DIAS-STP-96-17
hep-th/9606164

Non-Grassmann "Classicization" of Fermion Dynamics

T. Garavaglia*
Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

S. K. Kauffmann†
Unit 3 / 51-53 Darley Street, Mona Vale, NSW 2103, Australia

Abstract

A carefully motivated symmetric variant of the Poisson bracket in ordinary (not Grassmann) phase space variables is shown to satisfy identities which are in algebraic correspondence with the anticommutation postulates for quantized Fermion systems. "Symplecticity" in terms of this symmetric Poisson bracket implies generalized Hamilton's equations that can only be of Schrödinger type (e.g., the Dirac equation but not the Klein-Gordon or Maxwell equations). This restriction also excludes the old "four-Fermion" theory of beta decay.

PACS numbers: 03.70.+k 03.20.+i 03.65.-w

Keywords: Fermion, Poisson, classical, canonical, symplectic, anticommutation, quantization

July 1996

*E-mail: bronco@stp01.stp.dias.ie; Fax: +353 1 6680561; Tel: +353 1 6680748. Supported in part by Dublin Institute of Technology grant 79/85/8571.

†E-mail: skk@sydney.dialix.oz.au; Fax: +61 2 99992412; Tel: +61 2 99796301

Quantized Fermion dynamics, with its Pauli exclusion principle, no more possesses a limit of "large" quantum numbers than does elementary spin one-half quantum dynamics. Thus the notion of "classicizing" Fermion dynamics via a formal $\hbar \to 0$ limit [1] is physically unsound. In fact, the ostensible Fermion "classicization" developed in Refs. [1, 2] maneuvers shy of this trap by declining clean abandonment of quantum noncommutativity, which lingers on in the guise of anticommuting Grassmann phase space variables (the oxymoronic tag "anticommuting c-numbers" notwithstanding). However, the pointlessness of these physically mismotivated (by the $\hbar \to 0$ notion) and theoretically ambiguous (definitely not classical, but neither fully-fledged quantum) Grassmann phase space variables is illustrated by the familiar Dirac electron theory, whose second-quantized version describes a quantum Fermion dynamical system. Its "classicization" obviously ought to be the first-quantized Dirac electron theory, whose wave functions are pure c-number—*not* Grassmann variable—versions of the previously quantized electron fields, and whose dynamics *requires* that $\hbar \neq 0$ (e.g., in the familiar first-quantized Dirac equation).

Grassmann phase space variables are avoided here in favor of true c-number phase space variables, which are used to construct a heuristically compelling "symmetric" variant of the Poisson bracket (its definition specifically requires that $\hbar \neq 0$). This "symmetric" Poisson bracket satisfies phase-space vector component identities whose algebraic relation to the postulated *anticommutation* rules of quantized Fermion dynamics fully parallels the algebraic relation of the ordinary Poisson bracket phase-space vector component identities to the postulated *commutation* rules of quantized point particle and Boson dynamics.

Given this soundly based "symmetric" Poisson bracket, the structure of "classical" Fermion dynamics follows straightforwardly from the requirements of "symplecticity" with respect to it—the derivations can be carried out in perfect parallel with the well-known ones of ordinary (or Boson) classical dynamics [3]. For continuous one-parameter sequences of infinitesimal Fermion "canonical" transformations, one obtains the same natural generalization of Hamilton's equations as occurs in ordinary classical dynamics [3], but one also finds stringent constraints on the form of the "generalized Hamiltonian functions" or "canonical transformation generators" which are permitted to appear in these Fermion "classical" dynamical equations. Indeed, the restrictions on these generators are such that "classical" Fermion dynamics must be *linear* and described by a *Schrödinger* type of equation (which may possibly be inhomogeneous). The Dirac equation, which is of Schrödinger type, can describe a "classical" Fermion system, but the inherently non-Schrödinger (even though linear) Klein-Gordon and Maxwell equations cannot. Also the old "four-Fermion" theory of beta decay cannot describe a legitimate "classical" Fermion system (it is not thus forbidden under the Grassmann variable regime).

