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## On the Lagrangian Realization of the WZNW Reductions

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### Abstract

We develop a phase space path-integral approach for deriving the Lagrangian realization of the models defined by Hamiltonian reduction of the WZNW theory. We illustrate the uses of the approach by applying it to the models of non-Abelian chiral bosons,  $W$ -algebras and the GKO coset construction, and show that the well-known Sonnenschein's action, the generalized Toda action and the gauged WZNW model are precisely the Lagrangian realizations of those models, respectively.

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In the last few years it has become clear that the Wess-Zumino-Novikov-Witten (WZNW) theory [1] is acting as a ‘master theory’ for a large number of interesting conformally invariant models in  $1+1$  dimensions. A powerful method for extracting those models out of the WZNW theory is the method of Hamiltonian reduction, i.e., the reduction of Hamiltonian systems with symmetry [2]. The Hamiltonian reductions of the WZNW theory are defined by placing constraints on the conserved currents generating the left  $\times$  right Kac-Moody (KM) algebras, whereby the resulting models possess new (reduced) symmetry algebras realized explicitly in terms of the constrained currents [3]. For example, one can in this way obtain field theoretic models of  $W$ -algebras, non-linear extensions of the Virasoro algebra by conformal primary fields [4], realized as reduced KM algebras.

Once the constraints are given it is in principle straightforward to find the reduced Hamiltonian system. However, it is not obvious how to find the underlying Lagrangian from the Hamiltonian system, because the natural variables of the WZNW theory (and those of the reduced models) are not the canonical ones, which can be defined only locally and are also quite involved when expressed in terms of the conserved currents. The purpose of this paper is to present — through examples — a direct approach to the *Lagrangian realization* of the reduced WZNW models. The crucial point is that the passage to the Lagrangian becomes quite simple if one uses the natural, globally well-defined variables, rather than the local canonical variables, for describing the WZNW phase space which can be identified as the cotangent bundle of the loop group endowed with a modified symplectic form [5,6]. We shall illustrate the uses of our approach by deriving the Lagrangians of the reduced WZNW models of non-Abelian chiral bosons and the models of  $W$ -algebras. The respective actions will turn out to be the much-studied Sonnenschein’s action [7,8] and the generalized Toda action [3]. In these two cases one can in fact find the reduced variables for the Lagrangians, but in general it is not easy to choose them from the WZNW variables. This problem, however, can always be circumvented by considering a gauged system, i.e., a gauged WZNW model, if the constraints are first class and linear in the current. We shall obtain two types of gauged WZNW models, one of which yields the Lagrangian realization of reductions by general first class chiral constraints [3], and the other leads to the Lagrangian realization of the Goddard-Kent-Olive (GKO) construction [9]. The latter gauged WZNW model turns out to be equivalent to the usual gauged WZNW model considered in [10].

The WZNW theory is most frequently defined in the Lagrangian formalism [1], but for our purpose we need its Hamiltonian description, which we shall recall here briefly [5,6]. In general a Hamiltonian system is specified by manifold  $M$ , Poisson bracket  $\{, \}$  and Hamiltonian  $H$ . Using some irreducible matrix representation of the underlying finite dimensional simple Lie group  $G$  and its Lie algebra  $\mathcal{G}$ , the manifold (phase space) of the WZNW theory can be given as  $M = \{(g, J) | g(\sigma) \in G, J(\sigma) \in \mathcal{G}\}$ , where  $g(\sigma)$  and  $J(\sigma)$  are periodic, smooth functions of the space variable  $\sigma$ . On the phase space the fundamental Poisson brackets are \*

$$\begin{aligned} \{g(\sigma), g(\bar{\sigma})\} &= 0, \\ \{\text{Tr}(a J(\sigma)), g(\bar{\sigma})\} &= -a g(\sigma) \delta(\sigma - \bar{\sigma}), \\ \{\text{Tr}(a J(\sigma)), \text{Tr}(b J(\bar{\sigma}))\} &= \text{Tr}([a, b] J(\sigma)) \delta(\sigma - \bar{\sigma}) + 2\kappa \text{Tr}(ab) \delta'(\sigma - \bar{\sigma}), \end{aligned} \quad (1)$$

