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A PAIR HAMILTONIAN MODEL OF A NON-IDEAL BOSON GAS

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Abstract: The pressure in the thermodynamic limit of a non-ideal Boson gas whose Hamiltonian includes only diagonal and pairing terms can be expressed as the infimum of a functional depending on two measures on momentum space: a positive measure describing the particle density and a complex measure describing the pair density. In this paper we examine this variational problem with the object of determining when the model exhibits Bose-Einstein condensation. In addition we show that if the pairing term in the Hamiltonian is positive then it has no effect.

Resumé: Dans un modèle de gaz de Bosons en interaction dont l'hamiltonien ne contient que des termes diagonaux et des termes de paires, la limite thermodynamique de la pression est donnée par l'infimum d'une fonctionnelle dépendant de deux mesures sur l'espace des impulsions: une measure positive correspondant à la densité de particules et une mesure complexe décrivant la densité de paires. Dans cet article, nous étudions ce problème variationnel pour déterminer quand le modèle exhibe une condensation de Bose-Einstein. De plus, nous prouvons que si le terme de paires dans l'hamiltonien est positif, il est sans effet.

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§1. Introduction

Consider a system of identical bosons of mass m enclosed in a cube $\Lambda \subset \mathbb{R}^d$ of volume V centred at the origin. If the particle interaction is defined by a translation-invariant two-body potential $\Phi \in L^2(\mathbb{R}^d)$, then assuming periodic boundary conditions, the Hamiltonian of the system in the second-quantized form is given by:

$$H = T + \frac{1}{2V} \sum_{q,k,k' \in \Lambda^*} \tilde{\Phi}(q) a^*_{k+q} a^*_{k'-q} a_{k'} a_k, \tag{1.1}$$

where

$$\tilde{\Phi}(q) = \int_{\mathbb{R}^d} dx\ \Phi(x) e^{-iqx},$$

a^*_k and a_k are the boson creation and anihilation operators,

$$a^*_k = (V)^{-\frac{1}{2}} \int_\Lambda dx\ a^*(x) e^{ikx}, \qquad a_k = (V)^{-\frac{1}{2}} \int_\Lambda dx\ a(x) e^{-ikx},$$

$\Lambda^* = \{2\pi s / V^{1/d} : s \in \mathbb{Z}^d\}$ and $T = \sum_{k \in \Lambda^*} \epsilon(k) N_k$ with $N_k = a^*_k a_k$ and $\epsilon(k) = \|k\|^2/2m$.

One of the most interesting questions in the study of boson systems is the persistence of Bose-Einstein condensation in the presence of the interaction. For the Hamiltonian (1.1) this problem has so far been intractable; for this reason one is led to the study of model Hamiltonians which exhibit some fundamental properties of the original Hamiltonian (1.1) and which are at the same time simple enough so that they can be solved analytically. The only models which have been studied fully so far are "diagonal models ", that is ones in which the Hamiltonian can be expressed in terms of the occupation number operators N_k [1 - 6]. The next step is to include "pairing" terms $a^*_k a^*_{-k}$ and $a_k a_{-k}$. Let H^P be the "pair Hamiltonian" [7 - 10], that is the part of H in (1.1) which can be expressed in terms of diagonal and pairing terms; then H^P is given by

$$\begin{aligned} H^P = T &+ \frac{1}{2V} \sum_{k,k'} \tilde{\Phi}(0)\ a^*_k a^*_{k'} a_{k'} a_k + \frac{1}{2V} \sum_{k,k'(\neq \pm k)} \tilde{\Phi}(k'-k) a^*_{k'} a^*_k a_{k'} a_k \\ &+ \frac{1}{2V} \sum_{k,q(\neq \pm 0)} \tilde{\Phi}(q) a^*_{k+q} a^*_{-k-q} a_{-k} a_k. \end{aligned} \tag{1.2}$$

Three types of scattering interactions are taken into account in (1.2): forward scattering interaction: $q = 0$, exchange scattering interaction: $q = k' - k$ ($k' \neq \pm k$) and pair scattering interaction: $k' = -k$, similar to the interaction in the BCS model [11]. The restrictions in the sums are necessary to prevent duplication of terms.

If only the forward scattering terms ares included in (1.2) the model reduces to the mean-field model:

$$H^{MF} = T + \frac{1}{2V}\tilde{\Phi}(0)N(N-1), \tag{1.3}$$

where $N = \sum_{k\in\Lambda^*} N_k$; this model has been studied exhaustively [12].

Adding exchange scattering terms gives the Hamiltonian

$$H^S = T + \frac{1}{2V}\sum_{k,k'(k\neq\pm k')}\tilde{\Phi}(k'-k)N_{k'}N_k. \tag{1.4}$$

If the constraint $k \neq \pm k'$ is dropped this model corresponds to the "perturbed mean-field" model with Hamiltonian

$$H^{PMF} = T + \frac{1}{2V}\sum_{k,k'}\tilde{\Phi}(k'-k)N_{k'}N_k, \tag{1.5}$$

this model is the subject of [2] and [5].

The diagonal part of the "pair Hamiltonian"(1.2) is

$$H^{DP} = T + \frac{1}{2V}\tilde{\Phi}(0)N(N-1) + \frac{1}{2V}\sum_{k,k'(\neq\pm k)}\tilde{\Phi}(k-k')N_kN_{k'}. \tag{1.6}$$

If the constraint $k' \neq -k$ is removed, then (1.6) coincides with the "full diagonal Hamiltonian"

$$H^{FD} = T + \frac{1}{2V}\tilde{\Phi}(0)N(N-1) + \frac{1}{2V}\sum_{k,k'(\neq k)}\tilde{\Phi}(k-k')N_kN_{k'}\ , \tag{1.7}$$

treated recently in [6].

Here we study a modified version of (1.2) which contains pair scattering terms; more precisely, we consider the following pair Hamiltonian:

$$H^{uv} = T + \frac{1}{2V}\sum_{k,k'\in\Lambda^*} v(k,k')N_kN_{k'} + \frac{1}{2V}\sum_{k,k'\in\Lambda^*} u(k,k')a_k^*a_{-k}^*a_{-k'}a_{k'}. \tag{1.8}$$

Below we impose conditions on the $u(k,k')$ and $v(k,k')$ to ensure the existence of the grand canonical pressure in the thermodynamic limit.

In the series of papers [2 - 6] in which the diagonal models mentioned above were studied, the pressure in the thermodynamic limit was expressed as the supremum of a functional over the space of measure. The minimizing measure can be interpreted as the equilibrium distribution of the particles according to their momentum; in particular an atom in the measure is interpreted as the occurrence of Bose-Einstein condensation. The main technical tool used in these papers was Varadhan's Large deviation theory; this was possible because of the commutative nature of these models. These techniques were extended to non-commutative inhomogeneous mean-field models by Cegła, Lewis and Raggio [13], Duffield and Pulé [14, 15] and Raggio and Werner [16]. However and in all these cases the operators involved in the Hamiltonian are bounded. In the model under investigation in this paper the operators do not commute and moreover they are unbounded. We again give a variational formula for the pressure; the proof of this formula will be given in another paper. This time the variational formula is over two parameters: one parameter again describes the distribution of particles according to their momentum while the new parameter describes the pair density.

We should mention here that some models intermediate between the diagonal models and the pairing models have been studied; among these the best known is Bogoliubov's model [17, 18] which recently has been re-examined from the stability point of view [19].

Let $p_V^{uv}(\mu)$ be the pressure for the Boson gas with Hamiltonian given by (1.8). Then we have the following variational formula for the pressure in the thermodynamic limit $p^{uv}(\mu) = \lim_{V\to\infty} p_V^{uv}(\mu)$:

For $A \subset \mathbf{R}^d$ let

$$\nu_V(A) = \frac{1}{V}\sharp(A \cap \Lambda^*) \tag{1.9}$$

and let ν be the limit of the measure ν_V as V tends to ∞. Let M be the space of complex bounded measures on $\mathbf{R}^d$ and $M_+ \subset M$ the set of positive bounded measures. Let $t : \mathbf{R}^d \to \mathbf{R}^d$ be defined by $t(k) = -k$ and for $m \in M$ let $\tilde{m} \in M$ be defined by

$$\tilde{m} = \frac{1}{2}(m + m \circ t). \tag{1.10}$$

F is the set of pairs (m, n), with $m \in M_+$ and $n \in M$ satisfying:

(i) $n = n \circ t$;

(ii) n is absolutely continuous with respect to $\tilde{m}$;

(ii) if $\sigma(k) = \frac{dn}{d\tilde{m}}(k)$ then $|\sigma(k)| \leq 1$;

(iv) if $\rho(k) = \left(\frac{dm}{d\nu}\right)(k)$

$$(\rho(k) + \rho(-k))|\sigma(k)|^2 \leq (\rho(k) + \rho(-k) + 2) \tag{1.11}$$

and

$$\frac{1}{4}(\rho(k) + \rho(-k))^2|\sigma(k)|^2 \leq \rho(k)\rho(-k)) + \min(\rho(k), \rho(-k)). \tag{1.12}$$