Ordinary classical dynamics is usually discussed in terms of real-valued phase space vector variables of the form $(\vec{q}, \vec{p})$. However, its relation to the quantum theory and to Fermion systems is much more transparent if one changes these real phase space vector variables to the complex-valued dimensionless phase space vector variables $\vec{a} \equiv (\vec{q}/q_s + iq_s\vec{p}/\hbar)/\sqrt{2}$ and their complex conjugates $\vec{a}^* = (\vec{q}/q_s - iq_s\vec{p}/\hbar)/\sqrt{2}$, where q_s is a nonzero real-valued scale factor that has the same dimensions as the components of $\vec{q}$ (note also the obvious requirement that $\hbar \neq 0$). In terms of $\vec{a}$ and $\vec{a}^*$, $(\vec{q}, \vec{p}) = (q_s(\vec{a} + \vec{a}^*), -i\hbar(\vec{a} - \vec{a}^*)/q_s)/\sqrt{2}$. In terms of components of both of these types of phase space vector variable, the usual Poisson bracket of ordinary classical dynamics is

$$\{f, g\} \equiv \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k}\frac{\partial f}{\partial p_k} \right) = -\frac{i}{\hbar}\sum_k \left(\frac{\partial f}{\partial a_k}\frac{\partial g}{\partial a_k^*} - \frac{\partial g}{\partial a_k}\frac{\partial f}{\partial a_k^*} \right). \tag{1}$$

From the second Poisson bracket representation given in Eq. (1) above we abstract the "sub-bracket"

$$\{f \circ g\} \equiv \sum_k \frac{\partial f}{\partial a_k}\frac{\partial g}{\partial a_k^*}, \tag{2}$$

which we call the *ordered* Poisson bracket. We note that while $\{f \circ g\}$ is linear in each of its two argument functions f and g, it is neither antisymmetric nor symmetric (commutative) under their interchange. However, it *does* satisfy the identity $\{f \circ g\} = \{g^* \circ f^*\}^*$, which is in algebraic correspondence with the Hermitian conjugation formula for the *product of two Hilbert-space operators*, i.e., $\hat{f}\hat{g} = (\hat{g}^\dagger \hat{f}^\dagger)^\dagger$. This together with the fact that $\{f, g\} = -i(\{f \circ g\} - \{g \circ f\})/\hbar$, as follows from Eqs. (1) and (2), is a strong heuristic motivation for the usual quantum theoretic postulates that identify certain quantum operator commutators $\hat{f}\hat{g} - \hat{g}\hat{f}$ with the corresponding Poisson bracket expressions $i\hbar\{f, g\}$. The factor of $i\hbar$ which is involved can be eliminated by identifying these commutators directly with the corresponding *antisymmetric* Poisson brackets $\{f, g\}_- \equiv \{f \circ g\} - \{g \circ f\}$. As natural counterparts to these one has the *symmetric* Poisson brackets $\{f, g\}_+ \equiv \{f \circ g\} + \{g \circ f\}$, which are the obvious "classical" candidates to correspond to certain quantum operator *anticommutators* $\hat{f}\hat{g} + \hat{g}\hat{f}$, such as those which enter into the quantum postulates for Fermion systems. Bearing in mind that

$$\{f, g\}_\pm = \sum_k \left(\frac{\partial f}{\partial a_k}\frac{\partial g}{\partial a_k^*} \pm \frac{\partial g}{\partial a_k}\frac{\partial f}{\partial a_k^*} \right), \tag{3}$$

we readily calculate the symmetric and antisymmetric Poisson brackets for the components of $\vec{a}$ and $\vec{a}^*$:

$$\{a_i, a_j\}_\pm = 0 = \{a_i^*, a_j^*\}_\pm, \quad \{a_i, a_j^*\}_\pm = \delta_{ij} = \pm\{a_j^*, a_i\}_\pm. \tag{4}$$

