RIGOROUS BOUNDS FOR LOSS PROBABILITIES IN MULTIPLEXERS OF DISCRETE HETEROGENEOUS MARKOVIAN SOURCES

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ABSTRACT.

Exponential upper bounds of the form $\mathbb{P}[\text{queue} \geq b] \leq \varphi y^{-b}$ are obtained for the distribution of the queue length in a model of a multiplexer in which the input is a heterogeneous superposition of discrete Markovian on-off sources. These bounds are valid at all queue lengths, rather than just asymptotic in the limit $b \to \infty$. The decay constant y is found by numerical solution of a single transcendental equation which determines the effective bandwidths of the sources in the limit $b \to \infty$. The prefactor φ is given explicitly in terms of y. The bound provides a means to determine rigorous corrections to effective bandwidths for multiplexers with finite buffers.

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1. INTRODUCTION.

The problem of finding the queue length distribution in a queue with non-independent arrivals has attracted much attention recently due to applications in the design of multiplexers for the emergent asynchronous transfer mode (ATM) of data transmission in integrated services digital networks (ISDN). From the technological point of view it is required to guarantee sufficiently good quality of service: loss probabilities must be appropriately small and waiting times sufficiently short. The problem is resistant to simple exact treatment due to the nature of the arrival process. It is a superposition of sources which are typically bursty, in the sense that their activity is highly correlated into bursts rather than occurring independently at different times; and periodic (when viewed at the short time scales of the multiplexer output) either due to their origin (e.g. periodic sampling of voice traffic) or their occupation of periodic slots allocated for transmission. The goal of analysis is to provide mechanisms for design and performance prediction, and algorithms for allocation of resources during the operation of such devices. It is desirable that the results of such analysis be both robust (e.g. with respect to the uncertainties of modelling the sources) and conservative (i.e. that they should not overestimate the capacity of resources).

In this paper exponential upper bounds for the tail probabilities $\mathbb{P}[\text{queue} \geq b] \leq \varphi y^{-b}$ are obtained for a queue whose input is a heterogeneous super-position of discrete time Markovian on-off sources. In the model the queueing discipline is first-come first-served (FCFS) with an infinite buffer. With this queueing discipline the tail probabilities bound from above those of the corresponding model with a finite buffer of size b. This estimate is conservative in the sense that, because it is an upper bound, any calculation of resource allocation based upon it will overestimate rather than underestimate the resources required to provide a given quality of service. We stress that these bounds are valid *for all $b > 0$*, not just in the limiting regime $b \to \infty$, so that such calculations can be made for buffers of all lengths.

We specify the model precisely. The multiplexer has L Markovian sources. These sources are mutually independent and are divided into groups which we will label by $i = 1, 2, \ldots, I$. Within each group, the sources are statistically identical. Group i comprises L_i sources, and $\sum_i L_i = L$. In group i each source is as follows: a Markov chain on the state space $\{0, 1\}$, these states corresponding to the source configurations *silent* and *active*, respectively. The probability of transition from the silent state to the active state is a_i whereas the probability of the inverse transition is d_i. So the Markov transition matrix for one such line can be written as

$$T^i = \begin{pmatrix} 1 - a_i & d_i \\ a_i & 1 - d_i \end{pmatrix} \ .$$

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The quantity $a_i/(a_i + d_i)$ is called the *activity* of the source: this is the stationary probability that the source is active. In other words, T^i has stationary distribution $\mu_i = (d_i, a_i)/(a_i + d_i)$. Furthermore, the Markov chain is reversible: $T^i_{nm}\mu_m = T^i_{mn}\mu_n$ for all $n, m \in \{0, 1\}$. The mean lengths of silence and activity are $1/a_i$ and $1/d_i$ respectively. The quantity $1 - (a_i + d_i)$ can be used as a measure of the *burstiness* of the source, in terms of the correlations between the state ω_t of the source at successive times t and $t + 1$:

$$\mathbb{E}[\omega_t \omega_{t+1}] - \mathbb{E}[\omega_t]\mathbb{E}[\omega_{t+1}] = (1 - a_i - d_i)a_i d_i/(a_i + d_i)^2 \quad .$$