for  $a, b \in \mathcal{G}$ . Defining the 'right-current',

$$\tilde{J} = -g^{-1} J g + 2\kappa g^{-1} g', \quad (2)$$

which forms a KM algebra with the centre opposite in sign to the one of the KM algebra formed by the 'left-current'  $J$  in (1), we consider the Hamiltonian

$$H = \frac{1}{4\kappa} \int d\sigma \text{Tr}(J^2 + \tilde{J}^2). \quad (3)$$

This leads to the usual field equation

$$\dot{g} = \{g, H\} = \frac{1}{\kappa} J g - g' \quad \text{and} \quad \dot{J} = \{J, H\} = J'. \quad (4)$$

We shall also need the symplectic 2-form of the WZNW theory. It is known [5,6] to be given by

$$\omega = \int d\sigma \text{Tr} [\delta(J \delta g g^{-1}) + \kappa (\delta g g^{-1})(\delta g g^{-1})'], \quad (5)$$

where  $\delta$  is the functional exterior derivative. To see that the above Poisson brackets can indeed be derived from the symplectic 2-form (5), we first note that for a function  $F = F(g, J)$  on phase space a vector field  $X$  acts as

$$X(F) = (\delta F)(X) = \int d\sigma \text{Tr} \left( \frac{\delta F}{\delta g} X(g) + \frac{\delta F}{\delta J} X(J) \right), \quad (6)$$

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\* Convention:  $\kappa = -\frac{k}{4\pi}$  with  $k$  being the level of the KM algebra. Prime and dot stand for derivative with respect to the space,  $\sigma = x^1$ , and time,  $\tau = x^0$ ; and we use  $x^\pm = \frac{1}{2}(x^0 \pm x^1)$ .

where  $\frac{\delta F}{\delta g}$  and  $\frac{\delta F}{\delta J}$  are functional derivatives of  $F$  with respect to  $g$  and  $J$ . Then for two vector fields  $X$  and  $Y$  we have

$$\begin{aligned} \omega(X, Y) = \frac{1}{2} \int d\sigma \operatorname{Tr} & \left( X(J)Y(g)g^{-1} - Y(J)X(g)g^{-1} \right. \\ & \left. + J[X(g)g^{-1}, Y(g)g^{-1}] + 2\kappa(X(g)g^{-1})(Y(g)g^{-1})' \right). \end{aligned} \quad (7)$$

We associate the Hamiltonian vector field  $X_F$  to the function  $F$  by the formula,  $\delta F = 2\omega(\cdot, X_F)$ , which of course means

$$(\delta F)(Y) = 2\omega(Y, X_F), \quad \text{for } \forall Y. \quad (8)$$

Comparing the two sides of (8) given in terms of (6) and (7) and using the non-degeneracy of 'Tr', i.e., the non-degeneracy of the 2-form  $\omega$ , we find that  $X_F$  operates as

$$\begin{aligned} X_F(g) &= \frac{\delta F}{\delta J} g, \\ X_F(J) &= \left[ \frac{\delta F}{\delta J}, J \right] + 2\kappa \left( \frac{\delta F}{\delta J} \right)' - g \frac{\delta F}{\delta g}. \end{aligned} \quad (9)$$

Then, as usual, the Poisson bracket of two functions  $F = F(g, J)$  and  $H(g, J)$  on  $M$  is defined to be

$$\{F, H\} \equiv X_H(F) = -X_F(H) = 2\omega(X_H, X_F). \quad (10)$$

By using eqs.(7), (9) one can now easily obtain the Poisson bracket of any two functions on  $M$ . The fundamental Poisson brackets (1) arise as special cases of this general formula [6].

