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Geometry, the Renormalization Group and Gravity

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Abstract

We discuss the relationship between geometry, the renormalization group (RG) and gravity. We begin by reviewing our recent work on crossover problems in field theory. By crossover we mean the interpolation between different representations of the conformal group by the action of relevant operators. At the level of the RG this crossover is manifest in the flow between different fixed points induced by these operators. The description of such flows requires a RG which is capable of interpolating between qualitatively different degrees of freedom. Using the conceptual notion of course graining we construct some simple examples of such a group introducing the concept of a "floating" fixed point around which one constructs a perturbation theory. Our consideration of crossovers indicates that one should consider classes of field theories, described by a set of parameters, rather than focus on a particular one. The space of parameters has a natural metric structure. We examine the geometry of this space in some simple models and draw some analogies between this space, superspace and minisuperspace.

1. Introduction

The cosmopolitan nature of Charlie Misner's work is one of its chief features. It is with this in mind that we dedicate this article on the occasion of his 60th birthday. There are several recurring leitmotifs throughout theoretical physics; prominent amongst these would be geometry, symmetry, and fluctuations. Geometry clarifies and systematizes the relations between the quantities entering into a theory, e.g. Riemannian geometry in the theory of gravity and symplectic geometry in the case of classical mechanics. Symmetry performs a similar role, and in the case of continuous symmetries is often intimately tied to geometrical notions. For instance in the above examples Riemannian geometry and symplectic geometry are intimately related to

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the diffeomorphism and canonical groups respectively. Our third leitmotif, fluctuations, enters ubiquitously through the quantum principle, or classically in statistical physics. The key underlying idea here is that because of the fluctuations physics must be described in a probabilistic manner.

Having stated our prejudices let us be a little less ambitious than to consider all of theoretical physics and restrict our attention to field theory. We make no pretension to mathematical rigour taking the point of view that a field theory on a manifold $\mathcal{M}$ can be defined via a functional integral with a probability measure which is a functional of a set of possibly position dependent parameters $\{g^i\}$, e.g. coupling constants, masses, background fields etc. Physical quantities can be expressed as combinations of moments which in turn can be written as functions of the $\{g^i\}$. If we think of these parameters as coordinates on a parameter space $\mathcal{G}$ it is clear that physics should be invariant under changes in these coordinates. A particular type of coordinate change is engendered by a renormalization, e.g. between bare and renormalized g^i's. Other possible symmetry group transformations such as coordinate transformations on $\mathcal{M}$ or gauge transformations act as diffeomorphisms on $\mathcal{G}$. Here we are concerned exclusively with the behaviour under RG transformations, and hence under scale transformations. We investigate some geometrical structures on $\mathcal{G}$, in particular defining a metric and associated connection. We look at the change in the geometry under renormalization, thereby introducing all three of our leitmotifs. The geometry is a result of the fluctuations in the system, i.e the probabilistic description. Without fluctuations the metric is identically zero. The RG induces a flow on $\mathcal{G}$ the fixed points of which are of particular interest as they represent conformally invariant systems. This flow with respect to a given parameter can be either centrifugal or centripetal for a particular fixed point. If the former the parameter is said to be relevant, and irrelevant for the latter. The relevance or irrelevance can change according to the fixed point.

RG flows between different fixed points, i.e different conformal field theories, are especially interesting. The reason for this is the following: one of the most important tasks confronting a theory is to identify correctly the degrees of freedom (DOF) of a physical system. It is a fact of life that all physically relevant theories have qualitatively different effective DOF at different scales. For instance, in QCD the high energy DOF are quark, gluon DOF, whilst at low energy they are hadron, meson ... DOF. In gravity at low energy, gravitons are the low energy DOF, whereas at high energies, who knows...topological foam, strings .... The only thing that is reasonably certain is that it won't be gravitons. A closer to earth example would be liquid helium in a 3 dimensional (3D) slab geometry. For correlation lengths much less than the slab thickness helium atoms are the relevant DOF whereas in the opposite limit it is vortices. An example we will treat here is that of a $\lambda\phi^4$ theory on a manifold $S^1 \times R^{d-1}$ of size L. Suitably altered this model model can describe, amongst others, the Higgs model at finite temperature, the Casimir effect for an interacting quantum field theory or the critical behaviour of an Ising ferromagnet in a slab geometry. Here there is a change in DOF as the variable $x = mL$ changes, where m is the "mass" (inverse correlation length) in the physical system. As $x \rightarrow \infty$ the DOF are effectively d dimensional and as $x \rightarrow 0$, $d-1$ dimensional. We will also briefly discuss similar considerations in more realistic "cosmologies".

One of the first questions one must confront with a crossover problem is: how should one renormalize? If one accepts the fairly common point of view that renormalization means the consistent removal of ultraviolet (UV) divergences one generically finds a resultant RG which is independent of the parameter inducing the crossover, e.g. L in the above example. The β functions and anomalous dimensions of the problem are all then L independent. One also finds that the theory gives perturbative nonsense as $x \rightarrow 0$. The reason for this is relatively simple. Let us take a more physical picture of renormalization, as a "course graining" such as decimation/block spinning[1] [1]. Here we imagine integrating out DOF between one scale and another. For the finite system at scales $\ll L$ one would integrate out d dimensional DOF. However, as one course grains further one is eventually integrating out DOF with scales $\sim L$. In the finite direction there are no DOF with scales $> L$, therefore one cannot integrate them out. The only DOF left are $d-1$ dimensional and these are the physically relevant ones. So, a physically intuitive renormalization procedure takes into account the qualitatively changing nature of the effective DOF. It should be clear then why a L independent RG is badly behaved. Such a group is equivalent to integrating out only d dimensional DOF **for all scales**. The moral is that one should try to develop a RG that is capable of interpolating between qualitatively different DOF. In this paper we will show how this can be achieved in a wide class of crossover problems.

