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Brief Resume of Seiberg-Witten Theory

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Introduction

An outstanding problem in non-abelian gauge theory has been to make reliable predictions about the (non-perturbative) strong coupling region. (Another interesting problem has ben to find solvable models in more than two dimensions). In a recent paper [1] Seiberg and Witten (SW) have shown that the simplest non-trivial $N = 2$ supersymmetric theory provides at least a partial answer to these problems. First, they have shown that the local part of the effective Action is governed by a single analytic function F of a complex variable. Second they have made an Ansatz for the F that satisfies all the physical criteria and embodies electromagnetic duality, thus directly connecting the weak to the strong coupling regions. The correctness of their Ansatz is supported by some direct instanton computations [2]. The purpose of this note is to give a resume of the SW theory in the simplest possible mathematical terms.

1. N = 2 Supersymmetry.

We begin by recalling the essentials [3] of the $N = 2$ supersymmetry algebra and its Action. The algebra is

$$\{Q^i_\alpha, \bar{Q}^k_\beta\} = \delta_{ik}\sigma^\mu_{\alpha\beta}P_\mu \qquad \{Q^i_\alpha, Q^k_\beta\} = \epsilon_{ik}\epsilon_{\alpha\beta}Z \tag{1.1}$$

plus the hermitian conjugate of the second relation, where $i, k = 1, 2$ and Z is a central charge. This algebra is realized on the simplest possible non-trivial supermultiplet, namely

$$\Psi \supset \{\phi, \psi, A_\mu; F, D\} \tag{1.2}$$

where ϕ is a complex scalar field, ψ is a Dirac spinor A_μ is a gauge-field and F and D are complex and realdummy-fields respectively. This $N = 2$ superfield actually consists of two $N = 1$ superfields, namely

$$\Phi \supset \{\phi, q, F\} \quad \text{and} \quad V \supset \{A_\mu, f, D\} \quad \text{or} \quad W_\alpha \supset \{F_{\mu\nu}, f, D\} \tag{1.3}$$

where ϕ and V/W_α are chiral and vector multiplets respectively, the q

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and f fields being Weyl spinors of opposite chirality. Since A_μ belongs to the adjoint representation of the gauge group G and all the fields belong to the same multiplet they must all belong to the adjoint representation of G. The simplest SW model is for $G = SU(2)$ and we shall concentrate on this case.

2. N = 2 Super-Action.

The superaction for the $N = 2$ superfield just described is

$$\mathcal{A} = \text{Im Tr} \int d^4x d^2\theta_\alpha d^2\bar{\theta}_\beta \left(\Psi\right)^2 \tag{2.1}$$

On expanding this in terms of the $N = 1$ superfields it becomes

$$\mathcal{A} = \text{Im} \int d^4x d^2\theta_\alpha d^2\bar{\theta}_\beta \left(\bar{A}e^{-2g_oV}A\right) + \tau_o \int d^4x d^2\theta_\alpha \left(W_\alpha W_\alpha\right) \tag{2.2}$$

where

$$\tau_o = \frac{\theta_o}{2\pi} + \frac{4\pi i}{g_o^2} \tag{2.3}$$

the parameter g_o being the usual gauge-coupling constant and θ_o being the QCD-vacuum-angle (not to be confused with the usual supersymmetric Grassman variables). The exponential in the first term is just the supersymmetric generalization of the covariant derivative. Expanding (2.2) further in terms of conventional fields we obtain

$$\begin{aligned}\mathcal{A} = \text{tr} \int d^4x \Big\{ \frac{1}{2}(\phi^\dagger D^2\phi) + \bar{\psi}D\psi + g_o(\phi[\bar{\psi}, \gamma_5\psi]) + g_o^2[\phi^\dagger, \phi]^2 \Big\} \\ + \text{tr} \int d^4x \Big\{ \frac{1}{4g_o^2}F^{\mu\nu}F_{\mu\nu} + \frac{\theta_o}{32}\tilde{F}^{\mu\nu}F_{\mu\nu} \Big\}\end{aligned} \tag{3.4}$$

This Action will be immediately recognized as the standard action for a Quark-Gluon-Higgs system in which all the fields are in the adjoint representation and the coupling constants are reduced to g and θ by the supersymmetry. Thus it is not very exotic. Indeed it could be the QCD Action except for the fact that the quarks are in the adjoint and presence of the scalar field.