In order to set up the path-integral of this system, we define the measure  $\mathcal{D}J \mathcal{D}g$  by using an invariant volume form of the phase space, which is obtained by taking a suitable power of the symplectic 2-form  $\omega$ . From (5) it turns out to be quite simple:

$$\mathcal{D}J \mathcal{D}g = \prod \delta J \delta g g^{-1}, \quad (11)$$

where the product is over the group and the space-time. Another point to be noted is that the symplectic 2-form (5) is closed but *not exact*. Still, in general we can construct a first order Lagrangian if we extend the parameter region of the phase space, which is normally parametrized along one-dimensional curves by the time variable  $x^0$ , to a two-dimensional region by introducing a new parameter in such

a way that the extended region has the original region of time in its boundary [11]. Usually in the WZNW theory the extended region is chosen such that, if it is combined with the space  $x^1$  dimension, it becomes a three dimensional manifold  $B_3$  whose boundary is the  $1+1$  dimensional space-time itself. This construction allows us to formally write down the *phase space* path-integral of the WZNW theory,

$$Z = \int \mathcal{D}J \mathcal{D}g e^{i(\int \omega - \int dx^0 H)}. \quad (12)$$

From (5) the integration of (the pull-back of)  $\omega$  performed over the extended region reads

$$\int \omega = \int d^2x \operatorname{Tr} (J - \kappa \partial_1 g g^{-1})(\partial_0 g g^{-1}) - \frac{\kappa}{3} \int_{B_3} \operatorname{Tr} (dg g^{-1})^3. \quad (13)$$

Performing the Gaussian integration of the ‘momentum type’ variable  $J$  we obtain the *configuration space* path-integral,

$$Z = \int \mathcal{D}g e^{iS_{\text{WZ}}(g)}, \quad (14)$$

where  $S_{\text{WZ}}(g)$  is the usual WZNW action

$$S_{\text{WZ}}(g) = \frac{\kappa}{2} \int d^2x \operatorname{Tr} (\partial_+ g g^{-1})(\partial_- g g^{-1}) - \frac{\kappa}{3} \int_{B_3} \operatorname{Tr} (dg g^{-1})^3, \quad (15)$$

as expected. The expression of the phase-space path-integral (12) is the basis upon which the Hamiltonian reductions of the WZNW theory are to be implemented in the following.

### Non-Abelian chiral bosons

One of the simplest examples of the Hamiltonian reduction of the WZNW theory is that in which one of the chiral currents, say the left-current  $J$ , is entirely constrained to zero leaving the right-current  $\bar{J}$  alone. Naturally, the reduced theory is chiral and hence it can serve as a theory of (non-Abelian) chiral bosons. Let us examine the content of the reduced theory in detail.

We first note that the constraint surface,  $M_c \subset M$ , defined by

$$\operatorname{Tr} (a J(\sigma)) = 0, \quad \text{for } \forall a \in \mathcal{G}, \quad (16)$$

is nothing but the loop group  $LG$  of  $G$ . Since the constraints (16) satisfy

$$\{ \operatorname{Tr} (a J(\sigma)), \operatorname{Tr} (b J(\bar{\sigma})) \}_{|M_c} = 2\kappa \operatorname{Tr} (ab) \delta'(\sigma - \bar{\sigma}), \quad (17)$$

they are almost second class, apart from the set of zero-modes  $J_0 = 0$  in the Fourier expansion (the horizontal subalgebra) which is first class. This set of zero modes generates a *gauge symmetry*, and, accordingly, the *reduced phase space* is identified as  $M_{\text{red}} = M_c/G = LG/G$ . (The factorization is by the left-action of  $G$  generated by the zero modes.) To put it differently, one would need to use some (local) gauge fixing conditions\*  $\chi_0 = 0$  in order to reach  $M_{\text{red}}$  from  $M_c$ . The fact that the left- and the right-currents commute,

$$\{ \text{Tr}(a J(\sigma)), \text{Tr}(b \tilde{J}(\bar{\sigma})) \} = 0, \quad (18)$$

implies that  $\tilde{J}$ , which on the constraint surface reads

$$\tilde{J} = 2\kappa g^{-1} g', \quad (19)$$

is gauge invariant, and it is also easy to see that  $\tilde{J}$  provides a complete set of gauge invariant functions of  $g$ . It follows that the KM symmetry algebra of the right current survives the reduction, that is the Dirac brackets of the right-current,

