DIAS-STP-96-23

A Model of Continuous Polymers with Random Charges

E. Buffet[1,3] and J.V. Pulé[2,3]

[1]*School of Mathematical Sciences, Dublin City University, Dublin 9, Ireland.*
[2]*Mathematical Physics Department, University College, Belfield, Dublin 4, Ireland.*
[3]*School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland.*

Abstract

We study a model of polymers with random charges; the possible shapes of the polymer are represented by the sample paths of a Brownian motion, and the cumulative charge distribution along a polymer is modelled by a realisation of a Brownian bridge. Charges interact through a general positive-definite two-body potential. We study the infinite volume free energy density for a fixed realisation of the Brownian motion; this is not self-averaging, but shows on the contrary a sample dependence through the local time of the Brownian motion. We obtain an explicit series representation for the free energy density; this has a finite radius of convergence, but the free energy is nevertheless analytic in the inverse temperature in the physical domain.

Key Words

Polymers, random charge distribution, Brownian motion, Brownian local time.

PACS numbers: 36.20.E, 05.20, 05.40.

1 Introduction

The problem of describing the thermodynamic properties of polymers in random environments has received much attention in the theoretical physics literature [1,2,3,4,5,6,7]. Experimental evidence suggests that for a wide class of bio-polymers (such as proteins) there is no sample-to-sample regularity in the distribution of electric charges along the polymer; it is thus natural to view these charges as random variables. Taking the possible shapes of the polymer to be the sample paths of some stochastic process, one can then construct a precise mathematical model by specifying the interaction between the charges.

Various authors have studied specific models in the above class: Kantor and Kardar [3] consider a one-dimensional model with the following characteristics: the configurations of a finite polymer are described by the paths of a simple random walk $w_j, j = 1, \ldots, N$. The local charges along the polymer $q_j, j = 1, \ldots, N$ are taken to be realisations of N independent random variables obtaining the values ± 1 with equal probability. Finally the interaction has zero range, resulting in the Hamiltonian

$$H_N(w,q) = \sum_{1 \le i < j \le N} q_i q_j \delta_{w_i, w_j}. \tag{1}$$

The programme consists in studying the thermodynamic properties of this model for a fixed (but arbitrary) realisation of the charges, for instance by calculating the partition function as the conditional expectation

$$Z_n(\beta) = \mathbf{E}[e^{-\beta H_N(w,q)} | q].$$

The main questions are:

- does the limiting free energy $f(\beta) = \lim_{N \to \infty} \frac{1}{N} \log Z_N(\beta)$ depend on the realisation of the charge q?
- does the model show a phase transition from a collapsed state to an extended state?

However, it appears that this programme is very difficult to carry out, even for the simple Hamiltonian (1); hence a number of variations on the model have been considered by several authors: in [4] the charges are regarded as Gaussian random variables; more drastically, in [6] Derrida and Higgs choose for the polymer configurations those of a **directed** simple random walk (meaning that its increments take values $0, 1$ instead of ± 1). This last assumption restricts severely the self-overlapping structure of the polymer and results in a tractable problem, at least as far as the ground state of the model is concerned [6]. Finally, in [7] Martinez and Petritis introduce two modifications to the Derrida-Higgs model: first, the charge distribution is modelled by a Brownian bridge; more importantly, the programme outlined above is modified to the extent that the partition function is defined as a conditional expectation over the charge configuration for a fixed polymer configuration. The resulting limiting free energy is shown to be independent of the polymer configuration, and no phase transition occurs.

In this article, we elaborate on [7] in the following way: we describe the polymer configurations by the paths of a Brownian motion (instead of a directed random walk) and we allow the interaction between charges to be a general integrable two-body potential. Surprisingly enough, we can obtain a fairly explicit representation of the free energy in terms of a series involving the local time of Brownian motion; in particular, the free energy is **not** sample-independent ("self-averaging"). This series converges only for a finite range of temperatures; however, we show that the free energy is an analytic function of the inverse temperature β for all $\beta \ge 0$.