When $a_i + d_i = 1$ this covariance is 0 reflecting the fact that the ω_t are independent. We are interested in the case that there are positive correlations between the activities at successive times, which we call the

$$\text{bursty regime:} \qquad a_i + d_i < 1 \quad \text{for all } i \quad , \tag{1.1}$$

since in practice one tries to model source traffic whose bursts of activity and intervening silences are long compared with those of independent arrivals with the same activity. Typical values of a_i and d_i might be of the order 10^{-3} or smaller.

The queue operates as follows. Let z_t^i denote the number of lines in group i which are in the active state at each integral time t, and set $z_t = \sum_i z_t^i$. At each such time all active lines empty one cell into the buffer of the queue. The queue has a constant service rate s cells per period. Denoting by q_t the size of the queue at time t then we have the iteration

$$q_{t+1} = \max[0, q_t + z_t - s] \quad .$$

In what follows we obtain a bound on the queue length at time 0. Since the individual sources are reversible this quantity has the same distribution as $q := \lim_{t \to \infty} q_t$ where $q_t = \max[0, z_1 - s, z_1 + z_2 - 2s, \dots, z_1 + z_2 + \dots + z_t - ts]$.

In order that the queue does not permanently overload we require that the total activity over all inputs is less than the service rate of the queue, in other words that the

$$\text{stability condition:} \qquad \sum_i L_i a_i/(a_i + d_i) < s \tag{1.2}$$

is satisfied. The condition that loss probabilities from a finite buffer do not exceed a given proportion is more stringent. As we discuss below, much recent work has focused finding *effective bandwidths* σ_i for sources of various types i. σ_i is the amount of service capacity which must be allocated to each source of type i in a heterogeneous superposition if the loss probability for cells from *any* source in the

superposition is not to exceed a certain amount. (See [7,8,9] for discussions of effective bandwidths in general). In the asymptotic regime when the buffer size $b \to \infty$, this interpretation follows if one can show that the following linear constraint, the

$$\text{bandwidth condition:} \qquad \sum_i L_i \sigma_i < s \quad . \qquad (1.3)$$

implies that $\lim_{b\to\infty} b^{-1} \log \mathbb{P}[q \geq b] \leq \log y$ for the appropriate decay constant y.

The contribution of the present paper, stated in Theorem 1 below, is to establish exponential upper bounds $\mathbb{P}[q \geq b] \leq \varphi y^{-b}$ on the loss probability *for all* buffer sizes b, rather than asymptotic approximations for large b. This opens the way to finding rigorous constraints on the L_i at all finite queue lengths in order to guarantee sufficiently small loss probabilities, rather than in just the asymptotic case $b \to \infty$. We hope to investigate this application in a subsequent paper.

Theorem 1. *In the bursty regime of (1.1) when the stability condition (1.2) is satisfied, then for all $b > 0$, the tail $\mathbb{P}[q \geq b]$ of the queue length distribution is bounded above:*

$$\mathbb{P}[q \geq b] \leq \varphi(b, y) y^{-b} \qquad (1.4)$$

where φ is an explicit function of y and b which is polynomial in b, and y is the unique solution of the following implicit equation, the bandwidth equation:

$$\sum_i L_i \hat{\sigma}_i(y) = s \qquad (1.5)$$

where each $\hat{\sigma}_i(y)$ is an explicit function of y.

The detailed forms of φ and σ_i are given during the proof of this result in section 3. $\hat{\sigma}_i(y)$ is the effective bandwidth in the limiting regime $b \to \infty$ appropriate in order that loss probabilities have asymptotic decay rate y.

