corresponding to the piece $(\mathcal{P}_{\frac{1}{2}} + \mathcal{P}_1)$ of Γ, (3.73), by imposing the partial gauge fixing condition

$$\phi_{q_i}(x) = 0 \ , \qquad q_i \in (\mathcal{Q}_0 + \mathcal{Q}_{\frac{1}{2}}), \tag{3.80}$$

where the q_i form a basis of the space $(\mathcal{Q}_0 + \mathcal{Q}_{\frac{1}{2}})$ and the ϕ_q's are defined like in (2.3). This implies that the reduced phase space defined by the constraints in (3.79) is the same as the one determined by the original constraints (3.66). In conclusion, our purely first class constraints, (3.79), have the same physical content as Bershadsky's original mixed set of constraints, (3.66).

Finally, we give the relationship between Bershadsky's W_n^l-algebras and the $sl(2)$ systems. Having seen that the reduced KM phase spaces carrying the W_n^l-algebras can be realized by starting from the first class constraints in (3.79), it follows from (3.74) that the W_n^l-algebras coincide with particular $\mathcal{W}_S^\mathcal{G}$-algebras if and only if the space $\mathcal{D}_0$ is empty, i.e., for W_n^2 with n = odd. In order to establish the $\mathcal{W}_S^\mathcal{G}$ interpretation of W_n^l in the general case, we point out that the reduced phase space can be reached from (3.79) by means of the following two step process based on the $sl(2)$ structure. Namely, one can proceed by first fixing the gauge freedom corresponding to the piece $(\mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1})$ of Γ, and then fixing the rest of the gauge freedom. Clearly, the constraint surface resulting in the first step is the same as the one obtained by putting to zero those components of the highest weight gauge current representing $\mathcal{W}_S^\mathcal{G}$ which correspond to $\mathcal{D}_0$. The final reduced phase space is obtained in the second step by fixing the gauge freedom generated by the constraints belonging to $\mathcal{D}_0$, which we have seen to be the space of the upper triangular singlets of $\mathcal{S}$. Thus we can conclude that W_n^l can be regarded as a further reduction of the corresponding $\mathcal{W}_S^\mathcal{G}$, where the 'secondary reduction' is of the type mentioned at the end of Section 3.4. One can exhibit primary fields for the W_n^l-algebras and describe their structure in detail in terms of the underlying $\mathcal{W}_S^\mathcal{G}$-algebras by further analysing the secondary reduction, but this is outside the scope of the present paper, see [37].

4. Generalized Toda theories

Let us remind ourselves that, as has been detailed in the Introduction, the standard conformal Toda field theories can be naturally regarded as reduced WZNW theories, and as a consequence these theories possess the chiral algebras $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}} \times \tilde{\mathcal{W}}^{\mathcal{G}}_{\mathcal{S}}$ as their canonical symmetries, where $\mathcal{S}$ is the principal $sl(2)$ subalgebra of the maximally non-compact real Lie algebra $\mathcal{G}$. It is natural to seek for WZNW reductions leading to effective field theories which would realize $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}} \times \tilde{\mathcal{W}}^{\mathcal{G}}_{\mathcal{S}}$ as their chiral algebras *for any $sl(2)$ subalgebra $\mathcal{S}$ of any simple real Lie algebra.* The main purpose of this chapter is to obtain, by combining the results of sections 2.3 and 3.4, *generalized Toda theories* meeting the above requirement in the non-trivial case of the *half-integral* $sl(2)$ subalgebras of the simple Lie algebras. Before turning to describing these new theories, next we briefly recall the main features of those generalized Toda theories, associated to the *integral gradings* of the simple Lie algebras, which have been studied before [3,4,14-18]. The simplicity of the latter theories will motivate some subsequent developments.

4.1. Generalized Toda theories associated with integral gradings

The WZNW reduction leading to the generalized Toda theories in question is set up by considering an integral grading operator H of $\mathcal{G}$, and taking the special case

$$\Gamma = \mathcal{G}^H_{\geq 1} \qquad \text{and} \qquad \tilde{\Gamma} = \mathcal{G}^H_{\leq -1} \,, \tag{4.1}$$

and any non-zero

$$M \in \mathcal{G}^H_{-1} \qquad \text{and} \qquad \tilde{M} \in \mathcal{G}^H_{1} \,, \tag{4.2}$$

in the general construction given in Section 2.3. We note that by an integral grading operator $H \in \mathcal{G}$ we mean a diagonalizable element whose spectrum in the adjoint of $\mathcal{G}$ consists of integers and contains ± 1, and that $\mathcal{G}^H_n$ denotes the grade n subspace defined by H. In the present case $\mathcal{B}$ in (2.25b) is the subalgebra $\mathcal{G}^H_0$ of $\mathcal{G}$, and, because of the grading structure, the properties expressed by equation (2.34) hold. Thus the effective field equation reads as (2.37) and the corresponding action is given by the simple formula

$$I^H_{\text{eff}}(b) = S_{\text{WZ}}(b) - \int d^2x \, \langle b\tilde{M}b^{-1}, M \rangle \,, \tag{4.3}$$

where the field b varies in the little group G_0^H of H in G.

Generalized, or non-Abelian, Toda theories of this type have been first investigated by Leznov and Saveliev [1,3], who defined these theories by postulating their Lax potential,

$$\mathcal{A}_+^H = \partial_+ b \cdot b^{-1} + M \ , \qquad \mathcal{A}_-^H = -b\tilde{M}b^{-1} \ , \tag{4.4}$$

which they obtained by considering the problem that if one requires a $\mathcal{G}$-valued pure-gauge Lax potential to take some special form, then the consistency of the system of equations coming from the zero curvature condition becomes a non-trivial problem. In comparison, we have seen in Section 2.3 that in the WZNW framework the Lax potential originates from the chiral zero curvature equation (1.9), and the consistency and the integrability of the effective theory arising from the reduction is automatic.

It was shown in [3,4,16] in the special case when H, M and $\tilde{M}$ are taken to be the standard generators of an integral $sl(2)$ subalgebra of $\mathcal{G}$, that the non-Abelian Toda equation allows for conserved chiral currents underlying its exact integrability. These currents then generate chiral $\mathcal{W}$-algebras of the type $\mathcal{W}_S^{\mathcal{G}}$, for integrally embedded $sl(2)$'s.

By means of the argument given in Section 3.4, we can establish the structure of the chiral algebras of a wider class of non-Abelian Toda systems [18]. Namely, we see that if M and $\tilde{M}$ in (4.2) satisfy the non-degeneracy conditions

$$\mathrm{Ker}(\mathrm{ad}_M) \cap \mathcal{G}_{\geq 1}^H = \{0\} \qquad \text{and} \qquad \mathrm{Ker}(\mathrm{ad}_{\tilde{M}}) \cap \mathcal{G}_{\leq -1}^H = \{0\} \ , \tag{4.5}$$

then the left $\times$ right chiral algebra of the corresponding generalized Toda theory is isomorphic to $\mathcal{W}_{\mathcal{S}_-}^{\mathcal{G}} \times \tilde{\mathcal{W}}_{\mathcal{S}_+}^{\mathcal{G}}$, where $\mathcal{S}_-$ ($\mathcal{S}_+$) is an $sl(2)$ subalgebra of $\mathcal{G}$ containing the nilpotent generator M ($\tilde{M}$), respectively. The H-compatible $sl(2)$ algebras $\mathcal{S}_\pm$ occurring here are *not always integrally embedded* ones. Thus for certain *half-integral* $sl(2)$ algebras $\mathcal{W}_S^{\mathcal{G}}$ can be realized in a generalized Toda theory of the type (4.3). As we would like to have generalized Toda theories which possess $\mathcal{W}_S^{\mathcal{G}}$ as their symmetry algebra for an arbitrary $sl(2)$ subalgebra, we have to ask whether the theories given above are already enough for this purpose or not. This leads to the technical question as to whether for every half-integral $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ of $\mathcal{G}$ there exists an integral grading operator H such that $\mathcal{S}$ is an H-compatible $sl(2)$, in the sense introduced in Section 3.4. The answer to this question is negative, as proven in Appendix C, where the relationship between integral gradings and $sl(2)$ subalgebras is studied in detail. Thus we have to find new integrable conformal field theories for our purpose.

4.2. Generalized Toda theories for half-integral $sl(2)$ embeddings

In the following we exhibit a generalized Toda theory possessing the left $\times$ right chiral algebra $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}} \times \tilde{\mathcal{W}}^{\mathcal{G}}_{\mathcal{S}}$ for an arbitrarily chosen half-integral $sl(2)$ subalgebra $\mathcal{S} = \{M_-\,,\, M_0\,,\, M_+\}$ of the arbitrary but non-compact simple real Lie algebra $\mathcal{G}$. Clearly, if one imposes first class constraints of the type described in Section 3.4 on the currents of the WZNW theory then the resulting effective field theory will have the required chiral algebra. We shall choose the left and right gauge algebras in such a way to be dual to each other with respect to the Cartan-Killing form.

Turning to the details, first we choose a direct sum decomposition of $\mathcal{G}_{\frac{1}{2}}$ of the type in (3.51), and then define the *induced decomposition* $\mathcal{G}_{-\frac{1}{2}} = \mathcal{P}_{-\frac{1}{2}} + \mathcal{Q}_{-\frac{1}{2}}$ to be given by the subspaces

$$\mathcal{Q}_{-\frac{1}{2}} \equiv \mathcal{P}^{\perp}_{\frac{1}{2}} \cap \mathcal{G}_{-\frac{1}{2}} = [M_-\,,\, \mathcal{P}_{\frac{1}{2}}] \quad \text{and} \quad \mathcal{P}_{-\frac{1}{2}} \equiv \mathcal{Q}^{\perp}_{\frac{1}{2}} \cap \mathcal{G}_{-\frac{1}{2}} = [M_-\,,\, \mathcal{Q}_{\frac{1}{2}}]\,. \tag{4.6}$$

It is easy to see that the 2-form ω_{M_+} vanishes on the above subspaces of $\mathcal{G}_{-\frac{1}{2}}$ as a consequence of the vanishing of ω_{M_-} on the corresponding subspaces of $\mathcal{G}_{\frac{1}{2}}$. Thus we can take the left and right gauge algebras to be

$$\Gamma = (\mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}}) \quad \text{and} \quad \tilde{\Gamma} = (\mathcal{G}_{\leq -1} + \mathcal{P}_{-\frac{1}{2}})\,, \tag{4.7}$$

with the constant matrices M and $\tilde{M}$ entering the constraints given by M_- and M_+, respectively. The duality hypothesis of Section 2.3 is obviously satisfied by this construction.

In principle, the action and the Lax potential of the effective theory can be obtained by specializing the general formulas of Section 2.3 to the present particular case. In our case

$$\mathcal{B} = \mathcal{Q}_{\frac{1}{2}} + \mathcal{G}_0 + \mathcal{Q}_{-\frac{1}{2}}\,, \tag{4.8}$$

and the physical modes, which are given by the entries of b in the generalized Gauss decomposition $g = abc$ with $a \in e^{\Gamma}$ and $c \in e^{\tilde{\Gamma}}$, are now conveniently parametrized as

$$b(x) = \exp[q_{\frac{1}{2}}(x)] \cdot g_0(x) \cdot \exp[q_{-\frac{1}{2}}(x)]\,, \tag{4.9}$$

where $q_{\pm\frac{1}{2}}(x) \in \mathcal{Q}_{\pm\frac{1}{2}}$ and $g_0(x) \in G_0$, the little group of M_0 in G. Next we introduce some notation which will be useful for describing the effective theory.

