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On Hamiltonian Reductions of the Wess-Zumino-Novikov-Witten Theories

L. Fehér*, L. O'Raifeartaigh, P. Ruelle and I. Tsutsui

Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland

A. Wipf

Institut für Theoretische Physik
Eidgenössische Technische Hochschule
Hönggerberg, Zürich CH-8093, Switzerland

Abstract

The structure of Hamiltonian symmetry reductions of the Wess-Zumino-Novikov-Witten (WZNW) theories by first class Kac-Moody (KM) constraints is analyzed in detail. Lie algebraic conditions are given for ensuring the presence of exact integrability, conformal invariance and $\mathcal{W}$-symmetry in the reduced theories. A Lagrangean, gauged WZNW implementation of the reduction is established in the general case and thereby the path integral as well as the BRST formalism are set up for studying the quantum version of the reduction. The general results are applied to a number of examples. In particular, a $\mathcal{W}$-algebra is associated to each embedding of $sl(2)$ into the simple Lie algebras by using purely first class constraints. The primary fields of these $\mathcal{W}$-algebras are manifestly given by the $sl(2)$ embeddings, but it is also shown that there is an $sl(2)$ embedding present in every polynomial and primary KM reduction and that the W_n^l-algebras have a hidden $sl(2)$ structure too. New generalized Toda theories are found whose chiral algebras are the $\mathcal{W}$-algebras based on the half-integral $sl(2)$ embeddings, and the $\mathcal{W}$-symmetry of the effective action of those generalized Toda theories associated with the integral gradings is exhibited explicitly.

* Present address: Laboratoire de Physique Nucléaire, Université de Montréal, Montréal, Canada H3C 3J7. On leave from Bolyai Institute of Szeged University, H-6720 Szeged, Hungary.

Contents

1. Introduction

Due to their intimate relationship with Lie algebras, the various one- and two-dimensional Toda systems are among the most important models of the theory of integrable non-linear equations [1-19]. In particular, the standard conformal Toda field theories, which are given by the Lagrangean

$$\mathcal{L}_{\mathrm{Toda}}(\varphi) = \frac{\kappa}{2}\left(\sum_{i,j=1}^{l} \frac{1}{2|\alpha_i|^2} K_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j - \sum_{i=1}^{l} m_i^2 \exp\left\{\frac{1}{2}\sum_{j=1}^{l} K_{ij}\varphi^j\right\}\right), \tag{1.1}$$

where κ is a coupling constant, K_{ij} is the Cartan matrix and the α_i are the simple roots of a simple Lie algebra of rank l, have been the subject of many studies [1,3,4,8-13,19]. It has been first shown by Leznov and Saveliev [1,3] that the Euler-Lagrange equations of (1.1) can be written as a zero curvature condition, are exactly integrable, and possess interesting non-linear symmetry algebras [3,4,10,11,13,19]. These symmetry algebras are generated by chiral conserved currents, and are polynomial extensions of the chiral Virasoro algebras generated by the traceless energy-momentum tensor. The chiral currents in question are conformal primary fields, whose conformal weights are given by the orders of the independent Casimirs of the corresponding simple Lie algebra. Polynomial extensions of the Virasoro algebra by chiral primary fields are generally known as $\mathcal{W}$-algebras [20], which are expected to play an important role in the classification of conformal field theories and are in the focus of current investigations [20-29]. The importance of Toda systems in two-dimensional conformal field theory is in fact greatly enhanced by their realizing the $\mathcal{W}$-algebra symmetries.

It has been discovered recently that the conformal Toda field theories can be naturally viewed as Hamiltonian reductions of the Wess-Zumino-Novikov-Witten (WZNW) theory [12,13]. The main feature of the WZNW theory is its affine Kac-Moody (KM) symmetry, which underlies its integrability [30,31]. The WZNW theory provides the most 'economical' realization of the KM symmetry in the sense that its phase space is essentially a direct product of the left $\times$ right KM phase spaces. The WZNW $\rightarrow$ Toda Hamiltonian reduction is achieved by imposing certain first class, conformally invariant constraints on the KM currents, which reduce the chiral KM phase spaces to phase spaces carrying the chiral $\mathcal{W}$-algebras as their Poisson bracket structure [12,13]. Thus the $\mathcal{W}$-algebra is related to the phase space of the Toda theory in the same way as the KM algebra is related to the phase space of the WZNW theory. In the above manner, the $\mathcal{W}$-symmetry of the Toda theories becomes manifest by describing these theories as reduced WZNW theories. This way of looking at Toda theories has also numerous other advantages, described in detail in [13].

The constrained WZNW (KM) setting of the standard Toda theories ($\mathcal{W}$-algebras) allows for generalizations, some of which have already been investigated [14-18,26-29]. An important recent development is the realization that it is possible to associate a generalized $\mathcal{W}$-algebra to every embedding of the Lie algebra $sl(2)$ into the simple Lie algebras [16-18]. The standard $\mathcal{W}$-algebra, occurring in Toda theory, corresponds to the so called principal $sl(2)$. In fact, these generalized $\mathcal{W}$-algebras can be obtained from the KM algebra by constraining the current to the highest weight gauge, which has been originally introduced in [13] for describing the standard case. Another interesting development is the W_n^l-algebras introduced by Bershadsky [26] and further studied in [28]. It is known that the simplest non-trivial case W_3^2, which was originally proposed by Polyakov [27], falls into a special case of the $\mathcal{W}$-algebras obtained by the $sl(2)$ embeddings mentioned above. It has not been clear, however, as to whether the two classes of $\mathcal{W}$-algebras are related in general, or to what extent one can further generalize the KM reduction to achieve new $\mathcal{W}$-algebras.

In the present paper, we undertake the first systematic study of the Hamiltonian reductions of the WZNW theory, aiming at uncovering the general structure of the reduction and, at the same time, try to answer the above question. Various different questions arising from this main problem are also addressed (see Contents), and some of them can be examined on its own right. As this provides our motivation and in fact most of the later developments originate from it, we wish to recall here the main points of the WZNW $\rightarrow$ Toda reduction before giving a more detailed outline of the content.

To make contact with the Toda theories, we consider the WZNW theory*

$$S_{\mathrm{WZ}}(g) = \frac{\kappa}{2}\int d^2x\, \eta^{\mu\nu}\, \mathrm{Tr}\,(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g) - \frac{\kappa}{3}\int_{B_3} \mathrm{Tr}\,(g^{-1}dg)^3 \ , \tag{1.2}$$

for a simple, maximally non-compact, connected real Lie group G. In other words, we assume that the simple Lie algebra, $\mathcal{G}$, corresponding to G allows for a Cartan decomposition over the field of real numbers. The field equation of the WZNW theory can be written in the equivalent forms

$$\partial_- J = 0 \qquad \text{or} \qquad \partial_+ \tilde{J} = 0 \ , \tag{1.3}$$

where

$$J = \kappa\partial_+ g \cdot g^{-1} \ , \quad \text{and} \quad \tilde{J} = -\kappa g^{-1}\partial_- g \ . \tag{1.4}$$

* The KM level k is $-4\pi\kappa$. The space-time conventions are: $\eta_{00} = -\eta_{11} = 1$ and $x^{\pm} = \frac{1}{2}(x^0 \pm x^1)$. The WZNW field g is periodic in x^1 with period $2\pi r$.

These equations express the conservation of the left- and right KM currents, J and $\tilde{J}$, respectively. The general solution of the WZNW field equation is given by the simple formula

$$g(x^+, x^-) = g_L(x^+) \cdot g_R(x^-) \,, \tag{1.5}$$

where g_L and g_R are arbitrary G-valued functions, i.e., constrained only by the boundary condition imposed on g.

Let now M_-, M_0 and M_+ be the standard generators of the principal $sl(2)$ subalgebra of $\mathcal{G}$ [32]. By considering the eigenspaces $\mathcal{G}_m$ of M_0 in the adjoint of $\mathcal{G}$, $\mathrm{ad}_{M_0} = [M_0, \;]$, one can define a grading of $\mathcal{G}$ by the eigenvalues m. Under the principal $sl(2)$ this grading is an integral grading, in fact the spins occurring in the decomposition of the adjoint of $\mathcal{G}$ are the exponents of $\mathcal{G}$, which are related to the orders of the independent Casimirs by a shift by 1. It is also worth noting that the grade 0 part of

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_0 + \mathcal{G}_- \,, \qquad \mathcal{G}_\pm = \sum_{m=1}^{N} \mathcal{G}_{\pm m} \,, \tag{1.6}$$

is a Cartan subalgebra, and (by using some automorphism of the Lie algebra) one can assume that the generator M_0 is given by the formula $M_0 = \frac{1}{2}\sum_{\alpha>0} H_\alpha$, where H_α is the standard Cartan generator corresponding to the positive root α, and the generators $M_\pm$ are certain linear combinations of the step operators $E_{\pm\alpha_i}$ corresponding to the simple roots α_i, $i = 1, \ldots, \mathrm{rank}\,\mathcal{G}$.

The basic observation of [12,13] has been that the standard Toda theory can be obtained from the WZNW theory by imposing first class constraints which restrict the currents to take the following form:

$$J(x) = \kappa M_- + j(x), \qquad \text{with} \qquad j(x) \in (\mathcal{G}_0 + \mathcal{G}_+) \,, \tag{1.7a}$$

and

$$\tilde{J}(x) = -\kappa M_+ + \tilde{j}(x), \qquad \text{with} \qquad \tilde{j}(x) \in (\mathcal{G}_0 + \mathcal{G}_-) \,. \tag{1.7b}$$

(For clarity, we note that one should in principle include some dimensional constants in $M_\pm$ which are dimensionless, but such constants are always put to unity in this paper, for simplicity.) To derive the Toda theory (1.1) from the WZNW theory (1.2), one uses the generalized Gauss decomposition $g = g_+ \cdot g_0 \cdot g_-$ of the WZNW field g, where $g_{0,\pm}$ are from the subgroups $G_{0,\pm}$ of G corresponding to the Lie subalgebras $\mathcal{G}_{0,\pm}$, respectively. In this framework the Toda fields φ_i are given by the middle-piece of the Gauss decomposition,

$g_0 = \exp[\frac{1}{2}\sum_{i=1}^{l}\varphi_i H_i]$, which is invariant under the triangular KM gauge transformations belonging to the first class constraints (1.7). Note that here the elements $H_i \in \mathcal{G}_0$ are the standard Cartan generators associated to the simple roots. In fact, the Toda field equation can be derived directly from the WZNW field equation by inserting the Gauss decomposition of g into (1.3) and using the constraints (1.7). The effective action of the reduced theory, (1.1), can also be obtained in a natural way, by using the Lagrangean, gauged WZNW implementation of the Hamiltonian reduction [13].

In their pioneering work [1,3], Leznov and Saveliev proved the exact integrability of the conformal Toda systems by exhibiting chiral quantities by using the field equation and the special graded structure of the Lax potential $\mathcal{A}_\pm$, in terms of which the Toda equation takes the zero curvature form

$$[\partial_+ - \mathcal{A}_+\, ,\, \partial_- - \mathcal{A}_-] = 0\ . \tag{1.8}$$

In our framework the exact integrability of Toda systems is seen as an immediate consequence of the obvious integrability of the WZNW theory, which survives the reduction to Toda theory. In other words, the chiral fields underlying the integrability of the Toda equation are available from the very beginning, that is, they come from the fields entering the left $\times$ right decomposition of the general WZNW solution (1.5). Furthermore, the Toda Lax potential itself emerges naturally from the trivial, chiral Lax potential of the WZNW theory. To see this one first observes that the WZNW field equation is a zero curvature condition, since one can write for example the first equation in (1.3) as

$$[\partial_+ - J\, ,\, \partial_- - 0] = 0\ . \tag{1.9}$$

Using the constraints of the reduction, the Toda zero curvature condition (1.8) of [1,3] arises from (1.9) by conjugating this equation by $g_+^{-1}(x^+, x^-)$, namely by the inverse of the upper triangular piece of the generalized Gauss decomposition of the WZNW field g [18].

The $\mathcal{W}$-symmetry of the Toda theory appears in the WZNW setting in a very direct and natural way. Namely, one can interpret the $\mathcal{W}$-algebra as the KM Poisson bracket algebra of the gauge invariant differential polynomials of the constrained currents in (1.7). Concentrating on the left sector, the gauge transformations act on the current according to

$$J(x) \rightarrow e^{a(x^+)}\, J(x)\, e^{-a(x^+)} + \kappa(e^{a(x^+)})'\, e^{-a(x^+)}, \tag{1.10}$$

where $a(x^+) \in \mathcal{G}_+$ is an arbitrary chiral parameter function.* The constraints (1.7) are chosen in such a way that the following Virasoro generator

$$L_{M_0}(x) \equiv L_{\rm KM}(x) - {\rm Tr}\,(M_0 J'(x)), \quad \text{where} \quad L_{\rm KM}(x) = \frac{1}{2\kappa}{\rm Tr}(J^2(x)), \tag{1.11}$$

is gauge invariant, which ensures the conformal invariance of the reduced theory.

One obtains an equivalent interpretation of the $\mathcal{W}$-algebra by identifying it with the Dirac bracket algebra of the differential polynomials of the current components in certain gauges, which are such that a basis of the gauge invariant differential polynomials reduces to the independent current components after the gauge fixing. We call the gauges in question Drinfeld-Sokolov (DS) gauges [13], since such gauges has been used also in [5]. They have the nice property that any constrained current $J(x)$ can be brought to the gauge fixed form by a unique gauge transformation depending on $J(x)$ in a differential polynomial way. The most important DS gauge is the highest weight gauge [13], which is defined by requiring the gauge fixed current to be of the following form:

$$J_{\rm red}(x) = \kappa M_- + j_{\rm red}(x)\ , \qquad j_{\rm red}(x) \in {\rm Ker}({\rm ad}_{M_+})\ , \tag{1.12}$$

where ${\rm Ker}({\rm ad}_{M_+})$ is the kernel of the adjoint of M_+. In other words, $j_{\rm red}(x)$ is restricted to be an arbitrary linear combination of the highest weight vectors of the $sl(2)$ subalgebra in the adjoint of $\mathcal{G}$. The special property of the highest weight gauge is that in this gauge the conformal properties become manifest. Of course, the quantity $L_{\rm red}(x)$ obtained by restricting $L_{M_0}(x)$ in (1.11) to the highest weight gauge generates a Virasoro algebra under Dirac bracket. (Note that in our case $L_{\rm red}(x)$ is proportional to the M_+-component of $j_{\rm red}(x)$.) The important point is that, with the exception of the M_+-component, the spin s component of $j_{\rm red}(x)$ is in fact a primary field of conformal weight $(s+1)$ with respect to $L_{\rm red}(x)$ under the Dirac bracket. Thus *the highest weight gauge automatically yields a primary field basis of the $\mathcal{W}$-algebra*, from which one sees that the spectrum of conformal weights is fixed by the $sl(2)$ content of the adjoint of $\mathcal{G}$ [13].

In the above we arrived at the description of the $\mathcal{W}$-algebra as a Dirac bracket algebra by gauge fixing the first class system of constraints corresponding to (1.7). However, it is clear now that it would have been possible to define the $\mathcal{W}$-algebra as the Dirac bracket algebra of the components of $j_{\rm red}$ in (1.12) in the first place. Once this point is realized, a natural generalization arises immediately [16-18]. Namely, one can associate a classical

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* Throughout the paper, the notation $f' = 2\partial_1 f$ is used for every function f, including the spatial δ-functions. For a chiral function $f(x^+)$ one has then $f' = \partial_+ f$.