Let us set this work in context. The queue length distribution for homogeneous arrivals in a continuous-time Markov fluid-flow model has been treated some time ago in [1]. The corresponding heterogeneous problem has been examined in [11] but evaluation of bandwidths appears unfeasible at finite buffer lengths b. As far as we are aware, further results are confined to the following types. Firstly, one may consider a limiting regime in which $b \to \infty$ [6], in which case the analysis of [11] simplifies considerably. This approach is further developed in [16,4]. An exact treatment of heterogeneous N state Markov modulated arrival processes is given by matrix-geometric methods (see e.g. [12]) in [5]. Whereas one recovers the asymptotic decay constant y fairly easily, further detail of the distribution appear hard to access since the complexity of the algorithm is $O(L^{3(N-1)})$ for L sources. In

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the method of the present paper, the exponential decay rate of the tail probabilities are determined by numerical solution of a single transcendental equation, the rest of the bound being determined explicitly in terms of this rate. The second class of existing results use large deviation methods to find the decay constant y [10,13,15]. In practice this gives asymptotics for large queue lengths rather than bounds for all queue lengths.

The paper is organized as follows. In section 2 we recall result on bounds for queue lengths in homogeneous multiplexers obtain using martingale methods in [2,3]. These are used to bound the exponential moments $\mathbb{E}[u^q]$, where $u > 1$ and q is the queue length random variable (Theorem 2). The point of this is the following. In section 3 we return to the heterogeneous case and notionally divide up the total service rate s amongst the I groups of sources, allocating s_i to each group so that $\sum_i s_i = s$. Then since the sources are independent:

$$\mathbb{P}[q \geq b] \leq \mathbb{P}[\sum_i q_i \geq b] \leq u^{-b} \prod_i \mathbb{E}[u^{q_i}] \quad ,$$

where q_i denotes the queue length in a queue with the *homogeneous* arrivals from sources in group i, with service rate s_i. The bound of Theorem 1 is obtained by solving the variational problem in the quantities u and (s_i) (Propositions 2 and 3).

2. HOMOGENEOUS BOUNDS AND MOMENTS.

We first deal with the homogeneous case i.e. with only one group of statistically identical inputs. Thus we may temporarily dispense with the group index i. Define the excess work $X_t := z_1 + z_2 + \ldots + z_t - ts$ arriving up to time t, so that in this case of one group $q = \sup_{t \geq 1} X_t$. Let $\sigma = s/L$.

It is useful to define the functions

$$\hat{y}(x) := \frac{x(ax + 1 - a)}{(1 - d)x + d}$$

and

$$F(x) := \left(\frac{ax + d}{a + d} \right)^L \frac{1}{x^s} \quad \text{and} \quad G(x) := \frac{\hat{y}(x)^s}{(ax + 1 - a)^L}$$

Our bounds for heterogeneous multiplexers will be based upon the following results on the distributions of activities and queue lengths in homogeneous multiplexers which are proved using martingale methods (see e.g. [14]) in [2,3].

Proposition 1. *In the bursty regime $a + d < 1$:*

(1) One-time distribution: *Each z_t is binomially distributed $\mathbb{P}[z_t = k] = \binom{L}{k} a^k d^{L-k}$ $(a+d)^{-L}$ and hence $\mathbb{E}[x^{z_t - s}] = F[x]$.*

(2) Conditional distribution: $\mathbb{E}[x^{z_{t+1}} \mid z_t] = \left(x/\hat{y}(x)\right)^{z_t} \hat{y}(x)^s / G(x)$.