The operator Ad_{g_0} maps $\mathcal{G}_{-\frac{1}{2}}$ to itself and, by writing the general element u of $\mathcal{G}_{-\frac{1}{2}}$ as a two-component column vector $u = (u_1\ u_2)^t$ with $u_1 \in \mathcal{P}_{-\frac{1}{2}}$ and $u_2 \in \mathcal{Q}_{-\frac{1}{2}}$, we can designate this operator as a 2×2 matrix:

$$\mathrm{Ad}_{g_0|\mathcal{G}_{-\frac{1}{2}}} = \begin{pmatrix} X_{11}(g_0) & X_{12}(g_0) \\ X_{21}(g_0) & X_{22}(g_0) \end{pmatrix} , \tag{4.10}$$

where, for example, $X_{11}(g_0)$ and $X_{12}(g_0)$ are linear operators mapping $\mathcal{P}_{-\frac{1}{2}}$ and $\mathcal{Q}_{-\frac{1}{2}}$ to $\mathcal{P}_{-\frac{1}{2}}$, respectively. Analogously, we introduce the notation

$$\mathrm{Ad}_{g_0^{-1}|\mathcal{G}_{\frac{1}{2}}} = \begin{pmatrix} Y_{11}(g_0) & Y_{12}(g_0) \\ Y_{21}(g_0) & Y_{22}(g_0) \end{pmatrix} , \tag{4.11}$$

which corresponds to writing the general element of $\mathcal{G}_{\frac{1}{2}}$ as a column vector, whose upper and lower components belong to $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$, respectively.

The action functional of the effective field theory resulting from the WZNW reduction at hand reads as follows:

$$\begin{aligned} I^{S}_{\text{eff}}(g_0, q_{\frac{1}{2}}, q_{-\frac{1}{2}}) = S_{\mathrm{WZ}}(g_0) - \int d^2x \, \langle g_0 M_+ g_0^{-1} \, , \, M_- \rangle \\ + \int d^2x \, \left(\langle \partial_- q_{\frac{1}{2}} \, , \, g_0 \partial_+ q_{-\frac{1}{2}} g_0^{-1} \rangle + \langle \eta_{\frac{1}{2}} \, , \, X_{11}^{-1} \cdot \eta_{-\frac{1}{2}} \rangle \right) , \end{aligned} \tag{4.12a}$$

where the objects $\eta_{\pm\frac{1}{2}} \in \mathcal{P}_{\pm\frac{1}{2}}$ are given by the formulae

$$\eta_{\frac{1}{2}} = [M_+, q_{-\frac{1}{2}}] + Y_{12} \cdot \partial_- q_{\frac{1}{2}} \quad \text{and} \quad \eta_{-\frac{1}{2}} = [M_-, q_{\frac{1}{2}}] - X_{12} \cdot \partial_+ q_{-\frac{1}{2}} . \tag{4.12b}$$

The Euler-Lagrange equation of this action is the zero curvature condition of the following Lax potential:

$$\begin{aligned} \mathcal{A}^S_+ &= M_- + \partial_+ g_0 \cdot g_0^{-1} + g_0 (\partial_+ q_{-\frac{1}{2}} + X_{11}^{-1} \cdot \eta_{-\frac{1}{2}}) g_0^{-1} , \\ \mathcal{A}^S_- &= - g_0 M_+ g_0^{-1} - \partial_- q_{\frac{1}{2}} + Y_{11}^{-1} \cdot \eta_{\frac{1}{2}} . \end{aligned} \tag{4.13}$$

The above new (conformally invariant) effective action and Lax potential are among the main results of the present paper. Clearly, for an integrally embedded $sl(2)$ this action and Lax potential simplify to the ones given by equation (4.3) and (4.4).

The derivation of the above formulae is not completely straightforward, and next we wish to sketch the main steps. First, let us remember that, by (2.29a), to specialize the general effective action given by (2.40) and the Lax potential given by (2.32) to our situation, we should express the objects $\partial_+ c c^{-1}$ and $a^{-1}\partial_- a$ in terms of b by using the constraints on J and $\tilde{J}$, respectively. (In the present case it would be tedious to compute the inverse matrix of V_{ij} in (2.27), which would be needed for using directly (2.29b).) For this purpose it turns out to be convenient to parametrize the WZNW field g by using the grading defined by the $sl(2)$, i.e., as

$$g = g_+ \cdot g_0 \cdot g_- \quad \text{where} \quad g_+ = a \cdot \exp[q_{\frac{1}{2}}], \quad g_- = \exp[q_{-\frac{1}{2}}] \cdot c\,. \tag{4.14}$$

We recall that the fields a, c, g_0 and $q_{\pm\frac{1}{2}}$ have been introduced previously by means of the parametrization $g = abc$, with b in (4.9). Also for later convenience, we write $g_\pm$ as

$$g_+ = \exp[r_{\geq 1} + p_{\frac{1}{2}} + q_{\frac{1}{2}}] \quad \text{and} \quad g_- = \exp[r_{\leq -1} + p_{-\frac{1}{2}} + q_{-\frac{1}{2}}]\,. \tag{4.15}$$

Note that here and below the subscript denotes the grade of the variables, and $p_{\pm\frac{1}{2}} \in \mathcal{P}_{\pm\frac{1}{2}}$. In our case this parametrization of g is advantageous, since, as shown below, the use of the grading structure facilitates solving the constraints.

For example, the left constraint are restrictions on $J_{<0}$, for which we have

$$J_{<0} = (g_+ g_0 N g_0^{-1} g_+^{-1})_{<0} \quad \text{with} \quad N = \partial_+ g_- \cdot g_-^{-1}\,. \tag{4.16}$$

By considering this equation grade by grade, starting from the lowest grade, it is easy to see that the constraints corresponding to $\mathcal{G}_{\geq 1} \subset \Gamma$ are equivalent to the relation

$$N_{\leq -1} = g_0^{-1} M_- g_0\,. \tag{4.17}$$

The remaining left constraints set the $\mathcal{P}_{-\frac{1}{2}}$ part of $J_{-\frac{1}{2}}$ to zero, and to unfold these constraints first we note that

$$J_{-\frac{1}{2}} = [p_{\frac{1}{2}} + q_{\frac{1}{2}}\,,\, M_-] + g_0 \cdot N_{-\frac{1}{2}} \cdot g_0^{-1}\,, \quad \text{with} \quad N_{-\frac{1}{2}} = \partial_+ p_{-\frac{1}{2}} + \partial_+ q_{-\frac{1}{2}}\,. \tag{4.18}$$

By using the notation introduced in (4.10), the vanishing of the projection of J to $\mathcal{P}_{-\frac{1}{2}}$ is written as

$$[q_{\frac{1}{2}}\,,\, M_-] + X_{11} \cdot \partial_+ p_{-\frac{1}{2}} + X_{12} \cdot \partial_+ q_{-\frac{1}{2}} = 0\,, \tag{4.19}$$

and from this we obtain

$$\partial_+ p_{-\frac{1}{2}} = X_{11}^{-1} \cdot \left\{[M_-\,,\, q_{\frac{1}{2}}] - X_{12} \cdot \partial_+ q_{-\frac{1}{2}}\right\}\,. \tag{4.20}$$

Combining our previous formulae, finally we obtain that on the constraint surface of the WZNW theory

$$N = g_0^{-1} M_- g_0 + \partial_+ q_{-\frac{1}{2}} + X_{11}^{-1}(g_0) \cdot \left\{[M_-\,,\, q_{\frac{1}{2}}] - X_{12}(g_0) \cdot \partial_+ q_{-\frac{1}{2}}\right\}\,. \tag{4.21}$$

A similar analysis applied to the right constraints yields that they are equivalent to the following equation:

$$-g_+^{-1} \cdot \partial_- g_+ = -g_0 M_+ g_0^{-1} - \partial_- q_{\frac{1}{2}} + Y_{11}^{-1}(g_0) \cdot \left\{ [M_+ \,,\, q_{-\frac{1}{2}}] + Y_{12}(g_0) \cdot \partial_- q_{\frac{1}{2}} \right\} . \quad (4.22)$$

By using the relations established above, we can at this stage easily compute $b^{-1} T b = \partial_+ c c^{-1}$ and $b \tilde{T} b^{-1} = a^{-1} \partial_- a$ as well, and substituting these into (2.40), and using the Polyakov-Wiegmann identity to rewrite $S_{\mathrm{WZ}}(b)$ for b in (4.9), results in the action in (4.12) indeed. The Lax potential in (4.13) is obtained from the general expression in (2.32) by an additional 'gauge transformation' by the field $\exp[-q_{\frac{1}{2}}]$, which made the final result simpler. Of course, for the above analysis we have to restrict ourselves to a neighbourhood of the identity where the operators $X_{11}(g_0)$ and $Y_{11}(g_0)$ are invertible.

The choice of the constraints leading to the effective theory (4.12) guarantees that the chiral algebra of this theory is the required one, $\mathcal{W}_S^{\mathcal{G}} \times \tilde{\mathcal{W}}_S^{\mathcal{G}}$, and thus one should be able to express the $\mathcal{W}$-currents in terms of the local fields in the action. To this first we recall that in Section 3.1 we have given an algorithm for constructing the gauge invariant differential polynomials $W(J)$. The point we wish to make is that the expression of the gauge invariant object $W(J)$ in terms of the local fields in (4.12) is simply $W(\partial_+ b b^{-1} + T(b))$, where b is given by (4.9). Applying the reasoning of [40,18] to the present case, this follows since the function W is form-invariant under any gauge transformation of its argument, and the quantity $(\partial_+ b b^{-1} + T(b))$ is obtained by a (non-chiral) gauge transformation from J, namely by the gauge transformation defined by the field $a^{-1} \in e^{\Gamma}$, see equations (2.31-2). (In analogy, when considering a right moving $\mathcal{W}$-current one gauge transforms the argument $\tilde{J}$ by the field $c \in e^{\tilde{\Gamma}}$.) We can in principle compute the object $T(b)$, as explained in the above, and thus we have an algorithm for finding the formulae of the W's in terms of the local fields g_0 and $q_{\pm\frac{1}{2}}$.

The conformal symmetry of the effective theory (4.12) is determined by the left and right Virasoro densities $L_{M_0}(J)$ and $L_{-M_0}(\tilde{J})$, which survive the reduction. To see this conformal symmetry explicitly, it is useful to extract the *Liouville field* ϕ by means of the decomposition $g_0 = e^{\phi M_0} \cdot \hat{g}_0$, where $\hat{g}_0$ contains the generators from $\mathcal{G}_0$ orthogonal to M_0. One can easily rewrite the action in terms of the new variables and then its conformal symmetry becomes manifest since e^{ϕ} is of conformal weight $(1,1)$, $\hat{g}_0$ is conformal scalar, and the fields $q_{\pm\frac{1}{2}}$ have conformal weights $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively. This assignment of the conformal weights can be established in a number of ways, one can for example derive it from the corresponding conformal symmetry transformation of the WZNW field g in the gauged WZNW theory, see eq. (5.30). We also note that the action (4.12) can be made generally covariant and thereby our generalized Toda theory can be re-interpreted as a theory of two-dimensional gravity since ϕ becomes the gravitational Liouville mode [14].

We would like to point out the relationship between the generalized Toda theory given by (4.12) and certain non-linear integrable equations which have been associated to the half-integral $sl(2)$ subalgebras of the simple Lie algebras by Leznov and Saveliev, by using a different method. (See, e.g., equation (1.24) in the review paper in *J. Sov. Math.* referred to in [3].) To this we note that, in the half-integral case, one can also consider that WZNW reduction which is defined by imposing the left and right constraints corresponding to the subalgebras $\mathcal{G}_{\geq 1}$ and $\mathcal{G}_{\leq -1}$ of Γ and $\tilde{\Gamma}$ in (4.7). In fact, the Lax potential of the effective field theory corresponding to this WZNW reduction coincides with the Lax potential postulated by Leznov and Saveliev to set up their theory. Thus, in a sense, their theory lies between the WZNW theory and our generalized Toda theory which has been obtained by imposing a larger set of first class KM constraints. This means that the theory given by (4.12) can also be regarded as a reduction of their theory.

There is a certain freedom in constructing a field theory possessing the required chiral algebra $\mathcal{W}_S^{\mathcal{G}}$, for example, one has a freedom of choice in the halving procedure used here to set up the gauge algebra. The theories in (4.12) obtained by using different halvings in equation (3.51) have their chiral algebras in common, but it is not quite obvious if these theories are always completely equivalent local Lagrangean field theories or not. We have not investigated this 'equivalence problem' in general.