The outline of this article will be as follows: in section 2 we will give a short, intuitive exposition of renormalization and the RG with a view to the treatment of crossovers. In section 3 we will develop the concept of a RG that can interpolate between qualitatively different DOF, introducing the concept of a "floating" fixed point and illustrating our ideas with $\lambda\phi^4$ on $S^1 \times R^{d-1}$. In section 4 we will describe the beginnings of a geometrical framework for field theory wherein a much more general theory of crossovers may be built illustrating the concepts using a Gaussian model. Finally in section 5 we take an opportunity to make some speculative remarks and draw some conclusions.

2. Renormalization and the Renormalization Group

In this section as well as setting notation we would like to give an extremely brief and hopefully intuitive account of renormalization, hoping that the unconventional viewpoint will not prove unintelligible. As a concrete example consider a self interacting scalar field theory described by a partition function (generating functional) on

[1]Strictly speaking such renormalizations form a semigroup not a group.

$\mathcal{M} = R^d$

$$Z[m_B, \lambda_B, \Lambda] = \int [D\phi_B]_\Lambda e^{-\int_\Lambda d^dx L(\phi_B, m_B, \lambda_B)} \tag{1}$$

where

$$L = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m_B^2\phi_B^2 + \frac{\lambda_B}{4!}\Lambda^{4-d}\phi_B^4 \tag{2}$$

For the sake of making sense out of the theory we will assume there is always an UV cutoff Λ. In (1) we have a probability measure which is a function of two parameters and a cutoff. The parameters m_B and λ_B are good descriptors of the physics at scales $\sim \Lambda$. What this means is the following; if one could calculate the 2 and 4 point vertex functions exactly one would find them to be very complicated functions of m_B, λ_B and Λ. At scales $\sim \Lambda$, however, one would find that for $\lambda_B << 1$ $\Gamma^{(2)} \sim m_B$ and $\Gamma^{(4)} \sim \lambda_B$. On the other hand at scales $\kappa << \Lambda$ the parameters m_B and λ_B are in no way a good description of the associated correlation functions. Obviously as $\frac{\Lambda}{\kappa} \to \infty$ they get worse and worse. The deep underlying reason behind this is of course the existence of fluctuations. It is the dressing due to quantum or thermal fluctuations that changes the correlation functions as one changes scale. We emphasize though that if one can calculate in the theory exactly the bare parameters are as good as any others. What one would like is to describe the correlation functions using a more suitable set of parameters, in particular if we are considering physics at a scale $\sim \kappa$ it would seem to make good sense to describe the physics using new parameters m and λ which are a more natural description of the physics at this scale. An obvious natural choice would be to describe the physics at the scale κ in terms of the 2 and 4 point vertex functions at a scale κ' where $\kappa \sim \kappa'$. Thus one would require

$$\Gamma^{(2)}(k=0, m, \lambda, \kappa') = m^2 \qquad \Gamma^{(4)}(k=0, m, \lambda, \kappa') = \bar{\lambda} = \lambda\kappa'^{4-d} \tag{3}$$

The physics at the scale κ, i.e the correlation functions at that scale, would now be described in terms of the correlation functions at a nearby fiducial scale, κ'.

In the above we have loosely outlined the renormalization program for this model. Why renormalize? There are two answers to this, one perturbative, and one not. Perturbatively as $\frac{\Lambda}{\kappa}$ grows perturbation theory in terms of the bare coupling becomes worse and worse. This is the well known problem of "UV divergences". In terms of fluctuations the bare parameters are being perturbatively dressed by fluctuations between the scales Λ and κ. The recipe for getting round this problem is as outlined above; to perturb with a "small" coupling rather than a large one, i.e. the renormalized coupling. Thus one uses the value of $\Gamma^{(4)}$ at some scale κ' as one's small parameter. This perturbation theory is then reasonably well defined as long as κ is not too different from κ' as. In 4D, for example, the correction terms are proportional to powers of $\ln\frac{\kappa}{\kappa'}$. Thus it is perturbatively better to dress the correlation functions a small amount. The dressing between Λ and κ is large and therefore difficult to compute perturbatively whilst the dressing between κ and κ' is smaller. The optimum approach is to consider an infinitesimal dressing and to integrate the resulting differential equation. So, if one wishes to implement perturbation theory renormalization is essential. The non-perturbative reason is somewhat subtler and depends ultimately on whether one believes there is a fundamental cutoff or not. One puts it in to make mathematical sense of the theory and then asks if it can be sensibly removed again. It seems to be the case that this is only possible for special values of the bare parameters — their fixed point values. To understand this we must consider the RG.