2 The model

The three ingredients of our model are

(i) **a standard Brownian motion** $B_t, 0 \le t \le 1$; the polymer configurations at volume b are described by bB_t; it may be more usual to take the infinite volume limit by allowing the parameter t to run in $[0,a]$ and letting $a \to \infty$, but for Brownian motion the two view points are equivalent since in $bB_t = B_{tb^2}$ (in distribution);

(ii) **a standard Brownian bridge** $\alpha_t, 0 \le t \le 1$; the cumulative charge along the polymer is described by $\sqrt{b}\alpha_t$. In other words, $\sqrt{b}\alpha_a$ is the total charge carried by the portion of polymer parametrised by $0 \le t \le a$. The Brownian bridge boundary condition $\alpha_1 = 0$ ensures global neutrality. The idea of using a Brownian bridge to handle a charge constraint is borrowed from [7];

(iii) a two-body potential $h(x)$ specifying the interaction energy between two unit charges located x apart of each other. We will leave h fairly general, except for the following assumptions: h is even, bounded, integrable, finite at zero and positive-definite. It follows that h can be written as a convolution: for some square integrable function g,

$$h(x) = (g * g)(x) = \int_{-\infty}^{\infty} g(x-y)g(y)dy.$$

Our Hamiltonian is then

$$H_b = \frac{b}{2}\int_0^1\int_0^1 h(b(B_t - B_s))d\alpha_s d\alpha_t. \tag{2}$$

Because of the convolution property of h, this can be rewritten as

$$H_b = \frac{b}{2}\int_{-\infty}^{\infty} dx \left(\int_0^1 g(x - bB_t)d\alpha_t\right)^2. \tag{3}$$

But note that, because of the elementary properties of the Brownian bridge α_t (see [8]), the inner integrand,

$$J_b(x) = \int_0^1 g(x - bB_t)d\alpha_t \tag{4}$$

is for any fixed sample path $B_t(\omega)$ a Gaussian process (indexed by x) with zero mean and covariance

$$C_b(x,y) = \mathbf{E}[J_b(x)J_b(y)|B.]$$

$$= \int_0^1 g(x-bB_t)g(y-bB_t)dt - \left(\int_0^1 g(x-bB_t)dt\right)\left(\int_0^1 g(y-bB_t)dt\right). \tag{5}$$

Thus H_b is a superposition of squares of Gaussian random variables and the partition function is

$$Z_b(\beta) = \mathbb{E}[e^{-\beta H_b}|B.] = (\det(1+\beta b C_b))^{-\frac{1}{2}} \tag{6}$$

where C_b is the integral operator on $L^2(\mathbb{R})$ with kernel (5). Note that C_b is a trace-class operator so that the above determinant is well defined and can be evaluated as

$$\det(1+\beta bC_b) = \exp(Tr\log(1+\beta bC_b)).$$

Consequently, the free energy density is

$$\frac{1}{b}\log Z_b(\beta) = -\frac{1}{2b}Tr\log(1+\beta bC_b). \tag{7}$$

This formula is the basis of our study.

3 A local time representation of the free energy

Our first step consists in proving that when computing the limit of the free energy (7) as $b \to \infty$, one can replace the covariance C_b by the simpler form

$$K_b(x,y) = \int_0^1 g(x - bB_t)g(y - B_t)dt. \tag{8}$$

The main reason for this is the fact that K_b is a perturbation of C_b by a rank-one operator; indeed, define (as always for a fixed realisation $B_t(\omega)$ of the Brownian motion

$$j_b(x) = \int_0^1 g(x - bB_t)dt. \tag{9}$$

Then the kernels (5) and (8) obey

$$K_b(x,y) - C_b(x,y) = j_b(x)j_b(y) \tag{10}$$

so that the corresponding operators are related by

$$K_b - C_b = ||j_b||^2 P \tag{11}$$

where P is the orthogonal projection onto the normalised function $j_b(x)/||j_b||$. Use now the concavity of $x \to \log(1+\beta x)$ to write for every $x < y$:

$$0 \leq \log(1+\beta y) - \log(1+\beta x) \leq \beta \frac{y-x}{1+\beta x} \tag{12}$$

and thus

$$0 \leq Tr[\log(1+\beta b K_b) - \log(1+\beta b C_b)] \leq \beta b Tr[(K_b - C_b)(1+\beta b C_b)^{-1}]. \tag{13}$$

Using (11), the righthand side of (13) is

$$\beta b ||j_b||^2 Tr(P(1+\beta b C_b)^{-1}). \tag{14}$$

Evaluating the trace on an orthonormal basis having $j(x)/||j||$ as its first element, (14) becomes

$$\beta b (j_b, (1+\beta b C_b)^{-1} j_b) \leq \beta b ||j_b||^2. \tag{15}$$

In the last step, the fact that the covariance operator C_b is positive definite has been used to infer that $(1+\beta b C_b)^{-1} \leq 1$. So we have just proved:

Lemma 1 *For every realisation of the Brownian motion B_t,*

$$0 \leq \frac{1}{b} Tr[\log(1+\beta b K_b) - \log(1+\beta b C_b)] \leq \beta ||j_b||^2. \tag{16}$$

It remains to prove that $||j_b||^2$ tends to zero as $b \to \infty$. We will in fact prove a much more precise result using the concept of **Brownian local time** [8]: $L(t,x)$, is a doubly indexed family of random variables characterised by the following property: let $l : \mathbb{R} \to \mathbb{R}$ be a Borel function; then for almost all sample paths of Brownian motion,

$$\int_0^t l(B_s)dx = \int_{-\infty}^{\infty} l(x)L(t,x)dx. \tag{17}$$

The intuitive interpretation is that $L(t,x)dx$ represents the total time spent in $(x-dx, x+dx)$ by Brownian motion up to instant t. In the sequel we will make use only of the local time up to instant 1, so we simplify the notation by putting $L(1,x) = L(x)$.

Lemma 2 *For almost every sample path of Brownian motion,*

$$\lim_{b\to\infty} b\|j_b\|^2 = \left(\int_{-\infty}^{\infty} h(x)dx\right)\left(\int_{-\infty}^{\infty} L^2(x)dx\right).$$

Proof

$$b\|j_b\|^2 = b\int_{-\infty}^{\infty} j_b^2(x)dx = b\int_{-\infty}^{\infty}\left(\int_0^1 g(x-bB_t)dt\right)\left(\int_0^1 g(x-bB_s)ds\right)dx$$

$$= b\int_0^1\int_0^1 h(b(B_s-B_t))dsdt.$$

We use now a generalisation of (17), see [8] to write the above as

$$b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(b(x-y))L(x)L(y)dxdy.$$

Use the change of variables $b(x-y)=z$ to obtain:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(z)L(x)L(x-z/b)dxdz. \tag{18}$$

A fundamental property of local time is that it is continuous; hence the integrand tends to $h(z)L^2(x)$ as $b\to\infty$. Moreover it is known that

$$L_* = \sup_{x\in\mathbb{R}} L(x) \tag{19}$$

is almost surely finite [8]. Hence (18) is bounded by:

$$L_*\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(z)L(x)dxdz = L_*\left(\int_{-\infty}^{\infty} h(z)dz\right)\left(\int_{-\infty}^{\infty} L(x)dx\right) = L_*\int_{-\infty}^{\infty} h(z)dz. \tag{20}$$

The result follows by dominated convergence.

□

A trivial consequence of Lemma 2 is that $\|j_b\|$ tends to 0 as $b\to\infty$. Combining the two lemmas, we have:

Proposition 1 *For almost every realisation of Brownian motion,*

$$\lim_{b\to\infty}\frac{1}{b}Tr[\log(1+\beta bK_b)-\log(1+\beta bC_b)] = 0.$$

Hence, if the limiting free energy exists for any one of the two covariances K_b, C_b it does for the other one as well, and the two limits are the same.

So we can now substitute K_b for C_b in (7) and consider

$$f_b(\beta) = -\frac{1}{2b}Tr\log(1+\beta bK_b) = -\frac{1}{2b}\sum_{j=0}^{\infty}\log(1+\beta a_j^{(b)}) \tag{21}$$

where $0 \leq a_0^{(b)} \leq a_1^{(b)} \leq a_2^{(b)} \leq \ldots$ are the eigenvalues (repeated according to their multiplicity) of the compact self-adjoint symmetric operator bK_b on $L^2(\mathbb{R})$. Note that for each fixed j the series

$$f_b^{(j)}(\beta) = \log(1+\beta a_j^{(b)}) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (a_j^{(b)})^n \beta^n \tag{22}$$

converges in the region $|\beta| < (a_j^{(b)})^{-1}$. Hence for fixed b, all the series (22) have a common region of convergence $|\beta| < (\sup_j a_j^{(b)})^{-1} = \|bK_b\|^{-1}$. Moreover the convergence of $\sum_{j=1}^{\infty} f_b^{(j)}(\beta)$ is uniform in β within $|\beta| < r < \|bK_b\|^{-1}$ by Weierstrass' M-test because of the bound $|\log(1+\beta a_j^{(b)})| \leq C\beta a_j^{(b)} \leq Cra_j^{(b)}$ and of the fact that bK_b has a finite trace (see (24)). Hence we can change the order of summation to rewrite (21) as

$$f_b(\beta) = \frac{1}{2b}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\beta^n \sum_{j=0}^{\infty}(a_j^{(b)})^n = \frac{1}{2b}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\beta^n b^n Tr(K_b^n). \tag{23}$$

We analyse now the behaviour of (23) as $b \to \infty$.