A special case of this problem arises from the fact that one can expect that in some cases the theory in (4.12) is equivalent to one of the form (4.3). This is certainly so in those cases when for the half-integral $sl(2)$ of M_0 and $M_\pm$ one can find an integral grading operator H such that: (i) $[H\ , \ M_\pm] = \pm M_\pm$, (ii) $\mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1} = \mathcal{G}^H_{\geq 1}$, (iii) $\mathcal{P}_{-\frac{1}{2}} + \mathcal{G}_{\leq -1} = \mathcal{G}^H_{\leq -1}$, (iv) $\mathcal{Q}_{-\frac{1}{2}} + \mathcal{G}_0 + \mathcal{Q}_{\frac{1}{2}} = \mathcal{G}^H_0$, where one uses the M_0 grading and the H-grading on the left- and on the right hand sides of these conditions, respectively. By definition, we call the halving $\mathcal{G}_{\frac{1}{2}} = \mathcal{P}_{\frac{1}{2}} + \mathcal{Q}_{\frac{1}{2}}$ an *H-compatible halving* if these conditions are met. (We note in passing that an $sl(2)$ which allows for an H-compatible halving is automatically an H-compatible $sl(2)$ in the sense defined in Section 3.4, but, as shown in Appendix C, not every H-compatible $sl(2)$ allows for an H-compatible halving.) Those generalized Toda theories in (4.12) which have been obtained by using H-compatible halvings in the WZNW reduction can be rewritten in the simpler form (4.3) by means of a renaming of the variables, since in this case the relevant first class constraints are in the overlap of the ones which have been considered for the integral gradings and for the half-integral $sl(2)$'s to derive the respective theories. Since the form of the action in (4.3) is much simpler than the one in (4.12), it appears important to know the list of those $sl(2)$ embeddings which allow for an H-compatible halving, i.e., for which conditions (i) ... (iv) can be satisfied with some integral grading operator H and halving. We study this group theoretic question for the $sl(2)$ subalgebras of the maximally non-compact real forms of the classical Lie algebras in Appendix C. We show that the existence of an H-compatible halving is a very restrictive condition on the half-integral $sl(2)$ subalgebras of the symplectic and orthogonal Lie algebras, where such a halving exists only for the special $sl(2)$ embeddings listed at the end of Appendix C. In contrast, it turns out that for $\mathcal{G} = sl(n, R)$ an H-compatible halving can be found for every $sl(2)$ subalgebra, since in this case one can construct such a halving by proceeding similarly as we did in Section 3.5 (see (3.78)). This means that in the case of $\mathcal{G} = sl(n, R)$ any chiral algebra $\mathcal{W}_S^{\mathcal{G}}$ can be realized in a generalized Toda theory associated to an integral grading.

It is interesting to observe that those theories which can be alternatively written in both forms (4.3) and (4.12) allow for several conformal structures. This is so since in this case at least two different Virasoro densities, namely L_H and L_{M_0}, survive the WZNW reduction.

4.3. Two examples of generalized Toda theories

We wish to illustrate here the general construction of the previous section by working out two examples. First we shall describe a generalized Toda theory associated to the highest root $sl(2)$ of $sl(n+2, R)$. This is a half-integral $sl(2)$ embedding, but, as we shall see explicitly, the theory (4.12) can in this case be recasted in the form (4.3), since the corresponding halving is H-compatible. We note that the $\mathcal{W}$-algebras defined by these $sl(2)$ embeddings have been investigated before by using auxiliary fields in [29]. It is perhaps worth stressing that our method does not require the use of auxiliary fields when reducing the WZNW theory to the generalized Toda theories which possess these $\mathcal{W}$-algebras as their symmetry algebras, see also Section 5.3. According to the group theoretic analysis in Appendix C, the simplest case when a $\mathcal{W}_S^{\mathcal{G}}$-algebra defined by a half-integral $sl(2)$ embedding cannot be realized in a theory of the type (4.3) is the case of $\mathcal{G} = sp(4, R)$. As our second example, we shall elaborate on the generalized Toda theory in (4.12) which realizes the $\mathcal{W}$-algebra belonging to the highest root $sl(2)$ of $sp(4,R)$.

i) Highest root $sl(2)$ of $sl(n+2,R)$

In the usual basis where the Cartan subalgebra consists of diagonal matrices, the $sl(2)$ subalgebra $\mathcal{S}$ is generated by the elements

$$M_0 = \frac{1}{2}\begin{pmatrix} 1 & \cdots & 0 \\ 0 & 0_n & 0 \\ 0 & \cdots & -1 \end{pmatrix} \qquad \text{and} \qquad M_+ = M_-^t = \begin{pmatrix} 0 & \cdots & 1 \\ 0 & 0_n & 0 \\ 0 & \cdots & 0 \end{pmatrix}. \tag{4.23}$$

Note that here and below dots mean 0's in the entries of the various matrices. The adjoint of $sl(n+2)$ decomposes into one triplet, $2n$ doublets and n^2 singlets under this $\mathcal{S}$. It is convenient to parametrize the general element, g_0, of the little group of M_0 as

$$g_0 = e^{\phi M_0} \cdot e^{\psi T} \cdot \begin{pmatrix} 1 & \dots & 0 \\ 0 & \tilde{g}_0 & 0 \\ 0 & \cdots & 1 \end{pmatrix}, \quad \text{where} \quad T = \frac{1}{2+n}\begin{pmatrix} n & \cdots & 0 \\ 0 & -2I_n & 0 \\ 0 & \cdots & n \end{pmatrix} \tag{4.24}$$

is trace orthogonal to M_0 and $\tilde{g}_0$ is from $SL(n)$. We note that T and M_0 generate the centre of the corresponding subalgebra, $\mathcal{G}_0$. We consider the halving of $\mathcal{G}_{\pm\frac{1}{2}}$ which is defined by the subspaces $\mathcal{P}_{\pm\frac{1}{2}}$ and $\mathcal{Q}_{\pm\frac{1}{2}}$ consisting of matrices of the following form:

$$\begin{aligned} p_{\frac{1}{2}} &= \begin{pmatrix} 0 & p^t & 0 \\ 0 & 0_n & 0 \\ 0 & \cdots & 0 \end{pmatrix}, & q_{\frac{1}{2}} &= \begin{pmatrix} 0 & \cdots & 0 \\ 0 & 0_n & q \\ 0 & \cdots & 0 \end{pmatrix}, \\ p_{-\frac{1}{2}} &= \begin{pmatrix} 0 & \cdots & 0 \\ \tilde{p} & 0_n & 0 \\ 0 & \cdots & 0 \end{pmatrix}, & q_{-\frac{1}{2}} &= \begin{pmatrix} 0 & \cdots & 0 \\ 0 & 0_n & 0 \\ 0 & \tilde{q}^t & 0 \end{pmatrix}, \end{aligned} \tag{4.25}$$

where q and $\tilde{p}$ are n-dimensional column vectors and p^t and $\tilde{q}^t$ are n-dimensional row vectors, respectively. One sees that the $\mathcal{P}$ and $\mathcal{Q}$ subspaces of $\mathcal{G}_{\pm\frac{1}{2}}$ are invariant under the adjoint action of g_0, which means that the block-matrices in (4.10) and (4.11) are diagonal, and thus $\eta_{\pm\frac{1}{2}} = [M_\pm, q_{\mp\frac{1}{2}}]$. One can also verify that $X_{11} = e^{-\frac{1}{2}\phi-\psi}\tilde{g}_0$, and that using this the effective action (4.12) can be written as follows:

$$\begin{aligned} I_{\text{eff}}(g_0, q_{\frac{1}{2}}, q_{-\frac{1}{2}}) = S_{\text{WZ}}(g_0) - \int d^2x \Big[e^{\phi} - e^{-\frac{1}{2}\phi+\psi} \, (\partial_+\tilde{q})^t \cdot \tilde{g}_0^{-1} \cdot (\partial_- q) \\ + e^{\frac{1}{2}\phi+\psi} \, \tilde{q}^t \cdot \tilde{g}_0^{-1} \cdot q \Big], \end{aligned} \tag{4.26}$$

where dot means usual matrix multiplication. With respect to the conformal structure defined by M_0, e^{ϕ} has weights $(1,1)$, the fields q and $\tilde{q}$ have half-integer weights $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively, ψ and $\tilde{g}_0$ are conformal scalars. In particular, we see that ϕ is the Liouville mode with respect to this conformal structure.

In fact, the halving considered in (4.25) can be written like the one in (3.78), by using the integral grading operator H given explicitly as

$$H = M_0 + \frac{1}{2}T = \frac{1}{n+2}\begin{pmatrix} n+1 & 0 \\ 0 & -I_{n+1} \end{pmatrix}. \tag{4.27}$$

It is an *H-compatible halving* as one can verify that it satisfies the conditions (i)...(iv) mentioned at the end of Section 4.2, see also Appendix C. It follows that our reduced WZNW theory can also be regarded as a generalized Toda theory associated with the integral grading H. In other words, it is possible to identify the effective action (4.26) as a special case of the one in (4.3). To see this in concrete terms, it is convenient to parametrize the little group of H as

$$b = \exp(q_{\frac{1}{2}}) \cdot g_0 \cdot \exp(q_{-\frac{1}{2}}), \qquad \text{where} \qquad g_0 = e^{\Phi H} \cdot e^{\xi S} \cdot \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \tilde{g}_0 & 0 \\ 0 & \cdots & 1 \end{pmatrix}, \tag{4.28}$$

and $S = M_0 - (\frac{n+2}{2n})T$ is trace orthogonal to H. It is easy to check that by inserting this decomposition into the effective action (4.3) and using the Polyakov-Wiegmann identity one recovers indeed the effective action (4.26), with

$$\phi = \Phi + \xi \qquad \text{and} \qquad \psi = \frac{1}{2}\Phi - \frac{2+n}{2n}\xi. \tag{4.29}$$

The conformal structure defined by H is different from the one defined by M_0. In fact, with respect to the former conformal structure Φ is the Liouville mode and all other fields, including q and $\tilde{q}$, are conformal scalars.

ii) Highest root $sl(2)$ of $sp(4,R)$

We use the convention when the symplectic matrices have the form

$$g = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \qquad \text{where} \qquad B = B^t\,,\ C = C^t\,, \tag{4.30}$$

and the Cartan subalgebra is diagonal. The $sl(2)$ subalgebra $\mathcal{S}$ corresponding to the highest root of $sp(4, R)$ is generated by the matrices

$$M_0 = \frac{1}{2}(e_{11} - e_{33}), \quad M_+ = e_{13}\,, \quad \text{and} \quad M_- = e_{31}\,, \tag{4.31}$$

where e_{ij} denotes the elementary 4×4 matrix containing a single 1 in the ij-position. The adjoint of $sp(4)$ branches into $\underline{3} + 2 \cdot \underline{2} + 3 \cdot \underline{1}$ under $\mathcal{S}$. The three singlets generate an $sl(2)$ subalgebra different from $\mathcal{S}$, so that the little group of M_0 is $GL(1) \times SL(2)$. $GL(1)$ is generated by M_0 itself and the corresponding field is the Liouville mode. Using usual Gauss-parameters for the $SL(2)$, we can parametrize the little group of M_0 as

$$g_0 = e^{\phi M_0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\psi} + \alpha\beta e^{-\psi} & 0 & \alpha e^{-\psi} \\ 0 & 0 & 1 & 0 \\ 0 & \beta e^{-\psi} & 0 & e^{-\psi} \end{pmatrix}. \tag{4.32}$$

We decompose the $\mathcal{G}_{\pm\frac{1}{2}}$ subspaces (spanned by the two doublets) into their $\mathcal{P}$ and $\mathcal{Q}$ parts as follows

$$p_{\frac{1}{2}} + q_{\frac{1}{2}} = \begin{pmatrix} 0 & p & 0 & q \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -p & 0 \end{pmatrix}, \qquad p_{-\frac{1}{2}} + q_{-\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \tilde{p} & 0 & 0 & 0 \\ 0 & \tilde{q} & 0 & -\tilde{p} \\ \tilde{q} & 0 & 0 & 0 \end{pmatrix}. \tag{4.33}$$