$$\{ \text{Tr}(a \tilde{J}(\sigma)), \text{Tr}(b \tilde{J}(\bar{\sigma})) \}^* = \text{Tr}([a, b] \tilde{J}(\sigma)) \delta(\sigma - \bar{\sigma}) - 2\kappa \text{Tr}(ab) \delta'(\sigma - \bar{\sigma}), \quad (20)$$

are the same as the original Poisson brackets. On account of (19), the reduced phase space  $LG/G$  can be regarded as a submanifold of the space of the right-current — it is in fact one of the coadjoint orbits of the centrally extended loop group [12]. (One could obtain any coadjoint orbit as the reduced phase space by constraining  $J$  to other fixed values instead of zero.)

We next notice that the WZNW Hamiltonian  $H$  commutes weakly with the constraints defined by (16), i.e.,  $\{ H, \text{Tr}(a J(\sigma)) \}_{|M_c} = 0$ . This implies that the WZNW dynamics, defined by (4), leaves  $M_c$  invariant and thus it gives a natural projection on  $M_{\text{red}}$ . The Hamiltonian  $H$  in (3) reduces on  $M_c$  to

$$H_c = \frac{1}{4\kappa} \int d\sigma \text{Tr} \tilde{J}^2 = \kappa \int d\sigma \text{Tr}(g^{-1} g')^2, \quad (21)$$

which can be used to generate the dynamics of the reduced theory through the Dirac bracket. We then find that the gauge invariant object  $\tilde{J}$  obeys the chiral field equation

$$\partial_+ \tilde{J} = 0. \quad (22)$$

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\* The bundle  $LG \rightarrow LG/G$  is topologically non-trivial [12].

As for the group valued field  $g$ , the WZNW field equation (4) becomes the chiral equation  $\partial_+ g = 0$  upon restriction to  $M_c$ , but the chiral solutions can be subjected to arbitrary time dependent gauge transformations without changing their physical meaning, i.e., their projection on  $M_{\text{red}} = LG/G$ . In short,  $g$  is a chiral field in the reduced theory up to the gauge freedom inherent in it.

It is clear that the reduced theory inherits the conformal invariance of the WZNW theory, since the constraints (16) are invariant under the conformal transformations generated by the usual left and right Virasoro densities,

$$L(\sigma) = \frac{1}{2\kappa} \text{Tr} J^2(\sigma) \quad \text{and} \quad \tilde{L}(\sigma) = \frac{1}{2\kappa} \text{Tr} \tilde{J}^2(\sigma). \quad (23)$$

Evidently,  $\tilde{L}$ , which is manifestly gauge invariant, generates the conformal symmetry of the reduced theory (while  $L$  vanishes upon imposing the constraints). We have therefore shown that the present reduction gives rise to a conformally invariant theory of chiral bosons possessing a chiral field equation as well as a single KM algebra in a rather trivial way.

Having described the Hamiltonian system of chiral bosons, let us find the corresponding Lagrangian. We are going to read off the Lagrangian from the phase space path-integral of the reduced theory following the standard prescription [13] which implements the constraints in the path-integral (12). Namely, we define the phase space path-integral for our constrained WZNW theory by inserting the  $\delta$ -functions of the total, second class constraints,  $\phi = \{J_n, J_0, \chi_0\}$  ( $n \neq 0$ ), in (12) together with the associated determinant factors,

$$Z = \int \mathcal{D}J \mathcal{D}g \delta(\phi) \det^{\frac{1}{2}} |\{\phi, \phi\}| e^{i(\int \omega - \int dx^0 H)}. \quad (24)$$

More explicitly, the factors inserted read

$$\delta(\phi) \det^{\frac{1}{2}} |\{\phi, \phi\}| = \delta(J_n) \delta(J_0) \delta(\chi_0) \det |\{J_0, \chi_0\}| \det^{\frac{1}{2}} |\{J_n, J_m\}|. \quad (25)$$