For $k \in \mathbb{R}^d$, let

$$R(k) = \left\{ \frac{1}{4}(\rho(k) + \rho(-k) + 1)^2 - \frac{1}{4}(\rho(k) + \rho(-k))^2 |\sigma(k)|^2 \right\}^{\frac{1}{2}} + \frac{1}{2}\{\rho(k) - \rho(-k) - 1\}, \tag{1.13}$$

and for $x \geq 0$ let

$$s(x) = (1 + x)\ln(1 + x) - x \ln x; \tag{1.14}$$

then

$$p^{uv}(\mu) = - \inf_{(m,n) \in F} \mathcal{E}^{\mu}_{uv}(m, n). \tag{1.15}$$

where

$$\mathcal{E}^{\mu}_{uv}(m, n) = \int_{\mathbb{R}^d} (\epsilon(k) - \mu) m(dk) + \frac{1}{2} \int\int_{\mathbb{R}^d \times \mathbb{R}^d} v(k, k') m(dk) m(dk')$$

$$+ \frac{1}{2} \int\int_{\mathbb{R}^d \times \mathbb{R}^d} u(k, k') n(dk) n(dk') - \frac{1}{\beta} \int_{\mathbb{R}^d} s(R(k)) \nu(dk). \tag{1.16}$$

The variational formula (1.15) will be proved elsewhere. Here we restrict ourselves to the study of this variational formula. If (m, n) is a minimizer of $\mathcal{E}^{\mu}_{uv}$, then we can interpret m as the equilibrium *density of particles* and n as the equilibrium *density of pairs*. We identify the presence of an *atom* with respect to ν in m as the presence of a *Bose-Einstein condensate*. In examining the variational problem we are interested mainly in determining when Bose-Einstein condensation occurs and the value of n when this happens.

If the kernel u is of positive type then since $x \mapsto s(x)$ is increasing it is clear that for all allowed n

$$\mathcal{E}^{\mu}_{uv}(m, n) \geq \mathcal{E}^{\mu}_{uv}(m, 0), \tag{1.17}$$

and therefore

$$p^{uv}(\mu) = - \inf_{m \in M_+} \mathcal{E}^{\mu}_{uv}(m, 0). \tag{1.18}$$

Now we have proved in [2, 5] that for the perturbed meanfield model with Hamiltonian given by (1.6) the pressure $p^{PMF}(\mu)$ is given by

$$p^{PMF}(\mu) = - \inf_{m \in M_+} \mathcal{E}^{\mu}_{PMF}(m), \tag{1.19}$$

where

$$\mathcal{E}^{\mu}_{PMF}(m) = \mathcal{E}^{\mu}_{uv}(m, 0). \tag{1.20}$$

Thus if u is of positive type

$$p^{uv}(\mu) = p^{PMF}(\mu). \tag{1.21}$$

This result can be proved more directly; this we shall do in Section 2. In Section 3 we shall study the variational problem (1.15) in general when $u(k,k') \leq 0$ for all $k, k' \in \mathbb{R}^d$; in particular we shall prove the infimum is attained and that every minimizer satisfies the Euler-Lagrange equations for the problem. In Section 4 we shall study in detail the variational problem when u and v are constants.

§2. Positive u

In this section we consider the model with Hamiltonian defined in (1.8) in the case when u is a positive definite kernel and give a direct proof of the assertion (1.21). To be able to make use of the results in [2] we shall assume in this section that v satisfies the following condition:

$v : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is a bounded, continuous, positive definite function; there exists a continuous, strictly positive, symmetric function* $v_0 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *such that for all* $m \in M_+$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} v(k,k')m(dk)m(dk') \geq \int_{\mathbb{R}^d \times \mathbb{R}^d} v_0(k,k')m(dk)m(dk').$$

Proposition 1. *If the kernel u is bounded and positive definite then*

$$p^{uv}(\mu) = p^{PMF}(\mu)$$

for all $\mu \in \mathbb{R}$.

Proof: Since u is of positive type then clearly

$$H^{uv} \geq H^{PMF} \tag{2.1}$$

where H^{PMF} is as in (1.5); thus for $\mu \in \mathbb{R}$

$$p_V^{uv}(\mu) \leq p_V^{PMF}(\mu), \tag{2.2}$$

and

$$\limsup_{V\to\infty} p_V^{uv}(\mu) \leq \lim_{V\to\infty} p_V^{PMF}(\mu) = p^{PMF}(\mu). \tag{2.3}$$

To prove the upper bound let $\alpha < 0$ and let

$$\mathcal{D}_1 = \{t \in \mathcal{C}^b(\mathbb{R}^d) \ : \ \inf_{k\in\mathbb{R}^d}(\epsilon(k) - \alpha - t(k)) > 0\}, \tag{2.4}$$

where $\mathcal{C}^b(\mathbb{R}^d)$ is the space of continuous bounded functions on $\mathbb{R}^d$; for $t \in \mathcal{D}_1$ let

$$H^{t+\alpha} = \sum_{k\in\Lambda^*}(\epsilon(k) - \alpha - t(k))N_k. \tag{2.5}$$

By convexity we have that

$$\frac{1}{\beta V}\ln \operatorname{trace} e^{-\beta H^{uv}} \geq \frac{1}{\beta V}\ln \operatorname{trace} e^{-\beta H^{t+\alpha}} - \frac{1}{V}\langle(H^{uv} - H^{t+\alpha})\rangle_{t+\alpha}, \tag{2.6}$$

where $\langle A\rangle_{t+\alpha} = \operatorname{trace} e^{-\beta H^{t+\alpha}}A/\operatorname{trace} e^{\beta H^{t+\alpha}}$. Let

$$\rho(k;t,\alpha) = (\exp\beta(\epsilon(k) - t(k) - \alpha) - 1)^{-1}, \tag{2.7}$$

then

$$\langle N_k\rangle_{t+\alpha} = \rho(k;t,\alpha) \tag{2.8}$$

and

$$\langle N_k N_{k'}\rangle_{t+\alpha} = \rho(k't,\alpha)\rho(k';t,\alpha) \text{ if } k \neq k', \tag{2.9}$$

$$\langle N_k^2\rangle_{t+\alpha} = \rho(k;t,\alpha)(2\rho(k;t,\alpha) + 1). \tag{2.10}$$

We also have for $k \neq k'$ and $k \neq -k'$

$$\langle b_k^* b_{-k}^* b_{-k'} b_{k'}\rangle_{t+\alpha} = 0. \tag{2.11}$$

The first term in the right hand of the inequality (2.6) can be computed to give

$$\begin{aligned}\frac{1}{\beta V}\ln \operatorname{trace} e^{-\beta H^{t+\alpha}} &= -\frac{1}{\beta}\int_{\mathbb{R}^d}\ln(1 - e^{-\beta(k-t(k)-\alpha)})\nu_V(dk)\\ &= \int(\epsilon(k) - \mu)\nu_V(dk) - \frac{1}{\beta}\int_{\mathbb{R}^d} s(\rho(k;t,\alpha))\nu_V(dk).\end{aligned} \tag{2.12}$$

To compute $\frac{1}{V}\langle H^{uv}\rangle_{t+\alpha}$ we write H^{uv} in the form

$$H^{uv} = \sum_k(\epsilon(k) - \mu)N_k + \frac{1}{2V}\sum_k v(k,k)N_k^2$$

$$+\frac{1}{2V}\sum_{k\neq k'} v(k,k')N_k N_{k'} + \frac{1}{2V}u(0,0)(N_0^2 - N_0)$$

$$+\frac{1}{2V}\sum_{k\neq 0}(u(k,-k)+u(k,k))N_kN_{-k}+\frac{1}{2V}\sum_{\substack{k\neq k'\\ k\neq -k'}} b_k^*b_{-k}^*b_{-k'}b_k; \tag{2.13}$$

using (2.8), (2.9), (2.10) and (2.11) we then get

$$\frac{1}{V}\langle H^{uv}\rangle_{t+\alpha}=\int_{\mathbb{R}^d}(\epsilon(k)-\mu)\rho(k;t,\alpha)\nu_V(dk)+$$

$$\frac{1}{2}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} v(k,k')\rho(k;t,\alpha)\rho(k';t,\alpha)\nu_V(dk)\nu_V(dk')+\frac{c_V}{V}, \tag{2.14}$$

where

$$c_V=\frac{1}{2}\int_{\mathbb{R}^d} v(k,k)\rho(k;t,\alpha)(\rho(k;t,\alpha)+1)\nu_V(dk)+\frac{1}{V}u(0,0)\rho(0;t,\alpha)$$

$$+\frac{1}{2}\int_{\mathbb{R}^d}\{u(k,k)+u(k,-k)\}\rho(k;t,\alpha)\rho(-k;t,\alpha)\nu_V(dk). \tag{2.15}$$