Now the little group, or more precisely the $SL(2)$ generated by the three singlets, mixes the $\mathcal{P}$ and $\mathcal{Q}$ subspaces of $\mathcal{G}_{-\frac{1}{2}}$ so that the matrices X_{ij} and Y_{ij} in (4.10) and (4.11) possess off-diagonal elements:

$$X_{ij} = e^{-\frac{1}{2}\phi} \begin{pmatrix} e^{\psi} + \alpha\beta e^{-\psi} & \alpha e^{-\psi} \\ \beta e^{-\psi} & e^{-\psi} \end{pmatrix}, \qquad Y_{ij} = X_{ji}. \tag{4.34}$$

Inserting this into (4.12) yields the following effective action:

$$\begin{aligned} I^{S}_{\text{eff}}(g_0, q, \tilde{q}) = & S_{WZ}(g_0) - \int d^2x \Big[e^{\phi} - 2e^{-\frac{1}{2}\phi - \psi}(\partial_- q) \cdot (\partial_+ \tilde{q}) \\ & + 2e^{\frac{1}{2}\phi} \frac{(\tilde{q} + e^{-\frac{1}{2}\phi - \psi}\beta\partial_- q) \cdot (q + e^{-\frac{1}{2}\phi-\psi}\alpha\partial_+\tilde{q})}{e^{\psi} + \alpha\beta e^{-\psi}} \Big], \end{aligned} \tag{4.35}$$

for the Liouville mode ϕ, the conformal scalars ψ, α, β and the fields q, $\tilde{q}$ with weights $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively.

It is easy to see directly from its formula that it is impossible to obtain the above action as a special case of (4.3). Indeed, if the expression in (4.35) was obtained from (4.3) then the non-derivative term $\sim \tilde{q}\, q (e^{\psi} + \alpha\beta e^{-\psi})^{-1}$ could only be gotten from the second term in (4.3), but, since g_0 and b are matrices of unit determinant, this term could never produce the denominator in the non-derivative term in (4.35).

5. Quantum framework for WZNW reductions

In this chapter we study the quantum version of the WZNW reduction by using the path-integral formalism and also re-examine some of the classical aspects discussed in the previous chapters. We first show that the configuration space path-integral of the constrained WZNW theory can be realized by the gauged WZNW theory of Section 2.2. We then point out that the effective action of the reduced theory, (2.40), can be derived by integrating out the gauge fields in a convenient gauge, the physical gauge, in which the gauge degrees of freedom are frozen. A nontrivial feature of the quantum theory may appear in the path-integral measure. We shall find that for the generalized Toda theories associated with integral gradings the effective measure takes the form determined from the symplectic structure of the reduced theory. This means that in this case the quantum Hamiltonian reduction results in the quantization of the reduced classical theory; in other words, the two procedures, the reduction and the quantization, commute. We shall also exhibit the $\mathcal{W}$-symmetry of the effective action for this example. By using the gauged WZNW theory, we can construct the BRST formalism for the WZNW reduction in the general case. For conformally invariant reductions, this allows for computing the corresponding Virasoro centre explicitly. In particular, we derive here a nice formula for the Virasoro centre of $\mathcal{W}_S^{\mathcal{G}}$ for an arbitrary $sl(2)$ embedding. We shall verify that our result agrees with the one obtained in [16], in spite of the apparent difference in the structure of the constraints.

5.1. Path-integral for constrained WZNW theory

In this section we wish to set up the path-integral formalism for the constrained WZNW theory. For this, we recall that classically the reduced theory has been obtained by imposing a set of first-class constraints in the *Hamiltonian formalism*. Thus what we should do is to write down the path-integral of the WZNW theory first in phase space with the constraints implemented and then find the corresponding configuration space expression. The phase space path-integral can formally be defined once the canonical variables of the theory are specified. A practical way to find the canonical variables is the following [41]. Let us start from the WZNW action $S_{\rm WZ}(g)$ in (1.2) and parametrize the group element $g \in G$ in some arbitrary way, $g = g(\xi)$. We shall regard the parameters

ξ^a, $a = 1, ..., \dim G$, as the canonical coordinates in the theory. To find the canonical momenta, we introduce the 2-form $\mathcal{A} = \frac{1}{2}\mathcal{A}_{ab}(\xi)\, d\xi^a d\xi^b$ to rewrite the Wess-Zumino term as

$$\frac{1}{3}\mathrm{Tr}\,(dg\, g^{-1})^3 = d\mathcal{A}. \tag{5.1}$$

The 2-form $\mathcal{A}$ is well-defined only locally on G, since the Wess-Zumino 3-form is closed but not exact. Fortunately we do not need to specify $\mathcal{A}$ explicitly below. We next define $N_{ab}(\xi)$ by

$$\left(\frac{\partial g}{\partial \xi^a}\right) g^{-1} = N_{ab}(\xi) T^b, \tag{5.2}$$

where T^b are the generators of $\mathcal{G}$. The matrix N is easily shown to be non-singular, $\det N \neq 0$. Upon writing $S_{\mathrm{WZ}}(g) = \int d^2x\, \mathcal{L}(g)$, the canonical momentum conjugate to ξ^a is found to be

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial \partial_0 \xi^a} = \kappa \Big[N_{ab}(\xi)(\partial_0 g\, g^{-1})^b - \mathcal{A}_{ab}(\xi)\partial_1 \xi^b \Big]. \tag{5.3}$$

The Hamiltonian of the WZNW theory is then given by $H = \int dx^1 \mathcal{H}$ with

$$\mathcal{H} = \Pi_a \partial_0 \xi^a - \mathcal{L} = \frac{1}{2\kappa}\mathrm{Tr}\,\big[P^2 + (\kappa \partial_1 g\, g^{-1})^2\big], \tag{5.4}$$

where

$$P^a = (N^{-1})^{ab}(\Pi_b + \kappa \mathcal{A}_{bc}\partial_1 \xi^c). \tag{5.5}$$

Since $P = \kappa \partial_0 g\, g^{-1}$ in the original variables, the Hamiltonian density takes the usual Sugawara form as expected.

Classically, the constrained WZNW theory has been defined as the usual WZNW theory with its KM phase space reduced by the set of constraints given by (2.16), which in the canonical variables read

$$\begin{aligned} \phi_i &= \langle \gamma_i, P + \kappa(\partial_1 g\, g^{-1} - M) \rangle = 0, \\ \tilde{\phi}_i &= \langle \tilde{\gamma}_i, g^{-1} P g - \kappa(g^{-1}\partial_1 g + \tilde{M}) \rangle = 0, \end{aligned} \tag{5.6}$$

with the bases $\gamma_i \in \Gamma$, $\tilde{\gamma}_i \in \tilde{\Gamma}$. As in Section 2.2, no relationship is assumed here between the two subalgebras, Γ and $\tilde{\Gamma}$. Now we write down the phase space path-integral for the constrained WZNW theory. According to Faddeev's prescription [42] it is defined as

$$Z = \int d\Pi d\xi\, \delta(\phi)\delta(\tilde{\phi})\delta(\chi)\delta(\tilde{\chi}) \det|\{\phi,\chi\}| \det|\{\tilde{\phi},\tilde{\chi}\}| \times \exp\Big(i \int d^2x\, (\Pi_a \partial_0 \xi^a - \mathcal{H}) \Big), \tag{5.7}$$

where we implement the first class constraints by inserting $\delta(\phi)$ and $\delta(\tilde{\phi})$ in the path-integral. The δ-functions of χ and $\tilde{\chi}$ refer to gauge fixing conditions corresponding to the constraints, ϕ and $\tilde{\phi}$, which act as generators of gauge symmetries. By introducing Lagrange-multiplier fields, $A_- = A^i_-\gamma_i$ and $A_+ = A^i_+\tilde{\gamma}_i$, (5.7) can be written as

$$Z = \int d\Pi d\xi dA_+ dA_- \delta(\chi)\delta(\tilde{\chi}) \det|\{\phi,\chi\}| \det|\{\tilde{\phi},\tilde{\chi}\}| \\ \times \exp\Big(i\int d^2x \left[\mathrm{Tr}\left(\Pi\partial_0\xi + A_-\phi + A_+\tilde{\phi}\right) - \mathcal{H}\right]\Big). \tag{5.8}$$

By changing the momentum variable from Π_a to P^a in (5.5), the measure acquires a determinant factor, $d\Pi = dP \det N$, and the integrand of the exponent in (5.8) becomes

$$\begin{aligned} &\mathrm{Tr}\left(\Pi\partial_0\xi + A_-\phi + A_+\tilde{\phi}\right) - \mathcal{H} \\ &= \kappa\mathrm{Tr}\Big[-\frac{1}{2}\big(\frac{1}{\kappa}P\big)^2 + \frac{1}{\kappa}P(A_- + gA_+g^{-1} + \partial_0 g\, g^{-1}) - N^{-1}\mathcal{A}\,\partial_1\xi(\partial_0 g\, g^{-1}) \\ &\quad -\frac{1}{2}(\partial_1 g\, g^{-1})^2 + A_-(\partial_1 g\, g^{-1} - M) - A_+(g^{-1}\partial_1 g + \tilde{M})\Big]. \end{aligned} \tag{5.9}$$

Since the matrix $N(\xi)$ is independent of P, we can easily perform the integration over P provided that the remaining δ-functions and the determinant factors are also P-independent. We can choose the gauge fixing conditions, χ and $\tilde{\chi}$, so that this is true. (For example, the physical gauge which we will choose in the next section fulfills this demand.) Then we end up with the following formula of the configuration space path-integral:

$$Z = \int d\xi \det N\, dA_+ dA_- \delta(\chi)\delta(\tilde{\chi}) \det|\{\phi,\chi\}| \det|\{\tilde{\phi},\tilde{\chi}\}|\, e^{iI(g,A_-,A_+)}, \tag{5.10}$$

where $I(g, A_-, A_+)$ is the gauged WZNW action (2.18). We note that the measure for the coordinates in this path-integral is the invariant Haar measure,

$$d\mu(g) = \prod_a d\xi^a \det N = \prod_a (dg\, g^{-1})^a. \tag{5.11}$$

This is a consequence of the fact that the phase space measure in (5.7) is invariant under canonical transformations to which the group transformations belong.

The above formula for the configuration space path-integral means that the gauged WZNW theory provides the Lagrangian realization of the Hamiltonian reduction, which we have already seen on the basis of a classical argument in Section 2.2.

5.2. Effective theory in the physical gauge

Having seen how the constrained WZNW theory is realized as the gauged WZNW theory, we next discuss the effective theory which arises when we eliminate all the unphysical degrees of freedom in a particularly convenient gauge, the physical gauge. We shall rederive, in the path-integral formalism, the effective action which appeared in the classical context earlier in this paper. For this purpose, within this section we restrict our attention to the left-right dual reductions considered in Section 2.3. It, however, should be noted that this restriction is not absolutely necessary to get an effective action by the method given below. In this respect, it is also worth noting that Polyakov's 2-dimensional gravity action in the light-cone gauge can be regarded as an effective action in a non-dual reduction, which is obtained by imposing a constraint only on the left-current for $G = SL(2)$ [43,12]. We will not pursue the non-dual cases here.