One can think of renormalization as a mapping of correlation functions between two different "scales". These mappings have an abelian group structure and this group is known as the RG. The group action on $\mathcal{G}$ generates a flow. Of particular interest are the fixed points of this flow as they imply a system possesses scale invariance. The fundamental relation between bare and renormalized vertex functions is

$$\Gamma^{(N)}(k,m,\lambda,\kappa) = Z_{\phi}^{\frac{N}{2}}\Gamma_B^{(N)}(k,m_B,\lambda_B,\Lambda) \tag{4}$$

where the renormalized parameters are defined at some arbitrary scale κ, and Z_ϕ denotes a wavefunction renormalization factor. The bare theory's independence from κ leads to the RG equation

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta\frac{\partial}{\partial\lambda} + \gamma_{\phi^2}m^2\frac{\partial}{\partial m^2} - \frac{N}{2}\gamma_\phi\right)\Gamma^{(N)}(k_i,m^2,\lambda,\kappa) = 0 \tag{5}$$

where $\gamma_{\phi^2} = -\frac{\partial \ln Z_{\phi^2}}{\partial \ln\kappa}$, Z_{ϕ^2} being the renormalization constant associated with the operator ϕ^2 and $\gamma_\phi = \frac{\partial\ln Z_\phi}{\partial\ln\kappa}$ are the anomalous dimensions of the operators ϕ^2 and ϕ respectively. It is important to note that (5) results from an **exact** symmetry even though it expresses an apparent triviality, the reparameterization invariance of the correlation functions. Equation (5) can be solved by the method of characteristics and together with dimensional analysis yields

$$\Gamma^{(N)}(k_i,\lambda,m,\kappa) = (\kappa\rho)^{d-N\frac{(d-2)}{2}}\exp\left(\frac{N}{2}\int_\rho^1\frac{dx}{x}\gamma_\phi(x)\right)\Gamma^{(N)}\left(\frac{k_i}{\rho\kappa},\frac{m^2(\rho)}{\rho^2\kappa^2},\lambda(\rho),1\right) \tag{6}$$

where $\lambda(1)=\lambda$, $m(1)=m$, and ρ is arbitrary. $m(\rho)$ and $\lambda(\rho)$, the running mass and coupling satisfy

$$\rho\frac{d\lambda(\rho)}{d\rho} = \beta \qquad \rho\frac{dm^2(\rho)}{d\rho} = \gamma_{\phi^2}m^2(\rho) \tag{7}$$

Equation (5) tells us how $\Gamma^{(N)}$ gets dressed by fluctuations between the scales κ and $\kappa + d\kappa$, in terms of parameters which get dressed according to (7). Integrating this equation tells us how $\Gamma^{(N)}$ gets dressed by fluctuations between the scales κ and $\kappa\rho$. This dressing induces a flow on $\mathcal{G}$. Equation (6) is the exact solution of an exact equation which is a result of an exact symmetry. The fixed point of the coupling λ, λ^* is given by the solution of $\beta = 0$. Now, we can use our freedom in choosing ρ to eliminate the variable $m(\rho)$ in (6) via the condition $m^2(\rho) = \rho^2\kappa^2$. At the fixed point λ^* one can solve the equation for $m(\rho)$ to find $\rho \sim (\frac{m}{\kappa})^\nu$ where $\nu = (2-\gamma^*_{\phi^2})^{-1}$, $\gamma^*_{\phi^2}$ being the value of γ_{ϕ^2} at the fixed point. Similarly, defining $\eta = \gamma^*_\phi$ one finds for instance

$$\Gamma^{(2)}(k=0,m) \sim Am^{2\nu(2-\eta)} \qquad (8)$$

where A is some constant. Once again we emphasize that this is an exact result dependent only on the fact that a fixed point exists. The RG is not just about "improving perturbation theory". Of course, finding the fixed point and calculating A, ν and η is a different matter. In $d < 4$ dimensions for this model there are two known fixed points, the Gaussian fixed point $\lambda^* = 0$ and the Wilson-Fisher (WF) fixed point $\lambda^* \sim (4-d)$. At the Gaussian fixed point $\nu = \frac{1}{2}$ and $\eta = 0$ whilst at the other e.g. in 3D $\nu = 0.630$ and $\eta = 0.031$.

Physically the importance of the fixed point for λ is the following. λ like all other quantities gets dressed as a function of scale and therefore changes its value. At the point $\lambda = \lambda^*$ the coupling becomes completely insensitive to dressing and therefore has essentially dropped out of the problem. Obviously $m = 0$ is a fixed point for the mass. As fixed points essentially define a theory finding them is one of the main tasks of field theory. Returning now to a non-perturbative aspect of renormalization; in (4) we could instead of differentiating the bare vertex function with respect to κ have differentiated the renormalized function with respect to Λ. This yields an equation analogous to (5). If one can find a fixed point of this equation then one can take the cutoff $\Lambda \to \infty$ and thereby recover a continuum theory.