The quantum commutation and anticommutation relations which would algebraically correspond to Eqs. (4) are:

$$\hat{a}_i\hat{a}_j \pm \hat{a}_j\hat{a}_i = 0 = \hat{a}_i^\dagger\hat{a}_j^\dagger \pm \hat{a}_j^\dagger\hat{a}_i^\dagger, \quad \hat{a}_i\hat{a}_j^\dagger \pm \hat{a}_j^\dagger\hat{a}_i = \delta_{ij}\hat{I} = \pm(\hat{a}_j^\dagger\hat{a}_i \pm \hat{a}_i\hat{a}_j^\dagger). \tag{5}$$

When $\pm = -$, we recognize Eqs. (5) as the commutation relations of the ladder operators for independent quantum harmonic oscillators, while when $\pm = +$, we recognize Eqs. (5) as the anticommutation relations of the creation and annihilation operators for independent quantum Fermion system particle occupation states.

The canonical transformations of ordinary classical dynamics are mappings of the complex phase space vectors $\vec{a} \to \vec{A}(\vec{a}, \vec{a}^*)$ and $\vec{a}^* \to (\vec{A}(\vec{a}, \vec{a}^*))^*$ which preserve the *antisymmetric* Poisson bracket relations among the complex phase space vector components that are given by Eqs. (4) with $\pm = -$. In view of the algebraic correspondence with quantum Fermion systems established above, we may confidently define the the canonical transformations of Fermion system "classical" dynamics as those complex vector phase space mappings which preserve the *symmetric* Poisson bracket relations among the complex phase space vector components that are given by Eqs. (4) with $\pm = +$.

Specializing now to infinitesimal phase space transformations $\vec{a} \to \vec{A} = \vec{a} + \delta\vec{a}(\vec{a}, \vec{a}^*)$ in the manner of Guillemin and Sternberg [3], we readily calculate the antisymmetric and symmetric

Poisson brackets for the components of $\vec{A}$ and $\vec{A}^*$ to first order in $\delta\vec{a}$ and $\delta\vec{a}^*$ from Eq. (3):

$$\{A_i, A_j\}_\pm = \frac{\partial(\delta a_j)}{\partial a_i^*} \pm \frac{\partial(\delta a_i)}{\partial a_j^*}, \quad \{A_i^*, A_j^*\}_\pm = \frac{\partial(\delta a_i^*)}{\partial a_j} \pm \frac{\partial(\delta a_j^*)}{\partial a_i},$$

$$\{A_i, A_j^*\}_\pm = \delta_{ij} + \frac{\partial(\delta a_i)}{\partial a_j} + \frac{\partial(\delta a_j^*)}{\partial a_i^*} = \pm\{A_j^*, A_i\}_\pm. \tag{6}$$

If we now impose the requirement that this infinitesimal phase space vector transformation is *canonical* (i.e., that it preserves the antisymmetric or symmetric Poisson bracket relations among the complex phase space vector components given by Eqs. (4)), we obtain the three equations:

$$\frac{\partial(\delta a_j)}{\partial a_i^*} = \mp\frac{\partial(\delta a_i)}{\partial a_j^*}, \quad \frac{\partial(\delta a_j^*)}{\partial a_i} = \mp\frac{\partial(\delta a_i^*)}{\partial a_j}, \quad \frac{\partial(\delta a_i)}{\partial a_j} + \frac{\partial(\delta a_j^*)}{\partial a_i^*} = 0. \tag{7}$$

The last of Eqs. (7) is independent of the value of the $\mp$ symbol (i.e., of whether we deal with the infinitesimal canonical transformations of ordinary classical dynamics or those of Fermion system "classical" dynamics), and it is satisfied in particular for one-parameter infinitesimal $\delta\vec{a}$ which are of the form

$$\delta a_i = -\frac{i}{\hbar}(\delta\lambda)\frac{\partial G}{\partial a_i^*}, \tag{8}$$

where $\delta\lambda$ is a real-valued infinitesimal parameter and $G(\vec{a},\vec{a}^*)$ is a real-valued "generating function" whose dimension is that of action divided by the dimension of $\delta\lambda$. Because $\delta\lambda$ and $G(\vec{a},\vec{a}^*)$ are real, Eq. (8) implies that

$$\delta a_j^* = \frac{i}{\hbar}(\delta\lambda)\frac{\partial G}{\partial a_j}, \tag{9}$$

and we thus can readily verify that the last of Eqs. (7) is satisfied.

