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Decoupling of Heavy Masses in the Kaluza-Klein Approach *

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Abstract

Dimensional reduction in Kaluza-Klein theories is discussed. In particular the one-loop contribution of an infinite tower of heavy Kaluza-Klein modes to a Green function and coupling constant renormalization are studied in the case of a scalar model on a six-dimensional spacetime with two dimensions being compactified to a torus. It is shown that the contribution of heavy modes decouples in the limit of zero size of the space of extra dimensions. A renormalization scheme with manifest decoupling of heavy masses is described, and interpolation of the renormalization group (RG) β -functions between various dimensions is discussed.

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1 Introduction

The Kaluza-Klein idea [1] has become an important ingredient in many modern unification schemes, e.g. superstrings and supergravity (see [2] for reviews and extended lists of references). In general the Kaluza - Klein paradigm is based on the following assumptions: 1) spacetime has more than four dimensions with the extra dimensions being a compact space of small size L; and 2) field theory is formulated on a multidimensional (MD) space. Physical symmetries and effects observed in experiments are "projections" of MD symmetries and effects to four dimensions. Often MD spacetime is taken as the direct product $M_{(4)} \times K_{(d)}$, where $M_{(4)}$ is a four dimensional spacetime and $K_{(d)}$ is a compact d-dimensional manifold, usually a homogeneous space S/R (S is a compact Lie group and R is its subgroup) or, in the framework of string models, a Calabi-Yau manifold. In modern approaches one usually demands that the spacetime is a vacuum solution of the theory, and not just an ad hoc structure. In other words, the compactification of the space of extra dimensions is spontaneous.

Most investigations of MD models have dealt with their classical properties, many of which are now quite well understood. The main idea here is to re-interpret a MD model as an effective four dimensional model by choosing an appropriate subspace of field configurations in the original model. This subspace is usually distinguished by some symmetry properties and includes fields which are considered to be important at the classical level. The main tool here is the mode expansion, and we are going to illustrate it for a simple model.

Let us consider a one component scalar field on the $(4+d)$-dimensional manifold $E = M^4 \times T^d$. T^d is a d-dimensional torus of size L. In spite of its simplicity this model captures some interesting features of both classical and quantum properties of MD theories. The action is given by

$$S = \int_E d^4x d^dy [\frac{1}{2}(\frac{\partial\phi(x,y)}{\partial x})^2 + \frac{1}{2}(\frac{\partial\phi(x,y)}{\partial y})^2 - \hat{\lambda}\phi^4(x,y)/4!], \tag{1}$$

where $x^\mu, \mu = 0,1,2,3$, are coordinates on M^4 and $y^m, m = 1,2,...,d$ are coordinates on $T^d, 0 \le y^m < 2\pi L$. To help re-interpret this model in four-dimensional terms we make a Fourier expansion of the field $\phi(x,y)$,

$$\phi(x,y) = \sum_N \phi_N(x) Y_N(y),$$

where $Y_N(y)$ are eigenfunctions of the Laplace operator on the internal space

$$\frac{\partial^2}{\partial y^2} Y_N = -M_N^2 Y_N, \qquad \int d^dy Y_N^* Y_M = \delta_{M,N}.$$

For T^d $N = (N_1, N_2, ..., N_d)$ is a multi-index with integer entries, $N_m = 0, \pm 1, \pm 2, ...$, and the eigenfunctions and eigenvalues are given by

$$Y_N = \frac{1}{(2\pi L)^{d/2}} \exp(iyN/L),$$

where $yN = \sum_{m=1}^{d} y^m N_m$, and $M_N^2 = (N^2/L^2)$. Substituting the Fourier expansion into the action and integrating over y one obtains

$$\begin{aligned} S &= \int_{M^4} d^4x \{ \frac{1}{2} (\frac{\partial \phi_0(x)}{\partial x})^2 + \sum_{N \neq 0} [\frac{\partial \phi_N^*(x)}{\partial x^\mu} \frac{\partial \phi_N(x)}{\partial x_\mu} - M_N^2 \phi_N^*(x) \phi_N(x)] \\ &- \frac{\lambda_1}{4!} \phi_0^4(x) - \frac{\lambda_1}{2} \phi_0^2(x) \sum_{N \neq 0} \phi_N^*(x) \phi_N(x) - S'_{int}, \end{aligned} \quad (2)$$

where the four-dimensional coupling constant λ_1 is related to the MD one $\hat{\lambda}$ by $\lambda_1 = \hat{\lambda}/(2\pi L)^d$. The sums in (2) are over all nonzero multi-indices N with non-negative entries N_m, $\phi_N^* = \phi_{-N}$, and S'_{int} includes all terms containing third and fourth powers of ϕ_N with $N \neq 0$. We see that (2 includes one massless real scalar field $\phi_0(x)$ and an infinite set ("tower") of massive complex fields $\phi_N(x)$ with masses given by $M_N^2 = N^2/L^2$. These massive fields are usually referred to as Kaluza-Klein modes or pyrgons (from Greek $\pi\upsilon\rho\gamma o s$ for ladder).

The radius L is usually assumed small in comparison with currently accessible scales q^{-1}, say $q \approx q_{ew} \approx 100 GeV$. In many models the spontaneous compactification solutions give $L \approx L_{Planck} \approx 10^{-33} cm$. Some arguments that $L \geq 10^{-6} GeV^{-1}$ are presented in [3]. Thus, if the energy scales of our probes are less than the threshold energy for the creation of pyrgons it is reasonable to expect that one needs only to consider the massless field $\phi_0(x)$, i.e. only the first and the third terms in (2) are relevant. We say in this case that the model dimensionally reduces. The condition for extracting MD configurations corresponding to zero modes is very simple in our case and can be written as $\partial\phi(x,y)/\partial y^m = 0$. For MD theories on a spacetime with more complicated geometry $E = M^{(4)} \times S/R$ schemes of consistent dimensional reduction are more involved (see [4] and references therein for an extended review of the method).