The fact that there exist two fixed points for this theory means that one is really considering a class of field theories as a function of $x = \frac{\bar{\lambda}}{m^{4-d}}$. As $\bar{\lambda} \to 0$ one approaches the Gaussian theory, and as $m \to 0$ the WF fixed point. One crosses between them as a function of scale. The coupling $\bar{\lambda}$ is relevant in terms of RG flows with respect to the Gaussian fixed point. In other words a small perturbation from this fixed point induces a flow to larger length scales terminating at the WF fixed point. This is an example of crossover behaviour in field theory and describes a transition between qualitatively different DOF. For $x \ll 1$ the DOF are essentially non-interacting, whereas for $x \gg 1$ they are strongly interacting. The reader might legitimately enquire as to why, given that they are strongly interacting, one believes that perturbation theory can be used. This raises an important question: perturbation theory in terms of what coupling? In terms of $\bar{\lambda}$ straight perturbation theory breaks down as $m \to 0$ due to large dressings from the infrared (IR) regime as opposed to large dressings from the UV regime as was considered previously. The RG methodology tells you to ignore any differences between the UV and IR regimes. The essential problem is that of large dressings irrespective of whether the dressing arises from IR or UV fluctuations. Large dressings imply that one has used inadequate parameters to describe the physics, hence renormalization and the RG should be implemented. The correct parameter to perturb in is the running coupling constant which is a solution of the β function equation treated as a differential equation who's solution is a function of

x. The above is our first simple example of crossover behaviour in field theory. We would now like to proceed to other more pertinent examples showing some difficulties one encounters and their solution.

3. Crossover Behaviour in Field Theory

One of the main themes we have tried to emphasize in the introduction is that the effective degrees of freedom of a physical system are scale dependent. Here we take a simple but physically relevant paradigm to show the difficulties involved in trying to describe a qualitative change in the DOF of a system. We will try to emphasize a physical approach, stating in general only results, leaving the details in our other papers [2]. Consider a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\nabla\phi_B)^2 + \frac{1}{2}m_B^2\phi_B^2 + \sum_i \mu_B^i O_B^i \tag{9}$$

where $\sum_i \mu_B^i O_B^i$ represents schematically a relevant or set of relevant operators that induce a crossover from a fixed point associated with $\mu^i = 0$ to some others. For the moment we specify neither the symmetry of the order parameter or the dimensionality of the system. Some examples of relevant operators are the following: i) for $d < 4$, Gaussian$\rightarrow$WF fixed point as mentioned in the last section, $\mu_B^1 = \frac{\lambda_B}{4!}$, $O^1 = \phi_B^4$, $\mu_B^i(i \neq 1) = 0$; ii) quadratic symmetry breaking $(O(N) \rightarrow O(M))$, ϕ_B has an $O(N)$ symmetry, $\mu_B^1 = \frac{1}{2}\tau_B$, $O_B^1 = \sum_{i=1}^M(\phi_{iB})^2$, $\mu_B^2 = \frac{\lambda_B}{4!}$, $O_B^2 = \sum_{i=1}^N(\phi_{iB}^2)^2$; iii) uniaxial dipolar ferromagnets, where in Fourier space $\mu_B^1 = \frac{\alpha}{2}$, $O_B^1 = \frac{p_z^2}{p^2}\phi_B^2$, $\mu_B^2 = \frac{\lambda_B}{4!}$, $O_B^2 = \phi_B^4$. For the case of dimensional crossover one can determine the appropriate operators by Fourier transforming $\mathcal{L}$ with respect to the finite directions. One important common feature of the above is the introduction of an important new scale in each problem i.e. τ, the quadratic symmetry breaking term, α the strength of the dipole-dipole interactions and L the characteristic finite size scale. It is the existence of one or more new scales in a problem that makes a crossover much richer, more interesting and more complex than standard field theory. We call this generic length scale g. We also take this scale to be a physical scale and hence a RG invariant.

So, what does renormalization have to say about such systems? There is a widespread belief that renormalization just means getting rid of UV divergences. If we accept this belief and examine the above models one notices that the UV behaviour in these theories is independent of the parameter g, hence the UV divergences can be removed in a g independent way. We will give just one example of what happens if this philosophy is accepted. Consider $\lambda\phi^4$ on a manifold $S^1 \times R^3$ of size L. Using minimal subtraction gives for $mL \ll 1$ to one loop

$$\Gamma^{(2)} \rightarrow m^2\left(1 + \frac{\lambda}{32\pi^2}\ln\frac{m^2}{\kappa^2} + \frac{\lambda}{24m^2L^2} + O(\frac{\lambda^2}{m^3L^3})\right) \tag{10}$$

Obviously the perturbative corrections are large in this regime, in fact in the limit $Lm \to 0$ these corrections become infinite. From the point of view of renormalization this is no different than the bare vertices in the L independent theory getting a large dressing due to fluctuations. Here we've done a renormalization but still the vertex has a large L dependent dressing. Why is that? In implementing minimal subtraction we have really made an assumption, that parameters associated with the $L = \infty$ system will provide a good description of the physics when $L \to 0$. In this limit the system is effectively 3D and so one can hardly expect 4D parameters to be adequate. The total breakdown in perturbation theory above is a reflection of this fact. 3D $\lambda\phi^4$ theory has completely different DOF to 4D $\lambda\phi^4$ theory.

