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The renormalization group in curved space

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Abstract. The general features of the renormalization group (RG) in curved space are discussed. An explicit construction of such a group for Euclidean $\lambda\phi^4$ theory with $O(n)$ internal symmetry group on $R^{4-d} \times S^d$, where $d \leqslant 4$ and S^d has radius a, is given. This RG is generated by an infinitesimal generator which implements a coarse graining of the *true* degrees of freedom of the system. Thus it is more analogous to the block spin and lattice decimation RGs of statistical mechanics. Contrary to standard formulations of the RG in curved space which emphasize the ultraviolet physics it is explicitly a dependent.

There has been a large apparent discrepency between the RG viewed as a 'coarse graining', e.g. decimation or block spinning, and the RG used in field theory. This is due to the fact that the standard field theoretic RG concentrates on the ultraviolet properties of field theory. This is a reasonable approach for massless field theory in flat space where no physical length scale is preferred. In this case the distinction between the lattice and field theoretic RGs disappears for scales much larger than the lattice spacing. However, when an additional length scale enters, such as on $R^3 \times S^1$ [1], where L is the size of the S^1, then the natural RG is L dependent. As the scale of interest is much larger or smaller than L the RG is either that characteristic of R^3 or R^4 respectively. In a curved space setting the curvature is a natural length scale hence one would expect the RG equations to depend on it. We demonstrate that this is indeed the case by examining conformally coupled $O(n)$ $\lambda\phi^4$ theory on $R^{4-d} \times S^d$ (in the case of non-conformal coupling only m_a below changes). For $<\phi>= 0$, we employ the normalization conditions

$$\lambda = Z_\phi^2 \int \Gamma_B^{(4)}(x, x_1, x_2, x_3)\sqrt{g_1}dx_1\sqrt{g_2}dx_2\sqrt{g_3}dx_3\Big|_{m_a=\kappa}$$

$$1 = Z_\phi Z_{\phi^2} \int \Gamma_B^{(2,1)}(x, x_1, x_2)\sqrt{g_1}dx_1\sqrt{g_2}dx_2\Big|_{m_a=\kappa}$$

$$\kappa^2 = Z_\phi \int \Gamma_B^{(2)}(x, x_1)\sqrt{g_1}dx_1\Big|_{m_a=\kappa}$$

where m_a is the renormalized mass parameter. The renormalized physical mass in this geometry is

$$M_a^2 = \frac{\int G(x_1, x_2)\sqrt{g_1}dx_1}{\int G(x_1, x_2)\sigma^2(x_1, x_2)\sqrt{g_1}dx_1}$$

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where σ is the geodetic interval. M_a is an RG invariant. Wavefunction renormalization is then naturally defined by requiring

$$\frac{\int G(x_1, x_2)\, \sigma^2(x_1, x_2) \sqrt{g_1} dx_1}{\left(\int G(x_1, x_2) \sqrt{g_1} dx_1\right)^2}\Big|_{m_a=\kappa} = 1$$

λ, Z_ϕ and Z_{ϕ^2} are independent of x due to the symmetries of the space. The above correspond to the usual definitions in a flat space setting and in the absence of loop corrections agree with the free field parameters in the Lagrangian, up to numerical factors.

With these definitions the RG equation for $\Gamma^{(N)}$ is

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta(h, \kappa a)\frac{\partial}{\partial h} + \gamma_{\phi^2}(h, \kappa a)\, m_a^2 \frac{\partial}{\partial m_a^2} - \frac{N}{2}\gamma_\phi(h, \kappa a)\right)\Gamma^{(N)} = 0$$

and implements a course graining of the true degrees of freedom. To 2-loops

$$\beta(h) = -\epsilon(\kappa a)\, h + h^2 - \left(\frac{4(5n+22)}{(n+8)^2} f_1(\kappa a) - \frac{4(n+2)}{(n+8)^2} f_2(\kappa a)\right) h^3 + O(h^4)$$

$$\gamma_{\phi^2} = \frac{n+2}{n+8} h - \frac{6(n+2)}{(n+8)^2}\left(f_1(\kappa a) - \frac{1}{3} f_2(\kappa a)\right) h^2 + O(h^3)$$

$$\gamma_\phi = \frac{2(n+2)}{(n+8)^2} f_2(\kappa a)\, h^2 + O(h^3)$$

where $h = a_1(\kappa a)\lambda$, a_1 being the coefficient of the quadratic term in the β function for λ, and γ_ϕ, γ_{ϕ^2} are effective anomalous dimensions. $\epsilon(\kappa a) = d - \kappa(\partial/\partial\kappa)\ln[\sum_l d_l(1+\lambda_l)^{-(1+d/2)}]$ with $d_l = [(l+d-2)!/l!](2l+d-1)$ and $\lambda_l = \frac{1}{6}[l(l+d-1) + d(d-1)]/\kappa^2 a^2$ the degeneracies and eigenvalues of $(\Delta + \frac{1}{6}R)/\kappa^2$ on S^d, respectively. f_1 and f_2 are long and will be presented elsewhere [4]. For $d \leqslant 2$, $N > 1$ and $d < 2$, $N = 1$ there is a lower limit on κ corresponding to the fact that there is a maximum compton wavelength for particles in this geometry [2-3]. $\beta(h)$, γ_ϕ and γ_{ϕ^2} interpolate between those of R^4 and R^{4-d} as $m_a a$ ranges from ∞ to 0. This can be interpreted as the decoupling of the infinite tower of modes arising from the compact S^d. $\Gamma^{(N)}$ obeys finite size scaling in terms of the variable $M_a a$. For $d = 1$ and all n, and $d = 2$, $n \leqslant 1$ $\beta(h) = 0$ defines a floating fixed point which interpolates between the 4 and $4-d$ dimensional fixed points. The above results have applications to cosmological phase transitions and the structure of interacting quantum field theories in curved space (more details will be given in a later publication [4]).

References

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