Although classical dimensional reduction is well understood the same cannot be said of the quantum theory. Calculation of quantum corrections in the reduced theory without taking massive modes into account can be justified only in a formal sense. However, if the extra dimensions are taken seriously then quantum corrections from pyrgons must be accounted for, as first emphasized by Duff and Toms [5]. Quantum corrections in Kaluza-Klein theories were considered in [6], [7], [8] (see also [9], [10]). Relations between ultraviolet (UV) properties of a MD theory and the theory obtained from it by dimensional reduction were discussed by Fradkin and Tseytlin [11].

The important point about the contribution of pyrgons is the following; even if the momenta of the interacting particles are such that $q^2L^2 \ll 1$, so real pyrgons cannot be created, virtual pyrgons still contribute to loops. The contribution of a given diagram in (1) can be schematically presented as follows

$$\Gamma(q,L) = \Gamma_0(q) + \delta\Gamma(q,L), \qquad \delta\Gamma(q,L) = \sum_{N\neq 0} \Gamma_N(q,L), \tag{3}$$

where Γ_0 is the contribution of the diagram with only light particles, and Γ_N represents diagrams with at least one pyrgon loop. The sum in (3) may (and in fact often does) produce additional divergences. Even if all four dimensional divergencies are subtracted using four dimensional renormalizations, $\sum_{N\neq 0}\Gamma_N(q,L)$ may still diverge, the UV divergences are $(4+d)$-dimensional rather than four dimensional.

We will study in this paper the behaviour of $\Gamma(q,L)$ with L, primarily the limit $qL \to 0$, and the question of whether the contribution of heavy modes vanishes. This question is crucial for MD models as it addresses the detectability, at least in principle, of the MD structure of the Universe even at zero momenta. Since all previous experiments seem to be in agreement with a four dimensional spacetime picture, significant values of $\delta\Gamma(q,L)$ would immediately mean that MD models contradicted experimental data, at least in their standard formulation. So, the safe option for such models and the Kaluza-Klein approach in general is when $\delta\Gamma(q,L)$ tends to zero for qL going to zero. This means that pyrgons should decouple in this limit analogously to the decoupling of heavy quark contributions in QCD where such decoupling is guaranteed by the decoupling theorem [12], [13] (see also [14]). In our case this theorem cannot naively be applied as we will see.

To explain this let us first recall briefly the main statement of the theorem about the decoupling of heavy masses in four dimensions. Suppose that we have a theory with two fields $\phi_1(x)$ and $\phi_2(x)$ of masses m and M respectively, where $m << M$. Let the Lagrangian of the theory be $L(\phi_1, m, \phi_2, M, g_1, g_2)$. The decoupling theorem states that effects at energy scales much less than M are described by an effective low-energy theory with the Lagrangian $L^*(\phi^*, m^*, g^*)$, where ϕ^*, g^* and m^* are related to the fields and parameters of the original theories by finite renormalizations:

$$\begin{aligned} \phi^* &= z^{1/2}\phi_1, \qquad g^* = g^*(g_1, g_2, m, M) \\ m^* &= m^*(g_1, g_2, m, M). \end{aligned} \tag{4}$$

In particular Green's functions in the full theory are related to Green's functions in the low-energy theory by

$$\Gamma(q_1, ..., q_n; g_1, g_2, m, M, \mu) = z^{-n/2}\Gamma^*(q_1, ..., q_n; g^*, m^*, \mu)[1 + O(1/M^a)], \tag{5}$$

as $M \to \infty$ with $q_1, ..., q_n$ fixed. The corrections go to zero as a power of M times logarithms, typically $\sim M^{-2}$. Moreover, in the full theory renormalization schemes with manifest decoupling can be found for which the low-energy theory is obtained simply by deleting all heavy fields without changing the couplings and masses of the light fields, i.e. the relations (4) are trivial for such schemes. However, it is not at all obvious that the decoupling theorem holds true in a MD model, and indeed strong doubts about its validity in this case were expressed in literature (see [15], for example). The main criteria for the validity of the decoupling theorem in its standard version are: 1) the theory must be renormalizable; 2) there must be a finite number of fields with heavy masses. Neither of these conditions are fulfilled in our case, i.e. the model (2) in $(4+d)$ dimensions is non-renormalizable and contains an infinite tower of massive fields.

One purpose of the present paper is to show that the decoupling theorem is nevertheless true for MD theories, at least to one-loop order. We also study dimensional crossover, in particular we show that our model transforms from a non-renormalizable one to a renormalizable one as $L \to 0$. To exhibit these properties we consider (1) on $E = M^4 \times T^2$. We calculate the four-point vertex function Γ^4 and the RG β-functions to one-loop, demonstrating the decoupling of heavy masses and dimensional crossover. Generalization of our result to any number of extra dimensions is straightforward. We also believe that our results about the decoupling of Kaluza-Klein modes and dimensional crossover are rather generic and hold true in other MD models.

It should be mentioned here that similar questions were addressed in papers on theories with non-zero temperature. It was observed that quantum field theories could become simpler (dimensionally reduced) in the high temperature regime [16]. Such theories in the imaginary-time formalism are described by models on $R^3 \times S^1$ with the temperature $T = (2\pi L)^{-1}$ so that high temperatures correspond to the limit $L \to 0$. High-temperature dimensional reduction from four to three dimensions for QED, QCD (including quarks) and the ϕ^4 theory was discussed in [17] [18] [19]. Similar problems in the framework of statistical models were also considered in [19].