4. Text-Book Properties

The Action (3.4) is actually so normal that it embodies all the properties of Quantum gauge Theory that have surfaced over the past thirty years and could even be used as a model to teach quantum gauge theory. It might be worthwhile to list these properties:

1. It contains a gauge-field coupled to matter
2. It is asymptotically free
3. It is scale-invariant, but with a scale-anomaly

4. It has spontaneous symmetry breaking
5. It has central charges (Z and $\bar{Z}$)
6. It admits both instantons and monoples

Because of the supersymmetry it has some further special properties, whose significance will become clear later, namely,

7. It not only has a Montonon-Olive mass formula [4] for gauge-fields and monopoles but generalizes that formula

$$\text{from} \quad M = |v|(N_e + \frac{1}{g^2} n_m) \quad \text{to} \quad M = |Z| \quad \text{where} \quad Z = (a n_e + a_d n_m) \tag{4.1}$$

where n_e and n_m denote the gauge-field and monople charges respectively, and the coefficients a and a_d will be explained later.

8. It is symmetric with respect to a Z_4 symmetry which is the relic of the R-symmetry ($\theta_\alpha \to e^{i\epsilon}\theta_\alpha$) that survives the axial anomaly breakdown.
9. It has a holomorphic structure
10. It has a duality that connects the weak and strong coupling regimes
12. The duality generalizes to an $SL(2, Z)$ symmetry. In section 6 we explain these last three concepts in a little more detail.

5. Spontaneous Symmetry-Breaking

For $SU(2)$ this concept is very simple. From the form of the Higgs potential in (3.4) we see at once that there is a Higgs vacuum for $\phi = v\sigma$ where v is any complex number and σ is any fixed generator of $SU(2)$. Furthermore, for $v \neq 0$ this breaks the gauge-symmetry from $SU(2)$ to $U(1)$. For other gauge-groups G the corresponding statement is that $\mathbf{v}$ must lie in the Cartan subalgebra of G. On the other hand there is no spontaneous breakdown of supersymmetry. Thus the full breakdown is

$$SU(2) \quad \to \quad U(1) \qquad N = 2 \quad \text{supersymmetry unbroken} \tag{5.1}$$

Indeed it is the fact that the supersymmetry is unbroken that gives the model its nice properties, since otherwise the classical properties would not be preserved after quantization.

After the spontaneous breakdown the restriction of the $N = 1$ form of the classical Action (2.2) to the massless $U(1)$ fields takes the form

$$\mathcal{A} = \text{Im} \int d^4x d^2\theta_\alpha d^2\bar{\theta}_\beta \left(\bar{A}A\right) + \tau_o \text{Im} \int d^4x d^2\theta_\alpha \left(W_\alpha W_\alpha\right) \tag{5.2}$$

Since the adjoint representation of $U(1)$ is trivial this Action is a free-field one. However, in the quantum theory this does not mean that the effective Lagrangian is also free because, through the quantum fluctuations, the massive fields induce interaction term for the massless ones. The first great virtue of the SW model is that these interactions have a very specific form. In fact they show that, due to the $N = 2$ supersymmetry the *local* part of the effective Lagrangian can only be of the form

$$\mathcal{A} = \frac{1}{2}\int d^4x d^2\theta d^2\bar{\theta}\Big(\bar{A}A_d - \bar{A}_d A\Big) + \mathrm{Im}\int d^4x d^2\theta(\tau(A))\Big(W_\alpha W_\alpha\Big) \qquad (5.3)$$

where

$$A_d = F'(A) \quad \text{and} \quad \tau(A) = F''(A) \qquad (5.4)$$

for some function $F(A)$. Thus the effective Lagrangian is completely governed by the single function $F(A)$. Note that (5.3) is very similar to the classical Action which is the special case for which $F(A) = \frac{1}{2}\tau_o A^2$. As we shall see, the SW solution is actually a special Ansatz for the functional form of $F(A)$.