Lemma 3 *For almost every sample path of the Brownian motion*

$$\lim_{b\to\infty} b^{n-1} Tr(K_b^n) = h^{*n}(0) \int_{-\infty}^{\infty} L^n(x)dx$$

$$n = 1, 2, \ldots,$$

where h^{*n} is the n-fold convolution of h with itself (with the convention $h^{*1} = h$).

Proof

The result is trivial when $n = 1$:

$$TrK_b = \int_{-\infty}^{\infty} K_b(x,x)dx = \int_{-\infty}^{\infty}\left(\int_0^1 g(x-bB_t)^2 dt\right)dx = \int_0^1 h(0)dt = h(0). \tag{24}$$

For general n, we use the local time as in lemma 2 to obtain

$$b^{n-1}Tr(K_b^n) = b^{n-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} K_b(x_1,x_2)K(x_2,x_3)\ldots K_b(x_{n-1},x_n)K_b(x_n,x_1)dx_1dx_2\ldots dx_n$$

$$= b^{n-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\int_0^1\int_0^1\cdots\int_0^1 g(x_1-bB_{t_1})g(x_2-bB_{t_1})g(x_2-bB_{t_2})g(x_3-bB_{t_2})\right.$$
$$\ldots g(x_j-bB_{t_j})g(x_{j+1}-bB_{t_j})\ldots g(x_{n-1}-bB_{t_{n-1}})g(x_n-bB_{t_{n-1}})g(x_n-bB_{t_n})g(x_1-bB_{t_n})$$
$$\left. dt_1dt_2\ldots dt_n\right]dx_1dx_2\ldots dx_n$$

$$= b^{n-1}\int_0^1\int_0^1\cdots\int_0^1 h(b(B_{t_1}-B_{t_2}))h(b(B_{t_2}-B_{t_3}))\cdots h(b(B_{t_n}-B_{t_1}))dt_1dt_2\cdots dt_n$$

$$= b^{n-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(b(x_1-x_2))h(b(x_2-x_3))\cdots h(b(x_n-x_1)L(x_1)L(x_2)\ldots L(x_n)dx_1dx_2\ldots dx_n.$$

Define new variables as follows:

$$z_j = b(x_{j-1}-x_j) \qquad j = 2,3,\ldots,n.$$

The Jacobian of the change of variables $(x_1, x_2, x_3, \ldots, x_n) \to (x_1, z_2, z_3, \ldots, z_n)$ is b^{1-n} and we obtain:

$$b^{n-1}Tr(K_b^n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(z_2)h(z_3)\cdots h(z_n)h(-z_2 - z_3\cdots z_n)$$

$$L(x_1)(x_1 - z_2/b)\ldots L(x_1 - (z_2 + \cdots + z_n)/b)\ dx_1dz_2dz_3\ldots dz_n. \tag{25}$$

By the same dominated convergence argument as in lemma 2, this converges as $n \to \infty$ to

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(z_2)h(z_3)\cdots h(z_n)h(-z_2 - z_3\ldots z_n)dz_2dz_3\ldots dz_n\right)\left(\int_{-\infty}^{\infty} L^n(x_1)dx_1\right)$$

$$= h^{*n}(0)\int_{-\infty}^{\infty} L^n(x)dx.$$

□

Note that the integral of L^n is almost surely finite for all $n \geq 1$ since

$$\int_{-\infty}^{\infty} L^n(x)dx \leq (L_*)^{n-1}\int_{-\infty}^{\infty} L(x)dx = (L_*)^{n-1}.$$

Moreover since by assumption $0 \leq h(x) \leq C$, we see that $h^{*n}(0) \leq C(\int_{-\infty}^{\infty} h(x)dx)^{n-1} < \infty$. So each term of the series (23) converges as $b \to \infty$ to the corresponding term of

$$\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{\beta^n}{n}h^{*n}_{(0)}\int_{-\infty}^{\infty} L^n(x)dx. \tag{26}$$

Moreover, convergence also takes place for the series as a whole in the region

$$|\beta| < R = [L_* \limsup_{n\to\infty}(h^{*n}(0))^{1/n}]^{-1} \tag{27}$$

where L_* is defined in (19). This is because of the uniform upper bound

$$|(-1)^n\frac{\beta^n}{n}b^{n-1}Tr(K_b^n)| \leq \frac{\beta^n}{n}L_*^{n-1}h^{*n}(0) \tag{28}$$

which follows easily from (25).