$\mathcal{W}$-algebra to any $sl(2)$ subalgebra $\mathcal{S} = \{M_-,\ M_0,\ M_+\}$ of any simple Lie algebra $\mathcal{G}$, by defining it to be the Dirac bracket algebra of the components of $j_{\rm red}$ in (1.12), where one simply substitutes the generators $M_\pm$ of the *arbitrary* $sl(2)$ subalgebra $\mathcal{S}$ for those of the *principal* $sl(2)$. As we shall see in this paper, this Dirac bracket algebra is a polynomial extension of the Virasoro algebra by primary fields, whose conformal weights are related to the spins occurring in the decomposition of the adjoint of $\mathcal{G}$ under $\mathcal{S}$ by a shift by 1, in complete analogy with the case of the principal $sl(2)$. We shall designate the generalized $\mathcal{W}$-algebra associated to the $sl(2)$ embedding $\mathcal{S}$ as $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$.

With the main features of the WZNW $\rightarrow$ Toda reduction and the above definition of the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras at our disposal, now we sketch the philosophy and the outline of the present paper. We start by giving the most important assumption underlying our investigations, which is that we consider those reductions which can be obtained by imposing *first class* KM constraints generalizing the ones in (1.7). To be more precise, our most general constraints restrict the current to take the following form:

$$J(x) = \kappa M + j(x), \qquad \text{with} \qquad j(x) \in \Gamma^\perp\ , \tag{1.13}$$

where M is *some* constant element of the underlying simple Lie algebra $\mathcal{G}$, and $\Gamma^\perp$ is the subspace consisting of the Lie algebra elements trace orthogonal to *some* subspace Γ of $\mathcal{G}$. We note that earlier in (1.7a) we have chosen $\Gamma = \mathcal{G}_+$ and $M = M_-$, but we do not need any $sl(2)$ structure here. The whole analysis is based on requiring the first-classness of the system of linear KM constraints corresponding the pair (Γ, M) according to (1.13). However, this first-classness assumption is not as restrictive as one perhaps might think at first sight. In fact, as far as we know, our first class method is capable of covering all Hamiltonian reductions of the WZNW theory considered to date. The many technical advantages of using purely first class KM constraints will be apparent.

The investigations in this paper are organized according to three distinct levels of generality. At the most general level we only make the first-classness assumption and deduce the following results. First, we give a complete Lie algebraic analysis of the conditions on the pair (Γ, M) imposed by the first-classness of the constraints. We shall see that Γ in (1.13) has to be a subalgebra of $\mathcal{G}$ on which the Cartan-Killing form vanishes, and that every such subalgebra is solvable. The Lie subalgebra Γ will be referred to as the 'gauge algebra' of the reduction. For a given Γ, the first-classness imposes a further condition on the element M, and we shall describe the space of the allowed M's. Second, we establish a gauged WZNW implementation of the reduction, generalizing the one found previously in the standard case [13]. This gauged WZNW setting of the reduction will be first seen classically, but it will be also established in the quantum theory by considering the phase space path integral of the constrained WZNW theory. Third, the gauged WZNW framework will be used to set up the BRST formalism for the quantum Hamiltonian reduction in the general case. Fourth, by making the additional assumption that the left and right gauge algebras are dual to each other with respect to the Cartan-Killing form, we will be able to give a detailed local analysis of the effective theories resulting from the reduction. This duality assumption will also be related to the parity invariance of the effective theories, which is satisfied in the standard Toda case where the left and right gauge algebras are $\mathcal{G}_+$ and $\mathcal{G}_-$ in (1.6), respectively. In general, the WZNW reduction not only allows us to make contact with known theories, like the Toda theory in (1.1), where the simplicity and the large symmetry of the 'parent' WZNW theory are fully exploited for analyzing them, but also leads to new theories which are 'integrable by construction'.

At the next level of generality, we study the conformally invariant reductions. The basic idea here is that one can guarantee the conformal invariance of the reduced theory by exhibiting a Virasoro density such that the corresponding conformal action preserves the constraints in (1.13). Generalizing (1.11), we assume that this Virasoro density is of the form

$$L_H(x) = L_{\rm KM}(x) - \mathrm{Tr}\,(HJ'(x)) \ , \tag{1.14}$$

where H is *some* Lie algebra element, to be determined from the condition that L_H weakly commutes with the first class constraints. We shall describe the relations which are imposed on the triple of quantities (Γ, M, H) by this requirement, and thereby obtain a Lie algebraic sufficient condition for conformal invariance.

At the third level of generality, we deal with polynomial reductions and $\mathcal{W}$-algebras. The above mentioned sufficient condition for conformal invariance is a guarantee for L_H being a gauge invariant differential polynomial. We shall provide an additional condition on the triple of quantities (Γ, M, H) which allows one to construct out of the current in (1.13) a complete set of gauge invariant differential polynomials by means of a polynomial gauge fixing algorithm. The KM Poisson bracket algebra of the gauge invariant differential polynomials yields a polynomial extension of the Virasoro algebra generated by L_H. We shall prove that the existence of a complete set of primary fields in this algebra requires the existence of an element $M_+ \in \Gamma$ which together with $M_- \equiv M$ and $M_0 \equiv H$ forms an $sl(2)$ subalgebra of $\mathcal{G}$. This implies that the conformal weights of the primary fields are necessarily half integrals. The most important application of our sufficient condition for polynomiality concerns the $\mathcal{W}_S^{\mathcal{G}}$-algebras, for which the $sl(2)$ structure of the primary fields is manifest, as mentioned previously.

Let us remember that, for an arbitrary $sl(2)$ subalgebra $\mathcal{S}$ of $\mathcal{G}$, the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra can be defined as the Dirac bracket algebra of the highest weight current in (1.12) realized by purely second class constraints. However, we shall see in this paper that these second class constraints can be replaced by purely first class constraints even in the case of arbitrary, *integral or half-integral,* $sl(2)$ embeddings. Since the first class constraints satisfy our sufficient condition for polynomiality, we can realize the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra as the KM Poisson bracket algebra of the corresponding gauge invariant differential polynomials. After having our hands on first class KM constraints leading to the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras, we shall immediately apply our general construction to exhibiting reduced WZNW theories realizing these $\mathcal{W}$-algebras as their chiral algebras for arbitrary $sl(2)$-embeddings. In the non-trivial case of half-integral $sl(2)$-embeddings, these generalized Toda theories represent a new class of integrable models, which will be studied in some detail. It is also worth noting that realizing the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra as a KM Poisson bracket algebra of gauge invariant differential polynomials should in principle allow for quantizing it through the KM representation theory, for example by using the general BRST formalism which will be set up in this paper. As a first step, we shall give a concise formula for the Virasoro centre of this algebra in terms of the level of the underlying KM algebra.

The existence of purely first class KM constraints leading to the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra might be perhaps surprizing to the reader, since earlier in [16] it was claimed to be inevitably necessary to use at least some second class constraints from the very beginning, when reducing the KM algebra to $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ in the case of a half-integral $sl(2)$ embedding. Contrary to their claim, we will demonstrate that it is possible and in fact easy to obtain the appropriate first class constraints which lead to $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$. Roughly speaking, this will be achieved by discarding 'half' of those constraints which form the second class part in the mixed system of the constraints imposed in [16]. The mixed system of constraints can be recovered by a partial gauge fixing of our purely first class KM constraints. Similarly, Bershadsky's constraints [26], used to define the W_n^l-algebra, are also a mixed system in the above sense, i.e., it contains both first and second class parts. We can also replace these constraints by purely first class ones without changing the final reduced phase space. In this procedure we shall uncover the hidden $sl(2)$ structure of the W_n^l-algebras, namely, we shall identify them in general as further reductions of particular $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras.

The study of WZNW reductions embraces various subjects, such as integrable models, $\mathcal{W}$-algebras and their field theoretic realizations. We hope that the readers with different interests will find relevant results throughout this paper, and find an interplay of general considerations and investigations of numerous examples.

2. General structure of KM and WZNW reductions

The purpose of this chapter is to investigate the general structure of those reductions of the KM phase space and corresponding reductions of the full WZNW theory which can be defined by imposing *first class constraints setting certain current components to constant values*. In the rest of the paper, we assume that the WZNW group, G, is a connected real Lie group whose Lie algebra, $\mathcal{G}$, is a non-compact real form of a complex simple Lie algebra, $\mathcal{G}_c$. We shall first uncover the Lie algebraic implications of the constraints being first class, and also discuss a sufficient condition which may be used to ensure their conformal invariance. In particular, we shall see why the compact real form is outside our framework. We then set up a gauged WZNW theory which provides a Lagrangean realization of the WZNW reduction, for the case of general first class constraints. Finally, we shall describe the effective field theories resulting from the reduction in some detail in an important special case, namely when the left and right KM currents are constrained for such subalgebras of $\mathcal{G}$ which are dual to each other with respect to the Cartan-Killing form.

2.1. First class and conformally invariant KM constraints

Here we analyze the general form of the KM constraints which will be used subsequently to reduce the WZNW theory. The analysis applies to each current J and $\tilde{J}$ separately so we choose one of them, J say, for definiteness. To fix the conventions, we first note that the KM Poisson bracket reads

$$\{\langle u, J(x)\rangle\, ,\, \langle v, J(y)\rangle\}_{|x^0=y^0} = \langle [u,v], J(x)\rangle\delta(x^1-y^1) + \kappa\langle u,v\rangle\delta'(x^1-y^1), \tag{2.1}$$

where u and v are arbitrary generators of $\mathcal{G}$ and the inner product $\langle u\, ,\, v\rangle = \mathrm{Tr}\,(u\cdot v)$ is normalized so that the long roots of $\mathcal{G}_c$ have length squared 2. This normalization means that in terms of the adjoint representation one has $\langle u\, ,\, v\rangle = \frac{1}{2g}\mathrm{tr}\,(\mathrm{ad}_u\cdot \mathrm{ad}_v)$, where g is the dual Coxeter number. It is worth noting that $\langle u\, ,\, v\rangle$ is the usual matrix trace in the defining, vector representation for the classical Lie algebras A_l and C_l, and it is $\frac{1}{2}\times$ trace in the defining representation for the B_l and D_l series. We also wish to point out that the KM Poisson bracket together with all the subsequent relations which follow from it hold *in the same form* both on the usual canonical phase space and on the space of the classical solutions of the theory. This is the advantage of using equal time Poisson brackets and spatial δ-functions even on the latter space, where $J(x)$ depends on $x = (x^0, x^1)$ only through x^+ (see the footnote on page 7).

The KM reduction we consider is defined by requiring the constrained current to be of the following special form:

$$J(x) = \kappa M + j(x) , \qquad \text{with} \qquad j(x) \in \Gamma^{\perp} , \tag{2.2}$$

where Γ is some linear subspace and M is some element of $\mathcal{G}$. Equivalently, the constraints can be given as

$$\phi_\gamma(x) = \langle \gamma , J(x) \rangle - \kappa \langle \gamma , M \rangle = 0 , \qquad \forall \gamma \in \Gamma . \tag{2.3}$$

In words, our constraints set the current components corresponding to Γ to constant values. It is clear both from (2.2) and (2.3) that M can be shifted by an arbitrary element from the space $\Gamma^{\perp}$ without changing the actual content of the constraints. This ambiguity is unessential, since one can fix M, for example, by requiring that it is from some given linear complement of $\Gamma^{\perp}$ in $\mathcal{G}$, which can be chosen by convention.

In our method we assume that the above system of constraints is *first class*, and now we analyze the content of this condition. Immediately from (2.1), we have*

$$\{\phi_\alpha(x), \phi_\beta(y)\} = \phi_{[\alpha,\beta]}(x)\delta(x^1 - y^1) + \omega_M(\alpha,\beta)\delta(x^1 - y^1) + \langle \alpha, \beta \rangle \delta'(x^1 - y^1), \tag{2.4}$$

where the second term contains the restriction to Γ of the following anti-symmetric 2-form of $\mathcal{G}$:

$$\omega_M(u,v) \equiv \langle M , [u , v] \rangle , \qquad \forall u , v \in \mathcal{G} . \tag{2.5}$$

It is evident from (2.4) that the constraints are first class if, and only if, we have

$$[\alpha , \beta] \in \Gamma, \qquad \langle \alpha , \beta \rangle = 0 \quad \text{and} \quad \omega_M(\alpha , \beta) = 0, \quad \text{for} \quad \forall \alpha , \beta \in \Gamma. \tag{2.6}$$

This means that the linear subspace Γ has to be *a subalgebra on which the Cartan-Killing form and* ω_M *vanish.* It is easy to see that the three conditions in (2.6) can be equivalently written as

$$[\Gamma , \Gamma^{\perp}] \subset \Gamma^{\perp}, \qquad \Gamma \subset \Gamma^{\perp} \qquad \text{and} \qquad [M , \Gamma] \subset \Gamma^{\perp} , \tag{2.7}$$

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* For simplicity, we set κ to 1 in the rest of the paper, except in Chapter 5, where κ occurs in the formula of the Virasoro centre.

respectively. Subalgebras Γ satisfying $\Gamma \subset \Gamma^{\perp}$ exist in every real form of the complex simple Lie algebras except the compact one, since for the compact real form the Cartan-Killing inner product is (negative) definite.

We note that for a given Γ the third condition and the ambiguity in choosing M can be concisely summarized by the (equivalent) statement that

$$M \in [\Gamma\,,\,\Gamma]^{\perp}/\Gamma^{\perp}\,. \tag{2.8}$$

The constraints defined by the zero element of this factor-space are in a sense trivial. It is clear that, for a subalgebra Γ such that $\Gamma \subset \Gamma^{\perp}$, the above factor-space contains non-zero elements if and only if $[\Gamma,\Gamma] \neq \Gamma$. Actually this is always so because $\Gamma \subset \Gamma^{\perp}$ implies that Γ is a *solvable* subalgebra of $\mathcal{G}$. To prove this, we first note that if Γ is not solvable then, by Levi's theorem [33], it contains a semi-simple subalgebra, in which one can find either an $so(3,R)$ or an $sl(2,R)$ subalgebra. From this one sees that there exists at least one generator λ of Γ for which the operator ad_{λ} is diagonalizable with real eigenvalues. It cannot be that all eigenvalues of ad_{λ} are 0 since $\mathcal{G}$ is a simple Lie algebra, and from this one gets that $\langle\lambda\,,\,\lambda\rangle \neq 0$, which contradicts $\Gamma \subset \Gamma^{\perp}$. Therefore one can conclude that Γ is necessarily a solvable subalgebra of $\mathcal{G}$.

The second condition in (2.6) can be satisfied for example by assuming that every $\gamma \in \Gamma$ is a nilpotent element of $\mathcal{G}$. This is true in the concrete instances of the reduction studied in Chapters 3 and 4. We note that in this case Γ is actually a nilpotent Lie algebra, by Engel's theorem [33]. However, the nilpotency of Γ is not necessary for satisfying $\Gamma \subset \Gamma^{\perp}$. In fact, a solvable but not nilpotent Γ can be found in Appendix A.