6. Holomorphy and Duality

It is now easy to quantify what is meant by holomorphy and duality. Holomorphy is simply the statement that $F(A)$ depends only on A and not on $\bar{A}$. Duality means that the physics described by the effective Action (5.3) is invariant with respect to the duality transformation.

$$\begin{pmatrix} A \\ A_d \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} A \\ A_d \end{pmatrix} \qquad D_\alpha W_\alpha \to D_{\dot{\alpha}} W_{\dot{\alpha}} \qquad \tau(A) \to (\tau(A))^{-1} \qquad (6.1)$$

Note that the duality transformation is closely linked to the Legendre transform of $F(A)$ with respect A. By noting that in the free classical theory with $\theta_o = 0$ the transformation (6.1) reduces to

$$\vec{E} \to \vec{B} \quad \text{and} \quad g \to \frac{1}{g} \qquad (6.2)$$

we see that it is just the generalization of well-known Maxwell-Dirac (MD) duality. Thus the Action (5.3) not only generalizes MD duality but puts it into a genuine dynamical model. Furthermore, the duality generalizes to

$$\begin{pmatrix} A \\ A_d \end{pmatrix} \to \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} A \\ A_d \end{pmatrix} \quad \text{and} \quad \tau(A) \to \frac{p\tau(A) + q}{r\tau(A) + s} \qquad (6.3)$$

where the matrix with entries (p, q, r, s) is in $SL(2, Z)$. The integer-valuedness of the transformation follows from the requirement that, in the perturbation theory at least, it should change the θ angle only by multiples of 2π and should leave the mass-formula (4.1) form-invariant.

7. Perturbative and Non-Perturbative F(A)

Before going on to describe the S-W Ansatz for F(A) we consider the perturbative contribution $F_p(A)$ of $F(A)$. This turns out to be

$$F_p(A) = \hbar A^2 \ln\Big(A^2/\Lambda^2\Big) \qquad (7.1)$$

where Λ is the renormalization scale. This is evidently a classical plus a one-loop expression but it is correct to all orders in perturbation because [3] the energy momentum tensor is in the same multiplet as the axial current

$$\delta\Theta_{\mu\nu} = -\frac{\bar{\epsilon}}{4}(\sigma^{\mu\kappa}\partial_\kappa j^\mu + \sigma^{\nu\kappa}\partial_\kappa j^\mu)$$
$$\delta j^5_\mu = i\bar{\epsilon}\gamma_5 j_\mu \qquad \delta j^\alpha_\mu = \epsilon\Big(2\gamma^\nu\Theta_{\mu\nu} - i\gamma_5\gamma^\nu\partial_\nu j^5_\mu + \frac{i}{2}\epsilon_{\mu\nu\kappa\lambda}\gamma^\nu\partial^\kappa j^\lambda_5\Big) \tag{7.2}$$

and in $N = 2$ supersymmetry this situation is protected to all orders in perturbation. It follows that the quantum correction to the energy momentum tensor are similar to the axial anomaly, for which it is well-known that the one-loop result is exact to all orders.

For the non-perturbative part of F(A) the only solid a priori pieces of information are:

$$\mathrm{Im}(F''(A)) \geq 0 \qquad F_{np}(A) \neq 0 \qquad F_{np}(iA) = F_{np}(A) \tag{7.3}$$

and the fact that it vanishes for large A. The first relation from the convexity of the effective potential, specifically from the fact that $\mathrm{Im}(F''(A))$ is the coefficient of the kinetic term for the gauge-field, the second relation from 1-instanton computations and the third relation from the residue of R-invariance that is left after spontaneous symmetry-breaking. F(A) has to be guessed from this apparently meagre information.