The way out of this impasse is in many ways relatively simple — choose better renormalized parameters. Think back to the G-WF crossover discussed in the last section. The analog of the 4D theory there is the Gaussian theory and the analog of the 3D theory the WF fixed point theory. The analog of $\bar{\lambda}$ is L. L is a relevant parameter that causes a crossover from one fixed point to another. We managed to cope with the G-WF crossover, how so? Above we renormalized in an L independent way, the analog would be to renormalize in a $\bar{\lambda}$ independent manner. We could have certainly done this i.e renormalize the theory using only the counterterms appropriate for a Gaussian theory. For $\frac{\Lambda}{\kappa} \gg 1$ we would have found large dressings telling us that the Gaussian counterterms were not really sufficient to renormalize the theory. These large dressings occur because of the self interactions amongst the particles, because interacting DOF are qualitatively different to non-interacting ones. The correct thing to do was to choose renormalization conditions such as in (3) which were specified as functions of λ, i.e a good renormalization was dependent on λ the parameter that induces the crossover. In the case at hand we should therefore consider L dependent renormalization conditions such as

$$\Gamma^{(2)}(k=0,m,\lambda,L,\kappa) = m^2 \qquad \Gamma^{(4)}(k=0,m,\lambda,L,\kappa) = \lambda\kappa^{4-d} \tag{11}$$

These conditions imply that the β function and anomalous dimensions are all functions of $L\kappa$ as well as λ, i.e the RG itself is L dependent. An L independent RG tells you how parameters are dressed in the theory by L independent fluctuations whereas an L dependent one tells how things are dressed by L dependent ones. In the real physical system it is manifestly obvious that the real fluctuations in the system are L dependent and that consequently conditions such as (11) will yield parameters which are a more faithful representation of the physics. The moral is: if the DOF of a system can qualitatively change as a function of scale then it is clearly better if one can derive a RG which can follow such a change.

It should be clear how to implement this philosophy more generally. For a crossover caused by a relevant parameter g, one should impose normalization conditions at an arbitrary value of g thereby obtaining a g dependent RG equation. In such a crossover one is interpolating between different conformal field theories i.e. different representations of the conformal groups associated with the limits $\frac{g}{m} \to 0$ an ∞. Just as there are anomalous dimensions γ_ϕ and γ_{ϕ^2} which are characteristi conformal weights of the associated operators for a particular fixed point so define effective anomalous dimensions and critical exponents which are chara of the crossover system. Given that in d dimensions the dimension of the c ϕ^4 is canonically $4-d$ one can define an effective dimension d_{eff} via the $\frac{d\ln\Gamma^{(4)}}{d\ln m^2} = 4 - d_{eff} - 2\eta_{eff}$. What about the notion of a fixed point? For the sy size L true conformal symmetry is only realized in the limits $m \to 0$ and L which yields the d dimensional fixed point and $m \to 0$ $Lm \to 0$ which yields t dimensional one. The equation $\beta = 0$ as an algebraic equation still has some in these crossover systems. It does not, however, give a fixed point becau function is now explicitly scale dependent through the variable $L\kappa$. If one the β function as being the velocity of the RG flow in the λ direction the va is an equation that is satisfied only for a particular scale, not all scales as it s for a true fixed point. The β function equation is a differential equation ar be integrated. However, one can in fact define an effective or "floating" fi in the following manner. Consider the β function generically as

$$\kappa\frac{d\lambda}{d\kappa} = \beta(\lambda, L\kappa) = -(4-d)\lambda + a_1(L\kappa)\lambda^2 + a_2(L\kappa)\lambda^3 + O(\lambda^4)$$

where a_1 and a_2 are known functions (see [3]). Define a new coupling $h = a$

$$\kappa\frac{dh}{d\kappa} = -\varepsilon(L\kappa)h + h^2 + b(L\kappa)h^3 + O(h^4)$$

where $\varepsilon(L\kappa) = 4 - d - \frac{d\ln a_1}{d\ln\kappa}$ and $b(L\kappa)$ is a combination of a_2 and a $\beta(h, L\kappa) = 0$ yields a solution $h^* \equiv h^*(L\kappa)$. This is the floating fixed point. ∞ it yields the d dimensional fixed point and as $L\kappa \to 0$ the $d-1$ dimens point. Corresponding floating fixed points can be defined in all the crossove we have considered so far. The floating fixed point is the "small" parar which perturbation theory is implemented. A g dependent RG and a corre dependent RG improved perturbation theory allow for complete perturba to the crossover. The main reason for this is that such a RG can interpola the qualitatively different DOF in the problem. As a specific example we some one loop results [2] for the above finite size model. The fixed point $h^* = \varepsilon(\kappa L)$ where

$$4 - d_{eff} = \varepsilon(\kappa L) = 1 - \kappa\frac{d}{d\kappa}\ln\left(\sum_n\left(1 + \frac{4\pi^2 n^2}{L^2\kappa^2}\right)^{-\frac{3}{2}}\right)$$

$$\gamma_{\phi^2}(h^*) = \frac{h^*}{3} \qquad \gamma_\phi(h^*) = 0$$

These functions all interpolate in a smooth way between their 4D an $\varepsilon(\kappa L)$ is our "small" expansion parameter. It also yields (to first order) dimensionality of the system. It is worth noting here that the sole requirem finiteness of the correlation functions for all L is sufficient to determine the crossover.

So far we have outlined intuitively an approach to crossover behaviour and a it to an interesting class of problems. Our considerations were governed by t flows of the parameters. The natural arena for such flows is $\mathcal{G}$. Rather than co a particular crossover we would like to consider $\mathcal{G}$ more abstractly. This ma prove fruitful in cases where the relevant parameters are not a priori known.