The current components constrained in (2.3) are the infinitesimal generators of the KM transformations corresponding to the subalgebra Γ, which act on the KM phase space as

$$J(x) \longrightarrow e^{a^i(x^+)\gamma_i}\, J(x)\, e^{-a^i(x^+)\gamma_i} + (e^{a^i(x^+)\gamma_i})'\, e^{-a^i(x^+)\gamma_i}\,, \tag{2.9}$$

where the $a^i(x^+)$ are parameter functions and there is a summation over some basis γ_i of Γ. Of course, the first class conditions are equivalent to the statement that the constraint surface, consisting of currents of the form (2.2), is left invariant by the above transformations. From the point of view of the reduced theory, these transformations are to be regarded as gauge transformations, which means that the reduced phase space can be identified as the space of gauge orbits in the constraint surface. Taking this into account, we shall often refer to Γ as the gauge algebra of the reduction.

We next discuss a sufficient condition for the conformal invariance of the constraints. We assume that $M \notin \Gamma^{\perp}$ from now on. The standard conformal symmetry generated by the Sugawara Virasoro density $L_{\rm KM}(x)$ is then broken by the constraints (2.3), since they set some component of the current, which has spin 1, to a non-zero constant. The idea is to circumvent this apparent violation of conformal invariance by changing the standard action of the conformal group on the KM phase space to one which does leave the constraint surface invariant. One can try to generate the new conformal action by changing the usual KM Virasoro density to the new Virasoro density

$$L_H(x) = L_{\rm KM}(x) - \langle H, J'(x)\rangle, \tag{2.10}$$

where H is some element of $\mathcal{G}$. The conformal action generated by $L_H(x)$ operates on the KM phase space as

$$\begin{aligned}\delta_{f,H}\, J(x) &\equiv -\int dy^1\, f(y^+)\,\{L_H(y)\,,\, J(x)\}\\ &= f(x^+)J'(x) + f'(x^+)\big(J(x) + [H, J(x)]\big) + f''(x^+)H\ ,\end{aligned} \tag{2.11}$$

for any parameter function $f(x^+)$, corresponding to the conformal coordinate transformation $\delta_f\, x^+ = -f(x^+)$. In particular, $j(x)$ in (2.2) transforms under this new conformal action according to

$$\delta_{f,H}\, j(x) = f(x^+)j'(x) + f''(x^+)H + f'(x^+)\big(j(x) + [H, j(x)] + ([H, M] + M)\big), \tag{2.12}$$

and our condition is that this variation should be in $\Gamma^{\perp}$, which means that this conformal action preserves the constraint surface. From (2.12), one sees that this is equivalent to having the following relations:

$$H \in \Gamma^{\perp}, \qquad [H, \Gamma^{\perp}] \subset \Gamma^{\perp} \qquad \text{and} \qquad ([H, M] + M) \in \Gamma^{\perp}\ . \tag{2.13}$$

In conclusion, the existence of an operator H satisfying these relations is a *sufficient condition* for the conformal invariance of the KM reduction obtained by imposing (2.3). The conditions in (2.13) are equivalent to $L_H(x)$ being a gauge invariant quantity, inducing a corresponding conformal action on the reduced phase space. Obviously, the second relation in (2.13) is equivalent to

$$[H, \Gamma] \subset \Gamma\ . \tag{2.14}$$

An element $H \in \mathcal{G}$ is called *diagonalizable* if the linear operator ad_H possesses a complete set of eigenvectors in $\mathcal{G}$. By the eigenspaces of ad_H, such an element defines a grading of $\mathcal{G}$, and below we shall refer to a diagonalizable element as a *grading operator* of $\mathcal{G}$. In the examples we study later, conformal invariance will be ensured by the existence of a grading operator subject to (2.13).

If H is a grading operator satisfying (2.13) then it is always possible to shift M by some element of $\Gamma^{\perp}$ (i.e., without changing the physics) so that the new M satisfies

$$[H, M] = -M \ , \tag{2.15}$$

instead of the last condition in (2.13). It is also clear that if H is a grading operator then one can take graded bases in Γ and $\Gamma^{\perp}$, since these are invariant subspaces under ad_H. On re-inserting (2.15) into (2.12) it then follows that all components of $j(x)$ are primary fields with respect to the conformal action generated by $L_H(x)$, with the exception of the H-component, which also survives the constraints according to the first condition in (2.13).

As an example, let us now consider some arbitrary grading operator H and denote by $\mathcal{G}_m$ the eigensubspace corresponding to the eigenvalue m of ad_H. Then the graded subalgebra $\mathcal{G}_{\geq n}$, which is defined to be the direct sum of the subspaces $\mathcal{G}_m$ for all $m \geq n$, will qualify as a gauge algebra Γ for any $n > 0$ from the spectrum of ad_H. In this case $\Gamma^{\perp} = \mathcal{G}_{>-n}$ and the factor space $[\Gamma, \Gamma]^{\perp}/\Gamma^{\perp}$, which is the space of the allowed M's, can be represented as the direct sum of $\mathcal{G}_{-n}$ and that graded subspace of $\mathcal{G}_{<-n}$ which is orthogonal to $[\Gamma, \Gamma]$. It is easy to see that one obtains *conformally invariant* first class constraints by choosing M to be any *graded element* from this factor space. Indeed, if the grade of M is $-m$ then $L_{H/m}$ yields a Virasoro density weakly commuting with the corresponding constraints.

In summary, in this section we have seen that one can associate a first class system of KM constraints to any pair (Γ,M) subject to (2.6) by requiring the constrained current to take the form (2.2), and that the conformal invariance of this system of constraints is guaranteed if one can find an operator H such that the triple (Γ,M,H) satisfies the conditions in (2.13).

2.2. Lagrangean realization of the Hamiltonian reduction

We shall exhibit here a gauged WZNW theory providing the Lagrangean realization of those Hamiltonian reductions of the WZNW theory which can be defined by imposing first class constraints of the type (2.3) on the KM currents J and $\tilde{J}$ of the theory. It should be noted that, in the rest of this chapter, we do not assume that the constraints are conformally invariant.

To define the WZNW reduction, we can choose left and right constraints completely independently. We shall denote the pairs consisting of an appropriate subalgebra and a constant matrix corresponding to the left and right constraints as (Γ, M) and $(\tilde{\Gamma}, -\tilde{M})$, respectively. The reduced theory is obtained by first constraining the WZNW phase space by setting

$$\phi_i = \langle \gamma_i\,,\, J\rangle - \langle \gamma_i\,,\, M\rangle = 0, \qquad \text{and} \qquad \tilde{\phi}_i = -\langle \tilde{\gamma}_i\,,\, \tilde{J}\rangle - \langle \tilde{\gamma}_i\,,\, \tilde{M}\rangle = 0, \tag{2.16}$$

where the γ_i and the $\tilde{\gamma}_i$ form bases of Γ and $\tilde{\Gamma}$, respectively, and then factorizing the constraint surface by the canonical transformations generated by these constraints. One can apply this reduction either to the usual canonical phase space or to the space of solutions of the classical field equation. These are equivalent procedures since the two spaces in question are isomorphic. For later purpose we note that the constraints generate the following *chiral* gauge transformations on the space of solutions:

$$g(x^+, x^-) \longrightarrow e^{\gamma(x^+)} \cdot g(x^+, x^-) \cdot e^{-\tilde{\gamma}(x^-)}\ , \tag{2.17}$$

where $\gamma(x^+)$ and $\tilde{\gamma}(x^-)$ are arbitrary Γ and $\tilde{\Gamma}$ valued functions.

For completeness, we wish to mention here how the above way of reducing the WZNW theory fits into the general theory of Hamiltonian (symplectic) reduction of symmetries [34]. In general, the Hamiltonian reduction is obtained by setting the phase space functions generating the symmetry transformations through Poisson bracket (in other words, the components of the momentum map) to some constant values. The reduced phase space results by factorizing this constraint surface by the subgroup of the symmetry group respecting the constraints. The symmetry group we consider is the left $\times$ right KM group generated by $\Gamma \times \tilde{\Gamma}$ and our Hamiltonian reduction is special in the sense that the full symmetry group preserves the constraints. Of course, the latter fact is just a reformulation of the first-classness of our constraints.

We now come to the main point of the section, which is that the reduced WZNW theory, defined in the above by using the Hamiltonian picture, can be identified as the gauge invariant content of a corresponding gauged WZNW theory. This gauged WZNW interpretation of the reduction was pointed out in the concrete case of the WZNW $\rightarrow$ standard Toda reduction in [13], and we below generalize that construction to the present situation.

The gauged WZNW theory we are interested in is given by the following action functional:

$$I(g,A_-,A_+)\equiv S_{\rm WZ}(g)+\int d^2x\,\big(\langle A_-,\partial_+gg^{-1}-M\rangle +\langle A_+,g^{-1}\partial_-g-\tilde{M}\rangle+\langle A_-,gA_+g^{-1}\rangle\big)\,, \tag{2.18}$$

where the gauge fields $A_-(x)$ and $A_+(x)$ vary in Γ and $\tilde{\Gamma}$, respectively. The main property of this action is that it is invariant under the following *non-chiral* gauge transformations:

$$g\rightarrow \alpha g\tilde{\alpha}^{-1};\quad A_-\rightarrow \alpha A_-\alpha^{-1}+\alpha\partial_-\alpha^{-1};\quad A_+\rightarrow \tilde{\alpha}A_+\tilde{\alpha}^{-1}+(\partial_+\tilde{\alpha})\tilde{\alpha}^{-1}\,, \tag{2.19a}$$

where

$$\alpha=e^{\gamma(x^+,x^-)}\qquad\text{and}\qquad \tilde{\alpha}=e^{\tilde{\gamma}(x^+,x^-)}\,, \tag{2.19b}$$

for any $\gamma(x^+,x^-)\in\Gamma$ and $\tilde{\gamma}(x^+,x^-)\in\tilde{\Gamma}$. The proof of the invariance of (2.18) under (2.19) can proceed along the same lines as for the special case in [13]. In the proof one rewrites $S_{\rm WZ}(\alpha g\tilde{\alpha}^{-1})$ by using the well-known Polyakov-Wiegmann identity [35], and in this step one uses the fact that the WZNW action *vanishes* for fields in the subgroups of G with Lie algebras Γ or $\tilde{\Gamma}$. This is an obvious consequence of the relations $\Gamma\subset\Gamma^\perp$ and $\tilde{\Gamma}\subset\tilde{\Gamma}^\perp$. The other crucial point is that the terms in (2.18) containing the constant matrices M and $\tilde{M}$ are separately invariant under (2.19). It is easy to see that this follows from the third condition in (2.6). For example, under an infinitesimal gauge transformation belonging to $\alpha\simeq 1+\gamma$, the term $\langle A_-,M\rangle$ changes by

$$\delta\,\langle A_-,M\rangle=-\langle\partial_-\gamma,M\rangle+\omega_M(\gamma,A_-)\,, \tag{2.20}$$

which is a total divergence since the second term vanishes, as both A_- and γ are from Γ.

The Euler-Lagrange equation derived from (2.18) by varying g can be written equivalently as

$$\partial_-(\partial_+gg^{-1}+gA_+g^{-1})+[A_-,\partial_+gg^{-1}+gA_+g^{-1}]+\partial_+A_-=0\,, \tag{2.21a}$$

or

$$\partial_+(g^{-1}\partial_-g+g^{-1}A_-g)-[A_+,g^{-1}\partial_-g+g^{-1}A_-g]+\partial_-A_+=0\,, \tag{2.21b}$$

and the field equations obtained by varying A_- and A_+ are given by

$$\langle\gamma\,,\partial_+gg^{-1}+gA_+g^{-1}-M\rangle=0,\qquad \forall\,\gamma\in\Gamma\,, \tag{2.21c}$$

and

$$\langle\tilde{\gamma}\,,g^{-1}\partial_-g+g^{-1}A_-g-\tilde{M}\rangle=0,\qquad \forall\,\tilde{\gamma}\in\tilde{\Gamma}\,, \tag{2.21d}$$

respectively. We now note that by making use of the gauge invariance, A_+ and A_- can be set equal to zero simultaneously. The important point for us is that, as is easy to see, in the $A_\pm = 0$ gauge one recovers from (2.21) both the field equations (1.3) of the WZNW theory and the constraints (2.16). Furthermore, one sees that setting $A_\pm$ to zero is not a complete gauge fixing, the residual gauge transformations are exactly the chiral gauge transformations of equation (2.17).

The above arguments tell us that the space of gauge orbits in the space of classical solutions of the gauged WZNW theory (2.18) can be naturally identified with the reduced phase space belonging to the Hamiltonian reduction of the WZNW theory determined by the first class constraints (2.16). It can be also shown that the Poisson bracket induced on the reduced phase space by the Hamiltonian reduction is the same as the one determined by the gauged WZNW action (2.18). In summary, we see that the gauged WZNW theory (2.18) provides a natural Lagrangean implementation of the WZNW reduction.

2.3. Effective field theories from left-right dual reductions

The aim of this section is to describe the effective field equations and action functionals for an important class of the reduced WZNW theories. This class of theories is obtained by making the assumption that the left and right gauge algebras Γ and $\tilde{\Gamma}$ are *dual to each other* with respect to the Cartan-Killing form, which means that one can choose bases $\gamma_i \in \Gamma$ and $\tilde{\gamma}_j \in \tilde{\Gamma}$ so that

$$\langle \gamma_i, \tilde{\gamma}_j \rangle = \delta_{ij} \ . \tag{2.22}$$

This *technical* assumption allows for having a simple general algorithm for disentangling the constraints:

$$\phi_i = \langle \gamma_i,\, \partial_+ g\, g^{-1} - M \rangle = 0, \quad \text{and} \quad \tilde{\phi}_i = \langle \tilde{\gamma}_i,\, g^{-1}\partial_- g - \tilde{M} \rangle = 0, \tag{2.23}$$

which define the reduction. We shall comment on the physical meaning of the assumption at the end of the section, here we only point out that it holds, e.g., if one chooses Γ and $\tilde{\Gamma}$ to be the images of each other under a Cartan involution* of the underlying simple Lie algebra.

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* A Cartan involution σ of the simple Lie algebra $\mathcal{G}$ is an automorphism for which $\sigma^2 = 1$ and $\langle v, \sigma(v) \rangle < 0$ for any non-zero element v of $\mathcal{G}$.

For concreteness, let us consider the maximally non-compact real form which can be defined as the real span of a Chevalley basis H_i, $E_{\pm\alpha}$ of the corresponding complex Lie algebra $\mathcal{G}_c$, and in the case of the classical series A_n, B_n, C_n and D_n is given by $sl(n+1, R)$, $so(n, n+1, R)$, $sp(2n, R)$ and $so(n, n, R)$, respectively. In this case the Cartan involution is $(-1) \times$ transpose, operating on the Chevalley basis according to

$$H_i \longrightarrow -H_i \qquad E_{\pm\alpha} \longrightarrow -E_{\mp\alpha} . \tag{2.24}$$

It is obvious that $\langle v \,,\, v^t \rangle > 0$ for any non-zero $v \in \mathcal{G}$ and from this one sees that Γ^t is dual to Γ with respect to the Cartan-Killing form, i.e., (2.22) holds for $\tilde{\Gamma} = \Gamma^t$. It should also be mentioned that there is a Cartan involution for every non-compact real form of the complex simple Lie algebras, as explained in detail in [36].