Finally

$$\frac{1}{V}\langle H^{t+\alpha}\rangle_{t+\alpha}=\int_{\mathbb{R}^d}(\epsilon(k)-t(k)-\alpha)\rho(k;t,\alpha)\nu_V(dk). \tag{2.16}$$

Putting (2.12), (2.14) and (2.16) into (2.6) we obtain

$$p_V^{uv}\geq\int_{\mathbb{R}^d}(\mu-t(k)-\alpha)\rho(k;t,\alpha)\nu_V(dk)-\frac{1}{\beta}\int_{\mathbb{R}^d}\ln(1-e^{\beta(\epsilon(k)-t(k)-\alpha)}\nu_V(dk)+\frac{c_V}{V} \tag{2.17}$$

and thus since c_V is bounded,

$$\begin{aligned}\liminf_{V\to\infty} p_V^{uv}&\geq\int_{\mathbb{R}^d}(\mu-t(k)-\alpha)\rho(k;t,\alpha)\nu(dk)-\frac{1}{\beta}\int_{\mathbb{R}^d}\ln(1-e^{\beta(\epsilon(k)-t(k)-\alpha)}\nu(dk)\\&=-\mathcal{E}^{\mu}_{PMF}(m^{t,\alpha}),\end{aligned} \tag{2.18}$$

where $m^{t,\alpha}(dk)=\rho(k;t,\alpha)\nu(dk)$. It was proved in [2, Theorem 1] that for each $m\in M_+$ there is a sequence $\{t_n\}$ in $\mathcal{D}_1$ such that

$$\lim_{n\to\infty}\mathcal{E}^{\mu}_{PMF}(m^{t_n,\alpha})=\mathcal{E}^{\mu}_{PMF}(m). \tag{2.19}$$

Therefore from (2.18) we get

$$\liminf_{V\to\infty} p_V^{uv}\geq-\inf_{m\in M_+}\mathcal{E}^{\mu}_{PMF}(m)=p^{PMF}(\mu);$$

thus combining this with (2.2) we obtain

$$p^{uv}=\liminf_{V\to\infty} p_V^{uv}=p^{PMF}(\mu).$$ □

§3. The general variational problem

If m is a complex measure on $\mathbb{R}^d$, bounded or unbounded and $w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ we shall write $(wm)(k)$ for $\int_{\mathbb{R}^d} w(k,k')m(dk')$; also if $f : \mathbb{R}^d \to \mathbb{C}$ we shall denote the measure $f(k)m(dk)$ by fm. If $f : \mathbb{R}^d \to \mathbb{C}$ and m is a complex measure on $\mathbb{R}^d$ we shall write $\langle m, f\rangle$ for $\int_{\mathbb{R}^d} f(k)m(dk)$. With this notation we have

$$\mathcal{E}^{\mu}_{uv}(m,n) = \langle m, \epsilon - \mu\rangle + \frac{1}{2}\langle m, vm\rangle + \frac{1}{2}\langle n, u\bar{n}\rangle - \frac{1}{\beta}\langle \nu, s\circ R\rangle. \tag{3.1}$$

We shall make the following assumptions on u and v:

A1. *u is symmetric and $u(k,k') \le 0$ for all $k, k' \in \mathbb{R}^d$,*

A2. *v is a bounded, continuous, positive definite function; there is a number $\delta > 0$ such that $\langle m, (v+\tilde{u})m\rangle \ge \delta\|m\|^2$ for all $m \in M_+$, where*

$$\tilde{u}(k,k') = \frac{1}{4}\{u(k,k') + u(k,-k') + u(-k,k') + u(-k,-k')\};$$

A3. *there is a constant $C < \infty$ such that for all $k \in \mathbb{R}^d$ $(|u|\nu)(k) \le C$,*

A4. $\langle \nu, |u|\nu\rangle < \infty$.

Under the conditions (A1 - A4) we have that:

Proposition 2. *The functional $\mathcal{E}^{\mu}_{uv} : F \to \mathbb{R}$ is bounded below.*
Proof: For $(m,n) \in F$,

$$|\langle n, u\bar{n}\rangle| \le |\langle \sigma\tilde{m}, |u|\sigma\tilde{m}\rangle \le (\langle |\sigma|^2\tilde{m}, |u||\sigma|^2\tilde{m}\rangle)^{\frac{1}{2}}(\langle \tilde{m}, |u|\tilde{m}\rangle)^{\frac{1}{2}},$$

by the Schwarz inequality.
From (A1) and (A2) we get $\langle m(v-|\tilde{u}|)m\rangle \ge 0$ and therefore $\langle \tilde{m}, |u|\tilde{m}\rangle = \langle m, |\tilde{u}|m\rangle \le \langle m, vm\rangle$; and so

$$|\langle n, u\bar{n}\rangle| \le (\langle |\sigma|^2\tilde{m}, |u||\sigma|^2\tilde{m}\rangle)^{\frac{1}{2}}(\langle m, vm\rangle)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned}\langle m, vm\rangle + \langle n, u\bar{n}\rangle &\ge \langle mvm\rangle - |\langle n, u\bar{n}\rangle| \\ &\ge \{(\langle mvm\rangle)^{\frac{1}{2}} - (\langle |\sigma|^2\tilde{m}, |u||\sigma|^2\tilde{m}\rangle)^{\frac{1}{2}}\}(\langle m, vm\rangle;\end{aligned}$$

using the inequality $x^{\frac{1}{2}} - y^{\frac{1}{2}} \ge \frac{1}{2}(x-y)x^{-1}$ we then get

$$\langle m, vm\rangle + \langle n, u\bar{n}\rangle \ge \frac{1}{2}\{\langle m, vm\rangle - \langle |\sigma|^2\tilde{m}, |u||\sigma|^2\tilde{m}\rangle\}.$$

But by (ii) and (iv) we have

$$|\sigma|^2\tilde{m} \leq \nu + \tilde{m};$$

thus

$$\begin{aligned}\langle|\sigma|^2\tilde{m}, |u||\sigma|^2\tilde{m}\rangle &\leq \langle\tilde{m}|u|\tilde{m}\rangle + 2\langle\tilde{m}, |u|\nu\rangle + \langle\nu, |u|\nu\rangle\\ &\leq \langle m, |\tilde{u}|m\rangle + 2C\|m\| + \langle\nu, |u|\nu\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}\langle m, vm\rangle + \langle n, u\bar{n}\rangle &\geq \frac{1}{2}\{\langle m(v+\tilde{u})m\rangle - 2C\|m\| - \langle\nu, |u|\nu\rangle\}\\ &\geq \frac{1}{2}\{\delta\|m\|^2 - 2K\|m\| - C\}.\end{aligned}$$

Now let $\alpha < 0$, then

$$\begin{aligned}\mathcal{E}^{\mu}_{uv}(m,n) &= \langle m, \epsilon - \alpha\rangle + (\alpha - \mu)\|m\| + \frac{1}{2}\langle m, vm\rangle + \frac{1}{2}\langle n, u\bar{n}\rangle - \frac{1}{\beta}\langle\nu, s\circ R\rangle\\ &\geq \langle m, \epsilon - \alpha\rangle - \frac{1}{\beta}\langle\nu, s\circ R\rangle + \frac{1}{2}\{\delta\|m\|^2 - 2(K+\mu-\alpha)\|m\| - C.\end{aligned}$$

Let

$$A = -\frac{(K+\mu-\alpha)^2}{\delta} \leq \inf_{m\in M_+}\{\delta\|m\|^2 - 2(K+\mu-\alpha)\|m\|;$$

then since s is increasing

$$\mathcal{E}^{\mu}_{uv}(m,n) \geq I(m) + A \tag{3.2}$$

where $I(m) = \langle m, \epsilon - \alpha\rangle - \frac{1}{\beta}\langle\nu, s\circ\rho\rangle$.
Since

$$\inf_{m\in M_+} I(m) = -\frac{1}{\beta}\int_{\mathbb{R}^d}\ln(1 - e^{-\beta(\epsilon(k)-\alpha)})\nu(dk) > -\alpha$$

$\mathcal{E}^{\mu}_{uv}$ is bounded below.