The plan of the paper is as follows. In Sect. 2 we describe the model and a class of renormalization schemes which provide manifest decoupling of Kaluza-Klein modes. The four-point Green function to one loop order with dimensional regularization is calculated in Sect. 3. In this section the main result on decoupling of the heavy Kaluza-Klein modes for renormalized Green functions is demonstrated. The important issue of interchangebility of the limits $L \to 0$ and $\epsilon \to 0$, where ϵ is the regularization parameter, is discussed in particular in the context of cutoff regularization. In Sect. 4 RG β-functions are calculated and dimensional crossover of quantum properties is analysed. Some of the results presented here as well as their relation to statistical mechanics were published in [20].

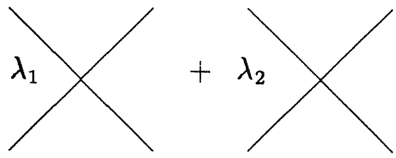

Figure 1

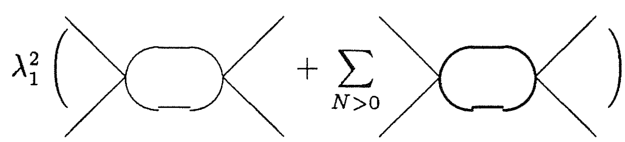

Figure 2

2 Description of the model and the renormalization scheme

Our simple model as mentioned is a one component scalar field on $E = M^4 \times S^1 \times S^1$ with the radii of the circles equal to L and with the action given by (1), which after a mode expansion is of the form (2) with one massless field ϕ_0 and an infinite tower of massive fields ϕ_N. We will consider one-loop diagrams with four external lines corresponding to massless particles. Only the interaction terms written down explicitly in (2) will be relevant for our computations.

As $\Gamma^4(q)$, where q is the total momentum of incoming particles, to one loop order is quadratically divergent, $\frac{\partial \Gamma^{(4)}}{\partial q^2}$ also diverges. Thus, the divergencies cannot be removed by renormalization of λ_{1B} alone. We must also add a $\lambda_{2B}\phi^2(x,y)\Box_{(4+d)}\phi^2(x,y)$, where $\Box_{(4+d)}$ is the D'Alambertian on E and λ_{2B} has mass dimension two. Of course, for calculation of other Green functions or higher order loop corrections other types of counterterms are necessary, we are not going to discuss them here. Thus, the Lagrangian that we will use for our calculation is

$$\begin{aligned} L &= \frac{1}{2}\left(\frac{\partial \phi_0(x)}{\partial x}\right)^2 + \sum_{N\neq 0}\left[\frac{\partial \phi_N^*(x)}{\partial x^\mu}\frac{\partial \phi_N(x)}{\partial x_\mu} - M_N^2 \phi_N^*(x)\phi_N(x)\right] \\ &- \frac{\lambda_{1B}}{4!}\phi_0^4(x) - \frac{\lambda_{1B}}{2}\phi_0^2(x)\sum_{N\neq 0}\phi_N^*(x)\phi_N(x) - \frac{\lambda_{20}}{4!}\phi_0^2(x)\Box\phi_0^2(x). \end{aligned} \tag{6}$$

We evaluate $\Gamma^{(4)}$ at the symmetric point where the external four momenta satisfy the equality $q_i q_j = q^2(\delta_{ij} - 1/4)$. The tree and one loop contributions are shown in Fig.'s 1 and 2 respectively. The thin line corresponds to the massless field, whereas the thick one with the label N on it corresponds to propagation of the field ϕ_N. Bars on lines of the second diagram in Fig. 1 correspond to derivatives with respect to external momenta. To one loop

$$\Gamma^{(4)}(q,L) = \lambda_{1B} + \lambda_{2B} q^2 + \lambda_{1B}^2 K(q,L), \tag{7}$$

$$K(q,L) = K_0(q) + \delta K(q,L) = K_0(q) + \sum_{N\neq 0} K_N(q,L), \tag{8}$$

where $K_0(q)$ corresponds to the first diagram and δK corresponds to the contribution of the sum term in Fig. 2. Here we assume that $\lambda_{2B} \sim \lambda_{1B}^2$, so that the one loop diagrams proportional to $\lambda_{1B}\lambda_{2B}$ or λ_{2B}^2 can be neglected. The consistency of this assumption is discussed in Sect. 4. Note that if we do not make this assumption control of the proliferation of divergencies becomes very hard.

To regularize the one-loop diagrams in Fig. 2 we use dimensional regularization. Keeping $\epsilon > 0$ automatically makes the sums in (8) regularized in the sense of ζ-function regularization. K_N can be written as

$$K_N = -\frac{3}{2}\int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \frac{1}{(k^2 + N^2/L^2)((q-k)^2 + N^2/L^2)}.$$

Evaluating this integral one gets the standard one-loop result

$$K_N(q,L) = -\frac{b_2(\epsilon)}{2\epsilon}\int_0^1 \frac{dt}{(q^2 t(1-t) + \frac{N^2}{L^2})^\epsilon} = \frac{1}{q^{2\epsilon}} I_N(qL) \tag{9}$$

where $b_2(\epsilon) = \frac{3\Gamma(1+\epsilon)}{(4\pi)^{2-\epsilon}}$ and is finite as $\epsilon \to 0$. Note that for $\epsilon \neq 0$, and $N \neq 0$ $K_N \to 0$ as $L \to 0$.