(3) Stability condition: *If $\sigma > a/(a+d)$ then there exists $x > 1$ with $G(x) = 1$ and $y = \hat{y}(x)$ with $1 < y < x$.*

(4) Upper Bounds: *For (x,y) as in (3) and $b > 0$,*

$$\mathbb{P}[q \geq b \mid z_1] \leq y^{-b} x^{z_1 - s} \quad \text{and hence} \quad \mathbb{P}[q \geq b] \leq y^{-b} F(x) \quad .$$

(5) Bound Prefactor: *For (x,y) as in (3), $F(x') < 1$ for $1 < x' < x$.*

Remarks: (1) follows from the fact that the probability for a single source to be active is $a/(a+d)$. (2) is proved in Proposition 2 of [2]. In (3), $s > La/(a+d)$ is the stability condition (1.2) applied to the homogeneous case. The properties of x and y and of F in (5) are determined in Proposition 3 and Theorem 2 of [2]. In (4) the second bound is proved in Theorem 1 of [2]. Bounds conditional on z_1 are determined in Theorem 3 of [3].

The upper bounds of Proposition 1(4) can be used in turn to bound the exponential moment of q itself: this will be needed to treat the heterogeneous case. Note the restriction that b be positive: this prevents us from simply using a change of variable to estimate $\mathbb{P}[u^q \geq p]$ for $p \leq 1 < u$. Recall the definition of the excess work $X_t = \sum_{t'=1}^{t}(z_{t'} - s)$ and set $Q = \sup_{t \geq 1} X_t$ so that $q = \max[0, Q]$.

Theorem 2. *With the assumptions of Proposition 1(3), let $1 < u < y$. Then*

$$\mathbb{E}[u^Q] \leq F(u) + \frac{u-1}{y-u} F(ux/y) \quad . \tag{2.1}$$

Proof:

$$\mathbb{E}[u^Q] = \sum_{b=-\infty}^{\infty} \mathbb{P}[Q = b] u^b = \sum_{z_1=0}^{L} \mathbb{P}[z_1] \sum_{b=z_1 - s}^{\infty} \mathbb{P}[Q = b \mid z_1] u^b \quad ,$$

since $Q \geq z_1 - s$, and so since $\mathbb{P}[Q = b] = \mathbb{P}[Q \geq b] - \mathbb{P}[Q \geq b + 1]$,

$$\sum_{z_1=0}^{L} \mathbb{P}[z_1] \left(\mathbb{P}[\bar{q} \geq 0 \mid z_1] u^{z_1 - s} + (1 - u^{-1}) \sum_{b=1}^{\infty} \mathbb{P}[\bar{q} \geq b \mid z_1] u^{z_1 - s + b} \right)$$

where $\bar{q} := q - (z_1 - s)$ which is equal to $\sup[0, z_2 - s, z_2 + z_2 - 2s, \ldots]$. From this two things follow. Firstly, $\bar{q} \geq 0$ and so $\mathbb{P}[\bar{q} \geq 0 \mid z_1] = 1$. Secondly, by stationarity

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of the z process, $\bar{q}$ has the same distribution as $q = \max[0, Q]$. Hence for $b > 0$

$$\mathbb{P}[\bar{q} \geq b \mid z_1] = \sum_{z_2=0}^{L} \mathbb{P}[\bar{q} \geq b \mid z_2]\mathbb{P}[z_2 \mid z_1]$$

$$\leq \sum_{z_2} y^{-b} x^{z_2 - s}\mathbb{P}[z_2 \mid z_1] \qquad \text{by Prop. 1(4)}$$

$$\leq y^{-b}(x/y)^{z_1 - s} \qquad \text{by Prop. 1(2) and 1(3).}$$

Thus

$$\mathbb{E}[u^Q] \leq \sum_{z_1=0}^{L} \mathbb{P}[z_1] \left(u^{z_1 - s} + (1 - u^{-1}) \sum_{b=1}^{\infty} (u/y)^b (xu/y)^{k-s} \right) \quad ,$$

and so the result follows from Prop. 1(1) after summing over b.