Equation (2.22) implies that the left and right gauge algebras do not intersect, and thus we can consider a direct sum decomposition of $\mathcal{G}$ of the form

$$\mathcal{G} = \Gamma + \mathcal{B} + \tilde{\Gamma} \,, \tag{2.25a}$$

where $\mathcal{B}$ is some linear subspace of $\mathcal{G}$. Here $\mathcal{B}$ is in principle an arbitrary complementary space to $(\Gamma + \tilde{\Gamma})$ in $\mathcal{G}$, but one can always make the choice

$$\mathcal{B} = (\Gamma + \tilde{\Gamma})^{\perp} \,, \tag{2.25b}$$

which is natural in the sense that the Cartan-Killing form is non-degenerate on this $\mathcal{B}$. Choosing $\mathcal{B}$ according to (2.25b) is especially well-suited in the case of the parity invariant effective theories discussed at the end of the section. We note that it might also be convenient if one can take the space $\mathcal{B}$ to be a subalgebra of $\mathcal{G}$, but this is not necessary for our arguments and is not always possible either.

We can associate a 'generalized Gauss decomposition' of the group G to the direct sum decomposition (2.25), which is the main tool of our analysis. By 'Gauss decomposing' an element $g \in G$ according to (2.25), we mean writing it in the form

$$g = a \cdot b \cdot c \,, \quad \text{with} \quad a = e^{\gamma}, \quad b = e^{\beta} \quad \text{and} \quad c = e^{\tilde{\gamma}}, \tag{2.26}$$

where γ, β and $\tilde{\gamma}$ are from the respective subspaces in (2.25).

There is a neighbourhood of the identity in G consisting of elements which allow a unique decomposition of this sort, and in this neighbourhood the pieces a, b and c can be extracted from g by algebraic operations. (Actually it is also possible to define b as a product of exponentials corresponding to subspaces of $\mathcal{B}$, and we shall make use of this freedom later, in Chapter 4.) We make the assumption that every G-valued field we encounter is decomposable as g in (2.26). It is easily seen that in this 'Gauss decomposable sector' the components of $b(x^+, x^-)$ provide a complete set of gauge invariant local fields, which are the local fields of the reduced theory we are after. Below we explain how to solve the constraints (2.23) in the Gauss decomposable sector of the WZNW theory. More exactly, for our method to work, we restrict ourselves to considering those fields which vary in such a Gauss decomposable neighbourhood of the identity where the matrix

$$V_{ij}(b) = \langle \gamma_i, b\tilde{\gamma}_j b^{-1} \rangle \tag{2.27}$$

is invertible. Due to the assumptions, the analysis given in the following yields a *local* description of the reduced theories. It is clear that for a global description one should use patches on G obtained by multiplying out the Gauss decomposable neighbourhood of the identity, but we do not deal with this issue here.

First we derive the field equation of the reduced theory by implementing the constraints directly in the WZNW field equation $\partial_-(\partial_+ gg^{-1}) = 0$. (This is allowed since the WZNW dynamics leaves the constraint surface invariant, i.e., the WZNW Hamiltonian weakly commutes with the constraints.) By inserting the Gauss decomposition of g into (2.23) and making use of the constraints being first class, the constraint equations can be rewritten as

$$\begin{aligned} &\langle \gamma_i,\, \partial_+ bb^{-1} + b(\partial_+ cc^{-1})b^{-1} - M \rangle = 0\,, \\ &\langle \tilde{\gamma}_i,\, b^{-1}\partial_- b + b^{-1}(a^{-1}\partial_- a)b - \tilde{M} \rangle = 0\,. \end{aligned} \tag{2.28}$$

With the help of the inverse of $V_{ij}(b)$ in (2.27), one can solve these equations for $\partial_+ cc^{-1}$ and $a^{-1}\partial_- a$ in terms of b,

$$\partial_+ cc^{-1} = b^{-1}T(b)b, \qquad \text{and} \qquad a^{-1}\partial_- a = b\tilde{T}(b)b^{-1}, \tag{2.29a}$$

where

$$\begin{aligned} T(b) &= \sum_{ij} V_{ij}^{-1}(b)\langle \gamma_j,\, M - \partial_+ bb^{-1} \rangle b\tilde{\gamma}_i b^{-1}, \\ \tilde{T}(b) &= \sum_{ij} V_{ij}^{-1}(b)\langle \tilde{\gamma}_i,\, \tilde{M} - b^{-1}\partial_- b \rangle b^{-1}\gamma_j b\,. \end{aligned} \tag{2.29b}$$

It is easy to obtain the effective field equation for the field $b(x^+, x^-)$ by using this explicit form of the constraints. This can be achieved for example by noting that, by applying the operator $\mathrm{Ad}_{a^{-1}}$ to equation (1.9) (i.e., by conjugating it by a^{-1}) the WZNW field equation can be written in the form

$$[\partial_+ - \mathcal{A}_+\,,\, \partial_- - \mathcal{A}_-] = 0 \tag{2.30}$$

with

$$\mathcal{A}_+ = \partial_+ b\, b^{-1} + b(\partial_+ c c^{-1})b^{-1} \quad \text{and} \quad \mathcal{A}_- = -a^{-1}\partial_- a\ . \tag{2.31}$$

Thus, by inserting the constraints (2.29) into the above form of the WZNW equation, we see that the field equation of the reduced theory is the zero curvature condition of the following Lax potential:

$$\mathcal{A}_+(b) = \partial_+ b\, b^{-1} + T(b) \quad \text{and} \quad \mathcal{A}_-(b) = -b\tilde{T}(b)b^{-1}\ . \tag{2.32}$$

More explicitly, the effective field equation reads

$$\partial_-(\partial_+ bb^{-1}) + [b\tilde{T}(b)b^{-1},\, T(b)] + \partial_- T(b) + b(\partial_+ \tilde{T}(b))b^{-1} = 0. \tag{2.33}$$

The expression on the left-hand-side of (2.33) in general varies in the full space $\mathcal{G}$, but not all the components represent independent equations. The number of the independent equations is the number of the independent components of the WZNW field equation minus the number of the constraints in (2.23), since the constraints automatically imply the corresponding components of the WZNW equation. Thus there are exactly as many independent equations in (2.33) as the number of the reduced degrees of freedom. In fact, the *independent field equations* can be obtained by taking the Cartan-Killing inner product of (2.33) with a basis of the linear space $\mathcal{B}$ in (2.25), and the inner product of (2.33) with the γ_i and the $\tilde{\gamma}_i$ vanishes as a consequence of the constraints in (2.23) together with the independent field equations. To see this one first recalls that the left-hand-side of (2.33) is, upon imposing the constaints, equivalent to $a^{-1}(\partial_- J)a$. Thus the inner product of this with Γ, and similarly that of $c(\partial_+ \tilde{J})c^{-1}$ with $\tilde{\Gamma}$, vanishes as a consequence of the constraints. From this, by using the identity $a^{-1}(\partial_- J)a = -bc(\partial_+ \tilde{J})c^{-1}b^{-1}$, one can conclude that the inner product of $a^{-1}(\partial_- J)a$ with $\tilde{\Gamma}$ also vanishes as a consequence of the constraints and the independent field equations.

At this point we would like to mention certain special cases when the above equations simplify. First we note that if one has

$$[\mathcal{B}\,, \Gamma] \subset \Gamma \qquad \text{and} \qquad [\mathcal{B}\,, \tilde{\Gamma}] \subset \tilde{\Gamma}\,, \tag{2.34}$$

then

$$T(b) = M - \pi_{\tilde{\Gamma}}(\partial_+ bb^{-1}) \quad \text{and} \quad \tilde{T}(b) = \tilde{M} - \pi_{\Gamma}(b^{-1}\partial_- b)\ , \tag{2.35}$$

where we introduced the operators

$$\pi_{\Gamma} = \sum_i |\gamma_i\rangle\langle\tilde{\gamma}_i| \qquad \text{and} \qquad \pi_{\tilde{\Gamma}} = \sum_i |\tilde{\gamma}_i\rangle\langle\gamma_i|\,, \tag{2.36}$$

which project onto the spaces Γ and $\tilde{\Gamma}$, and assumed that $M \in \tilde{\Gamma}$ and $\tilde{M} \in \Gamma$. (The latter assumption can be done without loss of generality due to the duality condition (2.22)). One obtains (2.35) from (2.29) by taking into account that in this case $V_{ij}(b)$ in (2.27) is the matrix of the operator Ad_b acting on $\tilde{\Gamma}$, and thus the inverse is given by $\mathrm{Ad}_{b^{-1}}$. The nicest possible situation occurs when $\mathcal{B} = (\Gamma + \tilde{\Gamma})^{\perp}$ is a *subalgebra* of $\mathcal{G}$ and also satisfies (2.34). In this case one simply has $T = M$ and $\tilde{T} = \tilde{M}$ and thus (2.33) simplifies to

$$\partial_-(\partial_+ b b^{-1}) + [b\tilde{M}b^{-1}\,,\, M] = 0\ . \tag{2.37}$$

The derivative term is now an element of $\mathcal{B}$ and by combining the above assumptions with the first class conditions $[M, \Gamma] \subset \Gamma^{\perp}$ and $[\tilde{M}, \tilde{\Gamma}] \subset \tilde{\Gamma}^{\perp}$ one sees that the commutator term in (2.37) also varies in $\mathcal{B}$, which ensures the consistency of this equation.

The effective field equation (2.33) is in general a non-linear equation for the field $b(x^+, x^-)$, and we can give a procedure which can in principle be used for producing its *general solution*. We are going to do this by making use of the fact that the space of solutions of the reduced theory is the space of the constrained WZNW solutions factorized by the chiral gauge transformations, according to equation (2.17). Thus the idea is to find the general solution of the effective field equation by first parametrizing, in terms of arbitrary chiral functions, those WZNW solutions which satisfy the constraints (2.23), and then extracting the b-part of those WZNW solutions by algebraic operations. In other words, we propose to derive the general solution of (2.33) by looking at the origin of this equation, instead of its explicit form.

To be more concrete, one can start the construction of the general solution by first Gauss-decomposing the chiral factors of the general WZNW solution $g(x^+, x^-) = g_L(x^+) \cdot g_R(x^-)$ as

$$g_L(x^+) = a_L(x^+) \cdot b_L(x^+) \cdot c_L(x^+), \quad g_R(x^-) = a_R(x^-) \cdot b_R(x^-) \cdot c_R(x^-). \tag{2.38}$$

Then the constraint equations (2.23) become

$$\partial_+ c_L c_L^{-1} = b_L^{-1} T(b_L) b_L \quad \text{and} \quad a_R^{-1} \partial_- a_R = b_R \tilde{T}(b_R) b_R^{-1}\ . \tag{2.39}$$

In addition to the the purely algebraic problems of computing the quantities T and $\tilde{T}$ and extracting b from $g = g_L \cdot g_R = a \cdot b \cdot c$, these first order systems of ordinary differential equations are all one has to solve to produce the general solution of the effective field equation. If this can be done by quadrature then the effective field equation is also integrable by quadrature. In general, one can proceed by trying to solve (2.39) for the functions $c_L(x^+)$

and $a_R(x^-)$ in terms of the arbitrary 'input functions' $b_L(x^+)$ and $b_R(x^-)$. Clearly, this involves only a *finite number of integrations whenever the gauge algebras* Γ *and* $\tilde{\Gamma}$ *consist of nilpotent elements of* $\mathcal{G}$. Thus in this case (2.33) is exactly integrable, i.e., its general solution can be obtained by quadrature.

We note that in concrete cases some other choice of input functions, instead of the chiral b's, might prove more convenient for finding the general solutions of the systems of first order equations on g_L and g_R given in (2.39) (see for instance the derivation of the general solution of the Liouville equation given in [12]).

It is natural to ask for the action functional underlying the effective field theory obtained by imposing the constraints (2.23) on the WZNW theory. In fact, the effective action is given by the following formula:

$$I_{\rm eff}(b) = S_{\rm WZ}(b) - \int d^2x\, \langle b\tilde{T}(b)b^{-1}\,,\, T(b)\rangle. \tag{2.40}$$

One can derive the following condition for the extremum of this action:

$$\langle \delta b b^{-1}, \partial_-(\partial_+ b b^{-1}) + [b\tilde{T}(b)b^{-1},\, T(b)] + \partial_- T(b) + b(\partial_+\tilde{T}(b))b^{-1}\rangle = 0. \tag{2.41}$$

It is straightforward to compute this, the only thing to remember is that the objects $b\tilde{T}b^{-1}$ and $b^{-1}Tb$ introduced in (2.29) vary in the gauge algebras Γ and $\tilde{\Gamma}$. The arbitrary variation of $b(x)$ is determined by the arbitrary variation of $\beta(x) \in \mathcal{B}$, according to $b(x) = e^{\beta(x)}$, and thus we see from (2.41) that the Euler-Lagrange equation of the action (2.40) yields exactly the independent components of the effective field equation (2.33), which we obtained previously by imposing the constraints directly in the WZNW field equation.

The effective action given above can be derived from the gauged WZNW action $I(g, A_-, A_+)$ given in (2.18), by eliminating the gauge fields $A_\pm$ by means of their Euler-Lagrange equations (2.21c-d). By using the Gauss decomposition, these Euler-Lagrange equations become equivalent to the relations

$$a^{-1}D_- a = b\tilde{T}(b)b^{-1}\,, \qquad \text{and} \qquad cD_+c^{-1} = -b^{-1}T(b)b\,, \tag{2.42}$$

where the quantities $T(b)$ and $\tilde{T}(b)$ are given by the expressions in (2.29b) and $D_\pm$ denotes the gauge covariant derivatives, $D_\pm = \partial_\pm \mp A_\pm$. Now we show that $I_{\rm eff}(b)$ in (2.40) can indeed be obtained by substituting the solution of (2.42) for $A_\pm$ back into $I(g, A_-, A_+)$ with $g = abc$. To this first we rewrite $I(abc, A_-, A_+)$ by using the Polyakov-Wiegmann identity [35] as

$$\begin{aligned} I(abc\,, A_-, A_+) = S_{\rm WZ}(b) - \int d^2x\, \Big(&\langle a^{-1}D_- a\,,\, b(cD_+c^{-1})b^{-1}\rangle \\ &+ \langle b^{-1}\partial_- b\,,\, cD_+c^{-1}\rangle - \langle \partial_+ b b^{-1}\,,\, a^{-1}D_- a\rangle + \langle A_-, M\rangle + \langle A_+, \tilde{M}\rangle\Big). \end{aligned} \tag{2.43}$$

This equation can be regarded as the gauge covariant form of the Polyakov-Wiegmann identity, and all but the last two terms are manifestly gauge invariant. The effective action (2.40) is derived from (2.43) together with (2.42) by noting, for example, that $\langle \partial_- aa^{-1}\, ,\, M\rangle$ is a total derivative, which follows from the facts that $a(x) \in e^{\Gamma}$ and $M \in [\Gamma\, ,\, \Gamma]^{\perp}$, by (2.8).

Above we have used the field equations to eliminate the gauge fields from the gauged WZNW action (2.18) on the ground that A_- and A_+ are not dynamical fields, but 'Lagrange multiplier fields' implementing the constraints. However, it should be noted that without further assumptions the Euler-Lagrange equation of the action resulting from (2.18) by means of this elimination procedure *does not* always give the effective field equation, which can always be obtained directly from the WZNW field equation. One can see this on an example in which one imposes constraints *only on one of the chiral sectors* of the WZNW theory. From this point of view, the role of our assumption on the duality of the left and right gauge algebras is that it guarantees that the effective action underlying the effective field equation can be derived from $I(g, A_-, A_+)$ in the above manner. To end this discussion, we note that for $g = abc$ the non-degeneracy of $V_{ij}(b)$ in (2.27) is equivalent to the non-degeneracy of the quadratic expression $\langle A_-\, ,\, gA_+g^{-1}\rangle$ in the components of $A_- = A_-^i\gamma_i$ and $A_+ = A_+^i\tilde{\gamma}_i$. This quadratic term enters into the gauged WZNW action given by (2.18), and its non-degeneracy is clearly important in the quantum theory, which we consider in Chapter 5.