□

We now make an additional assumption A5 which allows us to prove that the infimum of $\mathcal{E}^{\mu}_{uv}$ is attained in F:

A5. *There is a compact set $B \subset \mathbb{R}^d$ satisfying $t(B) = B$ such $u(x,y) < 0$ for $(x,y) \in B\times B$ and $u(x,y) = 0$ for $(x,y) \notin B\times B$.*

Proposition 3. *There exists $(m^*, n^*) \in F$ such that*

$$\mathcal{E}^{\mu}_{uv}(m^*, n^*) = \inf_{(m,n)\in F}\mathcal{E}^{\mu}_{uv}(m,n).$$

Proof: Let M be equipped with the narrow topology that is the weakest topology for which the mappings in $\mapsto \langle m, f\rangle$ are continuous for all $f \in \mathcal{C}^b(\mathbb{R}^d)$. Let $\alpha < 0$ and let

$$\mathcal{D} = \{(f,g) \in \mathcal{D}_1 \times \mathcal{C}^b(\mathbb{R}^d):$$

$$\inf_{k\in\mathbb{R}^d}((\epsilon(k) - \alpha - f(k))(\epsilon(k) - \alpha - f(-k)) - |g(k) + g(-k)|^2) > 0\}.$$

Define $C : \mathcal{D} \to \mathbb{R}$ by

$$C(f,g) = \frac{1}{2}\int_{\mathbb{R}^d}\{\epsilon(k) - \alpha - f(k) + \hat{E}(k) - 2\ln(1 - e^{\hat{E}_+(k)})\}\nu(dk)$$

where

$$\hat{E}(k) = -\{(\epsilon(k) - \alpha - \frac{1}{2}f(k) - \frac{1}{2}f(-k))^2 - |g(k) + g(-k)|^2\}^{\frac{1}{2}}$$

and

$$\hat{E}_+(k) = \hat{E}(k) + \frac{1}{2}(f(k) - f(-k)).$$

For $(m,n) \in M_+ \times M$ let

$$I(m,n) = \sup_{(f,g)\in\mathcal{D}}\{\langle m, f\rangle + \langle n, \bar{g}\rangle + \langle \bar{n}, g\rangle - C(f,g)\};$$

then for $(m,n) \notin F$, $I(m,n) = \infty$ and for $(m,n) \in F$

$$I(m,n) = \int_{\mathbb{R}^d}(\epsilon(k) - \alpha)m(dk) - \frac{1}{\beta}\langle \nu, s \circ R\rangle. \tag{3.3}$$

Since I is the supremum of a family of functions which are continuous in the product topology on $M_+ \times M$, I is lower semi-continuous in the product topology. Now $m \mapsto \langle m, vm\rangle$ is continuous and $n \mapsto \langle n, u\bar{n}\rangle$ is lower semi-continuous in the product topology (see [2]) and therefore if we define

$$\mathcal{E}^{\mu}_{uv}(m,n) = (\alpha - \mu)\|m\| + \frac{1}{2}\langle m, vm\rangle + \frac{1}{2}\langle n, u\bar{n}\rangle + I(m,n) \tag{3.4}$$

for $(m,n) \in M_+ \times M$ then $\mathcal{E}^{\mu}_{uv}$ is lower semi-continuous; clearly $\mathcal{E}^{\mu}_{uv}(m,n) = \infty$ for $(m,n) \notin F$ and for $(m,n) \in F$ the definition coincides with (1.16).

Let $e_0 = \inf_{(m,n)\in M_+\times M}\mathcal{E}^{\mu}_{uv}(m,n) = \inf_{(m,n)\in F}\mathcal{E}(m,n)$. Then $e_0 \leq \mathcal{E}^{\mu}_{uv}(0,0) = 0$; if $e_0 = 0$ then there is nothing to prove. Suppose $e_0 < 0$; we can find a sequence $\{(m_r, n_r)\}$ in $M_+\times M$ such that $(\mathcal{E}^{\mu}_{uv}(m_r, n_r) < 0$ and $\lim_{r\to\infty}\mathcal{E}^{\mu}_{uv}(m_r, n_r) = e_0$. Since $\mathcal{E}^{\mu}_{uv}$ is lower semi-continuous it is sufficient to prove that $\{(m_r, n_r)\}$ has a convergent subsequence. Since $\langle |n_r|, u|n_r|\rangle \leq \langle n_r, u\bar{n}_r\rangle$ we can assume that each

n_r is a positive measure; also because of assumption A5 and the fact that s is an increasing function we can assume that each m_r has support in B.

By the inequality (3.2)

$$I(m_r) \leq -A;$$

but it was proved in [2, Theorem 3] (see also [6]) that I has compact level sets in M_+; therefore $\{m_r\}$ has a convergent subsequence $\{m_{r_s}\}$ in M_+. Now

$$\begin{aligned}\|n_r\| = n_r(B) = \int_B \sigma_r(k)\tilde{m}_r(dk) &\leq \left\{\int_B \sigma_r(k)^2\tilde{m}_r(dk)\right\}^{\frac{1}{2}} \{\tilde{m}_r(B)\}^{\frac{1}{2}} \\ &\leq \{\nu(B) + \tilde{m}_r(B)\}^{\frac{1}{2}}\{\tilde{m}_r(B)\}^{\frac{1}{2}}\end{aligned}$$

by inequality (1.11).

Therefore

$$\|n_{r_s}\| \leq \{\nu(B) + \|m_{r_s}\|\}$$

and since B is compact and $\|m_{r_s}\|$ converges, $\{\|n_{r_s}\|\}$ is uniformly bounded. But $n_{r_s}(B^c) = 0$ and so n_{r_s} has a convergent subsequence. Thus we have proved that (m_r, n_r) has a convergent subsequence.

□

In the following proposition we collect together the properties of minimizers of $\mathcal{E}^{\mu}_{uv}$ which we shall need. If $m \in M_+$ we shall denote its singular part in the Lebesgue decomposition with respect to ν by m_s.

Proposition 4. *Let $(m,n) \in F$ be a minimizer of $\mathcal{E}^{\mu}_{uv}$ then*

(i) $\rho(k) > 0$, ν-a.e.

(ii) $(\rho(k) + \rho(-k))|\sigma(k)| < \rho(k) + \rho(-k) + 1$ ν-a.e.

(iii) $\sigma(k) = 0$ ν-a.e. *for* $k \in B^c$.

(iv) $|\sigma(k)| = 1$ $\tilde{m}_s$-a.e. *for* $k \in B$.

(v) Either $\sigma(k) = 0$ $\tilde{m}$-a.e. *for* $k \in B$ *or* $|\sigma(k)| > 0$ ν-a.e. *for* $k \in B$.

(vi) If $m_s(B) > 0$ *then* $|\sigma(k)| > 0$ ν-a.e. *for* $k \in B$.

Proof: (i) By (1.12) we have that if $\rho(k) = 0$ on a set of non-zero ν-measure then $\sigma(k) = 0$ and therefore $R(k) = \rho(k) = 0$ on this set; since $s'(0) = \infty$ this value of $\mathcal{E}(m,n)$ can be decreased (see [2] Lemma 5.2)

(ii) By (1.11) $(\rho(k)+\rho(-k))^2|\sigma(k)|^2 < (\rho(k)+\rho(-k))(\rho(k)+\rho(-k)+2) = (\rho(k)+\rho(-k)+1)^2 - 1$.

(iii) follows from the fact that s is increasing.

(iv) We know that $|\sigma(k)| \leq 1$; $\langle \nu, s \circ R \rangle$ is unchanged if σ is changed on a set of zero ν-measure. Now

$$\langle n, u\bar{n} \rangle = -\iint_{B\times B} |\sigma(k)||\sigma(k')||u(k,k')| \cos(\alpha(k) - \alpha(k'))\tilde{m}(dk)\tilde{m}(dk'),$$

where $\alpha(k) = \arg \sigma(k)$. Therefore $\alpha(k) = \text{const}$ $\tilde{m}$-a.e. and $|\sigma(k)| = 1$ $\tilde{m}_s$ - a.e. for $k \in B$.

(v) Let $A = \{k \in B : \sigma(k) = 0\}$; note that $\tilde{m}_s(A) = 0$ since $|\sigma(k)| = 1$ $\tilde{m}_s$ - a.e. for $k \in B$. Let $\hat{\sigma}(k) = \sigma(k) + \epsilon \mathbf{1}_A(k)\frac{\sqrt{\rho(k)\rho(-k)}}{\tilde{\rho}(k)}$ where $0 < \epsilon < 1$ and $\tilde{\rho}(k) = \frac{1}{2}(\rho(k) + \rho(-k))$; let $\hat{n}(dk) = \hat{\sigma}(k)\tilde{m}(dk)$, then

$$\begin{aligned}\mathcal{E}^{\mu}_{uv}(m,\hat{n}) - \mathcal{E}^{\mu}_{uv}(m,n) = &\frac{\epsilon^2}{2}\iint_{A\times A} \tilde{\rho}(k)u(k,k')\tilde{\rho}(k')\nu(dk)\nu(dk') \\ &- \epsilon\int_A \nu(dk)\frac{\sqrt{\rho(k)\rho(-k)}}{\tilde{\rho}(k)}\int_{B\setminus A} \tilde{m}(dk')|u(k,k')||\sigma(k')| \\ &+ \frac{1}{\beta}\int_A (s(R(k)) - s(\hat{R}(k)))\nu(dk),\end{aligned}$$

where $\hat{R}(k) = \{(\tilde{\rho}(k) + \frac{1}{2})^2 - \tilde{\rho}(k)^2|\hat{\sigma}(k)|^2\}^{\frac{1}{2}} + \frac{1}{2}(\rho(k) - \rho(-k) - 1)$.
Since s is concave we have for $k \in A$

$$\begin{aligned}&\frac{1}{\beta}(s(R(k)) - s(\hat{R}(k))) \leq \beta^{-1}(R(k) - \hat{R}(k))s'(R(k)) \\ &\leq \frac{R(k) - \hat{R}(k)}{\beta R(k)} = \frac{\epsilon^2 \rho(k)\rho(-k)}{\beta R(k)(R(k) + \hat{R}(k) - \rho(k) + \rho(-k) + 1)} \leq \frac{\epsilon^2}{\beta}\rho(-k)\end{aligned}$$

since $R(k) = \rho(k)$ for $k \in A$, $R(k) - \frac{1}{2}\rho(k) + \frac{1}{2}\rho(-k) > 0$ and $\hat{R}(k) - \frac{1}{2}\rho(k) + \frac{1}{2}\rho(-k) > 0$. Therefore

$$\frac{1}{\beta}\int_A (s(R(k)) - s(\hat{R}(k)))\nu(dk) \leq \frac{\epsilon^2}{\beta}\|m\|$$

and thus

$$\mathcal{E}^{\mu}_{uv}(m,\hat{n}) - \mathcal{E}^{\mu}_{uv}(m,n) \leq -\epsilon\int_A \nu(dk)\int_{B\setminus A} m(dk')|u(k,k')||\sigma(k')| + O(\epsilon^2).$$

