{ds_1} = 3\,\lambda^{(\text{LL})2}, \quad \frac{d\lambda^{(\text{LL})}}{ds_2} = \frac{N-1}{3}\,\lambda^{(\text{LL})2}. \tag{24}$$

They are easily integrated

$$\lambda^{(\text{LL})}(s_i) = \lambda \left(1 - 3\lambda s_1 - \frac{(N-1)}{3}\lambda s_2 \right)^{-1} \tag{25}$$

with the boundary condition $\lambda(s_i = 0) = \lambda$. Above, the s_1-term accounts for the running of λ due to the 'Higgs', the s_2-term for the evolution due to the 'Goldstones'.

Next we determine the running mass from

$$\frac{dm^{2(\text{LL})}}{ds_1} = \lambda^{(\text{LL})}\, m^{2(\text{LL})}. \tag{26}$$

This is easily solved

$$m^{2(\text{LL})}(s_i) = m^2 \left(\frac{\lambda^{(\text{LL})}(s_i)}{\lambda} \right)^{\frac{1}{3}} A(s_2). \tag{27}$$

The constant of integration $A(s_2)$ is obtained from the second m^2-equation

$$\frac{dm^{2(\text{LL})}}{ds_2} = \frac{N-1}{9} \left(1 + 2 \left(\frac{\lambda^{(\text{LL})}}{\lambda} \left(1 - \frac{(N+8)}{3}\lambda s_2 - 3t \right) \right)^{-1} \right) \lambda^{(\text{LL})}\, m^{2(\text{LL})}. \tag{28}$$

We finally find

$$m^{2(\text{LL})}(s_i) = m^2 \left(1 - 3\lambda s_1 - \frac{(N-1)}{3}\lambda s_2 \right)^{-\frac{1}{3}} \left(\frac{1 - \frac{(N+8)}{3}\lambda s_2 - 3t}{1 - 3t} \right)^{-\frac{2}{3}\frac{N-1}{N+8}}. \tag{29}$$

The boundary condition is chosen such that $m^2(s_i = 0) = m^2$.

In order to obtain the running cosmological constant we have to solve

$$\frac{d\Lambda^{(\text{LL})}}{ds_1} = \frac{1}{2} \left(m^{2(\text{LL})} \right)^2. \tag{30}$$

This yields the result

$$\Lambda^{(\text{LL})}(s_i) = \Lambda - \frac{1}{2} \left[\frac{\left(m^{2(\text{LL})}(s_i) \right)^2}{\lambda^{(\text{LL})}(s_i)} - \frac{m^4}{\lambda} \right] + B(s_2). \tag{31}$$

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To calculate the constant of integration $B(s_2)$ we turn to the second Λ-equation

$$
\frac{d\Lambda^{(\mathrm{LL})}}{ds_2} = \frac{2(N-1)}{3} \left(\frac{\lambda^{(\mathrm{LL})}}{\lambda} \left(1 - \frac{(N+8)}{3} \lambda s_2 - 3t \right) \right)^{-\frac{2}{3}} \left(m^{2\,(\mathrm{LL})} \right)^2
$$
$$
+ \frac{(4\pi)^2}{\hbar} \frac{1}{2} {}_2\beta_\lambda^{(\mathrm{LL})} \left(\frac{m^{2\,(\mathrm{LL})}}{\lambda^{(\mathrm{LL})}} \right)^2 - \frac{(4\pi)^2}{\hbar} {}_2\beta_{m^2}^{(\mathrm{LL})} \frac{m^{2\,(\mathrm{LL})}}{\lambda^{(\mathrm{LL})}} \tag{32}
$$

and obtain the final result

$$
\Lambda^{(\mathrm{LL})}(s_i) = -\frac{m^4}{2\lambda} \left[\left(1 - 3\lambda s_1 - \frac{N-1}{3}\lambda s_2 \right)^{\frac{1}{3}} \left(\frac{1 - \frac{N+8}{3}\lambda s_2 - 3t}{1 - 3t} \right)^{-\frac{4}{3}\frac{N-1}{N+8}} - 1 \right]
$$
$$
+ \frac{2m^4}{\lambda} \frac{N-1}{N-4} (1 - 3t)^{\frac{1}{3}} \left[\left(\frac{1 - \frac{N+8}{3}\lambda s_2 - 3t}{1 - 3t} \right)^{-\frac{N-4}{N+8}} - 1 \right] + \Lambda. \tag{33}
$$

Here the boundary condition is $\Lambda(s_i = 0) = \Lambda$. Due to the trivial ${}_i\beta_\varphi^{(\mathrm{LL})}$ the field parameter φ does not depend on s_i.