We mentioned at the beginning of the section that, considering a maximally non-compact $\mathcal{G}$, one can make sure that the duality assumption expressed by (2.22) holds by choosing Γ and $\tilde{\Gamma}$ to be the *transposes* of each other. Here we point out that this particular left-right related choice of the gauge algebras can also be used to ensure the *parity invariance* of the effective field theory. To this first we notice that, in the case of a maximally non-compact connected Lie group G, the WZNW action $S_{WZ}(g)$ is invariant under any of the following two 'parity transformations' $g \longrightarrow Pg$:

$$(P_1g)(x^0, x^1) \equiv g^t(x^0, -x^1)\ , \quad \text{and} \quad (P_2g)(x^0, x^1) \equiv g^{-1}(x^0, -x^1). \tag{2.44}$$

If one chooses $\tilde{\Gamma} = \Gamma^t$ and $\tilde{M} = M^t$ to define the WZNW reduction then the parity transformation P_1 simply interchanges the left and right constraints, ϕ and $\tilde{\phi}$ in (2.23), and thus the corresponding effective field theory is invariant under the parity P_1. The space $\mathcal{B} = (\Gamma + \tilde{\Gamma})^{\perp}$, i.e., the choice in (2.25b), is invariant under the transpose in this case, and thus the gauge invariant field b transforms in the same way under P_1 as g does in (2.44). Of course, the parity invariance can also be seen on the level of the gauged action

$I(g, A_-, A_+)$. Namely, $I(g, A_-, A_+)$ is invariant under P_1 if one extends the definition in (2.44) to include the following parity transformation of the gauge fields:

$$(P_1 A_\pm)(x^0, x^1) \equiv A^t_\mp(x^0, -x^1) \ . \tag{2.45}$$

The P_1-invariant reduction procedure does not preserve the parity symmetry P_2, but it is possible to consider reductions preserving just P_2 instead of P_1. In fact, such reductions can be obtained by taking $\tilde{\Gamma} = \Gamma$ and $\tilde{M} = M$.

Finally, it is obvious that to construct parity invariant WZNW reductions in general, for some arbitrary but non-compact real form $\mathcal{G}$ of the complex simple Lie algebras, one can use $-\sigma$ instead of the transpose, where σ is a Cartan involution of $\mathcal{G}$.

3. Polynomiality in KM reductions and the $\mathcal{W}_S^{\mathcal{G}}$-algebras

In the previous chapter we described the conditions for (2.2) defining first class constraints and for $L_H(J)$ in (2.10) being a gauge invariant quantity on this constraint surface. It is clear that the KM Poisson brackets of the gauge invariant differential polynomials of the current always close on such polynomials and δ-distributions. The algebra of the gauge invariant differential polynomials is of special interest in the conformally invariant case when it is a polynomial extension of the Virasoro algebra. This is particularly true if the algebra is primary, i.e., has a basis which consists of a Virasoro density and primary fields, since in that case it is a $\mathcal{W}$-algebra in the sense of Zamolodchikov [20]. In Section 3.1 we give two conditions, a non-degeneracy condition and a quasi-maximality condition, which allows one to construct out of the constrained current a complete set of gauge invariant differential polynomials by means of a differential polynomial gauge fixing algorithm. We call the KM reduction polynomial if such a polynomial gauge fixing algorithm is available, and also call the corresponding gauges Drinfeld-Sokolov (DS) gauges, since our construction is a generalization of the one given in [5]. The KM Poisson bracket algebra of the gauge invariant differential polynomials becomes the Dirac bracket algebra of the current components in the DS gauges, which we consider in Section 3.2. We then demonstrate that if this algebra is primary with respect to L_H then it is possible to find an $sl(2)$ subalgebra of $\mathcal{G}$ containing H and M. Using these results we show in Section 3.4 that the $\mathcal{W}_S^{\mathcal{G}}$-algebras of the Introduction can be derived from first class constraints that permit polynomiality and that they are manifestly primary. Thus we can realize these algebras as KM Poisson bracket algebras of gauge invariant differential polynomials, which in principle allows for quantizing them through the KM representation theory. The fact that we are led to the $\mathcal{W}_S^{\mathcal{G}}$-algebras rather naturally (though not quite uniquely) by the conditions of polynomiality and primariness indicates that these are important extended conformal algebras. The importance of the $\mathcal{W}_S^{\mathcal{G}}$-algebras is supported by the result of Section 3.5 as well, where we show that the W_n^l-algebras of [26] can be interpreted as further reductions of particular $\mathcal{W}_S^{\mathcal{G}}$-algebras. This makes it possible to exhibit primary fields for the W_n^l-algebras and to describe their structure in detail in terms of the corresponding $\mathcal{W}_S^{\mathcal{G}}$-algebras, which is the subject of [37]. It is not the concern of this paper, but we also mention for completeness that due to the secondary reduction the W_n^l-algebras are in general quite different from the $\mathcal{W}_S^{\mathcal{G}}$-algebras, since they are in a sense rational rather than polynomial [37].

3.1. A sufficient condition for polynomiality

Let us suppose that (Γ, M, H) satisfy the previously given conditions, (2.6) and (2.13), for

$$J(x) = M + j(x) \,, \qquad j(x) \in \Gamma^{\perp} \tag{3.1}$$

describing the constraint surface of conformally invariant first class constraints, where H is a *grading operator* and M is subject to

$$[H, M] = -M \,, \qquad M \notin \Gamma^{\perp} \,. \tag{3.2}$$

Then, as we shall show, the following two additional conditions:

$$\Gamma \cap \mathcal{K}_M = \{0\} \,, \qquad \text{where} \qquad \mathcal{K}_M = \mathrm{Ker}(\mathrm{ad}_M) \,, \tag{3.3}$$

and

$$\Gamma^{\perp} \subset \mathcal{G}_{>-1} \,, \tag{3.4a}$$

allow for establishing *a differential polynomial gauge fixing algorithm* whereby one can construct out of $J(x)$ in (3.1) a complete set of gauge invariant differential polynomials. We have called condition (3.3) the *non-degeneracy* condition since it means that ad_M cannot annihilate any element of Γ, and have called condition (3.4a) *quasi-maximal* because it requires the dimension of the gauge-algebra to be almost as large as permitted by the first class conditions*.

Before proving this result, we discuss some consequences of the conditions, which we shall need later. In the present situation Γ, $\Gamma^{\perp}$ and $\mathcal{G}$ are graded by the eigenvalues of ad_H, and first we note that (3.4a) is *equivalent* to

$$\mathcal{G}_{\geq 1} \subset \Gamma \,. \tag{3.4b}$$

Indeed, this follows from the fact that the spaces $\mathcal{G}_h$ and $\mathcal{G}_{-h}$ are dual to each other with respect to the Cartan-Killing form, which is a consequence of its non-degeneracy and invariance under ad_H. Of course, here and below the grading is the one defined by H, and we note that $\mathcal{G}_{\pm 1}$ are non-trivial because of (3.2). The condition given by (3.4a) plays a technical role in our considerations, but perhaps it can be argued for also physically, on the basis that it ensures that the conformal weights of the primary field components of $j(x)$ in

* This will be clear later, when we require primariness in addition to polynomiality.

(3.1) are *positive* with respect to L_H (2.10). Second, let us observe that in our situation M satisfying (3.2) is *uniquely determined*, that is, there is no possibility of shifting it by elements from $\Gamma^\perp$, simply because there are no grade -1 elements in $\Gamma^\perp$, on account of (3.4a). The non-degeneracy condition (3.3) means that the operator ad_M maps Γ into $\Gamma^\perp$ in an *injective* manner. By combining this with (3.2), (3.4a) and (2.7) we see that our gauge algebra Γ can contain only *positive* grades:

$$\Gamma \subset \mathcal{G}_{>0} \,. \tag{3.5}$$

This implies that every $\gamma \in \Gamma$ is represented by a nilpotent operator in any finite dimensional representation of $\mathcal{G}$, and that

$$\mathcal{G}_{\geq 0} \subset \Gamma^\perp \,. \tag{3.6}$$

It follows from (3.2) that $[H, \mathcal{K}_M] \subset \mathcal{K}_M$, which is telling us that $\mathcal{K}_M$ is also graded, and we see from (3.3) and (3.4b) that

$$\mathcal{K}_M \subset \mathcal{G}_{<1} \,. \tag{3.7}$$

Finally, we wish to establish a certain relationship between the dimensions of $\mathcal{G}$ and $\mathcal{K}_M$. For this purpose we consider an arbitrary complementary space $\mathcal{T}_M$ to $\mathcal{K}_M$, defining a linear direct sum decomposition

$$\mathcal{G} = \mathcal{K}_M + \mathcal{T}_M \,. \tag{3.8}$$

It is easy to see that for the 2-form ω_M we have $\omega_M(\mathcal{K}_M, \mathcal{G}) = 0$, and the restriction of ω_M to $\mathcal{T}_M$ is a *symplectic* form, in other words:

$$\omega_M(\mathcal{T}_M, \mathcal{T}_M) \qquad \text{is} \quad \text{non}-\text{degenerate} \,. \tag{3.9}$$

(We note in passing that $\mathcal{T}_M$ can be identified with the tangent space at M to the coadjoint orbit of G through M, and in this picture ω_M becomes the Kirillov-Kostant symplectic form of the orbit [34].) The non-degeneracy condition (3.3) says that one can choose the space $\mathcal{T}_M$ in (3.8) in such a way that $\Gamma \subset \mathcal{T}_M$. One then obtains the inequality

$$\dim(\Gamma) \leq \frac{1}{2}\dim(\mathcal{T}_M) = \frac{1}{2}\big(\dim(\mathcal{G}) - \dim(\mathcal{K}_M)\big) \,, \tag{3.10}$$

where the factor $\frac{1}{2}$ arises since ω_M is a symplectic form on $\mathcal{T}_M$, which vanishes, by (2.6), on the subspace $\Gamma \subset \mathcal{T}_M$.

After the above clarification of the meaning of conditions (3.3) and (3.4), we now wish to show that they indeed allow for exhibiting a complete set of gauge invariant differential polynomials among the gauge invariant functions. Generalizing the arguments of [5,13,15], this will be achieved by demonstrating that an arbitrary current $J(x)$ subject to (3.1) *can be brought to a certain normal form by a unique gauge transformation which depends on $J(x)$ in a differential polynomial way.*

A normal form suitable for this purpose can be associated to any *graded* subspace $\Theta \subset \mathcal{G}$ which is *dual to* Γ *with respect to the* 2*-form* ω_M. Given such a space Θ, it is possible to choose bases γ_h^i and θ_k^j in Γ and Θ respectively such that

$$\omega_M(\gamma_h^l, \theta_k^i) = \delta_{il}\delta_{hk}, \tag{3.11}$$

where the subscript h on γ_h^l denotes the grade, and the indices i and l denote the additional labels which are necessary to specify the base vectors at fixed grade. It is to be noted that, by definition, the subsript k on elements $\theta_k^j \in \Theta$ does not denote the grade, which is $(1-k)$. The normal (or *reduced*) form corresponding to Θ is given by the following equation:

$$J_{\rm red}(x) = M + j_{\rm red}(x) \quad \text{where} \quad j_{\rm red}(x) \in \Gamma^\perp \cap \Theta^\perp \, . \tag{3.12}$$

In other words, the set of reduced currents is obtained by supplementing the first class constraints of equation (2.3) by the gauge fixing condition

$$\chi_\theta(x) = \langle J(x), \theta\rangle - \langle M, \theta\rangle = 0 \, , \qquad \forall \theta \in \Theta \, . \tag{3.13}$$

We call a gauge which can be obtained in the above manner a Drinfeld-Sokolov (DS) gauge. It is not hard to see that the space $\mathcal{V} = \Gamma^\perp \cap \Theta^\perp$ is a graded subspace of $\Gamma^\perp$ which is disjoint from the image of Γ under the operator ${\rm ad}_M$ and is in fact complementary to the image, i.e., one has

$$\Gamma^\perp = [M, \Gamma] + \mathcal{V} \, . \tag{3.14}$$

It also follows from the non-degeneracy condition (3.3) that any graded complement $\mathcal{V}$ in (3.14) can be obtained in the above manner, by means of using some Θ. Thus it is possible to define the DS normal form of the current directly in terms of a complementary space $\mathcal{V}$ as well, as has been done in special cases in [5,13,18].

As the first step in proving that any current in (3.1) is gauge equivalent to one in the DS gauge, let us consider the gauge transformation by $g_h(x^+) = \exp[\sum_l a_h^l(x^+)\gamma_h^l]$ for some fixed grade h. Suppressing the summation over l, it can be written as*

$$j(x) \to j^{g_h}(x) = e^{a_h \cdot \gamma_h}(j(x) + M)e^{-a_h \cdot \gamma_h} + (e^{a_h \cdot \gamma_h})' e^{-a_h \cdot \gamma_h} - M \, . \tag{3.15}$$

* Throughout the chapter, all equations involving gauge transformations, Poisson brackets, etc., are to be evaluated by using a fixed time, since they are all consequences of equation (2.1). By this convention, they are valid both on the canonical phase space and on the chiral KM phase space belonging to space of solutions of the theory.

Taking the inner product of this equation with the basis vectors θ_k^i in (3.11) for all $k \leq h$, we see that there is no contribution from the derivative term. We also see that the only contribution from

$$e^{a_h \cdot \gamma_h} j(x) e^{-a_h \cdot \gamma_h} = j(x) + [a_h(x^+) \cdot \gamma_h, j(x)] + \dots \tag{3.16}$$

is the one coming from the first term, since all commutators containing the elements γ_h^l drop out from the inner product in question as a consequence of the following crucial relation:

$$[\gamma_h^l, \theta_k^i] \in \Gamma, \quad \text{for} \quad k \leq h, \tag{3.17}$$

which follows from (3.4b) by noting that the grade of this commutator, $(1 + h - k)$, is at least 1 for $k \leq h$. Taking these into account, and computing the contribution from those two terms in $j^{g_h}(x)$ which contain M by using (3.11), we obtain

$$\langle \theta_k^i, j^{g_h}(x) \rangle = \langle \theta_k^i, j(x) \rangle - a_h^i(x^+) \delta_{hk}, \quad \text{for all} \quad k \leq h. \tag{3.18}$$

We see from this equation that

$$\langle \theta_k^i, j(x) \rangle = 0 \quad \Longleftrightarrow \quad \langle \theta_k^i, j^{g_h}(x) \rangle = 0 \,, \quad \text{for} \quad k < h \,, \tag{3.19}$$

and

$$a_h^i(x^+) = \langle \theta_h^i, j(x) \rangle \quad \Rightarrow \quad \langle \theta_h^i, j^{g_h}(x) \rangle = 0 \,, \quad \text{for} \quad k = h. \tag{3.20}$$

These last two equations tell us that if the gauge-fixing condition $\langle \theta_k^i, j(x) \rangle = 0$ is satisfied for all $k < h$ then we can ensure that the same condition holds for $j^{g_h}(x)$ for the *extended range of indices* $k \leq h$, by choosing $a_h^i(x^+)$ to be $\langle \theta_h^i, j(x) \rangle$. From this it is easy to see that the DS gauge (3.13) can be reached by an iterative process of gauge transformations, and the gauge-parameters $a_h^i(x^+)$ are unique polynomials in the current at each stage of the iteration.