The renormalization scheme for which decoupling of pyrgons in the low-energy limit is manifest is formulated in terms of a subtraction operator R_L which is explicitly L dependent. R_L is an operator acting in both N space and q space and can be written as a matrix in both these spaces. Its matrix elements are

$$R_{MN}(q,\mu;L) = \delta_{MN}(1 - M^{(N)}(q,\mu;L)) \tag{10}$$

The action of $M^{(N)}$ on K_N is

$$M^{(N)}(q,\mu;L)K_N(q,L) = \sum_{n=0}^{\delta} \frac{(q^2-\mu^2)^n}{n!}\frac{\partial^n K_N}{\partial q^{2n}}|_{q^2=\mu^2} \tag{11}$$

where $1+\delta$ is the number of subtractions necessary to render $K(q,L)$ finite; e.g. for a logarithmic divergence

$$\begin{aligned} M^{(N)}(q,\mu;L)K_N(q,L) &= K_N(\mu,L) \\ R_{MN}(q,\mu;L)K_N(q,L) &= K_M(q,L) - K_M(\mu,L) \end{aligned} \tag{12}$$

Note that $M^{(N)}(q,\mu;L) \to 0$ ($N \neq 0$, $\epsilon \neq 0$) as $L \to 0$, hence in this limit $R_{L=0}$ has the same structure as it would have in strictly four dimensions, e.g.

$$R_{00}K_0(q, L=0) = K_0(q, L=0) - K_0(\mu, L=0)$$

for a four dimensional logarithmic divergence. For the case at hand

$$R_L K(q,L) = K(q,L) - K(\mu,L) - (q^2-\mu^2)\frac{\partial K(q,L)}{\partial q^2}|_{q^2=\mu^2} \tag{13}$$

If one uses an L independent subtraction scheme such as minimal subtraction which is appropriate for the $L=\infty$ system one finds

$$R_\infty K(q,L) = K(q,L) - K(\mu,L) - (q^2-\mu^2)\frac{\partial K(q,L=\infty)}{\partial q^2}|_{q^2=\mu^2} \tag{14}$$

It is vital to understand that $(R_0 - R_L)K(q,L) \to \infty$ due to the fact that one has underestimated the effect of pyrgon modes by overestimating their masses. In fact one has implicitly assumed they are infinitely massive. On the other hand as $L \to 0$ $(R_\infty - R_L)K(q,L) \to \infty$ due to the fact that one has overestimated their effect by underestimating their masses. In this case one has implicitly assumed they are massless. In reality they are neither infinitely massive or massless, they change their mass as L changes and a good renormalization scheme recognises this fact. For the subtraction operator R_L decoupling is manifest.

To formulate a particularly convenient renormalization prescription define an operator D as follows

$$\begin{aligned} D(\text{tree diagram}) &= \frac{\partial}{\partial q^2}(\text{tree diagram}), \\ DK_0(q) &= 0, \\ D\delta K(q,L) &= \frac{\partial}{\partial q^2}\delta K(q,L). \end{aligned} \tag{15}$$

This defines a slightly different R_L operator where $\frac{\partial}{\partial q^2}$ is replaced by D. We define dimensionless renormalized coupling constants λ_1 and λ_2 by the conditions

$$\begin{aligned} \lambda_1\mu^{2\epsilon} &= (M - q^2 D)\Gamma^{(4)}(q,L)|_{q^2=\mu^2}; \\ \lambda_2\mu^{-2+2\epsilon} &= D\Gamma^{(4)}(q,L)|_{q^2=\mu^2}, \end{aligned} \tag{16}$$

3 One-loop calculation and decoupling of Kaluza-Klein modes

Using the formulas (7) and (9) we compute the relations between renormalized and bare coupling constants for the renormalization scheme defined by eqs. (16):

$$\begin{aligned} \lambda_1\mu^{2\epsilon} &= \lambda_{1B} + \lambda_{1B}^2\mu^{2\epsilon}[K_0(\mu) + \frac{1}{2}A_1\delta K(\mu,L)] \\ &= \lambda_{1B} + \lambda_{1B}^2[I_0(\mu) + \frac{1}{2}\sum_{N\neq 0} A_{1+\epsilon}I_N(\mu,L)], \end{aligned} \tag{17}$$

$$\begin{aligned} \lambda_2\mu^{-2+2\epsilon} &= \lambda_{2B} + \lambda_{1B}^2\frac{\mu^{2\epsilon}}{2}A_0\delta K(\mu,L) \\ &= \lambda_{2B} + \lambda_{1B}^2\frac{1}{2}\sum_{N\neq 0} A_\epsilon I_N(\mu,L), \end{aligned} \tag{18}$$

where I_0 and $I_N(qL)$ are determined by (9) and we have introduced an operator $A_\nu = \mu\partial/\partial\mu - 2\nu$. Note the following useful identities $[A_{\nu_1}, A_{\nu_2}] = 0$ $A_\nu\mu^{-2\epsilon} = \mu^{-2\epsilon}A_{\nu+\epsilon}$. Inverting the expansions (17) and (18) to obtain the bare coupling constants in terms of the renormalized ones and substituting these expressions into (7) we get

$$\Gamma^{(4)}(q,L)\mu^{-2\epsilon} = \lambda_1 + \tfrac{q^2}{\mu^2}\lambda_2 - \lambda_1^2[(R_L K)(q,L)] \tag{19}$$
$$= \lambda_1 + \tfrac{q^2}{\mu^2}\lambda_2 - \lambda_1^2\{[(\tfrac{\mu^2}{q^2})^\epsilon - 1]I_0$$
$$+\sum_{N\neq 0}[(\tfrac{\mu^2}{q^2})^\epsilon I_N(qL) - I_N(\mu L) - (\tfrac{q^2}{\mu^2} - 1)\tfrac{A_\epsilon}{2} I_N(\mu L)]\}$$

It is easy to check that (20) is finite. Taking the limit $\epsilon = 0$ gives

$$\Gamma^{(4)}(q,L) = \lambda_1 + \frac{q^2}{\mu^2}\lambda_2 + \lambda_1^2\{\frac{b_2}{2}\log(\frac{q^2}{\mu^2}) - \frac{b_2}{2}\sum_{k=2}^{\infty}(-1)^k(\mu L)^{2k}$$
$$\times\frac{\Gamma(k)\Gamma(k+1)}{\Gamma(2k+2)}\zeta(k)[(\frac{q^2}{\mu^2})^k - 1 - k(\frac{q^2}{\mu^2} - 1)]\}, \tag{20}$$

where $b_2 = 3/16\pi^2$ and $\zeta(\nu) = \sum_{N\neq 0}\frac{1}{(N^2)^\nu}$ is the generalized zeta-function [21]. The expansion (20) is valid for $\mu L < 1$ and $(q^2/\mu^2) \sim O(1)$. The L independent term of order λ_1^2 is the standard result for $\lambda_1\phi^4$ theory in four dimensions in the momentum subtraction scheme, whereas the contribution of pyrgons is given by the sum.