4. Geometry of $\mathcal{G}$

In this section we wish to begin an investigation of some of the geometrical st that is inherent in the approach we are following. We will attempt to be as as possible to begin with, and consequently somewhat vague. As was seen preceeding sections it was essential, if one wished to have a controlled pertu expansion, to change from one set of parameters useful in one regime to a of parameters useful in another regime, for example the large mL and sm regimes respectively. We are therefore working on a coordinate patch and ch coordinates on this patch. The immediate question would appear to be wha are we working on, i.e. a coordinate patch of what?

Examining the functional integral

$$Z[\mathcal{M},\{\theta^i\},\Lambda]=\int[D\phi]_\Lambda e^{-S[\mathcal{M},\phi[\mathcal{M}],\{\theta^i\},\Lambda]}$$

we see that it defines a map from the space, $\mathcal{F}$, parameterized by $(\mathcal{M},\phi[\mathcal{M}]$ to a section of a line bundle over $\mathcal{G}$. $\mathcal{M}$ is the spacetime manifold, $\phi[\mathcal{M}]$ a on $\mathcal{M}$, $\{\theta^i\}$ are couplings between the fields and external sources and Λ p role of a regulator which will not be viewed as a true parameter of the the rather as either a reflection of a true underlying lattice or a device to contro problems, and assist in the definition of the functional integral. We choos $\{g^i\}$ discussed in previous sections to be local coordinates on $\mathcal{G}$.

Earlier we saw that explicit calculations required a change of paramete change of coordinates on $\mathcal{G}$, from bare parameters (coordinates) to renorm rameters (coordinates). If the object Z has any meaning it should have content in all coordinate systems. We will assume that Z is invariant un dinate transformations on $\mathcal{G}$ and therefore is a scalar. Now, when one is coordinate transformations, it is natural to examine what structure $\mathcal{G}$ pos can help one organize the analysis. Any structure $\mathcal{G}$ has must be induce or already exist in S. Ideally we would like our parameters to be related as possible to the moments of the probability distribution, as these are the experimentally accessible objects. We will assume that $\mathcal{G}$ is a topolog with a differentiable structure and possibly isolated singularities, and that Z can be considered a differentiable function on $\mathcal{G}$ away from such special points. Thus if we consider an infinitesimal variation in S of the form dS, where d is the exterior derivative operator on $\mathcal{G}$, we get an induced change in Z. If the sources, masses etc. are position dependent then $\mathcal{G}$ is infinite dimensional and analogous to superspace, which would suggest that a mini-superspace may be useful. Mini-superspace in this context means restricting our considerations to a small finite dimensional subspace of $\mathcal{G}$. It is primarily this situation that will concern us here.

It is convenient for the following to work with the functional integral as a normalized probability distribution, which we can achieve by dividing by $Z = e^{-W}$. We therefore get a normalized functional integral

$$\int [D\phi]_\Lambda e^{W-S} = 1 \tag{15}$$

Because it is normalized and d is restricted to $\mathcal{G}$, we have

$$\int [D\phi]_\Lambda d e^{W-S} = dW - < dS >= 0 \tag{16}$$

where $< A >$ means expectation value of A.

$$ds^2 =< (dW - dS) \otimes (dW - dS) > \tag{17}$$

defines a positive definite, symmetric, quadratic form on $\mathcal{G}$ arising from the positivity of the probability distribution or the convexity of the associated entropy functional. ds^2 plays the role of a metric on $\mathcal{G}$. An infinitesimal change in our parameters along some smooth curve in $\mathcal{G}$ defines a vector tangent to that curve and therefore we can express our metric as

$$g_{\mu\nu} =< \partial_\mu S \partial_\nu S > - \partial_\mu W \partial_\nu W \tag{18}$$

on the space satisfying $dW - < dS >= 0$. This metric is known as the Fisher information matrix [4] in probability theory and is used for comparing one probability distribution to another.

Let us begin with our most simple mini-superspace example, the Gaussian distribution, which corresponds to a free field theory in zero dimensions. We begin with a field ϕ coupled to an external source J described by the action

$$S[\phi, m^2, J] = \frac{1}{2} m^2 \phi^2 + J\phi \tag{19}$$

$\mathcal{M}$ is now a single point and we have a coordinate patch on $\mathcal{G}$ with coordinates (J, m^2). The generator of connected correlation functions is

$$W[J, m^2] = -\frac{1}{2}\frac{J^2}{m^2} + \frac{1}{2}\ln[\frac{m^2}{2\pi}] \tag{20}$$

The condition $dW- < dS >= 0$ gives

$$-(<\phi> + \frac{J}{m^2})dJ - \frac{1}{2}(<\phi^2> - \frac{J^2}{m^4} - \frac{1}{m^2})dm^2 = 0 \tag{21}$$

The corresponding metric on using (21) is

$$ds^2 = \frac{1}{m^2}dJ^2 - \frac{2}{m^2}\frac{J}{m^2}dJdm^2 + \frac{1}{m^2}((\frac{J}{m^2})^2 + \frac{1}{2}\frac{1}{m^2})(dm^2)^2 \tag{22}$$