If $\int_A \frac{\sqrt{\rho(k)\rho(-k)}}{\tilde{\rho}(k)}\nu(dk)\int_{B\setminus A}\tilde{m}(dk')|u(k,k')||\sigma(k')| \neq 0$, $\mathcal{E}^{\mu}_{uv}(m,\hat{n}) < \mathcal{E}^{\mu}_{uv}(m,n)$ if ϵ is sufficiently small contradicting the assumption that m is a minimizer; therefore

$$\int_A \frac{\sqrt{\rho(k)\rho(-k)}}{\tilde{\rho}(k)}\nu(dk)\int_{B\setminus A}\tilde{m}(dk')|u(k,k')||\sigma(k')| = 0.$$

Therefore either $\nu(A) = 0$ or $\tilde{m}(B \setminus A) = 0$
(vi) $\tilde{m}_s(B) = \tilde{m}_s(B \setminus A)$; therefore if $\tilde{m}_s(B) > 0$ then $\tilde{m}_s(B \setminus A) > 0$ and so by (v) $\nu(A) = 0$.

□

We shall now give a set of Euler-Lagrange equations for the variational problem under consideration. It is convenient to a introduce a new variable c where $c(k) = \tilde{\rho}(k)|\sigma(k)|$; we know that we can assume the $c(k) > 0$ ν-a.e. in B and $c(k)^2 < \tilde{\rho}(k)(\frac{1}{2} + \tilde{\rho}(k))$ ν-a.e. Since we can also assume that $|\sigma(k)| = 1$ $\tilde{m}_s$-a.e. our variational problem is equivalent to minimizing the following functional:

$$\begin{aligned}\tilde{\mathcal{E}}(m, c) = &\int_{\mathbb{R}^d} (\epsilon(k) - \mu) m(dk) + \frac{1}{2}\langle m, vm\rangle \\ &+ \frac{1}{2}\iint_{B\times B} c(k) u(k,k')\bar{c}(k')\nu(dk)\nu(dk') \\ &+ \frac{1}{2}\iint_{B\times B} c(k) u(k,k')\nu(dk)\tilde{m}_s(dk') \\ &+ \frac{1}{2}\iint_{B\times B} \bar{c}(k) u(k,k')\nu(dk)\tilde{m}_s(dk') \\ &+ \frac{1}{2}\iint u(k,k')\tilde{m}_s(dk)\tilde{m}_s(dk') - \frac{1}{\beta}\int_{\mathbb{R}^d} s(R(k))\nu(dk),\end{aligned}$$

where

$$R(k) = \{(\tilde{\rho}(k) + \frac{1}{2})^2 - |c(k)|^2\}^{\frac{1}{2}} + \frac{1}{2}(\rho(k) - \rho(-k) - 1).$$

By varying m_a, m_s and c we obtain the following Euler-Lagrange equations. Let

$$L(m,k) = \epsilon(k) - \mu - (vm)(k), \tag{3.5}$$

$$E(k) = \frac{1}{2}\{[L(m,k) + L(m,-k)]^2 - |(un)(k) + (un)(-k)|^2\}^{\frac{1}{2}} \tag{3.6}$$

and

$$E_{\pm}(k) = E(k) \pm \frac{1}{2}[L(m,k) - L(m,-k)]; \tag{3.7}$$

then we have:

Proposition 5. *If (m, n) is a minimizer of $\mathcal{E}^{\mu}_{uv}$ then (m, n) satisfies the following Euler-Lagrange equations:*

$$\begin{aligned}2\rho(k) + 1 = &\frac{1}{2}\left[\frac{L(m,k) + L(m,-k)}{2E(k)} + 1\right] \coth\frac{\beta}{2}E_+(k) \\ &+\frac{1}{2}\left[\frac{L(m,k) + L(m,-k)}{2E(k)} - 1\right] \coth\frac{\beta}{2}E_-(k) \qquad \nu\text{-a.e.};\end{aligned} \tag{3.8}$$

if $\sigma \not\equiv 0$, for $k \in B$

$$\frac{\sigma(k)\tilde{\rho}(k)}{(un)(k)+(un)(-k)} = -\frac{1}{8}\frac{\coth\frac{\beta}{2}E_+(k)+\coth\frac{\beta}{2}E_-(k)}{E(k)} \qquad \nu\text{−a.e.} \tag{3.9}$$

and

$$L(m,k)+\frac{1}{4}\{(un)(k)+(un)(-k)+(u\bar{n})(k)+(u\bar{n})(-k)\}=0 \qquad m_s\text{−a.e.} \tag{3.10}$$

We note that if (m,n) is a minimizer of $\mathcal{E}^{\mu}_{uv}$ then as a consequence of the Euler-Lagrange equations we have:

$$p^{uv}=\frac{1}{2}\langle m,vm\rangle+\frac{1}{2}\langle\bar{n},un\rangle-\frac{1}{\beta}\int_{\mathbb{R}^d}\ln(1-e^{-\beta E_+(k)})\nu(dk). \tag{3.11}$$

§4. The variational problem with u and v constant

We shall now study the variational problem in the special case when

(1) $v(k,k')=a>0$

(2) $B=\{k:\|k\|<r\}$

(3)

$$u(k,k')=\begin{cases}-\gamma & \text{if } (k,k')\in B\times B,\\ 0 & \text{otherwise,}\end{cases}$$

where $0<\gamma<a$.

It is clear in this case that if m is a minimizer of $\mathcal{E}^{\mu}_{uv}$ and $m_s\neq 0$ then m_s is concentrated at $k=0$, that is $\|m_s\|=m\{0\}$. Equations (3.8) and (3.9) now become

$$\rho(k)=\frac{1}{2}\left[\frac{(\epsilon(k)-\mu+a\|m\|)}{E(k)}\coth\frac{\beta}{2}E(k)-1\right] \tag{4.1}$$

and if $\sigma\neq 0$ for $k\in B$

$$\frac{\sigma(k)\rho(k)}{\gamma\int n(dk)}=\frac{\frac{1}{2}\coth\frac{\beta}{2}E(k)}{E(k)}, \tag{4.2}$$

where

$$\begin{aligned}E^2(k)&=(\epsilon(k)-\mu+a\|m\|)^2-\gamma^2\left(\int n(dk)\right)^2\\&=(\epsilon(k)-\mu+a\|m\|)^2 \text{ for } k\notin B.\end{aligned} \tag{4.3}$$

Here (3.10) becomes

$$a\|m\|-\mu=\gamma\int n(dk) \text{ if } m\{0\}\neq 0. \tag{4.4}$$

From (4.1) it is clear that

$$\epsilon(k) - \mu + a\|m\| \geq 0 \quad \nu - \text{ a.e.};$$

letting $k \to 0$ we get $a\|m\| - \mu \geq 0$. Also from (4.3) we see that for $k \in B$

$$\gamma \int n(dk) \leq \epsilon(k) - \mu + a\|m\|$$

and again letting $k \to 0$ we get

$$\gamma \int n(dk) \leq a\|m\| - \mu.$$

We introduce *new variables* $x \geq y \geq 0$, where $x = a\|m\| - \mu$ and $y = \gamma \int n(dk)$. In terms of x and y

$$E(k) = \begin{cases} ((\epsilon(k) + x)^2 - y^2)^{\frac{1}{2}} & \text{for } k \in B \\ \epsilon(k) + x & \text{for } k \notin B \end{cases}$$

and equations (4.1) and (4.2) can be re-written as

$$\rho(k) = \frac{1}{2}\left[\frac{\epsilon(k) + x}{E(k)} \coth \frac{\beta}{2} E(k) - 1\right], \tag{4.5}$$

$$\sigma(k)\rho(k) = \frac{1}{2} y \coth \frac{\frac{\beta}{2} E(k)}{E(k)}, \text{ for } k \in B. \tag{4.6}$$

Equation (4.4) implies that $x = y$ if $m_0 \equiv m\{0\} \neq 0$.