The LL running coupling $\lambda^{(\mathrm{LL})}(s_i)$, and therefore the running mass as well, have a Landau pole at $1 - 3\lambda s_1 - \frac{N-1}{3}\lambda s_2 = 0$ and clearly our approximation will break down before this pole is reached. Of more importance is the behaviour of the running cosmological constant as will be discussed at the end of the next section.

6   The LL RG improved $\overline{\mathrm{2MS}}$ effective potential

It is now an easy task to turn the results for the running two-scale parameters into a RG improved effective potential. $\mathcal{D}_i V = 0$ yields the identity

$$
V(\lambda, m^2, \varphi, \Lambda; \kappa_1, \kappa_2) = V(\lambda(s_i), m^2(s_i), \varphi(s_i), \Lambda(s_i); \kappa_1(s_1), \kappa_2(s_2)), \tag{34}
$$

with $\kappa_i(s_i)$ defined in (23). Next, we assume the validity of condition iv) from section 3. Hence, if

$$
\kappa_i(s_i)^2 = \mathcal{M}_i(s_j) \equiv m^2(s_j) + k_i \lambda(s_j)\varphi^2(s_j), \qquad k_1 = \frac{1}{2}, \; k_2 = \frac{1}{6} \tag{35}
$$

the loop-expansion of the EP should render a trustworthy approximation to the RHS of eqn. (34).

To proceed we have to determine the values of s_i fulfilling (35). Insertion of the $\kappa_i(s_i)$ from (35) into (23) yields a quite implicit set of equations

$$
s_i = \frac{\hbar}{2(4\pi)^2} \log \frac{\mathcal{M}_i(s_j)}{\kappa_i^2}. \tag{36}
$$

However, since we are meant to be summing consistently leading logarithms the explicit solution to this order is easily obtained

$$
s_1^{(\mathrm{LL})} = \frac{\hbar}{2(4\pi)^2} \log \frac{m^2 + \frac{1}{2}\lambda\varphi^2}{\kappa_1^2}, \quad s_2^{(\mathrm{LL})} = \frac{\hbar}{2(4\pi)^2} \log \frac{m^2 + \frac{1}{6}\lambda\varphi^2}{\kappa_2^2}. \tag{37}
$$

At scales $s_i^{(\mathrm{LL})}$ we can now approximate the RHS of eqn. (34) with the tree-level contribution as displayed in (2), hence

$$V^{(\mathrm{LL})}(\lambda, ...; \kappa_i) = \frac{\lambda^{(\mathrm{LL})}(s_i^{(\mathrm{LL})})}{24}\varphi^4 + \frac{1}{2}m^{2(\mathrm{LL})}(s_i^{(\mathrm{LL})})\varphi^2 + \Lambda^{(\mathrm{LL})}(s_i^{(\mathrm{LL})}). \tag{38}$$

Insertion of the various expressions for the running parameters yields the explicit, $O(N)$-invariant final result for the LL two-scale improved potential in $\overline{\mathrm{2MS}}$ [9]

$$\begin{aligned}
V^{(\mathrm{LL})} &= \frac{\lambda\varphi^4}{24}\left(1 - 3\lambda s_1^{(\mathrm{LL})} - \frac{N-1}{3}\lambda s_2^{(\mathrm{LL})}\right)^{-1} \\
&+ \frac{m^2\varphi^2}{2}\left(1 - 3\lambda s_1^{(\mathrm{LL})} - \frac{N-1}{3}\lambda s_2^{(\mathrm{LL})}\right)^{-\frac{1}{3}}\left(1 - \frac{\frac{N+8}{3}\lambda s_2^{(\mathrm{LL})}}{1-3t}\right)^{-\frac{2}{3}\frac{N-1}{N+8}} \\
&- \frac{m^4}{2\lambda}\left[\left(1 - 3\lambda s_1^{(\mathrm{LL})} - \frac{N-1}{3}\lambda s_2^{(\mathrm{LL})}\right)^{\frac{1}{3}}\left(1 - \frac{\frac{N+8}{3}\lambda s_2^{(\mathrm{LL})}}{1-3t}\right)^{-\frac{4}{3}\frac{N-1}{N+8}} - 1\right] \\
&+ 2\frac{N-1}{N-4}\frac{m^4}{\lambda}(1-3t)^{\frac{1}{3}}\left[\left(1 - \frac{\frac{N+8}{3}\lambda s_2^{(\mathrm{LL})}}{1-3t}\right)^{-\frac{N-4}{N+8}} - 1\right] + \Lambda.
\end{aligned} \tag{39}$$