From Eq. (8) or Eq. (9) we obtain the form of the equation which governs any continuous one-parameter trajectory of sequential infinitesimal canonical transformations in the complex vector phase space:

$$i\hbar\frac{da_i}{d\lambda} = \frac{\partial G}{\partial a_i^*} \quad \text{or} \quad -i\hbar\frac{da_i^*}{d\lambda} = \frac{\partial G}{\partial a_i}. \tag{10}$$

In the most general circumstance, G may have an explicit dependence on λ, i.e., it may be of the form $G(\vec{a},\vec{a}^*,\lambda)$. Bearing in mind the relation $(\vec{q},\vec{p}) = (q_s(\vec{a}+\vec{a}^*), -i\hbar(\vec{a}-\vec{a}^*)/q_s)/\sqrt{2}$ between the complex and real phase space vectors, Eq. (10) may be rewritten as the pair of real equations:

$$\frac{dq_i}{d\lambda} = \frac{\partial G}{\partial p_i}, \quad \frac{dp_i}{d\lambda} = -\frac{\partial G}{\partial q_i}, \tag{11}$$

which are the familiar generalized Hamilton's equations [3] that govern continuous one-parameter trajectories of sequential infinitesimal canonical transformations in the real $(\vec{q},\vec{p})$ vector phase space.

For the case of ordinary classical dynamics (for which the value of $\mp = +$ in Eqs. (7)), the first two of Eqs. (7) are satisfied identically for the one-parameter infinitesimal $\delta\vec{a}$ of the form given by Eqs. (8) and (9). However, for the case of Fermion system "classical" dynamics (for which the value of $\mp = -$), the first two of Eqs. (7) impose the following constraint on the real-valued "generating functions" $G(\vec{a},\vec{a}^*,\lambda)$ of the continuous one-parameter canonical transformation trajectories:

$$\frac{\partial^2 G}{\partial a_i \partial a_j} = 0 = \frac{\partial^2 G}{\partial a_i^* \partial a_j^*}. \tag{12}$$

Thus the "generating functions" of the continuous one-parameter trajectories of sequential infinitesimal canonical transformations in Fermion system "classical" dynamics are constrained to be constant or linear in each of $\vec{a}$ and $\vec{a}^*$, as well as real-valued. The most general form for such a "classical" Fermion system "generating function" is therefore

$$G(\vec{a}, \vec{a}^*, \lambda) = G_0(\lambda) + \sum_k (g_k(\lambda) a_k^* + g_k^*(\lambda) a_k) + \sum_{lm} G_{lm}(\lambda) a_l^* a_m, \tag{13}$$

where $G_0(\lambda)$ is real and $G_{lm}(\lambda)$ is a Hermitian matrix. Upon putting this constrained form for G into Eq. (10) for the continuous one-parameter trajectory of sequential infinitesimal canonical transformations which G generates, we arrive at

$$i\hbar \frac{da_i}{d\lambda} = g_i(\lambda) + \sum_j G_{ij}(\lambda) a_j, \tag{14}$$

which is a (possibly) inhomogeneous linear equation of matrix Schrödinger form. (If the $g_i(\lambda) = 0$, this is a general homogeneous type of Schrödinger equation, whereas if the $g_i(\lambda) = \hbar\delta_{ik}\delta(\lambda - \lambda')$, it is a general propagator type of Schrödinger equation.) Thus the "classical" dynamics of Fermion systems must be linear and describable by a Schrödinger type of equation.