It is clear that for $\mu L \to 0, qL \to 0$ and $q^2 \sim \mu^2$ the contribution of pyrgons vanishes. Thus, $\Gamma^{(4)}$ of the full theory to one loop is

$$\Gamma^{(4)}(q,L,\mu) = \Gamma^{(4)}_{4D}(q,\mu)(1 + O((\mu L)^4)) + \frac{q^2}{\mu^2}\lambda_2, \tag{21}$$

where $\Gamma^{(4)}_{4D}(q,\mu)$ is the four dimensional vertex function. (21) is one of the main results of the paper and shows that pyrgons decouple in the one-loop approximation, cf. (5). The last term in (21) proportional to λ_2 is not relevant in the limit $\frac{q^2}{\mu^2} \to 0$ as we will see later.

To understand the decoupling of Kaluza-Klein modes in more detail we will analyse the relations between the bare and renormalized coupling constants using the more "physical" cutoff regularization where the integrals over the loop four-momenta are bounded at $|k| = \Lambda$ and the sums over modes in loops are bounded by a number N_0. Removing the regulator means taking the limits $\Lambda \to \infty$ and $N_0 \to \infty$. We find to one loop

$$\lambda_1 = \lambda_{1B} - \lambda_{1B}^2\frac{b_2}{2}\{\int_0^1 dx[\log\frac{\Lambda^2 + \mu^2 x(1-x)}{\mu^2 x(1-x)} - \frac{\Lambda^2}{\Lambda^2 + \mu^2 x(1-x)}]$$
$$+ \sum_{N\neq 0}^{N_0}\int_0^1 dx[\log\frac{\Lambda^2 + \mu^2 x(1-x) + N^2/L^2}{\mu^2 x(1-x) + N^2/L^2} - \frac{\Lambda^2}{\Lambda^2 + \mu^2 x(1-x) + N^2/L^2}]$$

$$+ \quad \mu^2 \sum_{N\neq 0}^{N_0} \int_0^1 dx \frac{x(1-x)}{(\mu^2 x(1-x) + N^2/L^2)(1 + \frac{\mu^2 x(1-x)}{\Lambda^2} + \frac{N^2}{\Lambda^2 L^2})^2}\}, \qquad (22)$$

$$\lambda_2 \mu^{-2} \quad = \quad \lambda_{2B} - \lambda_{1B}^2 \frac{b_2}{2} \sum_{N\neq 0}^{N_0} \int_0^1 dx \frac{x(1-x)}{(\mu^2 x(1-x) + N^2/L^2)(1 + \frac{\mu^2 x(1-x)}{\Lambda^2} + \frac{N^2}{\Lambda^2 L^2})^2}\} (23)$$

where b_2 is the same as before. Let us now study the limits of these expressions. The first regime we consider is $\mu \ll \Lambda < \infty$ and $\mu L \ll 1$. It can be checked that for finite Λ all sums in the relations above are convergent, so we could put the regulator $N_0 = \infty$. Physically we can view this situation in the following way. Though formally we sum over all heavy modes in the one loop diagram in Fig. 2, since the loop four momenta is limited by the value $|k| = \Lambda$, the modes with masses larger than Λ do not give essential contributions, hence we do not see additional UV divergencies due to summation. So in the limit specified above but with ΛL finite (22) reduces to

$$\lambda_1 = \lambda_{1B} - \lambda_{1B}^2 \frac{b_2}{2}[(\log \frac{\Lambda^2}{\mu^2} + 1) + \frac{(\Lambda L)^4}{2}\zeta(2) + O((\Lambda L)^6)].$$

Now, if we take the limit $\Lambda L \to 0$, formally, we end up with the expression

$$\lambda_1 = \lambda_{1B} - \lambda_{1B}^2 \frac{b_2}{2}(\log \frac{\Lambda^2}{\mu^2} + 1)$$

which is exactly the one loop renormalization relation in four dimensional theory for UV cutoff regularization. The relation for λ_2 in this case takes the form

$$\lambda_2 = \lambda_{2B}\mu^2 + \lambda_{1B}^2(\mu L)^2[-\frac{(\Lambda L)^4}{6}\zeta(3) + O((\Lambda L)^6)],$$

which in the limit under consideration reduces to $\lambda_2 = \lambda_{2B}\mu^2$ which shows that the coupling constant λ_2 is not renormalized as expected in four dimensions. So, in the regime when the regulator Λ is finite the theory behaves like a four dimensional theory with a finite number of Kaluza-Klein modes and to which the standard decoupling theorem can be applied. This explains why we observe the decoupling of pyrgons and dimensional crossover from non-renormalizability to renormalizability when the size of the space of extra dimensions L vanishes.