Note that the second determinant factor in (25) is just a constant on account of (17). Then the integration of  $J$  yields the reduced path-integral,

$$Z = \int \mathcal{D}g \delta(\chi_0) \det |\{J_0, \chi_0\}| e^{iI_c(g)}, \quad (26)$$

where

$$\begin{aligned} I_c(g) &= -\kappa \int d^2x \text{Tr} (\partial_+ g g^{-1})(\partial_1 g g^{-1}) - \frac{\kappa}{3} \int_{B_3} \text{Tr} (dg g^{-1})^3 \\ &= S_{\text{WZ}}(g) - \frac{\kappa}{2} \int d^2x \text{Tr} (\partial_+ g g^{-1})^2. \end{aligned} \quad (27)$$



This is exactly the action of chiral bosons proposed by Sonnenschein [7]. The invariance of the action (27) under

$$g(x) \rightarrow g(x) R(x^-) \quad \text{and} \quad g(x) \rightarrow T(x^0) g(x), \quad (28)$$

for  $R(x^-), T(x^0) \in G$ , reflects the fact that there exists the aforementioned gauge symmetry generated by the zero-modes in addition to the usual right-symmetry which survived the reduction. Because of this, the field equation  $\partial_+(g^{-1}\partial_1 g) = 0$  (or  $\partial_1(\partial_+ g g^{-1}) = 0$ ) admits the solution

$$g(x) = g_0(x^0) g_R(x^-), \quad (29)$$

which is not quite chiral. However, it is evident that the non-chirality of  $g$  can always be eliminated by using the gauge freedom. The importance of the gauge invariance was recognized also in [8], where the action (27) has been derived by a coherent state path-integral method. Here we provided another perspective on the action of non-Abelian chiral bosons (27) by showing that it naturally arises from the Hamiltonian reduction of the WZNW theory, whereby some of its crucial physical properties, i.e., a chiral field equation, a single KM algebra, conformal invariance, are transparent.

### Models of $W$ -algebras

Another interesting example can be found by those WZNW reductions which yield field theoretic models of  $W$ -algebras. There is a natural way to associate a  $W$ -algebra to each embedding of the Lie algebra  $sl(2)$  into the simple Lie algebras, and these extended conformal algebras occur as symmetry algebras of (generalized) Toda theories (e.g., see [3] and references therein). It has been shown earlier by using an intermediate gauged WZNW theory that the Toda theory may be regarded as a reduced WZNW theory, belonging to left-right dual constraints which reduce the two chiral KM algebras to chiral  $W$ -algebras. Here we wish to derive the Toda theory directly from the WZNW theory.

For this purpose, let us consider a non-compact real Lie group  $G$  and choose a set of elements,  $\{M_-, M_0, M_+\}$ , which forms an  $sl(2)$  subalgebra of  $\mathcal{G}$ . Then the adjoint action of  $M_0$ ,  $\text{ad}_{M_0} = [M_0, \ ]$ , provides a grading of  $\mathcal{G}$  by its eigenvalues, i.e., by the  $sl(2)$ -spins. By using this grading, we can decompose the algebra  $\mathcal{G}$  into

the spaces of positive, zero and negative grades,

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_0 + \mathcal{G}_-. \quad (30)$$

For simplicity, we here assume that only integral spins occur in this decomposition. (This is true for example for the principal  $sl(2)$  subalgebra, which is relevant for the standard  $W$ -algebras.) Let us choose some bases  $\{\gamma_i\}$  and  $\{\tilde{\gamma}_i\}$  in  $\mathcal{G}_+$  and  $\mathcal{G}_-$ , and consider the reduction defined by the first class constraints,

$$\phi_i(\sigma) = \text{Tr } \gamma_i(J(\sigma) - \kappa M_-) = 0 \quad \text{and} \quad \tilde{\phi}_i(\sigma) = \text{Tr } \tilde{\gamma}_i(\tilde{J}(\sigma) + \kappa M_+) = 0. \quad (31)$$

Note that in this formula we have set to unity the dimensional constants which in principle occur in front of  $M_{\pm}$  to match the mass dimension. In spite of having the (hidden) mass parameters the conformal invariance can still be maintained by adopting the modified Virasoro density,

$$L_{M_0}(\sigma) = L(\sigma) - 2\text{Tr}(M_0 J'(\sigma)), \quad (32)$$

and similarly the right Virasoro density, which commute weakly with the constraints. Moreover, it is possible to find a set of gauge invariant differential polynomials of  $J$  consisting of the Virasoro density (32) and conformal primary fields which form a  $W$ -algebra. Accordingly, the reduced theory possesses the  $W$ -symmetry, which is larger than the conformal symmetry [3].