The SW-Ansatz

8. Preliminaries

SW begin by reducing the problem to one in complex analysis by considering only the vacuum value $A = v$ of the chiral scalar superfield and determining the functional form of F(v). Afterwards, (A) can be recovered by the simple substitution $F(v) \to F(v + \tilde{A}) \equiv F(A)$. This is analogous to the substitution

$$V(m, f, g) \quad \to \quad V(m + f\phi + g\phi^2, f + g\phi, g) = V_{eff}(\phi) \tag{8.1}$$

which is made to obtain the effective potential from the partition function P(m,f,g) of a standard renormalizable theory with a single scar field ϕ with masses m, and coupling constants f and g.
Next, they note that asymptotic freedom allows them to identify the perturbative region as the large scale one $v \to \infty$ and thus

$$\tau(v) \to \frac{i}{\pi}\ln(v) \quad \text{for} \quad v \to \infty \tag{8.2}$$

Since v is expected to be singular in the small scale (strong coupling) region one also postulates the existence of a universal (complex) parameter $u \in C$ normalized so that $a(u) \to u^2$ for $u \to \infty$. Assembling all this information they reduce the problem to the search for a function $\tau(u)$ such that

$$\mathrm{Im}(\tau(u)) \geq 0 \quad \text{and} \quad \tau(u)_{u\to\infty} \to \frac{i}{2\pi}\ln(u) \tag{8.3}$$

The procedure for choosing a $\tau(u)$ to satisfy (8.3) is actually rather similar that used by Veneziano in choosing his formula for the S-matrix S(s,t,u), where s,t and u are the invariant squares of the momenta. That is to say, instead of computing the function directly from the underlying theory one uses its properties (symmetries, boundary conditions etc.), to try to guess what it should be. Indeed duality (actually triality) plays here a role which is analogous to that played by crossing symmetry (symmetry in s,t and u) in the Veneziano case. But first one has to decide the general class of functions out of which the function $\tau(u)$ should be chosen. The standard class of functions which map the upper part C_+ of the complex plane into itself modulo subgroups of $SL(2,Z)$ is the class of *Fuchsian* functions [5] and the choice will be made out of these. So, to put the results in perspective, we digress for a moment to consider Fuchsiam functions or maps.

9. Fuchsian Maps

The Fuchsian maps $\tau(u)$ are maps from C_+ to $D = C_+/G$ where G is a subgroup of $SL(2,R)$ (restricted in our case to $SL(2,Z)$. Thus they map C_+ into fundamental domains $D \in C_+$ whose G-equivalent copies fill out C_+. The domains D are circular polygons (ones whose sides may be straight lines or circles) and in our case will stretch out to infinity. In general the polygons may not be of genus zero because the sides may be have to be identified in a non-trivial manner. The corners of the polygon correspond to points on the real axis in the u-variable and if the genus of the polygon is zero u can be extended to cover the whole complex plane, which compactifies to the Riemann sphere. Otherwise the compactification of the u-space leads to a Riemann surface of higher genus. The essential point is that these maps guarantee that $\mathrm{Im}(\tau(u)) \geq 0$.

9. The Schwarzian Derivatives

The Fuchsian functions $\tau(u)$ are apt to be complicated but a great simplification is achieved by considering not the functions themselves but their Schwarzian derivatives

$$S(\tau) = \frac{\tau'''}{\tau'} - \frac{3}{2}\left(\frac{\tau''}{\tau'}\right)^2 \tag{9.1}$$

In general the main property of Schwarzian derivatives is that they are invariant with respect to the modular transformations (6.1). However, in the case of Fuchsian functions they have the added advantage that they are simple meromorphic functions of the form

$$S(\tau(u)) = \sum_{i=1}^{i=n}\left\{\frac{1}{2}\frac{(1-\alpha_i^2)}{(u-a_i)^2} + \frac{\beta_i}{(u-a_i)}\right\} \qquad S(\tau(u)) \to \frac{1}{u^2} \quad u \to \infty \tag{9.2}$$

and this is why it is convenient to use $S(\tau)$ rather than τ itself. One $S(\tau)$ is known there is a simple and elegant way to recover τ from it, namely to write

$$\tau = \frac{y_1}{y_2} \quad \text{where} \quad y'' + \frac{1}{2}S(\tau(u))y = 0 \tag{9.3}$$