Note that R defined by (27) is almost surely finite and positive because $0 < L_* < \infty$ a.s. (see [8]) and , because of the positive-definiteness of h

$$R = (L_* \limsup_{n\to\infty}(h^{*n}(0))^{1/n})^{-1} = \left(L_*\int_{-\infty}^{\infty} h(x)dx\right)^{-1} \tag{29}$$

Moreover, R is the radius of convergence of the series (26) because, by continuity of $L(x)$

$$\lim_{n\to\infty}\left(\int_{-\infty}^{\infty} L^n(x)dx\right)^{1/n} = L_*. \tag{30}$$

Hence we have just proved:

Theorem 1 *For almost every realisation of Brownian motion, the free energy (7) converges as b tends to infinity to the series (26) for β in the range (27).*

Remarks

1. The free energy is **not** independent of the spatial configuration of the polymer, in contrast to [7].

2. Theorem 1 proves the existence of the free energy for a finite range of β. We will see in §4 that it exists for all $\beta \geq 0$.

3. Since the series (26) has a finite radius of convergence, it is tempting to conjecture that the model shows a phase transition. However one must be careful: after all, the series in the right hand side of (23) has also a finite radius of convergence, even though $f_b(\beta)$ defined by (21) is analytic for all $\beta \geq 0$; there the finite radius of convergence is associated to a singularity at $\beta = -R_b$. Observing that (26) is an alternating series that must have a singularity at $-R$ by Pringsheim's theorem [9], one is led to suspect that no singularity exists for $\beta > 0$. This sort of information could conceivably be extracted from the series representation (26) by showing that the function of β that it defines can be extended from $(-R, R)$ to a function on $(0, 2R)$ (with another series representation); however we find it more convenient to take a different starting point (see next section).

4. The inverse temperature R does however have a physical meaning of sorts: suppose that the interaction h is negative (so that charges of the same sign attract each other); such a system would be unstable and collapse at low temperature [3]. However it would exist at high enough temperature namely whenever

$$\beta \leq \left[L_* \int_{-\infty}^{\infty} -h(x)dx\right]^{-1} \tag{31}$$

4 Analyticity of the free energy

The idea behind our proof of the analyticity of the limiting free energy $f(\beta)$ for $\beta \geq 0$ is to exploit the analyticity of $f_b(\beta)$, see (21) and the uniform convergence of $f_b(\beta)$ as $b \to \infty$. For this purpose we rewrite $f_b(\beta)$ as follows:

$$f_b(\beta) = -\frac{1}{2}\int_0^{\infty} \log(1 + \beta x)\mu_b(dx) \tag{32}$$

where the measure μ_b is defined on Borel subsets A of $\mathbb{R}^+$ by:

$$\mu_b(A) = \frac{1}{b}\#\{j : a_j^{(b)} \in A\}. \tag{33}$$

In the above formula $0 \leq a_0^{(b)} \leq a_1^{(b)} \leq a_2^{(b)} \leq \ldots$ are, as before, the eigenvalues of bK_b.

We will control the behaviour of f_b for large b through that of μ_b, or rather the Laplace transform of μ_b; this will involve only known quantities such as $b^{n-1}Tr(K_b^n)$.

It turns out that it is more convenient to work with the modified measure

$$\hat{\mu}_b(dx) = \frac{x\mu_b(dx)}{h(0)}. \tag{34}$$

This is a probability measure because

$$\int_0^{\infty} \hat{\mu}_b(dx) = \frac{1}{h(0)}\int_0^{\infty} x\mu_b(dx) = \frac{1}{h(0)}TrK_b = 1. \tag{35}$$

The advantage of $\hat{\mu}_b$ over μ is that (32) is replaced by

$$f_b(\beta) = -\frac{1}{2}h(0)\int_0^\infty \frac{\log(1+\beta x)}{x}\hat{\mu}_b(dx) \tag{36}$$

which has a better behaved integrand.