To eliminate all the unphysical gauge degrees of freedom, we simply gauge them away from g, i.e., we gauge fix the Gauss decomposed g in (2.25) into the form

$$g = abc \rightarrow b. \tag{5.12}$$

More specifically, with the parametrization $a(x) = \exp[\sigma_i(x)\gamma_i]$, $c(x) = \exp[\tilde{\sigma}_i(x)\tilde{\gamma}_i]$ we define the *physical gauge* by

$$\chi_i = \sigma_i = 0, \qquad \tilde{\chi}_i = \tilde{\sigma}_i = 0. \tag{5.13}$$

We here note that for this gauge the determinant factors in (5.8) are actually constants. Now the effective action is obtained by performing the $A_\pm$ integrations in (5.10). The integration of A_- gives rise to the delta-function,

$$\prod_i \delta\Big(\langle\gamma_i, bA_+b^{-1} + \partial_+ b\, b^{-1} - M\rangle\Big), \tag{5.14}$$

with $\gamma_i \in \Gamma$ normalized by the duality condition (2.22). One then notices that the delta-function (5.14) implies exactly condition (2.29) with $\partial_+ c\, c^{-1}$ replaced by A_+. Hence, with the help of the matrix $V_{ij}(b)$ in (2.27) and $T(b)$ in (2.29), it can be rewritten as

$$(\det V)^{-1}\, \delta\Big(A_+ - b^{-1}T(b)b\Big). \tag{5.15}$$

Finally, the integration of A_+ yields

$$Z = \int d\mu_{\text{eff}}(b)\, e^{I_{\text{eff}}(b)}, \tag{5.16}$$

where $I_{\text{eff}}(b)$ is the effective action (2.40)*, and $d\mu_{\text{eff}}(b)$ is the effective measure given by

$$d\mu_{\text{eff}}(b) = (\det V)^{-1}\, d\mu(g)\delta(\sigma)\delta(\tilde{\sigma}) = (\det V)^{-1}\, \frac{d\mu(g)}{d\sigma\, d\tilde{\sigma}}\bigg|_{\sigma=\tilde{\sigma}=0}. \tag{5.17}$$

Of course, as far as the effective action is concerned, the path-integral approach should give the same result as the classical one, because the integration of the gauge fields is Gaussian and hence equivalent to the classical elimination of the gauge fields by their field equations. However, a non-trivial feature may arise at the quantum level when the effective path-integral measure (5.17) is taken into account. Let us examine the effective measure in the simple case where the space $\mathcal{B} = (\Gamma + \tilde{\Gamma})^{\perp}$, with which $b \in e^{\mathcal{B}}$, forms a subalgebra of $\mathcal{G}$ satisfying (2.34), and thus the effective action in (5.16) simplifies to

$$I_{\text{eff}}(b) = S_{\text{WZ}}(b) - \kappa \int d^2x\, \langle b\tilde{M}b^{-1}, M\rangle. \tag{5.18}$$

In this case, the 1-form appearing in the measure $d\mu(g)$ of (5.11),

$$dg\, g^{-1} = da\, a^{-1} + a(db\, b^{-1})a^{-1} + ab(dc\, c^{-1})b^{-1}a^{-1}, \tag{5.19}$$

turns out, in the physical gauge, to be

$$dg\, g^{-1}\big|_{\sigma=\tilde{\sigma}=0} = \gamma_i d\sigma_i + db\, b^{-1} + V_{ij}(b)\tilde{\gamma}_i d\tilde{\sigma}_j. \tag{5.20}$$

As a result, the determinant factor in (5.17) is cancelled by the one coming from (5.20), and the effective measure admits a simple form:

$$d\mu_{\text{eff}}(b) = db\, b^{-1}. \tag{5.21}$$

The point is that this is exactly the measure which is determined from the symplectic structure of the effective theory (5.18) obtained by the *classical* Hamiltonian reduction. This tells us that in this case the *quantum* Hamiltonian reduction results in the quantization of the reduced classical theory. In particular, since the above assumption for $\mathcal{B}$ is satisfied for the generalized Toda theories associated with integral gradings, we conclude that these generalized Toda theories are equivalent to the corresponding constrained (gauged) WZNW

* Actually, the effective action always takes the form (2.40) if one restricts the WZNW field to be of the form $g = abc$ with $a \in e^{\Gamma}$, $c \in e^{\tilde{\Gamma}}$ and b such that $V_{ij}(b)$ is invertible. The duality between Γ and $\tilde{\Gamma}$ is not necessary but can be used to ensure this technical assumption.

theories even at the quantum level, i.e., including the measure. This result has been established before in the special case of the standard Toda theory (1.1) in [44], where the measure $d\mu_{\rm eff}(b)$ is simply given by $\prod_i d\varphi^i$.

We end this section by noting that it is not clear whether the measure determined from the symplectic structure of the reduced classical theory is identical to the effective measure (5.17) in general. In the general case both measures in question could become quite involved and thus one would need some geometric argument to see if they are identical or not.

5.3. The $\mathcal{W}$-symmetry of the generalized Toda action $I^H_{\rm eff}(b)$

In the previous section we have seen the quantum equivalence of the generalized Toda theories given by (4.3) and the corresponding constrained WZNW theories. It follows from their WZNW origin that the generalized Toda theories possess conserved $\mathcal{W}$-currents. It is thus natural to expect that their effective actions, $I^H_{\rm eff}$ in (4.3) and $I^S_{\rm eff}$ in (4.12), allow for *symmetry transformations yielding the $\mathcal{W}$-currents as the corresponding Noether currents.* We demonstrate below that this is indeed the case on the example of the theories associated with integral gradings, when the action takes a simple form. We however believe that there are symmetries of the effective action corresponding to the conserved chiral currents inherited from the KM algebra for any reduced WZNW theory.

Let us consider a gauge invariant differential polynomial $W(J)$ in the constrained WZNW theory giving rise to the effective theory described by the action in (4.3). In terms of the generalized Toda field $b(x)$, this conserved $\mathcal{W}$-current is given by the differential polynomial

$$W_{\rm eff}(\beta) = W(M+\beta), \qquad \text{where} \qquad \beta \equiv \partial_+ b\, b^{-1}. \tag{5.22}$$

This equality [40,18] holds because the constrained current J and $(M+\beta)$ (which is, incidentally, just the Lax potential $\mathcal{A}^H_+$ in (4.4)) are related by a gauge transformation, as we have seen. By choosing some test function $f(x^+)$, we now associate to $W_{\rm eff}(\beta)$ the following transformation of the field $b(x)$:

$$\delta_W b(y) = \left[\int d^2x\, f(x^+)\frac{\delta W_{\rm eff}(x)}{\delta\beta(y)}\right]\cdot b(y)\ , \tag{5.23}$$

and we wish to show that $\delta_W b$ is a symmetry of the action $I^H_{\rm eff}(b)$. Before proving this, we notice, by combining the definition in (5.23) with (5.22), that $(\delta_W b)b^{-1}$ is a polynomial expression in f, β and their ∂_+-derivatives up to some finite order.

We start the proof by noting that the change of the action under an arbitrary variation δb is given by the formula

$$\begin{aligned} \delta I^H_{\text{eff}}(b) &= -\int d^2y\, \langle \delta b\, b^{-1}(y)\, ,\, b(y)\frac{\delta I^H_{\text{eff}}}{\delta b(y)}\rangle \\ &= -\int d^2y\, \langle \delta b\, b^{-1}(y)\, ,\, \partial_-\beta(y) + [b(y)\tilde{M} b^{-1}(y), M]\rangle\, . \end{aligned} \tag{5.24}$$

In the next step, we use the field equation to replace $\partial_-\beta$ by $-[b\tilde{M}b^{-1}, M]$ in the obvious equality

$$\partial_- W_{\text{eff}}(x) = \int d^2y\, \langle \frac{\delta W_{\text{eff}}(x)}{\delta\beta(y)}, \partial_-\beta(y)\rangle, \tag{5.25}$$

and then, from the fact that $\partial_- W_{\text{eff}} = 0$ on-shell, we obtain the following identity:

$$\int d^2y\, \langle \frac{\delta W_{\text{eff}}(x)}{\delta\beta(y)}, [b(y)\tilde{M} b^{-1}(y), M]\rangle = 0\, , \tag{5.26}$$

Of course, the previous argument only implies that (5.26) holds on-shell. However, we now make the crucial observation that (5.26) is an *off-shell identity*, i.e., it is valid for any field $b(x)$ not only for the solutions of the field equation. This follows by noticing that the object in (5.26) is a local expression in $b(x)$ containing only x^+-derivatives. In fact, any such object which vanishes on-shell has to vanish also off-shell, because one can find solutions of the field equation for which the x^+-dependence of the field b is prescribed in an arbitrary way at an arbitrarily chosen fixed value of x^-.

By using the above observation, it is easy to show that $\delta_W b$ in (5.23) is indeed a symmetry of the action. First, simply inserting (5.23) into (5.24), we have

$$\delta_W I^H_{\text{eff}}(b) = -\int d^2x\, f(x^+) \int d^2y\, \langle \frac{\delta W_{\text{eff}}(x)}{\delta\beta(y)}\, ,\, \partial_-\beta(y) + [b(y)\tilde{M} b^{-1}(y), M]\rangle. \tag{5.27}$$

We then rewrite this equation as

$$\delta_W I^H_{\text{eff}}(b) = -\int d^2x\, f(x^+)\partial_- W_{\text{eff}}(x), \tag{5.28}$$

with the aid of the identities (5.26) and (5.25). This then proves that

$$\delta_W I^H_{\text{eff}}(b) = 0\, , \tag{5.29}$$

since the integrand in (5.28) is a total derivative, thanks to $\partial_- f = 0$. One can also see, from equation (5.23), that W_{eff} is the Noether charge density corresponding to the symmetry transformation $\delta_W b$ of $I^H_{\text{eff}}(b)$.

5.4. BRST formalism for WZNW reductions

Since the constrained WZNW theory can be regarded as the gauged WZNW theory (2.18), one is naturally led to construct the BRST formalism for the theory as a basis for quantization. Below we discuss the BRST formalism based on the gauge symmetry (2.19) and thus return to the general situation of Section 5.1 where no relationship between the two subalgebras, Γ and $\tilde{\Gamma}$, is supposed.

Prior to the construction we here note how the conformal symmetry is realized in the gauged WZNW theory when there is an operator H satisfying the condition (2.13). (For simplicity, in what follows we discuss the symmetry associated to the left-moving sector.) In fact, with such H and a chiral test function $f^+(x^+)$ one can define the following transformation,

$$\begin{aligned} \delta g &= f^+ \partial_+ g + \partial_+ f^+ H g, \\ \delta A_- &= f^+ \partial_+ A_- + \partial_+ f^+ [H, A_-], \\ \delta A_+ &= f^+ \partial_+ A_+ + \partial_+ f^+ A_+, \end{aligned} \tag{5.30}$$

which leaves the gauged WZNW action $I(g, A_-, A_+)$ invariant. This corresponds exactly to the conformal transformation in the constrained WZNW theory generated by the Virasoro density L_H in (2.10), as can be confirmed by observing that (5.30) implies the conformal action (2.11) for the current with $f(x^+) = f^+(x^+)$. We shall derive later the Virasoro density as the Noether charge density in the BRST system.