In more detail, let us write the general element $g(a(x^+)) \in e^\Gamma$ of the gauge group as a product in order of descending grades, i.e., as

$$g(a(x^+)) = g_{h_n} \cdot g_{h_{n-1}} \cdots g_{h_1}, \quad \text{with} \quad g_{h_i}(x^+) = e^{a_{h_i}(x^+) \cdot \gamma_{h_i}} \,, \tag{3.21a}$$

where

$$h_n > h_{n-1} > \dots > h_1 \tag{3.21b}$$

is the list of grades occurring in Γ. Let us then insert this expression into

$$j \to j^g = g(j + M) g^{-1} + g' g^{-1} - M \,, \tag{3.22a}$$

and consider the condition

$$j^g(x) = j_{\rm red}(x) \ , \tag{3.22b}$$

with $j_{\rm red}(x)$ in (3.12), as an equation for the gauge-parameters $a_h(x^+)$. One sees from the above considerations that this equation is uniquely soluble for the components of the $a_h(x^+)$ and the solution is a differential polynomial in $j(x)$. This implies that the components of $j_{\rm red}(x)$ can also be uniquely computed from (3.22), and *the solution yields a complete set of gauge invariant differential polynomials of* $j(x)$, which establishes the required result. The above iterative procedure is in fact a convenient tool for computing the gauge invariant differential polynomials in practice [15]. We remark that, of course, any unique gauge fixing can be used to define gauge invariant quantities, but they are in general not polynomial, not even local in $j(x)$.

We also wish to note that an arbitrary linear subspace of $\mathcal{G}$ which is dual to $\mathcal{V}$ in (3.14) with respect to the Cartan-Killing form can be used in a natural way as the space of parameters for describing those current dependent KM transformations which preserve the DS gauge. In fact, it is possible to give an algorithm which computes the $\mathcal{W}$-algebra and its action on the other fields of the corresponding constrained WZNW theory by finding the gauge preserving KM transformations implementing the $\mathcal{W}$-transformations. This algorithm presupposes the existence of such gauge invariant differential polynomials which reduce to the current components in the DS gauge, which is ensured by the above gauge fixing algorithm, but it works without actually computing them. This issue is treated in detail in [13,18] in special cases, but the results given there apply also to the general situation investigated in the above.

3.2. The polynomiality of the Dirac bracket

It follows from the polynomiality of the gauge fixing that the components of the gauge fixed current $j_{\rm red}$ in (3.12) generate a differential polynomial algebra *under Dirac bracket*. In our proof of the polynomiality we actually only used that the graded subspace Θ of $\mathcal{G}$ is dual to the graded gauge algebra Γ with respect to ω_M and satisfies the condition

$$([\Theta\, , \Gamma])_{\geq 1} \subset \Gamma \ , \tag{3.23}$$

which is equivalent to the existence of the bases γ_h^l and θ_k^i satisfying (3.11) and (3.17). We have seen that this condition follows from (3.3) and (3.4), but it should be noted that it is a *more general condition*, since the converse is not true, as is shown by an example at the end of this section.

Below we wish to give a direct proof for the polynomiality of the Dirac bracket algebra belonging to the *second class* constraints:

$$c_\tau(x) = \langle \tau\,,\, J(x) - M\rangle = 0 \quad \text{where} \quad \tau \in \{\gamma_h^l\} \cup \{\theta_k^i\}\ . \tag{3.24}$$

The proof will shed a new light on the polynomiality condition. We note that for certain purposes second class constraints might be more natural to use than first class ones since in the second class formalism one directly deals with the physical fields. For example, the $\mathcal{W}_S^{\mathcal{G}}$-algebra mentioned in the Introduction is very natural from the second class point of view and can be realized by starting with a number of different first class systems of constraints, as we shall see in Section 3.4.

We first recall that, by definition, the Dirac bracket algebra of the reduced currents is

$$\begin{aligned}\{j^u_{\rm red}(x), j^v_{\rm red}(y)\}^* &= \{j^u_{\rm red}(x), j^v_{\rm red}(y)\}\\ &-\sum_{\mu\nu}\int dz^1 dw^1 \{j^u_{\rm red}(x), c_\mu(z)\}\Delta_{\mu\nu}(z,w)\{c_\nu(w), j^v_{\rm red}(y)\}\,,\end{aligned}\tag{3.25}$$

where, for any $u \in \mathcal{G}$, $j^u_{\rm red}(x) = \langle u, j_{\rm red}(x)\rangle$ is to be substituted by $\langle u, J(x) - M\rangle$ under the KM Poisson bracket, and $\Delta_{\mu\nu}(z,w)$ is the inverse of the kernel

$$D_{\mu\nu}(z,w) = \{c_\mu(z), c_\nu(w)\}\ , \tag{3.26}$$

in the sense that (on the constraint surface)

$$\sum_\nu \int dx^1\, \Delta_{\mu\nu}(z,x) D_{\nu\sigma}(x,w) = \delta_{\mu\sigma}\delta(z^1 - w^1). \tag{3.27}$$

To establish the polynomiality of the Dirac bracket, it is useful to consider the *matrix differential operator* $D_{\mu\nu}(z)$ defined by the kernel $D_{\mu\nu}(z,w)$ in the usual way, i.e.,

$$\sum_\nu D_{\mu\nu}(z) f_\nu(z) = \sum_\nu \int dw^1\, D_{\mu\nu}(z,w) f_\nu(w)\ , \tag{3.28}$$

for a vector of smooth functions $f_\nu(z)$, which are periodic in z^1. From the structure of the constraints in (3.24), $c_\tau = (\phi_\gamma, \chi_\theta)$, one sees that $D_{\mu\nu}(z)$ is a first order differential operator possessing the following block structure

$$D_{\mu\nu} = \begin{pmatrix} D_{\gamma\tilde\gamma} & D_{\gamma\theta}\\ D_{\tilde\theta\tilde\gamma} & D_{\tilde\theta\theta}\end{pmatrix} = \begin{pmatrix} 0 & E\\ -E^\dagger & F\end{pmatrix}, \tag{3.29}$$

where $E^\dagger$ is the formal Hermitian conjugate of the matrix E, $(E^\dagger)_{\theta\gamma} = (E_{\gamma\theta})^\dagger$. It is clear that the Dirac bracket in (3.25) is a differential polynomial in $j_{\rm red}(x)$ and $\delta(x^1-y^1)$ whenever the inverse operator $D^{-1}(z)$, whose kernel is $\Delta_{\mu\nu}(z,w)$, is a differential operator whose coefficients are differential polynomials in $j_{\rm red}(z)$. On the other hand, we see from (3.29) that the operator D is invertible if and only if its block E is invertible, and in that case the inverse takes the form

$$(D^{-1})_{\mu\nu} = \begin{pmatrix} (E^\dagger)^{-1}FE^{-1} & -(E^\dagger)^{-1} \\ E^{-1} & 0 \end{pmatrix} . \tag{3.30}$$

Since $E(z)$ and $F(z)$ are polynomial (even linear) in $j_{\rm red}(z)$ and in ∂_z and the inverse of $F(z)$ does not occur in $D^{-1}(z)$, it follows that $D^{-1}(z)$ is a polynomial differential operator if and only if $E^{-1}(z)$ is a polynomial differential operator.

To show that E^{-1} exists and is a polynomial differential operator we note that in terms of the basis of $(\Gamma + \Theta)$ in (3.24) the matrix E is given explicitly by the following formula:

$$E_{\gamma_h^m,\theta_k^n}(z) = \delta_{hk}\delta_{mn} + \langle [\gamma_h^m, \theta_k^n], j_{\rm red}(z)\rangle + \langle \gamma_h^m, \theta_k^n\rangle \partial_z \ . \tag{3.31}$$

The crucial point is that, by the grading and the property in (3.17), we have

$$E_{\gamma_h^m,\theta_k^n}(z) = \delta_{hk}\delta_{nm} \ , \qquad \text{for} \qquad k \leq h \ . \tag{3.32}$$

The matrix E has a block structure labelled by the (block) row and (block) column indices h and k, respectively, and (3.32) means that the blocks in the diagonal of E are unit matrices and the blocks below the diagonal vanish. In other words, E is of the form $E = 1 + \varepsilon$, where ε is a strictly upper triangular matrix. It is clear that such a matrix differential operator is polynomially invertible, namely by a *finite series* of the form

$$E^{-1} = 1 - \varepsilon + \varepsilon^2 + \ldots + (-1)^N \varepsilon^N \ , \qquad (\varepsilon^{N+1} = 0), \tag{3.33}$$

which finishes our proof of the polynomiality of the Dirac bracket in (3.25). One can use the arguments in the above proof to set up an algorithm for actually computing the Dirac bracket. The proof also shows that the polynomiality of the Dirac bracket is guaranteed whenever E is of the form $(1+\varepsilon)$ with ε being *nilpotent as a matrix*. In our case this was ensured by a special grading assumption, and it appears an interesting question whether polynomial reductions can be obtained at all without using some grading structure.

The zero block occurs in D^{-1} in (3.30) because the second class constraints originate from the gauge fixing of first class ones. We note that the presence of this zero block implies that the Dirac brackets of the gauge invariant quantities coincide with their original Poisson brackets, namely one sees this from the formula of the Dirac bracket by keeping in mind that the gauge invariant quantities weakly commute with the first class constraints.

Finally, we want to show that the polynomiality condition (3.23) is weaker than (3.3-4). More exactly, the non-degeneracy condition (3.3) is required by the very notion of the Θ space, (3.11), but (3.23) can hold without having the quasi-maximality condition (3.4). This is best seen by considering an example. To this let now $\mathcal{G}$ be the maximally non-compact real form of a complex simple Lie algebra. If $\{M_-, M_0, M_+\}$ is the principal $sl(2)$ embedding in $\mathcal{G}$, with commutation rules as in (3.36) below, we simply choose the one-dimensional gauge algebra $\Gamma \equiv \{M_+\}$ and take $M \equiv M_-$. The ω_M-dual to M_+ can be taken to be $\theta = M_0$, and then (3.23) holds. To show that conditions (3.4b) cannot be satisfied, we prove that a grading operator H for which $[H, M_-] = -M_-$ and $\mathcal{G}^H_{\geq 1} \subset \Gamma$, does not exist. First of all, $[H, M_-] = -M_-$ and $\langle M_-, M_+\rangle \neq 0$ imply $[H, M_+] = M_+$, and thus $\Gamma^H_{\geq 1} = \{M_+\}$. Furthermore, writing $H = (M_0 + \Delta)$, we find from $[H, M_\pm] = \pm M_\pm$ that Δ must be an $sl(2)$ singlet in the adjoint of $\mathcal{G}$. However, in the case of the principal $sl(2)$ embedding, there is no such singlet in the adjoint, and hence $H = M_0$. But then the condition $\mathcal{G}^{M_0}_{\geq 1} \subset \Gamma$ is not fulfilled.

3.3. An $sl(2)$ subalgebra of $\mathcal{G}$ from a primary field basis

The conditions given in Section 3.1 guarantee that the gauge invariant functions allow for a basis consisting of $n = \dim(\Gamma^\perp) - \dim(\Gamma)$ independent gauge invariant differential polynomials. The Poisson bracket algebra of the gauge invariant differential polynomials contains the Virasoro algebra generated by L_H. This extended conformal algebra will qualify as a $\mathcal{W}$-algebra in the sense of Zamolodchikov [20] if it has a *primary field basis*. By a primary field basis (with respect to the conformal structure defined by L_H) we mean a generating set W^i $(i = 1, \ldots, n)$ such that

$$W^1 = L_H\,, \qquad W^i: \quad \text{primary fields for} \quad i = 2, \ldots, n. \tag{3.34}$$

The existence of a primary field basis is not automatic. The intuitive reason for this is that the H-component of the constrained current $j(x)$ in (3.1) is not a primary field with respect to the conformal action generated by L_H:

$$\delta_{f,H}\, j = fj' + f'\big(j + [H, j]\big) + f'' H\ . \tag{3.35}$$

The purpose of this section is to prove the following theorem, which shows the importance of the $sl(2)$ subalgebras of the simple Lie algebras for describing the structure of the polynomial and primary KM reductions.

Theorem. *Consider conformally invariant first class constraints given by the triple (Γ, M, H) where H is a grading operator and $[H, M] = -M$. Suppose that the reduction is polynomial in the sense that the polynomial DS gauge fixing is available. (This is guaranteed if conditions (3.3-4) or (3.23) are satisfied.) Suppose furthermore that the reduced algebra has a primary field basis with respect to L_H. Then there exists an element $M_+ \in \Gamma$ such that $\{ M_- \equiv M,\ M_0 \equiv H,\ M_+ \}$ is an $sl(2)$* subalgebra of $\mathcal{G}$, *i.e., one has*

$$[M_0, M_\pm] = \pm M_\pm \ , \qquad [M_+, M_-] = 2M_0 \ . \tag{3.36}$$

We here present an indirect proof of the theorem. We start by assuming that the required $sl(2)$ generator M_+ does not exist, in spite of the assumed existence of the primary field basis (3.34). The non-existence of M_+ implies that

$$H \notin [M, \Gamma] \ . \tag{3.37}$$

Indeed, since Γ is graded by H, if there was some element $\tilde{M} \in \Gamma$ for which $H = [M\ ,\ \tilde{M}]$, then we could take M_+ to be the grade 1 component of $-2\tilde{M}$. On account of (3.37), we can choose a graded linear subspace $\mathcal{V}$ of $\Gamma^\perp$ which is disjoint from $[M, \Gamma]$ and satisfies $\Gamma^\perp = [M, \Gamma] + \mathcal{V}$ in such a way that

$$H \in \mathcal{V} \ . \tag{3.38}$$

As in Section 3.1, eqs. (3.12-14), we can associate a DS gauge to the complementary space $\mathcal{V}$, by requiring the gauge fixed current to lie on the gauge section $\mathcal{C}$ defined as follows:

$$\mathcal{C} = \{ J \mid J(x) = M + j_\mathcal{V}(x) \ , \quad j_\mathcal{V}(x) \in \mathcal{V} \}. \tag{3.39}$$

It will also be useful to consider the set of 'restricted configurations' $\mathcal{C}_0$ given by

$$\mathcal{C}_0 = \{ J \,|\, J(x) = M + h(x) H \ , \quad \forall h(x) \in R \} \ . \tag{3.40}$$

As for any DS gauge, we can find unique gauge invariant differential polynomials on the constraint surface, (3.1), which reduce to the components of $j_\mathcal{V}$ by restriction to $\mathcal{C}$. Combining this with the fact that we have $\mathcal{C}_0 \subset \mathcal{C}$, we obtain that if the W^i form a basis of the gauge invariant differential polynomials, then there exists a differential polynomial $P = P(W^1, W^2, \ldots, W^n)$ such that

$$P(W^1, W^2, \ldots, W^n)_{|\mathcal{C}_0} = h\ . \tag{3.41}$$

Let us now consider here the primary field basis, which is assumed to exist. For this basis we have

$$W^1_{|\mathcal{C}_0} = (\frac{1}{2}h^2 - h')\langle H, H\rangle \ \equiv w(h)\ , \tag{3.42}$$

and

$$W^i_{|\mathcal{C}_0} = 0 \qquad \text{for} \qquad i = 2, \ldots, n\ . \tag{3.43}$$

Equation (3.42) is the result of a straight substitution into the formula (2.10) of $W^1 = L_H$. For equation (3.43), we observe that $\mathcal{C}_0$ is an invariant submanifold under the conformal action (3.35), and thus the restriction of a primary field differential polynomial to $\mathcal{C}_0$ becomes a primary field differential polynomial of h. However, it is also easy to see from (3.35) that it is impossible to form a non-zero primary field differential polynomial from the field $h(x)$ alone, which leads to (3.43). The last three equations together imply that h is a differential polynomial of $w(h)$. This is clearly a contradiction, which completes the proof of the theorem.