Such functions τ are tailor-made for the S-W model where $\tau(u) = a'_d/a'$. All but the β-parameters in (9.2) have a simple geometrical meaning. The number n is the number of corners on the polygon D (excluding the point at infinity) and the parameters a_i and α_i are the locations of the corners and the internal angles of the polygon respectively. Because of the freedom of choosing the axes in u-space the number of independent parameters in $Q(u)$ is $3n-2$. The boundary condition puts two further restrictions on the on the β's and this reduces the number to $3n-4$. This already shows that $n \geq 2$. It is clear that the real choice in choosing a Fuchsian function is to choose n and then choose the $3n-4$ parameters in the meromorphic function Q(u).

10. S-W Choice

The question is: which Fuchsian function to choose? The S-W choice is made by adding two further inputs to the basic conditions, namely

1. Minimality: $\quad n = 2$
2. Duality: $\quad M_\infty = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \leftrightarrow \quad M_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Duality in this sense means that the monodromy matrix M_1 at one of the singularities is the dual of the monodromy matrix M_∞ at infinity. Physically, it means that the asymptotic freedom of g for large scales is the dual of the infra-red slavery of g (asymptotic freedom of g^{-1}) for low scales. It turns out that M_∞ and M_1 generate a monodromy group Γ_2 which includes the monodromy group of the second singularity, and consists of all matrices of the form

$$I + 2\begin{pmatrix} e & f \\ g & h \end{pmatrix} \qquad e, f, g, h \in Z \tag{10.1}$$

According to our previous analysis this determines a unique Fuchsian map (modulo the positions of the two singularities which are normalized to be $a_i = \pm 1$) and it turns out to be the map with $\alpha_i = 0$ and $\beta_i = \pm 1/4$. Thus with SW Ansatz equation (9.3) becomes

$$\tau(u) = \frac{y_1}{y_2} \quad \text{where} \quad y'' + \frac{1}{4}\left[\frac{3+u^2}{(u^2-1)^2}\right] y = 0 \tag{10.2}$$

There is actually a simplification in this case on changing to the variable a, where $a' = y$ because then (10.2) becomes

$$\tau = \frac{a'_1}{a'_2} \quad \text{where} \quad a'' + \frac{1}{4}\left(\frac{1}{u^2-1}\right) a = 0 \tag{10.3}$$

and the differential equation in (10.3) is just a hypergeometric equation. So finally $\tau(u)$ is simply the ratio of the derivatives of two simple hypergeometric functions. In fact the function $\tau^{-1}(u)$ is a well-known automorphic function called the elliptic modular function. Thus when all the smoke has cleared away it turns out that the SW Ansatz is to propose that $F(A)$ is such that $F''(A)$ is the inverse of the elliptic modular function!

11. Uniqueness

The S-W Ansatz is certainly not the only Ansatz compatible with the basic conditions

$$\mathrm{Im}(\tau(u)) \geq 0 \qquad \tau(u) \rightarrow \frac{i}{2\pi}\ln(u) \tag{11.1}$$

Indeed, even for two singularities, the Schwarzian could be

$$S(\tau(u)) = \frac{(1-\alpha^2)}{2(u-1)^2} + \frac{(1-\gamma^2)}{(u+1)^2} - \frac{1}{2}\frac{(2-\alpha^2-\gamma^2)}{(1-u^2)} \tag{11.2}$$

where $\alpha, \gamma = 0, \frac{1}{3}, \frac{1}{2}$, with monodromy groups called Γ_2, G_2 and G_θ. The solution with G_θ is ruled out because it is not R-invariant but G_2 remains as a reflexion-invariant alternative. For more than two singularities there are many more possibilities. For example a three-singularity reflexion-invariant solution with monodromy group $SL(2,Z)/Z_2$ is provided by $\tau(u) = J(u^2)$, where J is the standard modular function.