Note that this metric is not diagonal unless $J = 0$, however, a simple coordinate change allows us to diagonalize it, the appropriate choice of new coordinate being $\hat{\phi} = -\frac{J}{m^2}$ which is equivalent to starting with

$$S[\phi, \hat{\phi}, m^2] = \frac{1}{2}m^2\phi^2 - m^2\hat{\phi}\phi \tag{23}$$

$$W[\hat{\phi}, m^2] = -\frac{1}{2}m^2\hat{\phi}^2 + \frac{1}{2}\ln[\frac{m^2}{2\pi}] \tag{24}$$

the condition $dW- < dS >= 0$ now gives

$$m^2(<\phi> - \hat{\phi})d\hat{\phi} - \frac{1}{2}(<(\phi - \hat{\phi})^2> - \frac{1}{m^2})dm^2 = 0 \tag{25}$$

with metric

$$ds^2 = m^2 d\hat{\phi}^2 + \frac{1}{2}m^{-4}(dm^2)^2 \tag{26}$$

Observe that if m^2 were negative this metric would not be positive definite and if $m^2 = 0$ it would be highly singular. This is connected to stability, unitarity and convexity of W. It is not difficult to verify that this metric (in either coordinate system) has scalar curvature $R = -\frac{1}{2}$. Before discussing the meaning of this let us see what happens in a more realistic field theoretic setting.

Consider a Gaussian model on a compact manifold $\mathcal{M}$ of volume L^d, where $d \leq 4$ and

$$S[\phi, J, m^2, L, \Lambda] = \int_{\mathcal{M}} [\frac{1}{2}\phi(\Box + m^2)\phi + J\phi] \tag{27}$$

J and m^2 can be position dependent, and in fact generically are on a curved $\mathcal{M}$. A coordinate transformation equivalent to above gives

$$S[\phi, \hat{\phi}, m^2, L, \Lambda] = \int_{\mathcal{M}} [\frac{1}{2}\phi(\Box + m^2)\phi - \phi(\Box + m^2)\hat{\phi}] \tag{28}$$

with

$$W[\hat{\phi}, m^2, L, \Lambda] = -\frac{1}{2}\int_{\mathcal{M}} [\hat{\phi}(\Box + m^2)\hat{\phi}] + \frac{1}{2}Tr_{\Lambda}\ln[\frac{\Box + m^2}{\Lambda^2}] \tag{29}$$

For simplicity we assume $\hat{\phi}$ and m^2 are constant on $\mathcal{M}$, consequently, tr L as constants, $\mathcal{G}$ is a 2D mini-superspace. Keeping Λ finite ensures we problems.

Examining the condition $dW =< dS >$ we obtain the equation

$$\int_{\mathcal{M}}[< (\Box + m^2)\phi > -(\Box + m^2)\hat{\phi}]d\hat{\phi} - \frac{1}{2}[\int_{\mathcal{M}} < (\phi - \hat{\phi})^2 > -Tr_\Lambda(\frac{1}{\Box + ?}$$

This expression is finite without a cutoff only for $d = 1$ where $Tr(\frac{1}{\Box+n}$ and corresponds to the familiar situation of quantum mechanics. The r on this 2D space is

$$ds^2 = \int_{\mathcal{M}}[m^2 d\hat{\phi}^2] + \frac{1}{2}Tr_\Lambda\frac{1}{(\Box + m^2)^2}(dm^2)^2$$

This metric does not need a cutoff to be well-defined for $d < 4$, how $Tr(\frac{1}{\Box+m^2})^2$ is divergent and so our metric is not well-defined without

We can again look at the scalar curvature, which for the above metric $R = \frac{1}{4}Det^{-2}(g)\partial_{m^2}Det(g)$. Explicitly

$$R = -\frac{Tr_\Lambda(1 + \frac{\Box}{m^2})^{-3}}{\left(Tr_\Lambda(1 + \frac{\Box}{m^2})^{-2}\right)^2} + \frac{1}{2}\frac{1}{Tr_\Lambda(1 + \frac{\Box}{m^2})^{-2}}$$

For $d = 0$ this clearly reduces to the result for the Gaussian distribu $(S^1)^d$, $d = 1, 2$ or 3, the cutoff can be taken to zero giving in Fourier

$$R = -\frac{\sum_n (1 + (\frac{2\pi n}{mL})^2)^{-3}}{\left(\sum_n (1 + (\frac{2\pi n}{mL})^2)^{-2}\right)^2} + \frac{1}{2}\frac{1}{\sum_n (1 + (\frac{2\pi n}{mL})^2)^{-2}}$$

In the limit $mL \to 0$ the curvature reduces to the gaussian curvature in the limit $mL \to \infty$ it becomes

$$R = -\frac{1}{4}\frac{(2-d)}{\Gamma(\frac{4-d}{2})}(\frac{m^2L^2}{4\pi})^{-\frac{d}{2}} + \ldots$$

This is a nice example of a crossover in the context of the geometry and 3 the corrections in (33) are exponentially small while for $d = 1$ law. Interestingly for $2 < d < 4$ there is a crossover to positive requires R to pass through zero.

In a curved space setting for a conformally coupled free scalar field t and (31) for the metric and scalar curvature remain unchanged. W more general situation by including an additional coupling to the cu be associated with the mass term in a natural way. In the interacting as constant, i.e. we look only at the curvature of $\mathcal{G}$ in the $\hat{\phi}$, m^2 plane, one would find that the curvature depended on the scaling variable Lm where L is the characteristic length scale of the geometry and m is now the dressed mass of equation (11) in this geometry. In the case of a totally finite geometry the RG is of interest as physically there is a maximum length scale in the problem. Hence the RG can only flow so far before it stops. We also note that if we take a finite temperature field theory [5] then $T = \frac{1}{L}$ and the considerations of section 3 undergo a corresponding translation, we therefore have a temperature dependent RG. In a real cosmological setting one can imagine including in a RG picture various other effects, such as curvature, to get a quite detailed picture of how the universe cooled from the big bang. Naturally one would also wish to generalize to the case of non-constant curvature where one needs to consider a position dependent RG. More discussion of these interesting matters in the context of cosmology and the early universe will be discussed elsewhere [5].