There are various important checks on our result. By construction it reduces in the single-scale limits $N = 1$ and $N \to \infty$ to the well-known one-scale $\overline{\mathrm{MS}}$ results [14]. A non-trivial check is provided by expanding eqn. (39) to second order in $s_i^{(\mathrm{LL})}$. As required the result of this expansion coincides with the leading logarithmic terms in the explicit 2-loop effective potential as obtained in Ref. [13]. Furthermore, for $N \to \infty$ we recover in the LL approximation the next-to-large N expression for the EP as given in Ref. [15]. Finally, for $t = 0$ the result (39) has already been obtained using the $\overline{\mathrm{MS}}$ RG and a conjecture, proven up to two loops, for the boundary condition which becomes very involved in that approach [5].

We turn now to a discussion of the most important features of the result (39). In the broken phase ($m^2 < 0$) the tree-level minimum is at $\mathcal{M}_2 = 0$ or $s_2^{(\mathrm{LL})} \to -\infty$. Hence, as we approach it $\log(\mathcal{M}_2/\mathcal{M}_1)$ will become large. If we are prepared to trust eqn. (39) even in the *extreme* case of the tree minimum itself an intriguing property emerges.

As long as $N > 4$ the φ^4-and $m^2\varphi^2$-terms vanish and the Λ-term converges to a finite value. As the slope $\frac{dV^{(\mathrm{LL})}}{ds_2^{(\mathrm{LL})}}(s_2^{(\mathrm{LL})} \to -\infty) \searrow 0$ the EP takes its minimum in the broken phase at the tree-level value and becomes complex for even smaller φ^2-values.

But for $1 < N \le 4$ the Λ-term, and thence $V^{(\mathrm{LL})}$, diverges to minus infinity indicating that for these values of N *there is no stable vacuum in the broken phase*. Note especially that for $N = 4$, ie. the SM scalar boson content, there is still a divergence. It is softer than for $N = 2, 3$, however, as the penultimate term in eqn. (39) becomes a logarithm

$$V^{(\mathrm{LL})} = ... - \frac{m^4}{2\lambda}(1-3t)^{\frac{1}{3}}\log\left(1 - \frac{4\lambda s_2^{(\mathrm{LL})}}{1-3t}\right) + \Lambda. \tag{40}$$

7 Comment on NLL and Discussion

The method presented in the calculation of the LL two-scale effective potential is systematic. In fact, in Ref. [9] we have performed *a full analytic computation* of the NLL two-scale RG functions, of the corresponding NLL two-scale running parameters and finally of the NLL effective potential $V^{(\mathrm{NLL})}$. Our main result is that for $1 < N \leq 4$ *the vacuum instability in the broken phase persists* . Hence, it is not a simple artefact of the LL resummation performed in this paper.

The occurrence of a vacuum instability in the broken phase of the $O(N)$-model raises immediately the possibility of a similar outcome in a multi-scale analysis of the SM effective potential. As the method outlined generalizes naturally to problems with more than two scales we are in a position to investigate systematically the different possible scenarios. Because the SM analysis poses a many-scale problem and will become quite cumbersome it proves useful to study first the effects of adding either fermions as in a Yukawa-type model or gauging the simplest case of $N = 2$ as in the Abelian-Higgs model. The Yukawa case is either a two- or three-scale problem, depending on whether one includes Goldstone bosons or not. The Abelian-Higgs model in the Landau gauge will be a three-scale problem. In the three-scale case one has *three* integrability conditions $[\mathcal{D}_i, \mathcal{D}_j] = 0$ and three independent subsidiary conditions for free. They are analogous to $[\kappa_i \partial/\partial \kappa_i, \mathcal{D}_1] = 0$ which we used above. For the general n-scale problem one would have $\frac{1}{2}n(n-1)$ integrability conditions to be supplemented by $\frac{1}{2}n(n-1)$ subsidiary conditions. The question of whether fermions or gauge fields may stabilize the effective potential for small N in a full multi-scale analysis is under investigation.

Acknowledgments

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