The fact that H is an $sl(2)$ generator implies by (3.35) that the conformal weights of the primary field differential polynomials are half-integrals. It is worth stressing that we did not assume previously that the spectrum of the grading operator was half-integral. It is remarkable that this results from the purely classical considerations of polynomiality and primariness. The polynomiality assumption required in the theorem was that the polynomial DS gauge fixing is available. The non-degeneracy condition (3.3) is necessary for this. We also know that adding (3.4), or the weaker (3.23), to the non-degeneracy condition is sufficient for polynomiality. On the other hand, the existence of a primary field basis is a strong requirement which can be used to deduce further restrictions on the allowed triple (Γ, M, H) describing the conformally invariant reduction. The exact content of the 'DS gauge assumption' and the 'primariness assumption' requires further study, which we hope to present in a future publication.

3.4. First class constraints for the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras

In the previous section we have shown that it is possible to associate an $sl(2)$ subalgebra of $\mathcal{G}$ to any polynomial and primary KM reduction. Here we shall proceed in the opposite direction, and investigate those very natural $\mathcal{W}$-algebras which are manifestly based on the $sl(2)$ embeddings. Let $\mathcal{S} = \{M_-, M_0, M_+\}$ be an $sl(2)$ subalgebra of the simple Lie algebra $\mathcal{G}$:

$$[M_0, M_\pm] = \pm M_\pm , \qquad [M_+, M_-] = 2M_0 . \tag{3.44}$$

It was already pointed out in the Introduction that one can define an extended conformal algebra, denoted as $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$, by using any such $sl(2)$ embedding [16,18]. Namely, we defined the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra to be the Dirac bracket algebra generated by the components of the constrained KM current of the following special form:

$$J_{\rm red}(x) = M_- + j_{\rm red}(x) , \qquad \text{with} \qquad j_{\rm red}(x) \in \mathrm{Ker}(\mathrm{ad}_{M_+}) , \tag{3.45}$$

which means that $j_{\rm red}(x)$ is a linear combination of the $sl(2)$ highest weight states in the adjoint of $\mathcal{G}$. This definition is indeed natural in the sense that the conformal properties are manifest, since, as we shall see below, with the exception of the M_+-component the spin s component of $j_{\rm red}(x)$ turns out to be a primary field of conformal weight $(s+1)$ with respect to L_{M_0}. Before showing this, we shall present here *first class* KM constraints underlying the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra, which will be used in Chapter 4 to construct generalized Toda theories which realize the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras as their chiral algebras. We expect the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras to play an important organizing role in describing the (primary field content of) conformally invariant KM reductions in general. The theorem of the previous section clearly supports it, but we shall also give further arguments in favour of this idea later.

We wish to find a gauge algebra Γ for which the triple $(\Gamma, H = M_0, M = M_-)$ satisfies our sufficient conditions for polynomiality and (3.45) represents a DS gauge for the corresponding conformally invariant first class constraints. We start by noticing that the dimension of such a Γ has to satisfy the relation

$$\dim \mathrm{Ker}(\mathrm{ad}_{M_+}) = \dim \mathcal{W}^{\mathcal{G}}_{\mathcal{S}} = \dim \mathcal{G} - 2\dim \Gamma . \tag{3.46}$$

From this, since the kernels of $\mathrm{ad}_{M_\pm}$ are of equal dimension, we obtain that

$$\dim \Gamma = \frac{1}{2}\dim \mathcal{G} - \frac{1}{2}\dim \mathrm{Ker}(\mathrm{ad}_{M_-}) , \tag{3.47}$$

which means by (3.10) that we are looking for a Γ of *maximal* dimension. By the representation theory of $sl(2)$, the above equality is equivalent to

$$\dim \Gamma = \dim \mathcal{G}_{\geq 1} + \frac{1}{2}\dim \mathcal{G}_{\frac{1}{2}} , \tag{3.48}$$

where the grading is by the, in general half-integral, eigenvalues of ad_{M_0}. We also know, (3.4b) and (3.5), that for our purpose we have to choose the graded Lie subalgebra Γ of $\mathcal{G}$ in such a way that $\mathcal{G}_{\geq 1} \subset \Gamma \subset \mathcal{G}_{>0}$. Observe that the non-degeneracy condition (3.3) is automatically satisfied for any such Γ since in the present case $\mathrm{Ker}(\mathrm{ad}_{M_-}) \subset \mathcal{G}_{\leq 0}$, and $M_0 \in \Gamma^\perp$ is also ensured, which guarantees the conformal invariance, see (2.13).

It is obvious from the above that in the special case of an *integral* $sl(2)$ subalgebra, for which $\mathcal{G}_{\frac{1}{2}}$ is empty, one can simply take

$$\Gamma = \mathcal{G}_{\geq 1} \ . \tag{3.49}$$

For grading reasons,

$$\omega_{M_-}(\mathcal{G}_{\geq 1}, \mathcal{G}_{\geq 1}) = 0 \tag{3.50}$$

holds, and thus one indeed obtains first class constraints in this way.

One sees from (3.48) that for finding the gauge algebra in the non-trivial case of a *half-integral* $sl(2)$ subalgebra, one should somehow add half of $\mathcal{G}_{\frac{1}{2}}$ to $\mathcal{G}_{\geq 1}$, in order to have the correct dimension. The key observation for defining the required *halving* of $\mathcal{G}_{\frac{1}{2}}$ consists in noticing that the restriction of the 2-form ω_{M_-} to $\mathcal{G}_{\frac{1}{2}}$ is non-degenerate. This can be seen as a consequence of (3.9), but is also easy to verify directly. By the well known Darboux normal form of symplectic forms [34], there exists a (non-unique) direct sum decomposition

$$\mathcal{G}_{\frac{1}{2}} = \mathcal{P}_{\frac{1}{2}} + \mathcal{Q}_{\frac{1}{2}} \tag{3.51}$$

such that ω_{M_-} vanishes on the subspaces $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$ separately. The spaces $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$, which are the analogues of the usual momentum and coordinate subspaces of the phase space in analytic mechanics, are of equal dimension and dual to each other with respect to ω_{M_-}. The point is that the first-classness conditions in (2.6) are satisfied if we define the gauge algebra to be

$$\Gamma = \mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}} \ , \tag{3.52}$$

by using *any symplectic halving* of the above kind. It is obvious from the construction that the first class constraints,

$$J(x) = M_- + j(x) \qquad \text{with} \qquad j(x) \in \Gamma^\perp \ , \tag{3.53}$$

obtained by using Γ in (3.52) satisfy the sufficient conditions for polynomiality given in Section 3.1. With this Γ we have

$$\Gamma^\perp = \mathcal{G}_{\geq 0} + \mathcal{Q}_{-\frac{1}{2}} \ , \tag{3.54a}$$

where $\mathcal{Q}_{-\frac{1}{2}}$ is the subspace of $\mathcal{G}_{-\frac{1}{2}}$ given by

$$\mathcal{Q}_{-\frac{1}{2}} = [M_-, \mathcal{P}_{\frac{1}{2}}] \,. \tag{3.54b}$$

By combining (3.52) and (3.54) one also easily verifies the following direct sum decomposition:

$$\Gamma^{\perp} = [M_-, \Gamma] + \mathrm{Ker}(\mathrm{ad}_{M_+}) \,, \tag{3.55}$$

which is just (3.14) with $\mathcal{V} = \mathrm{Ker}(\mathrm{ad}_{M_+})$. This means that (3.45) is indeed nothing but the equation of a particular DS gauge for the first class constraints in (3.53), as required. This special DS gauge is called the *highest weight gauge* [13]. Similarly as for any DS gauge, there exists therefore a basis of gauge invariant differential polynomials of the current in (3.53) such that the base elements reduce to the components of $j_{\rm red}(x)$ in (3.45) by the gauge fixing. The KM Poisson bracket algebra of these gauge invariant differential polynomials is clearly identical to the Dirac bracket algebra of the corresponding current components, and we can thus realize the $\mathcal{W}_S^{\mathcal{G}}$-algebra as a KM Poisson bracket algebra of gauge invariant differential polynomials.

The second class constraints defining the highest weight gauge (3.45) are natural in the sense that in this case τ in (3.24) runs over the basis of the space $\mathcal{T}_{M_-} = [M_+ \,, \mathcal{G}]$ which is a natural complement of $\mathcal{K}_{M_-} = \mathrm{Ker}(\mathrm{ad}_{M_-})$ in $\mathcal{G}$, eq. (3.8).

In the second class formalism, the conformal action generated by L_{M_0} on the $\mathcal{W}_S^{\mathcal{G}}$-algebra is given by the following formula:

$$\delta^*_{f,M_0}\, j_{\rm red}(x) \equiv - \int dy^1\, f(y^+)\, \{L_{M_0}(y) \,, j_{\rm red}(x)\}^* \,, \tag{3.56}$$

where the parameter function $f(x^+)$ refers to the conformal coordinate transformation $\delta_f\, x^+ = -f(x^+)$, cf. (2.11), and $j_{\rm red}(x)$ is to be substituted by $J(x) - M_-$ when evaluating the KM Poisson brackets entering into (3.56), like in (3.25). To actually evaluate (3.56), we first replace L_{M_0} by the object

$$L_{\rm mod}(x) = L_{M_0}(x) - \frac{1}{2}\langle M_+ \,, J''(x)\rangle \,, \tag{3.57}$$

which is allowed under the Dirac bracket since the difference (the second term) vanishes upon imposing the constraints. The crucial point to notice is that $L_{\rm mod}$ weakly commutes with *all* the constraints defining (3.45) (not only with the first class ones) under the KM Poisson bracket. This implies that with $L_{\rm mod}$ the Dirac bracket in (3.56) is in fact identical to the original KM Poisson bracket and by this observation we easily obtain

$$\delta^*_{f,M_0}\, j_{\rm red}(x) = f(x^+)\, j'_{\rm red}(x) + f'(x^+)\big(j_{\rm red}(x) + [M_0, j_{\rm red}(x)]\big) - \frac{1}{2} f'''(x^+) M_+ . \tag{3.58}$$

This proves that, with the exception of the M_+-component, the $sl(2)$ highest weight components of $j_{\rm red}(x)$ in (3.45) transform as conformal primary fields, whereby the conformal content of $\mathcal{W}_S^{\mathcal{G}}$ is determined by the decomposition of the adjoint of $\mathcal{G}$ under S in the aforementioned manner. We end this discussion by noting that in the highest weight gauge $L_{M_0}(x)$ becomes a linear combination of the M_+-component of $j_{\rm red}(x)$ and a quadratic expression in the components corresponding to the singlets of S in $\mathcal{G}$. From this we see that $L_{M_0}(x)$ and the primary fields corresponding to the $sl(2)$ highest weight states give a basis for the differential polynomials contained in $\mathcal{W}_S^{\mathcal{G}}$, which is thus indeed a (classical) $\mathcal{W}$-algebra in the sense of the general idea in [20].

In the above we proposed a 'halving procedure' for finding *purely first class* constraints for which $\mathcal{W}_S^{\mathcal{G}}$ appears as the algebra of the corresponding gauge invariant differential polynomials. We now wish to clarify the relationship between our method and the construction in a recent paper by Bais *et al* [16], where the $\mathcal{W}_S^{\mathcal{G}}$-algebra has been described, in the special case of $\mathcal{G} = sl(n)$, by using a different method. We recall that the $\mathcal{W}_S^{\mathcal{G}}$-algebra has been constructed in [16] by adding to the first class constraints defined by the pair $(\mathcal{G}_{\geq 1}, M_-)$ the *second class* constraints

$$\langle u\,,\, J(x)\rangle = 0\;, \qquad \text{for} \quad \forall\, u \in \mathcal{G}_{\frac{1}{2}}\;. \tag{3.59}$$

Clearly, we recover these constraints by first imposing our complete set of first class constraint belonging to (Γ, M_-) with Γ in (3.52), and then partially fixing the gauge by imposing the condition

$$\langle u\,,\, J(x)\rangle = 0\;, \qquad \text{for} \quad \forall\, u \in \mathcal{Q}_{\frac{1}{2}}\;. \tag{3.60}$$

One of the advantages of our construction is that by using only first class KM constraints it is easy to construct generalized Toda theories which possess $\mathcal{W}_S^{\mathcal{G}}$ as their chiral algebra, for any $sl(2)$ subalgebra, namely by using our general method of WZNW reductions. This will be elaborated in the next chapter. We note that in [16] the authors were actually also led to replacing the original constraints by a first class system of constraints, in order to be able to consider the BRST quantization of the theory. For this purpose they introduced unphysical 'auxiliary fields' and thus constructed first class constraints in an extended phase space. However, in that construction one has to check that the auxiliary fields finally disappear from the physical quantities. Another important advantage of our halving procedure is that it renders the use of any such auxiliary fields completely unnecessary, since one can start by imposing a complete system of first class constraints on the KM phase space from the very beginning. We study some aspects of the BRST quantization in Chapter 5, and we shall see that the Virasoro central charge given in [16] agrees with the one computed by taking our first class constraints as the starting point.

The first class constraints leading to $\mathcal{W}_S^{\mathcal{G}}$ are not unique, for example one can consider an arbitrary halving in (3.51) to define Γ. We conjecture that these $\mathcal{W}$-algebras always occur under certain natural assumptions on the constraints. To be more exact, let us suppose that we have conformally invariant first class constraints determined by the pair (Γ, M_-) where M_- is a *nilpotent* matrix and the *non-degeneracy* condition (3.3) holds together with equation (3.47). By the Jacobson-Morozov theorem, it is possible to extend the nilpotent generator M_- to an $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$. It is also worth noting that the conjugacy class of $\mathcal{S}$ under the automorphism group of $\mathcal{G}$ is uniquely determined by the conjugacy class of the nilpotent element M_-. For this and other questions concerning the theory of $sl(2)$ embeddings into semi-simple Lie algebras the reader may consult refs. [32,33,38,39]. We expect that the above assumptions on (Γ, M_-) are sufficient for the existence of a complete set of gauge invariant differential polynomials and their algebra is isomorphic to $\mathcal{W}_S^{\mathcal{G}}$, where $M_- \in \mathcal{S}$. We are not yet able to prove this conjecture in general, but below we wish to sketch the proof in an important special case which illustrates the idea.