Turning to the construction of the BRST formalism, we first choose the space $\Gamma^* \subset \mathcal{G}$ which is dual to Γ with respect to the Cartan-Killing form (and similarly $\tilde{\Gamma}^*$ dual to $\tilde{\Gamma}$). Following the standard procedure [45] we introduce two sets of ghost, anti-ghost and Nakanishi-Lautrup fields, $\{c \in \Gamma,\ \bar{c}_+, B_+ \in \Gamma^*\}$ and $\{b \in \tilde{\Gamma},\ \bar{b}_-, B_- \in \tilde{\Gamma}^*\}$. The BRST transformation corresponding to the (left-sector of the) local gauge transformation (2.19) is given by

$$\begin{aligned} \delta_{\mathrm{B}} g &= -cg, & \delta_{\mathrm{B}} \bar{c}_+ &= iB_+, \\ \delta_{\mathrm{B}} A_- &= D_- c, & \delta_{\mathrm{B}} B_+ &= 0, \\ \delta_{\mathrm{B}} c &= -c^2, & \delta_{\mathrm{B}}(\text{others}) &= 0, \end{aligned} \tag{5.31}$$

with $D_\pm = \partial_\pm \mp [A_\pm, \quad]$. After defining the BRST transformation $\bar{\delta}_{\rm B}$ for the right-sector in an analogous way, we write the BRST action by adding a gauge fixing term and a ghost term to the gauged action,

$$I_{\rm BRST} = I(g, A_-, A_+) + I_{\rm gf} + I_{\rm ghost}. \tag{5.32}$$

The additional terms can be constructed by the manifestly BRST invariant expression,

$$\begin{aligned} I_{\rm gf} + I_{\rm ghost} &= -i\kappa(\delta_{\rm B} + \bar{\delta}_{\rm B}) \int d^2x \left(\langle \bar{c}_+, A_- \rangle + \langle \bar{b}_-, A_+ \rangle \right) \\ &= \kappa \int d^2x \left(\langle B_+, A_- \rangle + \langle B_-, A_+ \rangle + i \langle \bar{c}_+, D_- c \rangle + i \langle \bar{b}_-, D_+ b \rangle \right), \end{aligned} \tag{5.33}$$

where we have chosen the gauge fixing conditions as $A_\pm = 0$. Then the path-integral for the BRST system is given by

$$Z = \int d\mu(g)\, dA_+ dA_- dc\, d\bar{c}_+ db\, d\bar{b}_- dB_+ dB_-\, e^{iI_{\rm BRST}}, \tag{5.34}$$

which, upon integration of the ghosts and the Nakanishi-Lautrup fields, reduces to (5.10). (Strictly speaking, for this we have to generalize the gauge fixing conditions in (5.10) to be dependent on the gauge fields.) By this construction the nilpotency, $\delta_{\rm B}^2 = 0$, and the BRST invariance of the action, $\delta_{\rm B} I_{\rm BRST} = 0$, are easily checked.

It is, however, convenient to deal with the simplified BRST theory obtained by performing the trivial integrations of $A_\pm$ and $B_\pm$ in (5.34),

$$I_{\rm BRST}(g, c, \bar{c}_+, b, \bar{b}_-) = S_{\rm WZ}(g) + i\kappa \int d^2x \left(\langle \bar{c}_+, \partial_- c \rangle + \langle \bar{b}_-, \partial_+ b \rangle \right). \tag{5.35}$$

We note that this effective BRST theory is not merely a sum of a free WZNW sector and free ghost sector as it appears, but rather it consists of the two interrelated sectors in the physical space specified by the BRST charge defined below. At this stage the BRST transformation which leaves the simplified BRST action (5.35) invariant reads

$$\begin{aligned} &\delta_{\rm B} g = -cg, \qquad && \delta_{\rm B} \bar{c}_+ = -\pi_{\Gamma^*} \left[i(\partial_+ g\, g^{-1} - M) + (c\bar{c}_+ + \bar{c}_+ c) \right], \\ &\delta_{\rm B} c = -c^2, && \delta_{\rm B}(\text{others}) = 0, \end{aligned} \tag{5.36}$$

where $\pi_{\Gamma^*} = \sum_i |\gamma_i^*\rangle\langle\gamma_i|$ is the projection operator onto the dual space Γ^* with the normalized bases, $\langle \gamma_i, \gamma_j^* \rangle = \delta_{ij}$. From the associated conserved Noether current, $\partial_- j_+^{\rm B} = 0$, the BRST charge $Q_{\rm B}$ is defined to be

$$Q_{\rm B} = \int dx^+ j_+^{\rm B}(x) = \int dx^+ \langle c, \partial_+ g\, g^{-1} - M - c\bar{c}_+ \rangle. \tag{5.37}$$

The physical space is then specified by the condition,

$$Q_{\rm B}|{\rm phys}\rangle = 0. \tag{5.38}$$

In the simple case of the WZNW reduction which leads to the standard Toda theory, the BRST charge (5.37) agrees with the one discussed earlier [46].

In the case where there is an H operator which guarantees the conformal invariance, the BRST system also has the corresponding conformal symmetry,

$$\begin{aligned} \delta g &= f^+\partial_+ g + \partial_+ f^+ H g, & \delta b &= f^+\partial_+ b, \\ \delta c &= f^+\partial_+ c + \partial_+ f^+ [H, c], & \delta \bar{b}_- &= f^+\partial_+ \bar{b}_-, \\ \delta \bar{c}_+ &= f^+\partial_+ \bar{c}_+ + \partial_+ f^+ (\bar{c}_+ + [H, \bar{c}_+]), \end{aligned} \tag{5.39}$$

inherited from the one (5.30) in the gauged WZNW theory. If the H operator further provides a grading, one finds from (5.39) that the currents of grade $-h$ have the (left-) conformal weight $1-h$, except the H-component, which is not a primary field. Similarly, the ghosts c, $\bar{c}_+$ of grade h, $-h$ have the conformal weight h, $1-h$, respectively, whereas the ghosts b, $\bar{b}$ are conformal scalars. Now we define the total Virasoro density operator $L_{\rm tot}$ from the associated Noether current, $\partial_- j_+^{\rm C} = 0$, by

$$\int dx^+ j_+^{\rm C}(x) = \frac{1}{\kappa}\int dx^+ f^+(x^+) L_{\rm tot}(x). \tag{5.40}$$

The (on-shell) expression is found to be the sum of the two parts, $L_{\rm tot} = L_H + L_{\rm ghost}$, where L_H is indeed the Virasoro operator (2.10) for the WZNW part, and

$$L_{\rm ghost} = i\kappa\big(\langle \bar{c}_+, \partial_+ c\rangle + \partial_+\langle H, c\bar{c}_+ + \bar{c}_+ c\rangle\big), \tag{5.41}$$

is the part for the ghosts. The conformal invariance of the BRST charge, $\delta Q_{\rm B} = 0$, or equivalently, the BRST invariance of the total conformal charge, $\delta_{\rm B} L_{\rm tot} = 0$, are readily confirmed.

Let us find the Virasoro centre of our BRST system. The total Virasoro centre $c_{\rm tot}$ is given by the sum of the two contributions, c from the WZNW part and $c_{\rm ghost}$ from the ghost one. The Viraso centre from L_H is given by

$$c = \frac{k \dim \mathcal{G}}{k+g} - 12k\langle H, H\rangle, \tag{5.42}$$

where k is the level of the KM algebra and g is the dual Coxeter number. On the other hand, the ghosts contribute to the Virasoro centre by the usual formula,

$$c_{\rm ghost} = -2\sum_{\Gamma}\big[1 + 6h(h-1)\big], \tag{5.43}$$

where the summation is performed over the eigenvectors of ad_H in the subalgebra Γ. (One can confirm (5.43) by performing the operator product expansion with L_{ghost} in (5.41).)

5.5. The Virasoro centre in three examples

By elaborating on the general result of the previous section, we here derive explicit formulas for the total Virasoro centre in three interesting special cases of the WZNW reduction.

i) The generalized Toda theory $I^H_{\mathrm{eff}}(b)$

In this case the summation in (5.43) is over the eigenstates of ad_H with eigenvalues $h > 0$, since $\Gamma = \mathcal{G}^H_{>0}$. We can establish a concise formula for c_{tot}, (5.46) below, by using the following group theoretic facts.

First, we can assume that the grading operator $H \in \mathcal{G}$ is from the Cartan subalgebra of the complex simple Lie algebra $\mathcal{G}_c$ containing $\mathcal{G}$. Second, the scalar product $\langle\ ,\ \rangle$ defines a natural isomorphism between the Cartan subalgebra and the space of roots, and we introduce the notation $\vec{\delta}$ for the vector in root space corresponding to H under this isomorphism. More concretely, this means that we set $H = \sum_i \delta_i H_i$ by using an orthonormal Cartan basis, $\langle H_i, H_j\rangle = \delta_{ij}$. Third, we recall the *strange formula* of Freudenthal-de Vries [47], which (by taking into account the normalization of $\langle\ ,\ \rangle$ and the duality between the root space and the Cartan subalgebra) reads

$$\dim \mathcal{G} = \frac{12}{g} |\vec{\rho}|^2 \ , \tag{5.44}$$

where $\vec{\rho}$ is the Weyl vector, given by half the sum of the positive roots. Fourth, we choose the simple positive roots in such a way that the corresponding step operators, which are in general in $\mathcal{G}_c$ and not in $\mathcal{G}$, have non-negative grades with respect to H.

By using the above conventions, it is straightforward to obtain the following expressions

$$\begin{aligned} &\sum_{h>0} 1 = \dim\Gamma = \frac{1}{2}(\dim\mathcal{G} - \dim\mathcal{G}^H_0), \qquad \sum_{h>0} h = 2(\vec{\rho}\cdot\vec{\delta}), \\ &\sum_{h>0} h^2 = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}_H)^2 = g\langle H, H\rangle = g|\vec{\delta}|^2, \end{aligned} \tag{5.45}$$

for the corresponding terms in (5.43). Substituting these into (5.43) and also (5.44) into (5.42), one can finally establish the following nice formula of the total Virasoro centre [14]:

$$c_{\text{tot}} = c + c_{\text{ghost}} = \dim \mathcal{G}_0^H - 12\left|\sqrt{k+g}\,\vec{\delta} - \frac{1}{\sqrt{k+g}}\,\vec{\rho}\right|^2 . \tag{5.46}$$

In particular, in the case of the reduction leading to the standard Toda theory (1.1) the result (5.46) is consistent with the one directly obtained in the reduced theory [8,10].

ii) The $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra for half-integral $sl(2)$ embeddings

For $sl(2)$ embeddings the role of the H is played by M_0 and in the half-integral case we have $\Gamma = \mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}} = \mathcal{G}_{>0} - \mathcal{Q}_{\frac{1}{2}}$. It follows that the value of the total Virasoro centre can now be obtained by substracting the contribution of the 'missing ghosts' corresponding to $\mathcal{Q}_{\frac{1}{2}}$, which is $\frac{1}{2}\dim \mathcal{G}_{\frac{1}{2}}$, from the expression in (5.46). We thus obtain that in this case

$$c_{\text{tot}} = N_t - \frac{1}{2}N_s - 12\left|\sqrt{k+g}\,\vec{\delta} - \frac{1}{\sqrt{k+g}}\,\vec{\rho}\right|^2 , \tag{5.47a}$$

where

$$N_t = \dim \mathcal{G}_0 \ , \qquad \text{and} \qquad N_s = \dim \mathcal{G}_{\frac{1}{2}} \ , \tag{5.47b}$$

are the number of tensor and spinor multiplets in the decomposition of the adjoint of $\mathcal{G}$ under the $sl(2)$ subalgebra $\mathcal{S}$, respectively. We note that, as proven by Dynkin [39], it is possible to choose a system of positive simple roots so that the grade of the corresponding step operators is from the set $\{0, \frac{1}{2}, 1\}$, and that $\vec{\delta}$ is $(\frac{1}{2}\times)$ the so called *defining vector* of the $sl(2)$ embedding in Dynkin's terminology.