What distinguishes the S-B solution is that it carries the dual symmetry that was used by S-W as input. The point is that Γ_2 is an *invariant* subgroup of SL(2,Z) and

$$\mathrm{SL(2,Z)}/\Gamma_2 = P_3 \tag{11.3}$$

where P_3 is the permutation group of order 3. Mathematically the permutation group P_3 interchanges the point at infinity and the two singularities and physically it interchanges gauge-fields, monopoles and dyons. The original duality input emerges as the symmetry between the the gauge-field at infinity and the monople at one of the singularities.

12. Correctness

Since the S-W Ansatz is not unique one has to check whether it is, in fact the correct choice. In principle this can be done by making direct computations of the non-pertubative part of the Action using instanton computations. In practice this has been done [2] only for arbitrary gauge-groups in the 1-instanton configurations and for $SU(2)$ in the 2-instanton configurations and in these cases the SW Ansatz agrees with direct computations.

The general idea of these computations is as follows: For any Fuchsian function satisfying the boundary conditions and R-invariance we have the asymptotic expansion

$$F(v) = v^2 + \hbar v^2 \ln\Big(\frac{v}{\Lambda}\Big)^2 + v^2 \sum c_m \Big(\frac{\Lambda}{v}\Big)^{4m} \tag{12.1}$$

In the direct instanton computations $F(v)$ is supposed to be the partition function and the contributions of the various powers in m are supposed to come from the instanton sectors of topological charge m. The idea, therefore, is to compute the partition function in an m-instanton background under the assumption that the scalar field has a non-zero value v on the sphere at infinity. What one does in practice is to first choose a background for the gauge and fermion fields by the conditions

$$F_{\mu\nu} = F^*_{\mu\nu} \qquad \gamma^\mu D_\mu \psi = 0 \tag{12.2}$$

where F^* denotes the dual of F. These fields are parametrized by the $8m$ ADHM parameters ρ, ν for self-dual gauge-fields, where ρ. in particular, denotes the size of the instanton, and the 8m parameters $\eta, \bar{\eta}$ for their zero-mode fermion fields. One then postulates that the scalar field is given by

$$D^2\phi = [\bar{\psi}, \psi] \qquad \phi(\infty) = v \tag{12.3}$$

and is thus a functional of these $16n$ parameters. Finally one postulates that long-range part of the partition function comes only from the surface term $\mathrm{tr}(\int \phi^\dagger D_r \phi)$ in the Action (3.4) and that the short range part drops out because the bosonic and fermionic contributions cancel on account of supersymmetry. In that case the partition function evidently takes the form

$$P(v) = \Lambda^{4n} e^{\frac{8n\pi^2}{g^2}} \int d\rho d(\nu) \rho^{4n-3} d\rho d\nu d\bar{\eta} d\eta e^{\int d\vec{\Omega}.(\phi, \vec{D}\phi)} \tag{12.4}$$

For dimensional reasons these integrals are of the form

$$P(v) = \Lambda^{4m} e^{\frac{8m\pi^2}{g^2}} \int \rho^{4n-3} d\rho d(\nu) e^{-(a\rho)^2 f(\nu)} = a^2 \left(\frac{\Lambda}{v}\right)^{4m} e^{\frac{8m\pi^2}{g^2}} e_m \tag{12.5}$$

where the e_m's are dimensionless constants. It is these constants that are to be compared with the SW constants c_m and, as mentioned earlier, the coefficients computed up to now are in agreement. So the instanton computations provide reasonably strong support for the correctness of the SW Ansatz. As a check on the sensitivity of the result we have computed c_1 for the alternative $n = 2$ Ansatz and it is different from the SW one.
It must be admitted that the validity of the instanton computations as described above is not quite clear, because the equations (12.2) and (12.3) are regarded only as approximations and the background configurations described by them are neither solutions of the classical field equations nor supersymmetric-invariant. However, we hope to present a more convincing argument for the validity of the instanton computations in a later paper.

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