Let us now consider the second regime when $\Lambda/\mu \to \infty, \Lambda L \to \infty$, so that $\Lambda L >> N_0$, and μL is finite. Then the renormalization relation (22) takes the form

$$\begin{aligned} \lambda_1 \quad &= \quad \lambda_{1B} - \lambda_{1B}^2 \frac{b_2}{2}[(\ln \frac{\Lambda^2}{\mu^2} + 1)(\zeta_{N_0}(0) + 1) \\ &+ \quad \zeta'_{N_0}(0) + \ln(\mu^2 L^2)\zeta_{N_0}(0) - \frac{\mu^4 L^4}{60}\zeta(2) O((\mu L)^6)], \end{aligned}$$

where $\zeta_{N_0}(0)$ and $\zeta'_{N_0}(0)$ denote the regularized sums $\sum_{N\neq 0}^{N_0} 1$ and $\sum_{N\neq 0}^{N_0}(-\log N^2)$ respectively. These sums diverge when the regularization is removed and thus the renormalization of λ_1 absorbs both four dimensional (related to $\Lambda \to \infty$) and six dimensional (related to the divergent sums) UV divergencies. For the same limit the renormalization relation for λ_2 is the following

$$\lambda_2 = \lambda_{2B}\mu^2 - \lambda_{1B}^2(\mu L)^2[\frac{1}{6}\zeta_{N_0}(1) + \frac{(\mu L)^2}{30}\zeta(2) + O((\mu L)^4)], \tag{24}$$

Here $\zeta_{N_0}(1) = \sum_{N\neq 0}^{N_0} 1/N^2$ is the divergent sum when the regulator $N_0 \to \infty$. (24) is Λ independent and contains only the divergences corresponding to the two extra dimensions. As we see, the limit $\Lambda \to \infty$ taken first corresponds to going to short distancies in the theory where spacetime is essentially six dimensional so that the divergent term does not depend on the size L and thus is not sensitive to details of compactification. In this regime all heavy modes are excited as virtual particles and the final result for the renormalization relations depends on the regulator N_0 explicitly. Taking the limit of small L in this case does not mean the shrinking of two extra dimensions to zero as it was assumed in the derivation that $\Lambda L \gg 1$ and we are therefore always in six dimensions. So, the contribution of pyrgons to the renormalization relations is non-vanishing and we do not see dimensional crossover in this regime. The point we have been trying to make here is that the limits $L \to 0$ and "regulator"(ϵ, Λ^{-1}, N_0^{-1}) $\to 0$ do not commute. The moral of this is that it is very dangerous to make statements about dimensional reduction by looking only at regularized but unrenormalized quantities. We stress that since the expressions (20)-(20) do not depend on the regularization, the statement about the decoupling of pyrgons from $\Gamma^{(4)}$ in the limit $\mu L \to 0$, $qL \to 0$ with $\frac{q^2}{\mu^2} \sim O(1)$ holds true. In fact if one goes to RG improved perturbation theory using an L dependent RG as in the next section one finds that the decoupling is also completely renormalization point independent. It is important to emphasize that dimensional reduction if it occurs is a physical thing; in particular it is renormalization scheme independent. However, what is certainly true is that some renormalization schemes make it manifest while others do not as emphasized in the introduction. For instance, it is extremely difficult to see dimensional reduction using minimal subtraction. Since the equations (17, 18) do not contain pyrgon contributions in the limit $\mu L = 0$ the renormalization scheme (16) is the analog of the scheme with manifest decoupling described in the introduction.

4 Renormalization group equations

In this section we calculate the RG equations for the coupling constants and examine their solutions. The β-functions for the coupling constants $\lambda_1(\mu)$ and $\lambda_2(\mu)$ defined by the renormalization prescriptions (16) at the subtraction point μ are the following

$$\beta_1(\lambda_1) = \mu\frac{\partial\lambda_1}{\partial\mu} = -2\epsilon\lambda_1 + \lambda_1^2[-2\epsilon I_0 - J(\mu L)], \tag{25}$$

$$\beta_2(\lambda_2) = \mu\frac{\partial\lambda_2(\mu)}{\partial\mu} = (2-2\epsilon)\lambda_2 + \lambda_1^2 J(\mu L), \tag{26}$$

where we found it convenient to define $J(\mu L) = \frac{1}{2}\sum_{N>0} A_\epsilon A_{1+\epsilon} I_N$. For $\epsilon = 0$ the expression $-2\epsilon I_0$ in the r.h.s. of (25) is finite and equals $(-b_2) = 3/16\pi^2$ which is the usual one-loop coefficient of the β-function in the four dimensional $\lambda_1\phi^4/4!$ theory. $J(\mu L)$ for $\epsilon = 0$ here is

$$J(\mu L) = b_2\sum_{l=2}^{\infty}(-)^l\frac{(\Gamma(l+1))^2}{\Gamma(2l+2)}(l-1)\zeta(l)(\mu L)^{2l} \tag{27}$$

and behaves like $(\mu L)^4$ for $\mu L \to 0$. The solutions of these RG equations are

$$\lambda_1(\mu) = \frac{\lambda_1(\mu_0)}{1-\lambda_1(\mu_0)b_2\ln\frac{\mu}{\mu_0} - \lambda_1(\mu_0)\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}J(\mu' L)}, \tag{28}$$

$$\lambda_2(\mu) = (\frac{\mu}{\mu_0})^2\lambda_2(\mu_0) - \mu^2\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'^3}\lambda_1^2(\mu')J(\mu' L). \tag{29}$$

$\lambda_1(\mu)$ has a pole at some point $\mu = \mu^*$ which is L dependent. For $L = 0$ $\mu^* = \mu_0\exp(1/b_2\lambda_1(\mu_0))$. Thus our calculations are meaningful for $\mu \ll \mu^*$. The blow-up of the coupling constants at $\mu = \mu^*$ is a remnant of the Landau ghost in the theory and reflects the inconsistency of the quartic self-interacting scalar model (see [22]). As $J(\mu L)$ vanishes for $\mu L \to 0$ (28) goes to the standard expession for the running coupling constant in the four-dimensional model in the one-loop approximation and the solution of (29) becomes independent of $\lambda_1(\mu)$. This reflects the fact that a counterterm for λ_2 is not necessary in four dimensions. $\lambda_2 = 0$ is a fixed point in this limit. This demonstrates the dimensional crossover from non-renormalizability on $M^4 \times T^2$ to renormalizability on M^4 in terms of the RG equations and their solutions.