Consider the case when $m_0 = 0$. In this case we can integrate (4.5) to get

$$x + \mu = \frac{a}{2} \int_{\mathbb{R}^d} \left[\frac{\epsilon(k) + x}{E(k)} \coth \frac{\beta}{2} E(k) - 1\right] \nu(dk) \tag{4.7}$$

and integrating (4.5) we get if $y \neq 0$

$$\alpha = \frac{a}{2} \int_B \frac{1}{E(k)} \coth \frac{\beta}{2} E(k) \nu(dk), \tag{4.8}$$

where $\alpha = \frac{a}{\gamma} > 1$. Let $\rho_c = \int \frac{1}{e^{\beta\epsilon(k)} - 1} \nu(dk)$. It is straightforward to check that if $\mu < a\rho_c$ then the equation

$$x + \mu = \frac{a}{2} \int_{\mathbb{R}^d} \left[\coth \beta \frac{(\epsilon(k) + x)}{a} - 1\right] \nu(dk)$$

has a unique solution $x(\mu)$; then

$$m(dk) = \frac{1}{2}\left[\coth\beta\frac{(\epsilon(k)+x(\mu))}{2} - 1\right]\nu(dk), \qquad n = 0 \tag{4.9}$$

is a solution of the Euler-Lagrange equations.

Let us consider now the case when $m_0 \neq 0$. In this case $x = y$ and we know from Proposition 4 (vi) that $y \neq 0$. Letting

$$\tilde{E}(k) = \begin{cases} ((\epsilon(k)+x)^2 - x^2)^{\frac{1}{2}} & \text{for } k \in B \\ \epsilon(k) + x & \text{for } k \notin B \end{cases}$$

and integrating equations (4.5) and (4.6) we now get

$$x + \mu - am_0 = \frac{1}{2}a\int_{\mathbb{R}^d}\left[\frac{\epsilon(k)+x}{\tilde{E}(k)}\coth\beta\frac{\tilde{E}}{2}(k) - 1\right]\nu(dk) \tag{4.10}$$

$$\frac{\alpha x - am_0}{x} = \frac{1}{2}a\int_B \frac{1}{\tilde{E}(k)}\coth\beta\frac{\tilde{E}}{a}\nu(dk). \tag{4.11}$$

Equivalently

$$\begin{aligned}(\alpha-1)x - \mu = {} & \frac{1}{2}a\int_B\left[1 - \frac{\epsilon(k)}{\tilde{E}(k)}\coth\beta\frac{\tilde{E}(k)}{2}\right]\nu(dk) \\ & - a\int_{B^c}\frac{1}{e^{\beta(\epsilon(k)+x)}-1}\nu(dk)\end{aligned} \tag{4.12}$$

and

$$m_0 = \frac{1}{a}(x+\mu) - \frac{1}{2}\int_{\mathbb{R}^d}\left[\frac{\epsilon(k)+x}{\tilde{E}(k)}\coth\beta\frac{\tilde{E}}{2}(k) - 1\right]\nu(dk). \tag{4.13}$$

Therefore, the Euler-Lagrange equations have a solution with $m_0 > 0$ if and only if (4.12) has a solution with the right-hand side of (4.13) strictly positive.

The expression (3.13) for the pressure now becomes:

$$p^{uv}(\mu) = \frac{1}{2a}(x+\mu)^2 - \frac{1}{2\gamma}|y|^2 - \frac{1}{\beta}\int^{\mathbb{R}^d}\ln(1 - e^{-\beta E(k)})\nu(dk).$$

$E(k)$ is the spectrum of the elementary excitations of the model; we note that if $m_0 = 0$ then it is possible that $x > y$ and

$$\lim_{k\to 0} E(k) = (x^2 - y^2)^{\frac{1}{2}} > 0,$$

while if $m_0 \neq 0$ then $x = y$ and

$$E(k) = \tilde{E}(k) \sim (2\epsilon(k)x)^{\frac{1}{2}}$$

for small k. This means that in our model with the interaction defined by $u(k,k')$ and $v(k,k')$ as in (1.8), the *occurrence* of Bose-Einstein condensation *produces a phonon spectrum* for small k, while the *absence* of condensation creates a *gap*. It was observed in [8, 20, 21] that for the pair Hamiltonian (1.2) there is a gap in the spectrum. However, in contrast with our model, in this case the *gap* appears when Bose-Einstein condensation *occurs* and various attempts were made to rectify this unphysical behaviour of the model (1.2) [10, 22-26]. Our model is thus more satisfactory from the physical point of view; this is achieved at a price, namely a deformation of the original interaction (1.2) to the model (1.8) and a particular choice of the kernels $u(k,k')$ and $v(k,k')$.

To study the problem further we shall make the following definitions

$$I_1(x,y) = \frac{1}{2}a\int_B \frac{1}{E(k)}\coth\frac{\beta}{2}E(k)\nu(dk) \quad x \geq y \geq 0, \tag{4.14}$$

$$I_2(x,y) = \frac{1}{2}a\int_{\mathbb{R}^d}\left[\frac{\epsilon(k)+x}{E(k)}\coth\frac{\beta}{2}E(k) - 1\right]\nu(dk) \quad x \geq y \geq 0. \tag{4.15}$$

Let $x_1(y;\alpha)$ be the solution of $I_1(x,y) = \alpha$ as an equation in x for the values of y for which it exists ($x_1(y,\alpha)$ exists for all α) and similarly let $x_2(y;\mu)$ be the solution of $I_2(x,y) = x + \mu$ for the values of y and μ for which it exists. The properties of I_1, I_2, x_1 and x_2 are given in the appendix.

Let $\tilde{I}_1(x) = I_1(x,x)$ and $\tilde{I}_2(x) = I_2(x,x)$; let $\mu_0 = \sup_{x\geq 0}(\tilde{I}_2(x) - x)$. Since $\tilde{I}_2(0) = a\rho_c$, $\mu_0 \geq a\rho_c$. Finally let

$$A(x) = \frac{1}{2}a\int_B\left[1 - \frac{\epsilon(k)}{\tilde{E}(k)}\coth\frac{\beta\tilde{E}(k)}{2}\right]\nu(dk) - a\int_{B^c}\frac{1}{e^{\beta(\epsilon(k)+x)}-1}\nu(dk). \tag{4.16}$$

$x \mapsto A(x)$ is strictly increasing and strictly concave; $A'(0) = \infty$, $A(0) = -a\rho_c$ and $A(x) \to \frac{1}{2}aK$ as $x \to \infty$ where $K = \nu(B)$.

Consider the equation (see figure 1)

$$(\alpha - 1)x - \mu = A(x). \tag{4.17}$$

If $\mu > a\rho_c$ this equation has a unique solution. For $\alpha > 1$, let $\mu_1(\alpha)$ be the unique value of μ $-\frac{1}{2}aK < \mu < a\rho_c$ such that (4.17) has a unique solution. Then for fixed α, we have the following:

(i) if $\mu > a\rho_c$, (4.17) has a unique solution,

(ii) if $a\rho_c \geq \mu > \mu_1(\alpha)$, (4.17) has two solutions,

(iii) if $\mu < \mu_1(\alpha)$, (4.17) has no solutions.

We remark that if x^* is a solution of (4.17), then by (4.13) x^* corresponds to a solution of the Euler-Lagrange equations if

$$x^* + \mu - \check{I}_2(x^*) \geq 0.$$

Let $\check{x}_1(\alpha)$ be the solution of $x_1(y,\alpha) = y$ that is the value of x (or y) when $x_1(y,\alpha)$ hits $x = y$. Then $\check{I}_1(\check{x}_1(\alpha)) = \alpha$ and therefore at $x = \check{x}_1(\alpha)$, $\frac{d\check{I}_1}{dx}(x)\frac{dx}{d\alpha} = 1$; but $\frac{d\check{I}_1}{dx}(x) < 0$ and therefore $\frac{dx}{d\alpha} < 0$ so that $\alpha \mapsto \check{x}_1(\alpha)$ is decreasing. Also as $\alpha \to 0$, $\check{x}_1(\alpha) \to \infty$ and as $\alpha \to \infty$, $\check{x}_1(\alpha) \to 0$.

If x satisfies $\check{I}_2(x) = x + \mu$, then $x < \check{x}_1(\alpha)$ implies that $A(x) > (\alpha - 1)x - \mu$ and $x > \check{x}_1(\alpha)$ implies that $A(x) < (\alpha - 1)x - \mu$. This follows from the identity

$$A(x) = x\check{I}_1(x) - \check{I}_2(x) \tag{4.18}$$

and the fact that $\check{I}_1$ is decreasing.