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We note that no further inequalities are used in obtaining this bound than are used in obtaining the bounds in Proposition 1(4). So this is probably the best bound which can be achieved on $\mathbb{E}[u^Q]$ by these methods. A greater but simpler bound is obtained by noting that (with the same conditions on u, x and y)

$$\mathbb{E}[u^Q] = \sum_{b=-\infty}^{\infty} u^b \mathbb{P}[Q = b]$$

$$\leq \mathbb{P}[Q \leq 0] + (1 - u^{-1}) \sum_{b=1}^{\infty} u^b \mathbb{P}[Q \geq b]$$

$$\leq 1 + F(x)(1 - u^{-1}) \sum_{b=1}^{\infty} (u/y)^b$$

$$= 1 + \frac{u - 1}{y - u} F(x) \quad . \tag{2.2}$$

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3. HETEROGENEOUS UPPER BOUNDS.

The basic idea for dealing with the heterogeneous model is that we can notionally divide up the service rate s amongst the groups. Each group i is independently serviced at rate s_i with $\sum_i s_i = s$. For each such choice of the s_i we obtain an upper bound on the queue length by combining Theorem 2 with Chebychev's inequality. We must then optimize this bound over all possible choices of the (s_i). Let $\mathcal{A}(s)$ denote the set of $(s_i)_{i \in I}$ with $\sum_i s_i = s$ and $s_i > L_i a_i/(a_i + d_i)$. Define the excess work in group i:

$$X_t^i = \sum_{t'=1}^{t}(z_{t'}^i - s_i) \quad ,$$

so that in accordance with the previous definition, the excess work due to arrivals from all groups at time t is $X_t = \sum_i X_t^i$. Let $Q^i = \sup_{t \geq 1} X_t^i$ and as before $Q = \sup_{t \geq 1} X_t$.

We will need "i-versions" of functions used in the previous section: F_i, G_i and y_i are defined analogously to the homogeneous case using a_i, d_i, s_i, L_i and $\sigma_i := s_i/L_i$ in place of a, d, s, L and σ.

Proposition 2. *For all i, let the burstiness condition $a_i + d_i < 1$ be satisfied, and also a stability condition $\sigma_i > a_i/(a_i + d_i)$ for each i individually. so that we can choose (x_i, y_i) such that $y_i = \hat{y}_i(x_i)$ and $G_i(x_i) = 1$. Then for $b > 0$*

$$\mathbb{P}[q \geq b] \leq \inf_{u:\ 1 < u < y_i \forall i}\ \inf_{(s_i) \in \mathcal{A}(s)} u^{-b} \prod_i \left(F_i(u) + \frac{u-1}{y_i - u} F_i(ux_i/y_i) \right) \qquad (3.1)$$

Proof: Since $b > 0$, $\mathbb{P}[q \geq b] = \mathbb{P}[Q \geq b]$. Let $(s_i) \in \mathcal{A}(s)$, and $u \geq 1$. Then

$$\mathbb{P}[Q \geq b] \leq \mathbb{P}[\sum_i Q^i \geq b]$$

$$\leq u^{-b} \mathbb{E}[u^{\sum_i Q^i}] \qquad \text{(Chebychev's inequality)}$$

$$= u^{-b} \prod_i \mathbb{E}[u^{Q^i}] \qquad \text{(independent sources)}$$

Bounding each term in the product by Theorem 2, subject therefore to the stated conditions on a_i, d_i, σ_i, x_i and y_i, and taking the infimum over $u > 1$ and $\mathcal{A}(s)$ one obtains the right hand side of (3.1). ∎