In the large μL limit ($M^4 \times T^2$ goes to M^6) and $J(\mu L) \sim \frac{\pi b_2}{6}(\mu L)^2$. This function diverges because the volume of the internal space T^2 was absorbed into the coupling constants when we did the dimensional reduction and this volume diverges when $\mu L \to \infty$. Thus the couplings (λ_1, λ_2 which are natural for the four-dimensional limit ($\mu L \to 0$) are inappropriate for the six-dimensional one ($\mu L \to \infty$). One can define six-dimensional couplings $\bar\lambda_1 = (\mu L)^2\lambda_1$, $\bar\lambda_2 = (\mu L)^2\lambda_2$ and check that the RG equations and solutions for $(\bar\lambda_1(\mu), \bar\lambda_1(\mu))$ do not contain any divergent terms for $\mu L > 0$. To have coupling constants which smoothly interpolate between the four-dimensional and six-dimensional limits we define

$$h_1(\mu) = \frac{1}{b_2(\epsilon)}\lambda_1(\mu)S(\mu L), \quad h_2(\mu) = \frac{1}{b_2(\epsilon)}\lambda_2(\mu)S(\mu L) \tag{30}$$

with $S(\mu L) = 1 + J(\mu L)/b_2(\epsilon)$. These have β - functions

$$\begin{aligned} \beta_1(h_1) &= -2\epsilon(\mu L)h_1 + b_2 h_1^2, & (31) \\ \beta_2(h_2) &= (2 - 2\epsilon(\mu L))h_2 - b_2 h_1^2 \frac{J(\mu L)}{S(\mu L)}, & (32) \end{aligned}$$

where the function $\epsilon(\mu L) = \epsilon - \mu(\partial S(\mu L)/\partial\mu)/(2S(\mu L))$ interpolates between ϵ and $\epsilon - 1$. In the limit $\mu L \to 0$ the coupling constants (h_1, h_2) equal (λ_1, λ_2) and the equations (31), (32) coincide with (25), (26). For the six-dimensional limit the definitions (30) coincide with slight modifications of $(\bar{\lambda}_1, \bar{\lambda}_2)$, and the RG equations have the form

$$\begin{aligned} \beta(h_1) &= (2 - 2\epsilon)h_1 + b_2 h_1^2, \\ \beta(h_2) &= (4 - 2\epsilon)h_2 - b_2 h_1^2. \end{aligned}$$

These are the natural six dimensional RG equations for this system. Thus the equations (31), (32) interpolate, in what we believe to be a natural way, between the four and six dimensional theories. The non-triviality of the RG equations for the coupling h_2 in six dimensions reflects the non-renormalizability of the theory with the infinite tower of Kaluza-Klein modes.

The solutions of these equations can obviously be obtained from the solutions (28), (29) by changing the variables according to (30)

$$\begin{aligned} h_1(\mu) &= \frac{h_1(\mu_0)\frac{S(\mu l)}{S(\mu_0 L)}}{1 - \frac{b_2^2 h_1(\mu_0)}{S(\mu_0 L)} \ln\frac{\mu}{\mu_0} - \frac{b_2}{S(\mu_0 L)} h_1 \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} J(\mu' L)}, \\ h_2(\mu) &= (\frac{\mu}{\mu_0})^2 \frac{S(\mu L)}{S(\mu_0 L)} h_2(\mu_0) - b_2 \mu^2 S(\mu L) \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'^3} h_1^2(\mu') \frac{J(\mu' L)}{S^2(\mu' L)}. \end{aligned}$$

The interpolation of these formulas between the corresponding solutions for four and six dimensional theories can be easily checked.

A few remarks are in order here. Since in the four-dimensional limit the coupling h_2 does not get renormalized it seems natural to choose the initial value $h_2(\mu_0) = 0$ in which case the theory reduces exactly to the four-dimensional one. In our one-loop calculations we assumed that $\lambda_2 \leq \lambda_1^2$. If $h_2(\mu_0) = 0$ or is fine tuned to be small then there is an interval $\mu_0 \leq \mu \leq \bar{\mu} \ll \mu^*$ where $h_2(\mu)$ remains small relative to h_1^2 and our assumption is valid. We would like to mention that the initial assumption of imposing the relationship $\lambda_2 \sim \lambda_1^2$ is similar to the fine tuning of Coleman and Weinberg in the case of scalar electrodynamics [23].

5 Conclusion

We have demonstrated to one-loop order that a non-renormalizable theory can reduce to a renormalizable one as the extra dimensions shrink to zero size. We believe that the formlism herein can be extended to arbitrary order, in particular using the R_L subtraction operator introduced in Section 2 and its associated RG. The physical criterion for dimensional reduction to occur is that all physical mass scales (masses, momenta etc) be much less than L^{-1}. Though we have only addressed the question in the case of the four point function in this work, similar analysis can be carried out for a general N-point function. We can summarize these results as the decoupling of compact dimensions via the decoupling of an infinite tower of Kaluza-Klein modes. Zero or light modes give the leading contribution to physical amplitudes and the RG equations in the limit $\mu L \to 0$, i.e. when heavy modes cannot be seen experimentally. We presented a renormalization scheme in which the decoupling of heavy modes is manifest. As far as the relations between bare and renormalized coupling constants are concerned we concluded that the form of the relations depends on the order in which the limits $L \to 0$ and $\epsilon \to 0$. We stress again however, that physical quantities cannot depend on this apparent ambiguity.