We now solve the variational problem in some regions of the α-μ plane; we have not been able to exhaust the α-μ plane, however our results give an indication of what can occur. In figure 2 we have labeled the regions of the α-μ plane we can deal with. par**Region A:** $\mu > \mu_0$. This is the simplest case. We have in this case that for all $x > 0$

$$x + \mu - I_2(x,y) \geq x + \mu - \check{I}_2(x) \geq \mu - \mu_0 > 0,$$

since $y \mapsto I_2(x,y)$ is decreasing; therefore $x + \mu - I_2(x,y)$ has no solution for any y. On the other hand if x^* is the unique solution of (4.17) then

$$m_0^* \equiv \frac{1}{a}\{x^* + \mu - \check{I}_2(x^*)\} \geq \frac{\mu - \mu_0}{a} > 0;$$

this means that $\mathcal{E}^{\mu}_{uv}$ has a *unique minimizer* (m^*, n^*) where

$$\begin{aligned} m^*(dk) &= m_0^*\delta_0 + \rho^*(k)\nu(dk) \\ n^*(dk) &= m_0^*\delta_0 + \sigma^*(k)\rho^*(k)\nu(dk) \end{aligned} \tag{4.19}$$

where

$$\rho^*(k) = \frac{1}{2}\left[\frac{\epsilon(k) + x^*}{\check{E}(k,x^*)} \coth \frac{1}{2}\beta\check{E}(k,x^*) - 1\right] \tag{4.20}$$

and

$$\sigma^*(k)\rho^*(k) = \frac{1}{2}x^* \coth \frac{1}{2}\beta\check{E}(k,x^*). \tag{4.21}$$

Before we examine the other regions we make some general remarks. For each $\alpha > 1$ there is a unique value of μ such that

$$A(\tilde{x}_1(\alpha)) - (\alpha - 1)\tilde{x}_1(\alpha) + \mu = 0;$$

let $\mu_2(\alpha)$ be this value of μ. Note that $\mu_1(\alpha) \leq \mu_2(\alpha) \leq \mu_0$. The shape of the curve $\mu_2(\alpha)$ is given in figure 2. Let $\mu > \mu_2(\alpha_0)$ and let (α_1, μ), (α_2, μ) with $1 \leq \alpha_1 \leq \alpha_2 \leq \infty$ be the endpoints of the segment parallel to the α-axis which is contained in $\mu \geq \mu_2(\alpha)$ and contains the point (α_0, μ). Let $x_U = \tilde{x}_1(\alpha_1)$ and $x_L = \tilde{x}_1(\alpha_2)$; since both x_U and x_L satisfy

$$\tilde{I}_2(x) = x + \mu,$$

if $x_L < x < x_U$ then $\tilde{I}_2(x) < x + \mu$ (figure 3). Also since $\alpha_1 < \alpha_0 < \alpha_2$, $x_L < \tilde{x}_1(\alpha) < x_U$ and thus

$$A(x_U) - (\alpha - 1)x_U + \mu < 0 < A(x_L) - (\alpha - 1)x_L + \mu.$$

Therefore there is a solution x^* of

$$A(x^*) - (\alpha - 1)x^* + \mu = 0,$$

satisfying $x_L < x^* < x_U$ and thus

$$x^* + \mu - \tilde{I}_2(x^*) > 0.$$

This means that there is a solution of the Euler-Lagrange equations of the form (4.19).

Region B: We have seen that above that in this region there is a solution of the Euler-Lagrange equations with $m_0 > 0$. Here $\alpha_2 = \infty$ so that $x_L = 0$ and for $x < x_U$, $\tilde{I}_2(x) < x + \mu$; therefore since $x_1(y, \alpha)$ and $x_2(y, \mu)$ are increasing they never intersect. Since for $\mu > a\rho_c$ a solution with $y = 0$ and $m_0 = 0$ is *not possible*, the situation is as in region A.

Region C: Similar arguments show that there is a *unique solution* of the Euler-Lagrange equations with $n \neq 0$ and $m_0 = 0$.

Regions D, E, F: Again the above argument shows that there is a solution of the Euler-Lagrange equations with $m_0 \neq 0$ but we *cannot exclude other solutions*.

Region G: If $\mu < \mu_1(\alpha)$ then $A(x) = (\alpha - 1)x - \mu$ has no solutions and therefore there is no solution with $m_0 \neq 0$. In general *we cannot determine whether* $n = 0$.

Region H: Here $\mu < \mu_3(\alpha)$ where $\mu_3(\alpha)$ is the value of μ which satisfies $\tilde{x}_1(\alpha) = x_2(0,\mu)$. Since $\tilde{x}_1(\alpha) < x_2(0,\mu)$, $x_1(y,\alpha)$ and $x_2(y,\mu)$ cannot intersect. Therefore the Euler-Lagrange equations have *only one solution* with $n = 0$.

Finally we remark that if $\alpha < x_2(0,\mu) + aK + \mu$, then

$$I_1(x_1(0,\alpha),0) = \alpha < x_2(0,\mu) + aK + \mu = I_2(x_2(0,\mu),0) + aK = I_1(x_2(0,\mu),0).$$

Therefore $x_1(0,\alpha) > x_2(0,\mu)$, and so $x_1(y,\alpha)$ and $x_2(y,\mu)$ cross and the Euler-Lagrange equations have a solution with $n \neq 0$. It is possible to check that in this region the solution with $n = 0$ does not correspond to a minimizer. As $\mu \to -\frac{1}{2}aK$, $\alpha_0(\mu) \to 1$, therefore if $\frac{1}{2}aK > 1$ it is possible to satisfy $\alpha_0(\mu) < \alpha < x_2(0,\mu) + aK + \mu$.

Concluding Remarks:

(a) The presence of Bose-Einstein condensation $m_0 \neq 0$ causes abnormal pairing $n \neq 0$ (the Hamiltonian (1.8) is gauge invariant). In this case there is no gap in the spectrum and we expect both the one-particle and two-particle reduced density matrices to display off-diagonal long-range order (ODLRO).

(b) There is a region where $m_0 = 0$ while $n \neq 0$; in this case we expect ODLRO to occur in the two-particle reduced density matrices but not in the one-particle reduced density matrices. There is the possibility of a gap in the spectrum of excitations.

(c) For small μ (region H) we do not expect ODLRO and the model (1.8) is equivalent to (1.5); there is no gap in the spectrum.

The possibility of "two-stage" condensation, that is, condensation in the one-particle and two-particle states was discussed in [27]; there the model displays a similar behaviour.

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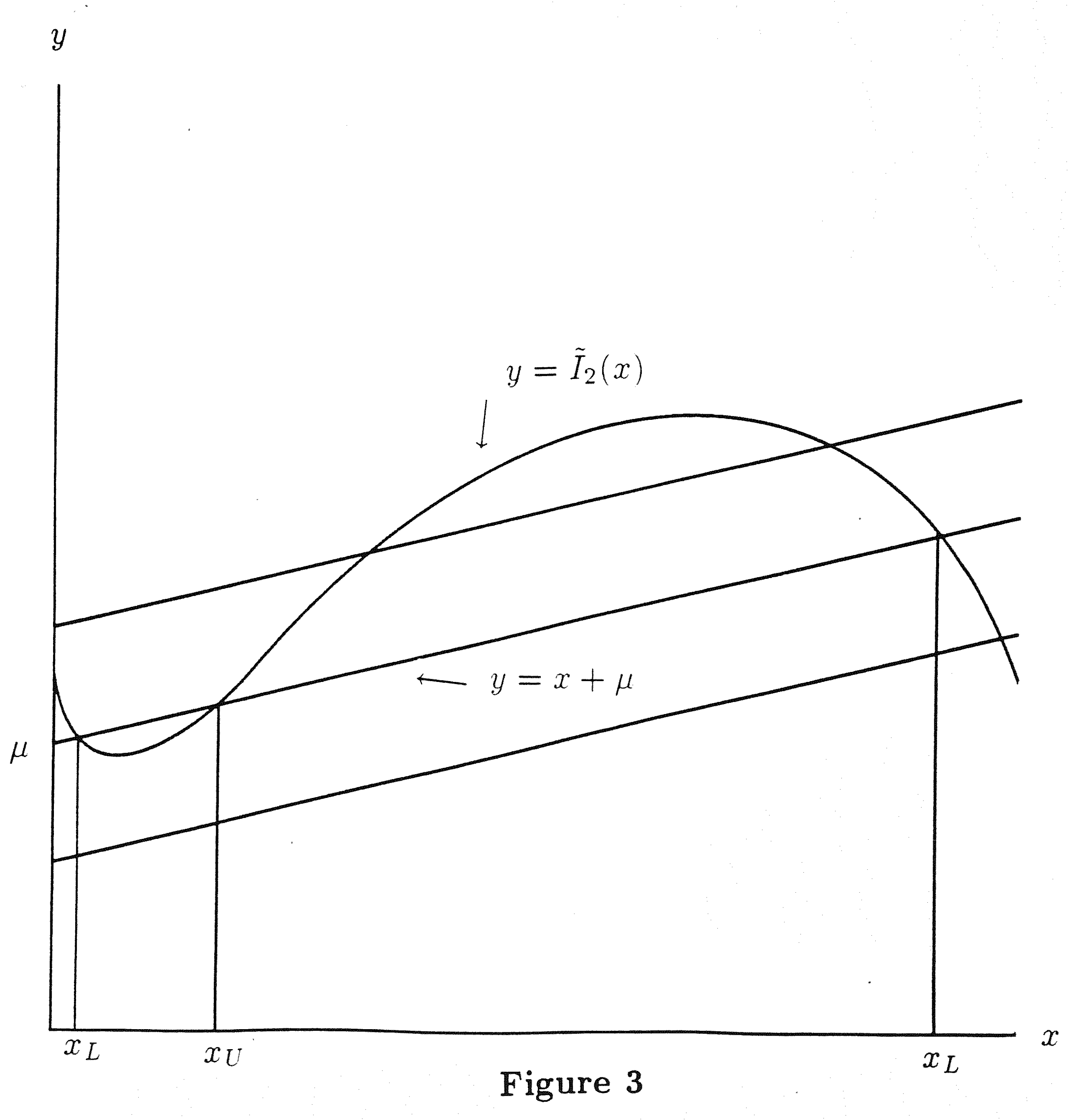