As before, we shall derive the reduced theory by starting with the phase space path-integral with first class constraints,

$$Z = \int \mathcal{D}J \mathcal{D}g \delta(\phi) \delta(\tilde{\phi}) \delta(\chi) \delta(\tilde{\chi}) \det |\{\phi, \chi\}| \det |\{\tilde{\phi}, \tilde{\chi}\}| e^{i(\int \omega - \int dx^0 H)}, \quad (33)$$

where we have inserted delta-functions of gauge fixing conditions  $\chi$  and  $\tilde{\chi}$  corresponding to  $\phi$  and  $\tilde{\phi}$ . In order to describe the theory in terms of reduced variables, we associate to (30) a ‘generalized Gauss decomposition’ of the group  $G$ ,

$$g = g_+ g_0 g_-, \quad \text{with} \quad g_+ = e^{\beta_+}, \quad g_0 = e^{\beta_0}, \quad g_- = e^{\beta_-}, \quad (34)$$

where  $\beta_{0,\pm}$  are from the respective subalgebras in (30). We restrict ourselves to considering the ‘big cell’ of the phase space where  $g$  is Gauss decomposable. Then, by using the gauge transformations generated by the first class constraints, we can choose the ‘physical gauge’,

$$g = g_+ g_0 g_- \rightarrow g_0, \quad (35)$$

in which the determinant factors in (33) are constants. Also, in this gauge the second delta-function can be written as

$$\delta(\tilde{\phi}_i) = (\det V(g_0))^{-1} \delta(\text{Tr } \tilde{\gamma}_i (J - g_0 M_+ g_0^{-1})), \quad (36)$$

where we have defined  $V_{ij}(g_0) = \text{Tr}(\gamma_i g_0 \tilde{\gamma}_j g_0^{-1})$ . The determinant factor appearing in (36) is cancelled by the factor arising from the measure  $\mathcal{D}g$  computed in the physical gauge, which in fact is a result ensured by the construction of the path-integral reduction [13]. We may now perform the  $J$ -integration by two steps as follows. We first decompose  $J$  according to (30) as

$$J(x) = J_+(x) + J_0(x) + J_-(x), \quad (37)$$

and carry out the  $J_+$ - and  $J_-$ -integrations. Then we find

$$Z = \int \mathcal{D}J_0 \mathcal{D}g_0 e^{i(\int \omega_0 - \int d^2x H_0)}, \quad (38)$$

where

$$H_0 = \int d^2x \text{Tr} \left[ \frac{1}{4\kappa} (J_0^2 + \tilde{J}_0^2) + \kappa g_0 M_+ g_0^{-1} M_- \right]. \quad (39)$$

Here  $\tilde{J}_0$  is defined similarly to (2), and  $\int \omega_0$  is given exactly by (13) with  $g$  and  $J$  replaced by  $g_0$  and  $J_0$ , respectively. We thus see from the reduced Hamiltonian (39) and the phase space path-integral (38) that the reduced theory (i.e., the generalized Toda theory) is nothing but a WZNW theory based on  $\mathcal{G}_0$  plus a potential term. Finally, the integration of the momentum variable  $J_0$  yields the configuration space path-integral,

$$Z = \int \mathcal{D}g_0 e^{I_{\text{Toda}}(g_0)}. \quad (40)$$

The action appearing in (40),

$$I_{\text{Toda}}(g_0) = S_{\text{WZ}}(g_0) - \kappa \int d^2x \text{Tr} (g_0 M_+ g_0^{-1} M_-), \quad (41)$$

is the generalized Toda action [3] we have been after.