As has been mentioned in Section 3.4, Bais *et al* [16] (see also [29]) studied a similar reduction of the KM algebra for half-integral $sl(2)$ embeddings where all the current components corresponding to $\mathcal{G}_{>0}$ are constrained from the very beginning. In their system, the constraints (3.59) of $\mathcal{G}_{\frac{1}{2}}$, being inevitably second-class, are modified into first-class by introducing an auxiliary field to each constraint of $\mathcal{G}_{\frac{1}{2}}$. Accordingly, the auxiliary fields give rise to the extra contribution $-\frac{1}{2}\dim \mathcal{G}_{\frac{1}{2}}$ in the total Virasoro centre. It is clear that adding this to the sum of the WZNW and ghost parts (which is of the form (5.46) with M_0 substituted for H), renders the total Virasoro centre of their system identical to that of our system, given by (5.47). This result is natural if we recall the fact that their reduced phase space (after complete gauge fixing) is actually identical to ours. It is obvious that our method, which is based on purely first-class KM constraints and does not require auxiliary fields, provides a simpler way to reach the identical reduced theory.

iii) The W_n^l-algebras

By using the results of Section 3.5 we can easily compute the Virasoro centre of the W_n^l-algebras. We consider the conformal structure given by L_{M_0}, where M_0 is the $sl(2)$ generator (3.68), and introduce ghosts for the first class constraints defined by Γ (3.74). The contribution to the Virasoro centre from L_{M_0} is given by

$$c = \frac{(n^2-1)k}{k+n} - km(m+1)[3n-(2m+1)l]. \tag{5.48}$$

Taking into account the multiplicities of the grades in Γ, we find from (5.43)

$$\begin{aligned} c_{\text{ghost}} &= -2\dim\mathcal{D}_0 + \dim\mathcal{P}_{\frac{1}{2}} - 2\sum_{i=1}^{m}[1+6i(i-1)]\dim\mathcal{G}_i \\ &= -(m^3+4m^2+3m+1)l^2 + [n(2m^3+3m^2+6m+2)+1]l \\ &\quad - n^2(3m^2+2)\,. \end{aligned} \tag{5.49}$$

The result disagrees with the one obtained for W_n^2 in [26], where instead of our L_{M_0} a different L_H was adopted for defining the conformal structure and (instead of performing the symplectic halving) a set of auxiliary fields was introduced to render the constraints first class. This disagreement is not surprizing because of the ambiguity in defining the conformal structure of W_n^l, i.e., in choosing the H in (2.10), which eventually reflects in the value of the Virasoro centre. In addition, there is also an arbitrariness in the number of the auxiliary fields introduced, and the Virasoro centre agrees only when one uses the minimal number of the fields (with the same H).

6. Discussion

The main purpose of this paper has been to study the general structure of the Hamiltonian reductions of the WZNW theory. Considering the number of interesting examples resulting from the reduction, this problem appears important for the theory of two-dimensional integrable systems and in particular for conformal field theory.

Our most important result perhaps is that we established the gauged WZNW setting of the Hamiltonian reduction by first class constraints in full generality. It was then used here to set up the BRST formalism in the general case, and for obtaining the effective actions for the left-right dual reductions. We hope that the general framework we set up will be useful for further studies of this very rich problem.

The other major concern of the paper has been to investigate the $\mathcal{W}$-algebras and their field theoretic realizations arising from the WZNW reduction. We found first class KM constraints leading to the $\mathcal{W}_S^{\mathcal{G}}$-algebras which allowed us to construct generalized Toda theories realizing these interesting extended conformal algebras. We believe that the $sl(2)$-embeddings underlying the $\mathcal{W}_S^{\mathcal{G}}$-algebras are to play an important organizing role in general for understanding the structure, especially the primary field content, of the conformally invariant reduced KM systems. This is quite a natural idea since we have seen that the presence of an $sl(2)$ embedding can be exhibited in every polynomial and primary KM reduction and that the W_n^l-algebras are nothing but further reductions of $\mathcal{W}_S^{\mathcal{G}}$-algebras belonging to particular $sl(2)$-embeddings (see also [37]). The importance of $sl(2)$ structures in classifying $\mathcal{W}$-algebras have been advocated in the recent preprint [50] as well, on the basis of different arguments. In our study of $\mathcal{W}$-algebras we employed two (apparently) new methods, which are likely to have a wider range of applicability than what we exploited here. The first is *the method of symplectic halving* whereby we constructed purely first class KM constraint for the $\mathcal{W}_S^{\mathcal{G}}$ as well as for the W_n^l-algebras. The second is what we call *the* $sl(2)$*-method*, which can be summarized by saying that if one has conformally invariant first class constraints given by some (Γ, M_-) with M_- nilpotent, then one should build the $sl(2)$ containing M_- and try to analyse the system in terms of this $sl(2)$. We used this method to investigate, in the non-degenerate case, the generalized Toda sytems belonging to integral gradings, and also to provide the $\mathcal{W}_S^{\mathcal{G}}$-interpretation of the W_n^l-algebras.

We wish to remark here that, as far as we know, the technical problem concerning the inequivalence of those $\mathcal{W}_S^{\mathcal{G}}$-algebras which belong to group theoretically inequivalent $sl(2)$

embeddings has not been tackled yet.

It is well known [22] that the standard $\mathcal{W}$-algebras can be identified as the second Poisson bracket structure of the KdV type hierarchies of Drinfeld-Sokolov [5]. This fact leads to the question whether there is a relationship between $\mathcal{W}$-algebras and integrable hierachies also in more general cases [16,17,28,48,49].

We gave a general local analysis of the effective theories arising in the left-right dual case of the reduction, and investigated in particular the generalized Toda theories obtained by the reduction in some detail. In the case of the generalized Toda theories associated with the integral gradings we exhibited the way in which the $\mathcal{W}$-symmetry operates as an ordinary symmety of the action, and demonstrated that the quantum Hamiltonian reduction is consistent with the canonical quantization of the reduced classical theory. It would be nice to have the analogous problems under control also in more general cases. In our analysis we restricted the considerations to Gauss-decomposable fields. The fact that the Gauss decomposition may break down can introduce apparent singularities in the local description of the effective theories, but the WZNW description is inherently global and remains valid for non Gauss-decomposable fields as well [12,13]. It is hence an interesting problem to further analyze the global (topological) aspects of the phase space of the reduced WZNW theories.

We should also note that it is possible to remove the technical assumption of left-right duality. In particular, the study of purely chiral WZNW reductions could be of importance, as they are likely to give natural generalizations of Polyakov's 2d gravity action [43,12].

In this paper we assumed the existence of a gauge invariant Virasoro density L_H, of the form given by (2.10), for obtaining conformally invariant reductions. However, the example of Appendix A indicates that there is another class of conformally invariant reductions where the form of the surviving Virasoro density is different from that of an L_H. The study of this novel way of preserving the conformal invariance may open up a new perspective on conformal reductions of the WZNW theory.

There are many further interesting questions related to the Hamiltonian reductions of the WZNW theory, which we could not mention in this paper. We hope to be able to present those in future publications.

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Appendix A: A solvable but not nilpotent gauge algebra

In all the cases of the reduction we considered in Chapters 3 and 4, the gauge algebra Γ was a graded nilpotent subalgebra of $\mathcal{G}$. On the other hand, we have seen in Section 2.1 that the first-classness of the constraints implies that Γ is solvable. We want here to discuss a constrained WZNW model for which the gauge algebra is *solvable but not nilpotent.* Interestingly enough, it turns out that in this example no H satisfying (2.13) exists which would render the constraints conformally invariant. However, conformal invariance can still be maintained, showing clearly that the existence of such an H is only a sufficient but not a necessary condition.

We choose the Lie algebra $\mathcal{G}$ to be $sl(3,R)$ and the gauge algebra Γ as generated by the following three generators

$$\gamma_1 = E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = E_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.1a}$$

$$\gamma_3 = \frac{1}{\sqrt{3}}(2H_1 + H_2) + \frac{1}{2}(E_{\alpha_2} - E_{-\alpha_2}) = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} \end{pmatrix}, \tag{A.1b}$$

where the Cartan-Weyl generators are normalized by $[H_i, E_{\pm\alpha_i}] = \pm E_{\pm\alpha_i}$ and $[E_{\alpha_i}, E_{-\alpha_i}] = 2H_i$, for the simple positive roots α_i. Note that, being diagonalizable over the complex numbers, γ_3 is not a nilpotent operator. The algebra of Γ is

$$[\gamma_1, \gamma_2] = 0, \quad [\gamma_1, \gamma_3] = -\frac{\sqrt{3}}{2}\gamma_1 + \frac{1}{2}\gamma_2, \quad [\gamma_2, \gamma_3] = -\frac{1}{2}\gamma_1 - \frac{\sqrt{3}}{2}\gamma_2. \tag{A.2}$$

It is easy to verify that Γ is a solvable, not-nilpotent Lie algebra. It qualifies as a gauge algebra since $\mathrm{Tr}\,(\gamma_i \gamma_j) = 0$.

It is readily checked that the spaces $\Gamma^\perp$ and $[\Gamma,\Gamma]^\perp$ are given by

$$\begin{aligned} \Gamma^\perp &= \mathrm{span}\{H_2, E_{\alpha_1}, E_{\alpha_1+\alpha_2}, 2H_1 + \sqrt{3}E_{\alpha_2}, 2H_1 - \sqrt{3}E_{-\alpha_2}\}, \\ [\Gamma,\Gamma]^\perp &= \mathrm{span}\{H_1, H_2, E_{\alpha_1}, E_{\alpha_1+\alpha_2}, E_{\alpha_2}, E_{-\alpha_2}\}. \end{aligned} \tag{A.3}$$

Thus $[\Gamma,\Gamma]^\perp/\Gamma^\perp$, which is the space of the M's leading to first class constraints, is one-dimensional, and we can take

$$M = \mu Y \equiv \frac{\mu}{\sqrt{3}}(4H_1 + 2H_2) = \frac{\mu}{\sqrt{3}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{A.4}$$

without loss of generality.

The next question is the conformal invariance. As discussed in Section 2.1, a sufficient condition for conformal invariance is provided by the existence of a (modified) Virasoro density $L_H = L_{KM} - \partial_x \langle H, J(x)\rangle$ weakly commuting with the constraints. For this to work, the generator H must satisfy the three conditions in (2.13). However, it is an easy matter to show that those conditions are contradictory in the present case, and therefore no such H exists.

The above analysis can also be carried out for the simpler gauge algebra spanned by γ_3 only. This gauge algebra is obviously nilpotent, since it is Abelian. Nevertheless, the previous conclusion remains: There exists no H which would render the first class constraints conformally invariant, for any $M \neq 0$ from $[\Gamma, \Gamma]^{\perp}/\Gamma^{\perp}$. This shows the importance of the gauge generators being nilpotent operators, rather than the gauge algebra being nilpotent. It would be interesting to know whether there is always an H satisfying (2.13) for gauge algebras consisting of nilpotent operators.

Although there is no H such that the constraints are preserved by L_H, we can nevertheless construct another Virasoro density Λ which does preserve the constraints. It is given by

$$\Lambda(x) = L_{KM}(x) - \mu \langle \gamma_3^t, J(x)\rangle. \tag{A.5}$$

For M given in (A.4), the constraints read

$$\langle \gamma_1, J(x)\rangle = \langle \gamma_2, J(x)\rangle = 0\,, \qquad \langle \gamma_3, J(x)\rangle = \mu\,, \tag{A.6}$$

and are checked to weakly commute with Λ: $\{\Lambda(x), \langle \gamma_i, J(y)\rangle\} \approx 0$ on the constraint surface (A.6). (Note that, when going from L_{KM} to Λ, we have not changed the conformal central charge, which is classically zero.) Therefore we expect the reduced theory to be invariant under the conformal transformation generated by Λ being its Noether charge density. We now proceed to show that it is indeed the case. Before doing this, we display the form of Λ on the constraint surface:

$$\Lambda(x) = T_1^2(x) + T_2^2(x)\,, \tag{A.7a}$$

$$T_1 = \frac{1}{2}\langle E_{\alpha_2} + E_{-\alpha_2}, J\rangle\,, \qquad T_2 = \langle H_2, J\rangle. \tag{A.7b}$$

Following the analysis of Section 2.3, we take the left and right gauge algebras to be dual to each other ($\langle \gamma_i, \tilde{\gamma}_j\rangle = \delta_{ij}$)

$$\Gamma = \mathrm{span}\{\gamma_1, \gamma_2, \gamma_3\}, \quad \tilde{\Gamma} = \mathrm{span}\{\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3\} = \mathrm{span}\{\gamma_1^t, \gamma_2^t, \gamma_3^t\}, \tag{A.8}$$

and consider $M = \mu Y$ and $\tilde{M} = \nu Y^t = \nu Y$. We write the $SL(3,R)$ group elements as $g = a \cdot b \cdot c$, with $a \in \exp \Gamma$, $b \in \exp \mathcal{H}$ and $c \in \exp \tilde{\Gamma}$, with $\mathcal{H} = \text{span}\{Y, H_2\}$ the Cartan subalgebra. We did not conform to the general prescription given in Section 2.3, which required to write $g = abc$ with $b \in \exp \mathcal{B}$ for a space $\mathcal{B}$ complementary to $\Gamma + \tilde{\Gamma}$ in $\mathcal{G}$, eqs.(2.25-26). Had we done that, the resulting effective action would have looked much more complicated. Here, we simply take a set of coordinates in which the action looks simple.