Figure 3

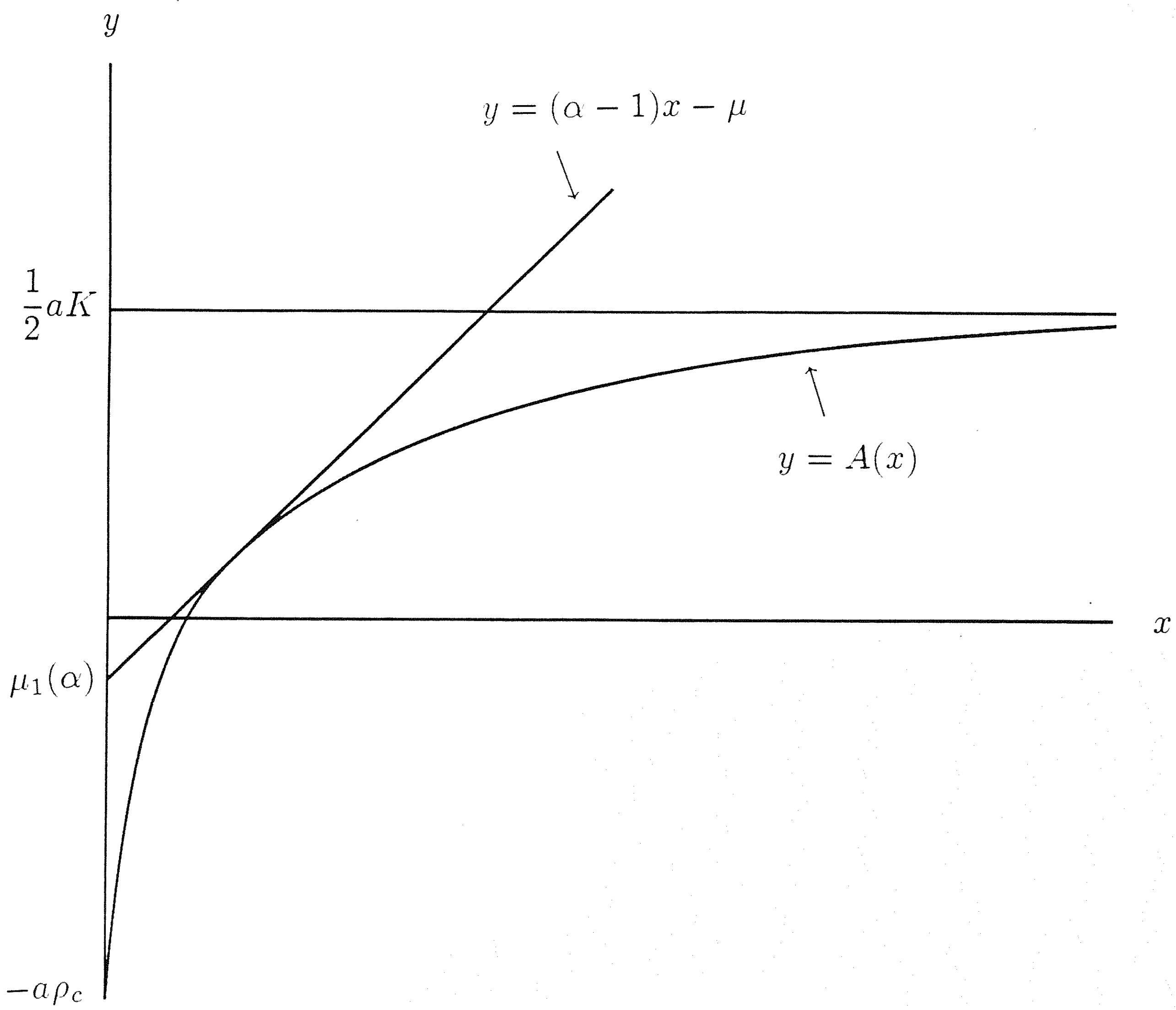

Figure 1

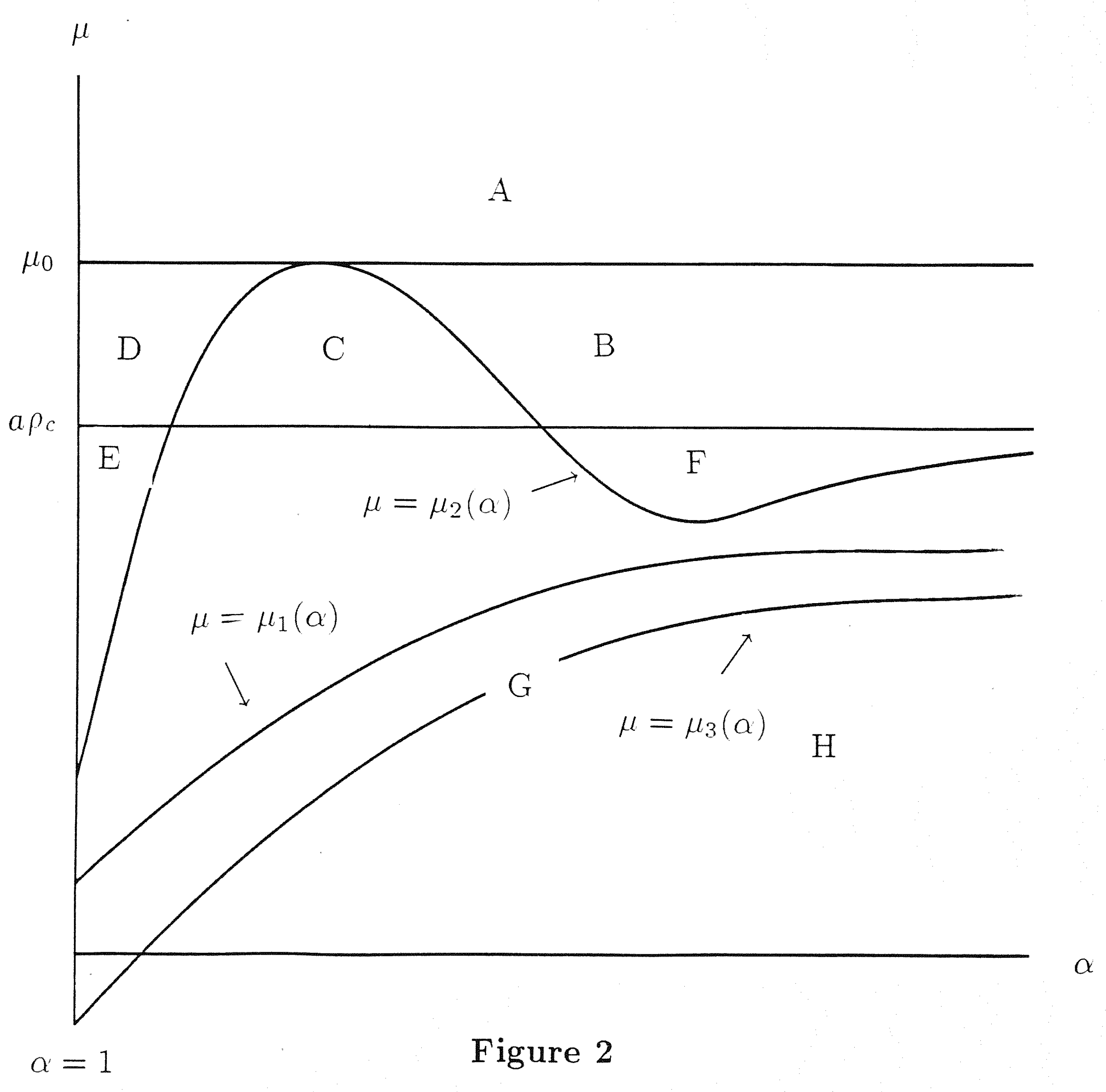
μ
A
μ_0
D
C
B
$a\rho_c$
E
F
$\mu = \mu_2(\alpha)$
$\mu = \mu_1(\alpha)$
G
$\mu = \mu_3(\alpha)$
H
α
$\alpha = 1$

Figure 2