As far as the Hamiltonian reduction leading to a  $W$ -algebra is concerned, one does not need to take the constraints in the dual form as in (31); one can take the constraints on the left- and right-currents independently. One could also generalize the reduction by considering arbitrary chiral first class constraints, although the

reduced theory might no longer possess a  $W$ -algebra. For those general cases the above procedure to reach the reduced Lagrangian may not be easily carried out, since the separation of the variables into reduced and ignorable ones is difficult in general. (Note that in the Toda case it is the grading structure and the dual nature of the constraints which permitted the simple  $J$ -integration by the use of the Gauss decomposition, which also provided us the reduced variables in the physical gauge.) However, even in those cases we can at least have a Lagrangian realization by a gauge theory, that is, by the gauged WZNW theory.

To see this, let us choose two subalgebras  $\Gamma$  and  $\tilde{\Gamma}$  of  $\mathcal{G}$  independently and consider the generalized chiral constraints,

$$\phi_i(\sigma) = \text{Tr} \gamma_i(J(\sigma) - \kappa M) = 0 \quad \text{and} \quad \tilde{\phi}_i(\sigma) = \text{Tr} \tilde{\gamma}_i(\tilde{J}(\sigma) + \kappa \tilde{M}) = 0, \quad (42)$$

where  $\{\gamma_i\}$  and  $\{\tilde{\gamma}_i\}$  are bases from  $\Gamma$  and  $\tilde{\Gamma}$ , respectively, and  $M$  and  $\tilde{M}$  are some constant elements of  $\mathcal{G}$ . The first-classness of the constraints requires that

$$\text{Tr}(\gamma_i \gamma_j) = 0 \quad \text{and} \quad \text{Tr}(M[\gamma_i, \gamma_j]) = 0, \quad (43)$$

and a similar relation for the constraints on the right-current. (The first relation implies that such constraints are possible only for a non-compact group  $G$ .) The phase space path-integral is given similarly as in (33) and we first exponentiate  $\delta(\phi)$  and  $\delta(\tilde{\phi})$  by introducing two independent Lagrange multiplier fields  $A_- \in \Gamma$  and  $A_+ \in \tilde{\Gamma}$ . By choosing appropriate gauge fixing conditions  $\chi$  and  $\tilde{\chi}$  (e.g., the physical gauge defined analogously to the Toda case) the remaining determinant factors and  $\delta$ -functions can be taken to be  $J$ -independent and thus we can integrate  $J$  out as usual. This way we arrive at the the following effective path-integral

$$Z = \int \mathcal{D}g \mathcal{D}A_- \mathcal{D}A_+ \delta(\chi) \delta(\tilde{\chi}) \det |\{\phi, \chi\}| \det |\{\tilde{\phi}, \tilde{\chi}\}| e^{iI_{\text{eff}}(g, A_-, A_+)}, \quad (44)$$

where

$$I_{\text{eff}}(g, A_-, A_+) = S_{\text{WZ}}(g) + \kappa \int d^2x \text{Tr} [A_- (\partial_+ g g^{-1} - M) + A_+ (g^{-1} \partial_- g - \tilde{M}) + A_- g A_+ g^{-1}]. \quad (45)$$

The gauged WZNW action (45), which is invariant under

$$g \rightarrow \alpha g \tilde{\alpha}^{-1}, \quad A_- \rightarrow \alpha A_- \alpha^{-1} + \partial_- \alpha \alpha^{-1}, \quad A_+ \rightarrow \tilde{\alpha} A_+ \tilde{\alpha}^{-1} + \partial_+ \tilde{\alpha} \tilde{\alpha}^{-1}, \quad (46)$$

for any  $\alpha(x) \in e^\Gamma$ ,  $\tilde{\alpha}(x) \in e^{\tilde{\Gamma}}$ , provides a Lagrangian realization of the WZNW reduction by the general chiral first class constraints, although it is not given in terms of the reduced variables alone.