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To find the optimal value of the variational expression (3.1) seems an intricate problem. However, if one is interested in the case that b is large, one expects that the optimal bound will involve making the u close to $\inf_i y_i$. In fact we can find an expression for the maximum possible value of u, and the values of s_i and y_i to which it corresponds. By Prop. 1(3), each choice of s_i fixes x_i and y_i through the conditions $y_i = \hat{y}_i(x_i)$ and $G_i(x_i) = 1$. Now the function $\hat{y}_i$ is invertible: since $\hat{y}_i(x)((1-d_i)x+d_i) = x(a_ix+1-a_i)$ then solving the appropriate quadratic equation and taking the positive root in order to get a positive quantity, one finds

$$x_i = \hat{y}_i^{-1}(y_i)$$
$$:= (2a_i)^{-1}\left(a_i - 1 + (1 - d_i)y_i + \left(4a_id_iy_i + (a_i - 1 + (1 - d_i)y_i)^2\right)^{1/2}\right) .$$

Eliminating x_i from the condition $G_i(x_i) = 1$ then finally we express s_i in terms of y_i:

$$s_i = \hat{s}_i(y_i) := L_i \log(a_i\hat{y}_i^{-1}(y_i) + 1 - a_i)/\log y_i \quad .$$

This motivates the following technical result whose proof we defer to an appendix.

Lemma 1.
(1) $\hat{y}_i^{-1}$ is strictly increasing from $[1,\infty)$ onto $[1,\infty)$.
(2) $x \mapsto p_i(x) := \log(a_ix + 1 - a_i)/\log\hat{y}_i(x)$ is strictly increasing on $(1,\infty)$ and extends by continuity to take the value $a_i/(a_i + d_i)$ at $x = 1$.
Hence $y \mapsto \hat{s}_i(y)$ is strictly increasing on $[1,\infty)$ and takes the value $L_ia_i/(a_i + d_i)$ at $y = 1$.

This enables us to find the supremum of possible values of u in the following sense, and hence the upper bound for $\mathbb{P}[q \geq b]$.

Proposition 3. *Assume the burstiness condition (1.1) and stability condition (1.2) are satisfied. Then*
$$y := \sup_{(s_i)\in\mathcal{A}(s)} \sup \{y' \; : \; y' < y_i \; \forall \; i\}$$
is the unique solution of the equation

$$\sum_i \hat{s}_i(y) = s \quad . \tag{3.2}$$

Proof: Since by Lemma 1 the $\hat{s}_i$ are increasing and $\hat{s}_i(y_i) = s_i$

$$\sup_{(s_i)\in\mathcal{A}(s)} \sup \{y' \; : \; y' < y_i \; \forall \; i\} = \sup_{(s_i)\in\mathcal{A}(s)} \sup \{y' \; : \; \hat{s}_i(y') < s_i \; \forall \; i\}$$
$$\leq \sup_{(s_i)\in\mathcal{A}(s)} \sup \{y' \; : \; \sum_i \hat{s}_i(y') < s\} \quad . \tag{3.3}$$

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Again by Lemma 1, $\sum_i \hat{s}_i$ is strictly increasing on $[1, \infty)$ and $\sum_i \hat{s}_i(1) = \sum_i L_i a_i/(a_i + d_i) < s$. Hence the supremum on the right hand side of (3.3), which we denote y^*, is the unique solution of the equation $\sum_i \hat{s}_i(y^*) = s$. By the inequality of (3.3), $y^* \geq y$ and so in particular $y^* > 1$. According to Lemma 1, $\hat{s}_i$ is increasing, so $s_i^* := \hat{s}_i(y^*) > \hat{s}_i(1) = L_i a_i/(a_i + d_i)$. Consequently so that $(s_i^*) \in A(s)$ and so $y^* \leq y$, which combined with the reverse inequality means that $y = y^*$.