We derived RG equations for couplings h_1 and h_2 which interpolated smoothly between four and six dimensions in the limits $\mu L = 0$ and $\mu L \to \infty$ respectively. It is in this sense that we have dimensional crossover from six to four dimensions, from non-renormalizability to renormalizability.

As it was mentioned in the Introduction the problems here are similar to those on dimensional reduction in four dimensional finite temperature theories where $L = T^{-1}$. It was shown in many papers that in the limit $T \to \infty$ the nonstatic modes (corresponding to nonzero modes in our terminology) decouple in the one loop approximation and the theories reduce to effective three-dimensional ones. However, it was argued in [18] that decoupling fails at higher orders in perturbation theory due to the fact that loop corrections in the large temperature limit can induce large masses $\sim T$. It should be patently obvious that if $m > T$ (in the language of this paper the "light" particle is heavier than the lightest pyrgon) dimensional reduction cannot occur. A quantitative measure of dimensional reduction is $\epsilon(\mu L)$ which defines an effective dimension $d_{eff}(\mu L) = 6 - \epsilon(\mu L)$. The question of whether decoupling occurs to all orders is a question of great importance for Kaluza-Klein theories and will be discussed in a future paper. If dimensional reduction is to be a property of a Kaluza-Klein model then certainly it will place severe constraints on the parameters of the model. Such constrained models might be supersymmetric models as they have no hierarchy problem [24].

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References

[1] Th. Kaluza *Sitzungsber. Preuss. Akad. Wiss. Math.* **K1** (1921) 966.
O. Klein *Z. Phys.* **37** (1926) 895.

[2] M.J. Duff, B.E.W. Nilsson and C.N. Pope *Phys. Rep.* **C 130** (1986) 1. M.B. Green, J.H. Schwarz and E. Witten *Superstring Theory* (Cambridge University Press: New York, 1987).

[3] E.W. Kolb and R. Slansky *Phys. Lett.* **B135** (1984) 378.

[4] Yu.A. Kubyshin, J.M. Mourão, G. Rudolph and I.P. Volobujev *Dimensional Reduction of gauge theories, spontaneous compactification and model building, Lecture Notes in Physics* **349** (Springer-Verlag, Berlin, 1989).

[5] M.J. Duff and D.J. Toms. *in* Proc. 2nd Quantum Gravity Seminar, eds. M.A. Markov and P.C. West (Plenum Press, 1984); preprint Ref. TH.3248-CERN (1982).

[6] S.D. Unwin *Phys. Lett.* **103B** (1981) 18.

[7] T. Appelquist, A. Chodos *Phys. Rev.* **D28** (1983) 772; *Phys. Rev. Lett.* **50** (1983) 141 T. Appelquist, A. Chodos and E. Myers *Phys. Lett.* **127B** (1983) 51.

[8] J.S. Dowker *Phys. Rev.* **D29** (1984) 2773.
N.Marcus and A. Sagnotti *Nucl. Phys.* **B256** (1985) 77.

[9] T. Appelquist, A. Chodos and P.G.O. Rubin. *Modern Kaluza-Klein theories* (Addison-Wesley Publ/ Comp., Menlo Park, 1987).

[10] S. Weinberg *in* Inner space/outer space, eds. E. Kolb et al. (University of Chicago Press, Chicago, 1986).

[11] E.S. Fradkin and A.A. Tseytlin *Nucl. Phys.* **B227** (1983) 252.

[12] T. Appelquist and J. Carrazone *Phys. Rev.* **D11** (1975) 2856.

[13] K. Symanzyk *Comm. Math. Phys.* **34** (1973) 7.

[14] J.C. Collins *Renormalization* (Cambridge University Press, Cambridge, 1984).

[15] I. Antoniadis *Phys. Lett.* **B246** (1990) 377.

[16] L. Dolan and R. Jackiw *Phys. Rev.* **D9** (1974) 3320. O. Kalashnikov and V. Klimov *Sov. J. Nucl. Phys.* **53** (1979) 669; 33 (1981) 443. D. Gross, R. Pisarski and L. Yaffe *Rev. Mod. Phys.* **53** (1981) 43. R. Jackiw and S. Templeton *Phys. Rev.* **D23** (1981) 43
T. Appelquist and R. Pisarski *Phys. Rev.* **D23** (1981) 2305.

[17] R.F. Alvarez-Estrada *Phys. Rev.* **D36** (1987) 2411, *Physica* **A 158** (1989) 178.

[18] N.P. Landsman *Nucl. Phys.* **B322** (1989) 498.

[19] Denjoe O'Connor, C.R. Stephens and B.L. Hu *Ann. Phys.* **190** (1989) 310. Denjoe O'Connor and C.R. Stephens *Nucl. Phys.* **B260** (1991) 297; Finite size scaling and the RG. Preprint Imperial/TP/89/90/36, Imperial College; Dimensional Reduction and the non-triviality of $\lambda\phi^4$ in four dimensions at high temperature, Preprint THU-92/02.

[20] Yu. Kubyshin, D. O'Connor and C. Stephens. Preprint JINR E2-91-497, JINR, Dubna (1991); Preprints Nos DIAS-STP-91-36, DIAS-STP-91-37, DIAS, Dublin (1991).

[21] S.W. Hawking *Commun. Math. Phys.* **55** (1977) 133.

[22] N.N. Bogoliubov and D.V. Shirkov *Introduction to the theory of quantized fields* (Interscience, N.Y., 1959).

[23] S. Coleman, E.J. Weinberg, *Phys. Rev.* **D7** (1973) 2887.

[24] P. Candelas, G. Horowitz, A. Strominger and E. Witten *Nucl. Phys.* **B258** (1985) 46.