Lemma 4 *For every realisation of Brownian motion, the measure $\hat{\mu}_b$ converges weakly to a probability measure $\hat{\mu}$ as $b \to \infty$.*

Proof

It suffices to prove that the Laplace transform of $\hat{\mu}_b$

$$\Phi_b(\lambda) = \int_0^\infty e^{-\lambda x}\hat{\mu}_b(dx) \tag{37}$$

converges as $b \to \infty$ for all $\lambda \geq 0$ to a function $\Phi(\lambda)$ such that $\lim_{\lambda\to 0}\Phi(\lambda) = 1$ [10]. But

$$\Phi_b(\lambda = \frac{1}{h(0)}\int_0^\infty xe^{-\lambda x}\mu_b(dx) \tag{38}$$

$$= \frac{1}{h(0)}\frac{1}{b}Tr(bK_b e^{-\lambda b K_b}) \tag{39}$$

$$= \frac{1}{h(0)}\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}b^n Tr(K_b^{n+1}) \tag{40}$$

For the last step, we invoke the same argument as when going from (21) to (23). Finally, use lemma 3 and the estimate (28) to show that (40) converges as $b \to \infty$ to

$$\Phi(\lambda) = \frac{1}{h(0)}\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}h_{(0)}^{*(n+1)}\int_{-\infty}^{\infty} L^{n+1}(x)dx. \tag{41}$$

The above is analytic in λ; in particular

$$\lim_{\lambda\to 0}\Phi(\lambda) = \Phi(0) = 1.$$

□

In order to state and prove the main result of this section we move to the complex plane β; this is in order to make use of the special properties of sequences of holomorphic functions.

Theorem 2 *The limit as b tends to infinity of $f_b(\beta)$ exists and is holomorphic in a neighbourhood of the positive real axis $0 < \beta < \infty$. Moreover, all the derivatives of $f_b(\beta)$ converge to those of the infinite-volume free energy density $f(\beta)$.*

Proof

Denote the integrand of (36) by

$$u_\beta(x) = \frac{1}{x}\log(1+\beta x) \tag{42}$$

and let $B(\beta_0)$ be the disc of radius $\beta_0/2$ centered at β_0.

Consider any $\beta_0 > 0$. One can prove that both the real and the imaginary parts of $u_\beta(x)$ have x-derivatives which are bounded uniformly in x and β in the region $x > 0, \beta \in B(\beta_0)$.

It follows from this that the family of functions u_β is **uniformly equicontinuous** in that region, i.e.

$$\forall \epsilon > 0, \exists \delta : |u_\beta(x) - u_\beta(y)| < \epsilon \quad \forall\, \beta \in B(\beta_0), \quad |x - y| < \delta. \tag{43}$$

Hence (36), which is the expectation of u_β with respect to the probability measure $\hat{\mu}_b$, converges **uniformly** with respect to β as b tends to infinity to

$$f(\beta) = -\frac{1}{2}h(0)\int_0^\infty u_\beta(x)\hat{\mu}(dx) \tag{44}$$

by virtue of the corollary in chapter VIII.I of [10]. Finally, since $f_b(\beta)$ converges uniformly in $B(\beta_0)$ as b tends to infinity we appeal to theorem 7.10.1 of [9] to conclude that $f(\beta)$ is holomorphic in $B(\beta_0)$. Since $\beta_0 > 0$ is arbitrary and the neighbourhood of the origin is covered by Theorem 1, the result follows.

□

Acknowledgments: this work has been partially supported by the European Commission under the Human Capital and Mobility Scheme, EU contract CHRX-CT93-0411.

References

1 Obukhov, S.P.: Configurational statistics of a disordered polymer chain, J. Phys. A.19, 3655-3664 (1986).

2 Garel, T. and Orland H.: Mean-field model for protein folding, Europhys. Lett. 6, 307-310 (1988).

3 Kantor, Y. and Kardar M.: Polymers with random self-interactions, Europhys. Lett. 14, 421-426 (1991).

4 Derrida, B., Griffiths R.B. and Higgs P.G.: A model of directed random walks with random self-interactions, Europhys. Lett. 18, 361-366 (1992).

5 Wittmer, J. Johner A. and Joanny J.F.: Random and alternating polyampholytes Europhys. Lett. 24, 263-268 (1993).

6 Derrida B. and Higgs P.G.: Low-temperature properties of directed random walks with random self interactions, J. Phys. A27, 5485-5493 (1994).

7 Martinez, S. and Petritis D.: Thermodynamics of a Brownian bridge polymer model in a random environment, J. Phys. A, 29, 1267-1279 (1996).

8 Revuz, D. and Yor M.:Continuous martingales and Brownian motion, Berlin, Springer 1991.

9 Hille, E: Analytic function theory, vol. 1, New York, Chelsea 1962.

10 Feller, W.: An introduction to probability theory and its applications vol. I, 3rd edition, New York, John Wiley 1968.