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Proof of Theorem 1. Applying Proposition 3 with $y_i = y$ for all i and hence $x_i = \hat{y}_i^{-1}(y)$, then equation (3.2) yields equation (1.5):

$$\sum_i L_i \hat{\sigma}_i(y) = s$$

where

$$\hat{\sigma}_i(y) := \hat{s}_i(y)/L_i = \log(a_i \hat{y}_i^{-1}(y) + 1 - a_i)/\log y \quad . \tag{3.4}$$

The bound of (3.1) becomes

$$\mathbb{P}[q \geq b] \leq \sup_{1 \leq u < y} u^{-b} \prod_i \left(F_i(u) + \frac{u-1}{y-u} F_i(x_i u/y) \right) \quad . \tag{3.5}$$

We pick out the dominant behaviour in (3.5) for large b as being that as u approaches y, so that the dominant contribution to (3.5) is proportional to $u^{-b}(y-u)^{-I}$. By differentiation this expression is minimized by $u = yb/(b+I)$, which upon substitution in (3.5) yields the bound (1.1)

$$\mathbb{P}[q \geq b] \leq \varphi(y, b)y^{-b}$$

with

$$\varphi(y, b) = (1 + I/b)^b \prod_i \left(F_i\left(\frac{yb}{b+I}\right) + \left(F_i\left(\frac{x_i b}{b+I}\right)\right)(by - I - b)/Iy \right) \quad . \tag{3.6}$$

The stated prefactor polynomial in b can be obtained by using the bound (2.2) in the same manner. This amounts to replacing $F_i(\frac{yb}{b+I})$ by 1 and $F_i(\frac{x_i b}{b+I})$ by $F_i(x_i)$ in (3.6). Noting that $(1 + I/b)^b \leq e^I$ then one can take

$$\varphi(y, b) = \prod_i e\left((1 - F_i(x_i)) + (1 + b/I)(1 - y^{-1})F_i(x_i)\right) \quad . \tag{3.7}$$

■

Note that we have not found the infimum (3.1), rather we have found a bound on $\mathbb{P}[q \geq b]$, which holds for all b, but which is expected to be the optimal one obtainable from (3.1) as $b \to \infty$. Concerning the size of the prefactor, we note from Proposition 1(5) that all the occurrences of the F_i give a quantity less than 1.

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Appendix: Proof of Lemma 1.

(1) We omit the index i for convenience. By direct calculation $+\infty > \hat{y}'(x) = (x^2 a(1-d) + 2adx + d(1-a))/((1-d)x + d)^2 > 0$. Thus $\hat{y}^{-1}$ is strictly increasing.

(2) Setting $h_a(x) := \log(1 + a(x-1))$ then

$$p(x) := h_a(x)/(h_1(x) + h_a(x) - h_{1-d}(x)) \quad .$$

Let $x' > x > 1$. Then

$$p(x') - p(x) = \frac{\int_1^{x'} dt'\, h_a'(t')}{\int_1^{x'} h_1'(t') + h_a'(t') - h_{1-d}'(t')} - \frac{\int_1^{x} dt\, h_a'(t)}{\int_1^{x} h_1'(t) + h_a'(t) - h_{1-d}'(t)}$$

the denominator of which is positive, the numerator being

$$\int_1^{x'} dt' \int_1^{x} dt\, h_a'(t')\big(h_1'(t) - h_a'(t)\big) - h_a'(t)\big(h_1'(t') - h_a'(t')\big)$$

$$= \int_x^{x'} dt' \int_1^{x} dt\, h_a'(t')\big(h_1'(t) - h_a'(t)\big) - h_a'(t)\big(h_1'(t') - h_a'(t')\big)$$

$$= \int_x^{x'} dt' \int_1^{x} dt \left(\frac{1}{v(t)} - \frac{1}{v(t')}\right) \frac{ad}{(1 + a(t'-1))(1 + a(t-1))} \quad (A.1)$$

where $v(t) := t((1-d)t - d)/(at - 1 - a)$. But just as in part (1) we find that v is an increasing function, and since $t' > t$ throughout the integral (A.1) we see that $p(x') > p(x)$ as required. Furthermore $\lim_{x \to 1} p(x) = a/(a+d)$.

Taking the composition (with indices restored) $\hat{s}_i(y) = L_i p_i\big(\hat{y}_i^{-1}(y)\big)$ we obtain the stated result.

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