Let us assume that we have conformally invariant first class constraints described by (Γ, M_-, H) subject to the sufficient conditions for polynomiality given in Section 3.1, such that H is an *integral grading operator* of $\mathcal{G}$. We note that these are exactly the assumptions satisfied by the constraints in the non-degenerate case of the generalized Toda theories associated to integral gradings [18]. In this case equation (3.47) is actually automatically satisfied as a consequence of the non-degeneracy condition (3.3). One can also show that it is possible to find an $sl(2)$ algebra $\mathcal{S} = \{M_-, M_0, M_+\}$ for which in addition to $[H, M_-] = -M_-$ one has

$$[H, M_0] = 0 \qquad \text{and} \qquad [H, M_+] = M_+ \ , \tag{3.61}$$

and that for this $sl(2)$ algebra the relation

$$\mathrm{Ker}(\mathrm{ad}_{M_+}) \subset \mathcal{G}^H_{\geq 0} \tag{3.62}$$

holds, where the superscript indicates that the grading is defined by H. For the $sl(2)$ subject to (3.61) the latter property is in fact equivalent to $\mathrm{Ker}(\mathrm{ad}_{M_-}) \subset \mathcal{G}^H_{\leq 0}$, which is just the non-degeneracy condition as in our case $\Gamma = \mathcal{G}^H_{>0}$. The proof of these statements is given in Appendix B.

We introduce a definition at this point, which will be used in the rest of the paper. Namely, we call an $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ an *H-compatible* $sl(2)$ from now on if there exists an integral grading operator H such that $[H, M_\pm] = \pm M_\pm$ is satisfied together with the non-degeneracy condition. The non-degeneracy condition can be expressed in various equivalent forms, it can be given for example as the relation in (3.62), and its (equivalent) analogue for M_-.

Turning back to the problem at hand, we now point out that by using the H-compatible $sl(2)$ we have the following direct sum decomposition of $\Gamma^\perp = \mathcal{G}^H_{\geq 0}$:

$$\mathcal{G}^H_{\geq 0} = [M_-, \mathcal{G}^H_{>0}] + \mathrm{Ker}(\mathrm{ad}_{M_+}). \tag{3.63}$$

This means that the set of currents of the form (3.45) represents a DS gauge for the present first class constraints. This implies the required result, that is that the $\mathcal{W}$-algebra belonging to the constraints defined by $\Gamma = \mathcal{G}^H_{>0}$ together with a non-degenerate M_- is isomorphic to $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ with $M_- \in \mathcal{S}$. In this example both $L_H(x)$ and $L_{M_0}(x)$ are gauge invariant differential polynomials. Although the spectrum of ad_H is *integral* by assumption, in some cases the H-compatible $sl(2)$ is embedded into $\mathcal{G}$ in a *half-integral* manner, i.e., the spectrum of ad_{M_0} can be half-integral in certain cases. We shall return to this point later. We further note that in general it is clearly impossible to build such an $sl(2)$ out of M_- for which H would play the role of M_0. It follows from the theorem proved in the previous section that in those cases there is no full set of primary fields with repect to L_H which would complete this Virasoro density to a generating set of the corresponding differential polynomial $\mathcal{W}$-algebra. We have seen that such a conformal basis is manifest for $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$, which seems to indicate that in the present situation the conformal structure defined by the $sl(2)$, L_{M_0}, is preferred in comparison to the one defined by L_H.

We also would like to mention an interesting general fact about the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras, which will be used in the next section. Let us consider the decomposition of $\mathcal{G}$ under the $sl(2)$ subalgebra $\mathcal{S}$. In general, we shall find singlet states and they span a Lie subalgebra in the Lie subalgebra $\mathrm{Ker}(\mathrm{ad}_{M_+})$ of $\mathcal{G}$. Let us denote this zero spin subalgebra as $\mathcal{Z}$. It is easy to see that we have the semi-direct sum decomposition

$$\mathrm{Ker}(\mathrm{ad}_{M_+}) = \mathcal{Z} + \mathcal{R}, \qquad [\mathcal{Z}, \mathcal{R}] \subset \mathcal{R}, \qquad [\mathcal{Z}, \mathcal{Z}] \subset \mathcal{Z}, \tag{3.64}$$

where $\mathcal{R}$ is the linear space spanned by the rest of the highest weight states, which have non-zero spin. It is not hard to prove that the subalgebra of the original KM algebra which belongs to $\mathcal{Z}$, survives the reduction to $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$. In other words, the Dirac brackets of the $\mathcal{Z}$-components of the highest weight gauge current, $j_{\rm red}$ in (3.45), coincide with their original KM Poisson brackets, given by (2.1). Furthermore, this $\mathcal{Z}$ KM subalgebra acts on the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra by the corresponding original KM transformations, which preserve the highest weight gauge:

$$J_{\rm red}(x) \rightarrow e^{a^i(x^+)\zeta_i}\, J_{\rm red}(x)\, e^{-a^i(x^+)\zeta_i} + (e^{a^i(x^+)\zeta_i})'\, e^{-a^i(x^+)\zeta_i}, \tag{3.65}$$

where the ζ_i form a basis of $\mathcal{Z}$. In particular, one sees that the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra inherites the semi-direct sum structure given by (3.64) [16]. The point we wish to make is that it is possible to *further reduce* the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra by applying the general method of conformally invariant KM reductions to the present $\mathcal{Z}$ KM symmetry. In principle, one can generate a huge number of new conformally invariant systems out of the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras in this way, i.e., by applying conformally invariant constraints to their singlet KM subalgebras. For example, if one can find a subalgebra of $\mathcal{Z}$ on which the Cartan-Killing form of $\mathcal{G}$ vanishes, then one can consider the obviously conformally invariant reduction obtained by constraining the corresponding components of $j_{\rm red}$ in (3.45) to zero. We do not explore these 'secondary' reductions of the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras in this paper. However, their potential importance will be highlighted by the example of the next section.

Finally, we note that, for a half-integral $sl(2)$, one can consider (instead of using Γ in (3.52)) also those conformally invariant first class constraints which are defined by the triple (Γ, M_0, M_-) with any graded Γ for which $\mathcal{G}_{\geq 1} \subset \Gamma \subset (\mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}})$. The polinomiality conditions of Section 3.1 are clearly satisfied with any such 'quasi-maximal but non-maximal' Γ, and the corresponding extended conformal algebras are in a sense between the KM and $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras. However, it does not automatically follow that these algebras have a primary field basis, although we verified this in some examples.

3.5. The $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ interpretation of the W^l_n-algebras

The W^l_n-algebras are certain conformally invariant reductions of the $sl(n,R)$ KM algebra introduced by Bershadsky [26] using a mixed set of *first class and second class* constraints. It is known [16] that the simplest non-trivial case W^2_3, originally proposed by Polyakov [27], coincides with the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra belonging to the highest root $sl(2)$ of $sl(3,R)$. The purpose of this section is to understand whether or not these reduced KM systems fit into our framework, which is based on using *purely first class* constraints, and to uncover their possible connection with the $\mathcal{W}_S^{\mathcal{G}}$-algebras in the general case. (In this section, $\mathcal{G} = sl(n,R)$.) In fact, we shall construct here purely first class KM constraints leading to the W_n^l-algebras. The construction will demonstrate that the W_n^l-algebras can in general be identified as *further reductions of particular* $\mathcal{W}_S^{\mathcal{G}}$*-algebras*. The secondary reduction process is obtained by means of the singlet KM subalgebras of the relevant $\mathcal{W}_S^{\mathcal{G}}$-algebras, in the manner mentioned in the previous section.

By definition [26], the KM reduction yielding the W_n^l-algebra is obtained by constraining the current to take the following form:

$$J_{\rm B}(x) = M_- + j_{\rm B}(x), \qquad j_{\rm B}(x) \in \Delta^\perp, \tag{3.66}$$

where Δ denotes the set of all strictly upper triangular $n \times n$ matrices and

$$M_- = e_{l+1,1} + e_{l+2,2} + \ldots + e_{n,n-l}, \tag{3.67}$$

the e's being the standard $sl(n,R)$ generators ($l \leq n-1$), i.e., M_- has 1's all along the l-th slanted line below the diagonal. The current in (3.66) corresponds to imposing the constraints $\phi_\delta(x) = 0$ for all $\delta \in \Delta$, like in (2.3). Generally, these constraints comprise first and second class parts, where the first class part is the one belonging to the subalgebra $\mathcal{D}$ of Δ defined by the relation $\omega_{M_-}(\mathcal{D},\Delta) = 0$, (see (2.4)). The second class part belongs to the complementary space, $\mathcal{C}$, of $\mathcal{D}$ in Δ. In fact, for $l=1$ the constraints are the usual first class ones which yield the standard $\mathcal{W}$-algebras, but the second class part is non-empty for $l>1$. The above KM reduction is so constructed that it is conformally invariant, since the constraints weakly commute with the Virasoro density $L_{H_l}(x)$, see (2.10), where $H_l = \frac{1}{l}H_1$ and H_1 is the standard grading operator of $sl(n,R)$, for which $[H_1\,,\, e_{ik}] = (k-i)e_{ik}$.

We start our construction by extending the nilpotent generator M_- in (3.67) to an $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$. In fact, parametrizing $n = ml + r$ with $m = [\frac{n}{l}]$ and $0 \leq r < l$, we can take

$$M_0 = \mathrm{diag}\Big(\overbrace{\frac{m}{2},\cdots}^{r\ \text{times}},\overbrace{\frac{m-1}{2},\cdots}^{(l-r)\ \text{times}},\cdots,\overbrace{-\frac{m}{2},\cdots}^{r\ \text{times}}\Big), \tag{3.68}$$

where the mutiplicities, r and $(l-r)$, occur alternately and end with r. The meaning of this formula is that the fundamental of $sl(n,R)$ branches into l irreducible representations under $\mathcal{S}$, r of spin $\frac{m}{2}$ and $l-r$ of spin $\frac{m-1}{2}$. The explicit form of M_+ is a certain linear combination of the e_{ik}'s with $(k-i) = l$, which is straightforward to compute.

We describe next the first and the second class parts of the constraints in (3.66) in more detail by using the grading defined by M_0. We observe first that in terms of this grading the space Δ admits the decomposition

$$\Delta = \Delta_0 + \mathcal{G}_{\frac{1}{2}} + \mathcal{G}_1 + \mathcal{G}_{>1} \ . \tag{3.69}$$

From this and the definition of ω_{M_-}, the subalgebra $\mathcal{D}$ comprising the first class part can also be decomposed into

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{G}_{>1} \ , \tag{3.70}$$

where

$$\mathcal{D}_0 = \mathrm{Ker}\,(\mathrm{ad}_{M_-}) \cap \Delta_0 \tag{3.71}$$

is the set of the $sl(2)$ singlets in Δ, and $\mathcal{D}_1$ is a subspace of $\mathcal{G}_1$ which we do not need to specify. By combining (3.69) and (3.70), we see that the complementary space $\mathcal{C}$, to which the second class part belongs, has the structure

$$\mathcal{C} = \mathcal{Q}_0 + \mathcal{G}_{\frac{1}{2}} + \mathcal{P}_1 \ , \tag{3.72}$$

where the subspace $\mathcal{Q}_0$ is complementary to $\mathcal{D}_0$ in Δ_0, and $\mathcal{P}_1$ is complementary to $\mathcal{D}_1$ in $\mathcal{G}_1$. The 2-form ω_{M_-} is non-degenerate on $\mathcal{C}$ by construction, and this implies by the grading that the spaces $\mathcal{Q}_0$ and $\mathcal{P}_1$ are symplectically conjugate to each other, which is reflected by the notation.

We shall construct a gauge algebra, Γ, so that Bershadsky's constraints will be recovered by a partial gauge fixing from the first class ones belonging to Γ. As a generalization of the halving procedure of the previous section, we take the following ansatz:

$$\Gamma = \mathcal{D} + \mathcal{P}_{\frac{1}{2}} + \mathcal{P}_1 \ , \tag{3.73}$$

where $\mathcal{P}_{\frac{1}{2}}$ is defined by means of some symplectic halving $\mathcal{G}_{\frac{1}{2}} = \mathcal{P}_{\frac{1}{2}} + \mathcal{Q}_{\frac{1}{2}}$, like in (3.51). It is important to notice that this equation can be recasted into

$$\Gamma = \mathcal{D}_0 + \mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1} \ , \tag{3.74}$$

which would be just the familiar formula (3.52) if $\mathcal{D}_0$ was not here. By using (3.67) and (3.68), $\mathcal{D}_0$ can be identified as the set of $n \times n$ block-diagonal matrices, σ, of the following form:

$$\sigma = \text{block-diag}\{\Sigma_0, \sigma_0, \Sigma_0, \ldots.., \Sigma_0, \sigma_0, \Sigma_0\}, \tag{3.75}$$

where the Σ_0's and the σ_0's are identical copies of strictly upper triangular $r \times r$ and $(l-r) \times (l-r)$ matrices respectively. This implies that

$$\dim \mathcal{D}_0 = \frac{1}{4}[l(l-2) + (l-2r)^2] , \tag{3.76}$$

which shows that $\mathcal{D}_0$ is non-empty except when $l = 2$, $r = 1$, which is the case of W_n^2 with $n =$ odd. The fact that $\mathcal{D}_0$ is in general non-empty gives us a trouble at this stage, namely, we have now no guarantee that the above Γ is actually a *subalgebra* of $\mathcal{G}$. By using the grading and the fact that $\mathcal{D}_0$ is a subalgebra, we see that Γ in (3.74) becomes a subalgebra if and only if

$$[\mathcal{D}_0 , \mathcal{P}_{\frac{1}{2}}] \subset \mathcal{P}_{\frac{1}{2}}. \tag{3.77}$$

We next show that it is possible to find such a 'good halving' of $\mathcal{G}_{\frac{1}{2}}$ for which $\mathcal{P}_{\frac{1}{2}}$ satisfies (3.77).

For this purpose, we use yet another grading here. This grading is provided by using the particular diagonal matrix, $H \in \mathcal{G}$, which we construct out of M_0 in (3.68) by first adding $\frac{1}{2}$ to its half-integral eigenvalues, and then substracting a multiple of the unit matrix so as to make the result traceless. In the adjoint representation, we then have $\mathrm{ad}_H = \mathrm{ad}_{M_0}$ on the tensors, and $\mathrm{ad}_H = \mathrm{ad}_{M_0} \pm 1/2$ on the spinors. We notice from this that the H-grading is an integral grading. In fact, the relationship between the two gradings allows us to define a good halving of $\mathcal{G}_{\frac{1}{2}}$ as follows:

$$\mathcal{P}_{\frac{1}{2}} \equiv \mathcal{G}_{\frac{1}{2}} \cap \mathcal{G}_1^H , \qquad \text{and} \qquad \mathcal{Q}_{\frac{1}{2}} \equiv \mathcal{G}_{\frac{1}{2}} \cap \mathcal{G}_0^H . \tag{3.78}$$

Since M_- is of grade -1 with respect to both gradings, the spaces given by (3.78) clearly yield a sympectic halving of $\mathcal{G}_{\frac{1}{2}}$ with respect to ω_{M_-}. That this is a good halving, i.e., it ensures the condition (3.77), can also be seen easily by observing that $\mathcal{D}_0$ has grade 0 in the H-grading, too. Thus we obtain the required subalgebra Γ of $\mathcal{G}$ by using this particular $\mathcal{P}_{\frac{1}{2}}$ in (3.74).

Let us consider now the first class constraints corresponding to the above constructed gauge algebra Γ, $\phi_\gamma(x) = 0$ for $\gamma \in \Gamma$, which bring the current into the form

$$J_\Gamma(x) = M_- + j_\Gamma(x) , \qquad j_\Gamma(x) \in \Gamma^\perp . \tag{3.79}$$

It is easy to verify that the original constraint surface (3.66) can be recovered from (3.79) by a partial gauge fixing in such a way that the residual gauge transformations are exactly the ones belonging to the space $\mathcal{D}$. In fact, this is achieved by fixing the gauge freedom