### GKO coset construction

For our final example we consider a coset theory obtained by the GKO construction [9]. The GKO construction is designed to produce representations of the Virasoro algebra based on the coset  $G/H$ , where  $H$  is a subgroup of a simple (or semi-simple) group  $G$ . If it is to be realized in the WZNW context,  $H$  will be a diagonal subgroup of the left  $\times$  right group  $G \times G$ . Let us find the constraints which lead to the GKO coset construction. Actually, we need not look far; the constraints are given by

$$\phi_i(\sigma) = \text{Tr } \gamma_i(J(\sigma) + \tilde{J}(\sigma)) = 0, \quad \text{for } \gamma_i \in \mathcal{H}, \quad (47)$$

where  $\mathcal{H} \subset \mathcal{G}$  is the Lie algebra of  $H$ . These constraints are first class and indeed generate the symmetry corresponding to the diagonal subgroup  $H$ .

It is also easy to find the modified Virasoro density which commutes (strongly) with the constraints (47),

$$L^{G/H}(\sigma) = L(\sigma) - L^H(\sigma), \quad (48)$$

where  $L^H = \frac{1}{2\kappa} \text{Tr}_{(H)} J^2$  is the usual Sugawara construction of the Virasoro density with the summation taken only over the subalgebra  $\mathcal{H}$ . Thus the reduced theory is invariant under the conformal transformations generated by  $L^{G/H}$ , which is the Virasoro density used in the GKO construction.

The Lagrangian realization of the reduced theory purely in terms of reduced variables is difficult to find in general, since the constraints involve both of the left- and right-currents simultaneously and thus it is hard to disentangle them to select the reduced variables for the Lagrangian. Therefore we again content ourselves with deriving a gauged WZNW theory, similarly as in the previous example. As before, we first introduce a Lagrange multiplier field  $A \in \mathcal{H}$  and write the phase space path-integral as

$$\begin{aligned} Z &= \int \mathcal{D}J \mathcal{D}g \delta(\phi) \delta(\chi) \det |\{\phi, \chi\}| e^{i(\int \omega - \int dx^0 H)} \\ &= \int \mathcal{D}J \mathcal{D}g \mathcal{D}A \delta(\chi) \det |\{\phi, \chi\}| e^{i(\int \omega - \int dx^0 H - A \cdot \phi)}, \end{aligned} \quad (49)$$

where  $A \cdot \phi = \int dx^2 \text{Tr} A(x)\phi(x)$  and  $\chi = 0$  is a gauge fixing condition. Then, by the  $J$ -integration we find

$$Z = \int \mathcal{D}g \mathcal{D}A \delta(\chi) \det |\{\phi, \chi\}| e^{iI_{\text{eff}}(g,A)}, \quad (50)$$

where

$$I_{\text{eff}}(g, A) = S_{\text{WZ}}(g) - \kappa \int d^2x \text{Tr} [A(\partial_+ g g^{-1}) - A(g^{-1} \partial_- g) + AgAg^{-1} - A^2]. \quad (51)$$

The action (51) is invariant under the gauge transformation,

$$g \rightarrow h g h^{-1}, \quad A \rightarrow h A h^{-1} + \partial_0 h h^{-1}, \quad (52)$$

for  $h = h(x^0) \in H$ , but it is not invariant under the gauge transformation for fully space-time dependent  $h(x)$ . However, if one writes  $A = A_0$  one finds that the effective action (51) is nothing but the ‘axial gauge’  $A_1 = 0$  version of the gauged WZNW action considered in [10], which is invariant under the full  $h(x)$ . In summary, we have shown that the action (51) provides the Lagrangian realization of the GKO construction, and that its gauge invariant content is the same as that of the (fully-)gauged WZNW theory of [10].

In this paper we presented a direct approach to the Lagrangian realizations of the WZNW reductions and thereby derived Sonnenschein’s action of non-Abelian chiral bosons, the generalized Toda action, and the two types of gauged WZNW models implementing the general chiral first class constraints and the GKO construction. This approach should be applicable in general to any WZNW reduction, and could be used to extract further interesting (known, or unknown) models out of the WZNW theory.

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