The generating functions of the continuous one-parameter canonical transformation trajectories are usually considered to be *observables* of classical theory when they have no explicit dependence on the parameter. Thus the most general "observable" of Fermion system "classical" dynamics must have the form of G in Eq. (13), but with G_0, g_k, and G_{lm} having no λ-dependence. However, when this "classical" Fermion theory is quantized by passing (with $\pm = +$) from the "symmetric" Poisson bracket relations of Eqs. (4) to the anticommutation relations of Eqs. (5), it often happens (particularly in local field theories) that the "inhomogeneous" $\sum_k (g_k a_k^* + g_k^* a_k)$ term of an "observable" G is not really, in fact, a bona fide observable. Even at the present "classical" level it is always possible to effectively suppress this "inhomogeneous" part of an "observable" if the Hermitian matrix G_{lm} is not singular. This is done by making the canonical transformation

$$a_i \to A_i = a_i + \sum_j \left(G^{-1}\right)_{ij} g_j. \tag{15}$$

It is easily verified that the transformed A_i of Eq. (15) also satisfy the "symmetric" Poisson bracket relations (with $\pm = +$) of Eqs. (4). In terms of these A_i, Eq. (14), specialized to "observables", becomes

$$i\hbar \frac{dA_i}{d\lambda} = \sum_j G_{ij} A_j, \tag{16}$$

which is of homogeneous Schrödinger equation form, while Eq. (13), specialized to "observables", becomes

$$G(\vec{A}, \vec{A}^*) = G_0 - \sum_{lm} \left(G^{-1}\right)_{lm} g_l^* g_m + \sum_{lm} G_{lm} A_l^* A_m, \tag{17}$$

which has no "inhomogeneous" term.

The Dirac equation, which is of Schrödinger type, can of course describe a "classical" Fermion system, but the Klein-Gordon and Maxwell equations, although they are linear, turn out not to be of Schrödinger type. For example, in one spatial dimension a discretized version of the Klein-Gordon equation is

$$\ddot{q}_i - (c/(2\Delta x))^2 (q_{i+2} - 2q_i + q_{i-2}) + (mc^2/\hbar)^2 q_i = 0. \tag{18}$$

This can be replaced by the first-order equation pair

$$\dot{q}_i = p_i, \quad \dot{p}_i = (c/(2\Delta x))^2(q_{i+2} - 2q_i + q_{i-2}) - (mc^2/\hbar)^2 q_i, \tag{19}$$

which is a version of Hamilton's equations for the particular Hamiltonian (time evolution generating function and observable)

$$H(\vec{q}, \vec{p}) = \frac{1}{2}\sum_k \left(p_k^2 + (c/(2\Delta x))^2(q_{k+1} - q_{k-1})^2 + (mc^2/\hbar)^2 q_k^2\right). \tag{20}$$

The constraint given by Eqs. (12) on Fermion system "classical" generating functions G in the complex vector phase space translates in the real $(\vec{q}, \vec{p})$ vector phase space into the two real-valued constraint equations:

$$q_s^2 \frac{\partial^2 G}{\partial q_i \partial q_j} = \left(\frac{\hbar}{q_s}\right)^2 \frac{\partial^2 G}{\partial p_i \partial p_j}, \quad \frac{\partial^2 G}{\partial q_i \partial p_j} = -\frac{\partial^2 G}{\partial q_j \partial p_i}, \tag{21}$$

where the scale factor q_s is real and nonzero. For the discretized Klein-Gordon Hamiltonian of Eq. (20) we have that

$$q_s^2 \frac{\partial^2 H}{\partial q_i \partial q_{i+2}} = -(cq_s/(2\Delta x))^2 \neq 0 \quad \text{and} \quad \frac{\partial^2 H}{\partial p_i \partial p_{i+2}} = 0, \tag{22}$$

which is *not* in accord with the constraint on "classical" Fermion system generating functions that is given by the first of Eqs. (21). Thus the Klein-Gordon equation is not of Schrödinger type and cannot describe a "classical" Fermion system.

It is quite clear as well that the old "four-Fermion" theory of beta decay is inherently nonlinear and thus cannot describe a "classical" Fermion system (there is no such objection under the Grassmann variable regime).

Acknowledgment

The authors wish to thank S.K. Dutt, E.H. Kerner and M.J. Morgan for discussions.

References

[1] R. Casalbuoni, Nuovo Cimento **33**A, 115 (1976).

[2] R. Casalbuoni, Nuovo Cimento **33**A, 389 (1976).

[3] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics* (Cambridge University Press, Cambridge, 1984).