5. Conclusions and Speculations

The main aim of this paper has been to try to stimulate thought along certain directions. There are certain problems that have remained intractable for many years now: the confinement problem in QCD and quantum gravity to name but two. We do not claim to have solutions to problems such as these. We do claim, however, that such theories exhibit certain key, common features, the chief one being that the DOF in the problem are radically different at different energy scales. We would also claim that if this metamorphosis could be understood then a quantitative understanding of the theory would probably follow.

The question of how systems behave under changes in scale is most naturally addressed using the field theoretic RG, a consequence of an exact symmetry. However, there are, as pointed out here, different, inequivalent representations of the RG. If one has a field theory parametrized by a set of parameters $P \equiv \{g^i\}$ corresponding to a point in $\mathcal{G}$ it might occur that different subsets of the parameters, relevant for describing the theory at different scales, are taken into one another by the RG flow on $\mathcal{G}$. If one's renormalization depends only on a subset of the parameters one is restricting one's flow to take place only in a subspace $\mathcal{T}$ of $\mathcal{G}$. The resultant RG, $RG_{\mathcal{T}}$, depends only on a subset K of the parameters and the RG flows take place only on $\mathcal{T}$. If any of the $P-K$ parameters are relevant in the RG sense then the true RG flows of the theory, $RG_{\mathcal{G}}$, thought of as true scale changes, will wish to flow off $\mathcal{T}$ into $\mathcal{G}$. However, the use of $RG_{\mathcal{T}}$ does not allow for such flows. Such a state of affairs would be shown up by the perturbative unreliability of the results based on $RG_{\mathcal{T}}$. If none of the parameters K are relevant then there should be no problem. However, one can only say what parameters are relevant when one knows the full fixed point structure of the theory! In principle it is obviously better to work with $RG_{\mathcal{G}}$. If a certain parameter was important then one has made sure that its effects are treated properly, and if it wasn't then that will come out of the analysis. There can be no danger, except for extra work, from keeping a parameter in, but there can be severe problems if it is left out. In the problems treated in this paper, although non-trivial, they were easy in the sense that the parameter space $\mathcal{G}$ was obvious. In the finite size case there were really 3 parameters m, λ and L. An L independent RG was equivalent to working on a 2 dimensional space which wasn't big enough to capture the physics. What about QCD, or gravity? After all, in QCD without fermions there appears to be only one parameter! There is another length scale in QCD, the confinement scale, however it is not manifest in the original Lagrangian, it comes out dynamically. This length scale is the analog of L. As we don't know how it really originates we arrive at a Catch 22 situation. Our suggestion in such cases would be the following: there are in most, if not all, of these type of problems important classical field configurations; instantons, monopoles, vortices etc. which are very important at one scale and not at another. What one should do is derive a RG which is explicitly dependent on such classical backgrounds just as we have shown in section 4 that one should have a RG that is explicitly dependent on one's background spacetime. This is contrary to the standard view which tries to make a clean split between the background (associated with IR effects) and renormalization of fluctuations (which are usually taken to be associated with UV effects). Although there may be scales where such an artificial split is sensible it will certainly be true that there will be scales where it manifestly is not. We hope it is clear from the above that when we talk about a parameter space it can be something quite complicated such as that of the standard model, a very pertinent example of crossover behaviour. We hope that we have convinced the reader that there is a lot to be said for developing a RG that can interpolate between different DOF.

We have considered here a class of problems that can be treated so as to yield perturbatively the full crossover behaviour. In section 4 we started to outline the most basic geometrical elements of a more general framework for treating crossovers. Our view was that a theory could be described by a set of moments of a probability distribution that was a function of a set of parameters. The idea was then to look at geometrical structures on $\mathcal{G}$ to see: i) whether some non-perturbative results could be gained in this way, and ii) whether through the geometry one could obtain a better, geometrical understanding of crossovers. It is obvious that in the more general setting we are at a very rudimentary stage indeed. We do believe however, that there are deep and important things to be learned from this approach.

The geometry we looked at in section 4 was ordinary Riemannian geometry based on a metric and a connection. There are many questions to be asked. For instance, is the connection we introduced the only relevant one? It would appear that symplectic and contact geometry also play an important role. There exists a symplectic form on the "phase" space composed of the $\{g^i\}$ and their Legendre transform conjugates which are expectation values of operators. There are also obvious connections with the trace anomaly that we will not go into here. From a more physical point of view one would imagine the intuition from lattice decimation could be extended to local decimations which would lead to a position dependent RG. In this setting the relevant geometry may be Weyl geometry. One may even speculate [6] on cosmological expansion as a form of natural decimation, where we are continuously decimating to scales larger and larger than the Planck scale, or equivalently we are following an RG flow further and further into the IR. One of the problems in GR is the origin of time. There is from the cosmological expansion of the universe a natural pinning of time to energy scale, is this an accident? It may be that the direction of time is due to gravity having an IR fixed point and that we are only observing its RG flow as time.

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