The reduction yields an effective theory for the group-valued field b, of which the effective action is given by (2.40) with (2.29b). Using the parametrization $b = \exp(\alpha Y) \cdot \exp(2\beta H_2)$, the explicit form of the effective action is

$$I_{\text{eff}}(\alpha,\beta) = \int d^2x \left\{ \partial_+\alpha\partial_-\alpha + \partial_+\beta\partial_-\beta - \frac{(\partial_+\alpha - \mu)(\partial_-\alpha - \nu)}{\cosh^2\beta} \right\}. \tag{A.9}$$

By inspection, we see that this effective action is going to be conformally invariant if the field β is a scalar, and if the transformation of α is such that $\mu - \partial_+\alpha$ and $\nu - \partial_-\alpha$ are (1,0) and (0,1) vectors respectively. It implies that, under a conformal transformation $x^\pm \longrightarrow x^\pm - f^\pm(x^\pm)$, the fields α and β transform as

$$\begin{aligned} \delta\alpha &= f^+ (\partial_+\alpha - \mu) + f^- (\partial_-\alpha - \nu), \\ \delta\beta &= f^+ \partial_+\beta + f^- \partial_-\beta. \end{aligned} \tag{A.10}$$

We now want to show our previous claim: the action (A.9) is conformally invariant under the conserved Virasoro density $\Lambda(x)$, which reproduces the f^+-transformations (A.10) by Poisson brackets. (The f^--transformations could also be realized by constructing the corresponding Virasoro density $\tilde{\Lambda}$ in the right-handed sector in a similar way.) For this, we first note that in terms of the reduced variables α and β the two current components T_1 and T_2 of (A.7b) read

$$T_1 = -(\mu - \partial_+\alpha)\tanh\beta\,, \quad \text{and} \quad T_2 = \partial_+\beta. \tag{A.11}$$

These expressions can be obtained as follows. Writing $g = a \cdot b \cdot c$ and using the constraints (2.29b), the constrained current reads

$$J = a[T(b) + \partial_+ b \cdot b^{-1}]a^{-1} + \partial_+ a \cdot a^{-1}, \tag{A.12}$$

with $T(b)$ given by (2.29). Although neither T_1 nor T_2 is gauge invariant, the quantity we want to compute, $\Lambda(x)$, is gauge invariant. As a result, it cannot depend on the gauge variables contained in a. Hence we can just as well put $a = 1$ in (A.12). Doing that, the definitions (A.7b) yield (A.11). We thus find the following expression for Λ:

$$\Lambda = (\mu - \partial_+\alpha)^2 \tanh^2\beta + (\partial_+\beta)^2 . \tag{A.13}$$

It is an easy matter to show, by using the field equations obtained from the action (A.9),

$$\begin{aligned} &\sinh^2\beta\, \partial_+\partial_-\alpha + \tanh\beta \left[\partial_+\beta(\partial_-\alpha - \nu) + \partial_-\beta(\partial_+\alpha - \mu)\right] = 0 , \\ &\cosh^2\beta\, \partial_+\partial_-\beta - \tanh\beta\, (\partial_-\alpha - \nu)(\partial_+\alpha - \mu) = 0 , \end{aligned} \tag{A.14}$$

that Λ is indeed chiral, satisfying

$$\partial_-\Lambda = 0 . \tag{A.15}$$

Moreover one also checks the following Poisson brackets

$$\begin{aligned} \{\Lambda(x), \alpha(y)\} &= -(\partial_+\alpha - \mu)\, \delta(x^1 - y^1) , \\ \{\Lambda(x), \beta(y)\} &= -(\partial_+\beta)\, \delta(x^1 - y^1), \end{aligned} \tag{A.16}$$

which reproduce the transformations (A.10). Thus the density Λ features all what is expected from the Noether charge density associated with the conformal symmetry.

Finally, we present here for completeness the general solution of the equations of motion (A.14). Along the lines of Section 2.3, it can be obtained as follows:

$$\begin{aligned} \alpha &= (\eta_L + \eta_R) + \tan^{-1}\left[\frac{\sinh(\theta_L - \theta_R)}{\sinh(\theta_L + \theta_R)} \tan(\lambda_L - \rho_R)\right] + \mu x^+ + \nu x^- , \\ \cosh(2\beta) &= \cosh(2\theta_L)\cosh(2\theta_R) + \sinh(2\theta_L)\sinh(2\theta_R)\cos(2(\lambda_L - \rho_R)), \end{aligned} \tag{A.17}$$

where $\{\eta_L, \lambda_L, \theta_L\}$ and $\{\eta_R, \rho_R, \theta_R\}$ are arbitrary functions of x^+ and x^- only, respectively, and the three functions of each chirality are related by the equations,

$$\partial_+\eta_L + \partial_+\lambda_L \cosh(2\theta_L) = 0 , \qquad \partial_-\eta_R + \partial_-\rho_R \cosh(2\theta_R) = 0 . \tag{A.18}$$

Appendix B: H-compatible $sl(2)$ and the non-degeneracy condition

Our purpose in this technical appendix is to analyse the notion of the H-compatible $sl(2)$ subalgebra, which has been introduced in Section 3.4. We recall that the $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ of the simple Lie algebra $\mathcal{G}$ is called H-compatible if H is an integral grading operator, $[H\,,\,M_\pm] = \pm M_\pm$, and $M_\pm$ satisfy the non-degeneracy conditions

$$\mathrm{Ker}(\mathrm{ad}_{M_\pm}) \cap \mathcal{G}^H_\mp = \{0\}. \qquad (B.1)$$

Note that the second property in this definition is equivalent to the fact that $\mathcal{S}$ commutes with $(H - M_0)$. We prove here the results stated in Section 3.4, and also establish an alternative form of the non-degeneracy condition, which will be used in Appendix C.

Let us first consider an arbitrary (not necessarily integral) grading operator H of $\mathcal{G}$ and some non-zero element M_- from $\mathcal{G}^H_{-1}$. We wish to show that to each such pair (H, M_-) there exists an $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ for which $M_+ \in \mathcal{G}^H_{+1}$. To exhibit the $\mathcal{S}$-triple in question, we need the Jacobson-Morozov theorem, which has already been mentioned in Section 3.4. In addition, we shall also use the following lemma, which can be found in [33] (Lemma 7 on page 98, attributed to Morozov).

Lemma: Let $\mathcal{L}$ be a finite-dimensional Lie algebra over a field of characteristic 0 and suppose $\mathcal{L}$ contains elements h and e such that $[h\,,\,e] = -e$ and $h \in [\mathcal{L}\,,\,e]$. Then there exists an element $f \in \mathcal{L}$ such that

$$[h\,,\,f] = f \quad \text{and} \quad [f\,,\,e] = 2h\,. \qquad (B.2)$$

Turning to the proof, we first use the Jacobson-Morozov theorem to find generators (m_-, m_0, m_+) in $\mathcal{G}$ completing $m_- \equiv M_-$ to an $sl(2)$ subalgebra. We then decompose the elements m_0 and m_+ into their components of definite grade, i.e., we write

$$m_0 = \sum_n m_0^n \quad \text{and} \quad m_+ = \sum_n m_+^n\,, \qquad (B.3)$$

where n runs over the spectrum of the grading operator H. Since M_- is of grade -1, it follows from the $sl(2)$ commutation relations that

$$[m_0^0\,,\,M_-] = -M_- \quad \text{and} \quad [m_+^1\,,\,M_-] = 2m_0^0\,, \qquad (B.4)$$

and these relations tell us that $h = m_0^0$ and $e = M_-$ satisfy the conditions of the above lemma. Thus there exists an element f satisfying (B.2), which we can write as $f = \sum_n f^n$ by using the H-grading again. The proof is finished by verifying that $M_+ \equiv f^1$ and $M_0 \equiv m_0^0$ together with M_- span the required $sl(2)$ subalgebra of $\mathcal{G}$.

From now on, let H be an *integral* grading operator. For an element $M_\pm$ of grade ± 1, respectively, the pair $(H, M_\pm)$ is called *non-degenerate* if it satisfies the corresponding condition in (B.1).

We claim that if $S = \{M_-, M_0, M_+\}$ is an $sl(2)$ for which the generators $M_\pm$ are from $\mathcal{G}^H_{\pm 1}$, then the non-degeracy of the pairs (H, M_-) and (H, M_+) are *equivalent statements*. This will follow immediately from the $sl(2)$ structure if we prove that the *non-degeneracy of the pair $(H, M_\pm)$ is equivalent to the following equality*:

$$\dim \mathrm{Ker}(\mathrm{ad}_{M_\pm}) = \dim \mathcal{G}_0^H \ . \tag{B.5}$$

It is enough to prove this latter statement for a pair (H, M_-), since then for a pair (H, M_+) it can be obtained by changing H to $-H$. To prove this let us first rearrange the identity

$$\dim \mathcal{G} = \dim \mathrm{Ker}(\mathrm{ad}_{M_-}) + \dim [M_-, \mathcal{G}] \tag{B.6}$$

by using the grading as

$$\begin{aligned} \dim \mathrm{Ker}(\mathrm{ad}_{M_-}) - \dim \mathcal{G}_0^H =& \left\{\dim \mathcal{G}_+^H - \dim [M_-, \mathcal{G}_+^H]\right\} \\ &+ \left\{\dim \mathcal{G}_-^H - \dim [M_-, \mathcal{G}_0^H + \mathcal{G}_-^H]\right\} \end{aligned} . \tag{B.7}$$

Since both terms on the right hand side of this equation are non-negative, we see that

$$\dim \mathrm{Ker}(\mathrm{ad}_{M_-}) \geq \dim \mathcal{G}_0^H \ , \tag{B.8}$$

and equality is achieved here if and only if

$$\dim \mathcal{G}_+^H = \dim [M_-, \mathcal{G}_+^H] \quad \text{and} \quad [M_-, \mathcal{G}_0^H + \mathcal{G}_-^H] = \mathcal{G}_-^H \ . \tag{B.9}$$

On the other hand, we can show that the two equalities in (B.9) are actually equivalent to each other. To see this, let us assume that the second equality in (B.9) is not true. This is clearly equivalent to the existence of some non-zero $u \in \mathcal{G}_+^H$ such that $\langle u \, , [M_-, \mathcal{G}_0^H + \mathcal{G}_-^H]\rangle = \{0\}$. By the invariance and the non-degeneracy of the Cartan-Killing form, this is in turn equivalent to $[M_-, u] = 0$, which means that the first equality in (B.9) is not true. By noticing that the first equality in (B.9) is just the non-degeneracy condition for the pair

(H, M_-), we can conclude that the non-degeneracy condition is indeed equivalent to the equality in (B.5).

We wish to mention a consequence of the results proven in the above. To this let us consider a non-degenerate pair (H, M_-). By our more general result, we know that there exists such an $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ for which M_+ is from $\mathcal{G}^H_{+1}$. The point to mention is that this $\mathcal{S}$ is an H-compatible $sl(2)$ subalgebra, as has already been stated in Section 3.4. In fact, it is now easy to see that this follows from the equivalence of (B.1) with (B.5) by taking into account that the kernels of $\mathrm{ad}_{M_\pm}$ are of equal dimension by the $sl(2)$ structure.

Appendix C: H-compatible $sl(2)$ embeddings and halvings

In Section 3.4, we showed that, given a triple (Γ, M, H) satisfying the conditions for first-classness, conformal invariance and polynomiality (eqs. (2.6), (2.13) and (3.2-4)), the corresponding $\mathcal{W}$-algebra is isomorphic to $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$, provided that H is an integral grading operator. Here $\mathcal{S} = \{M_-, M_0, M_+\}$ is some $sl(2)$ subalgebra containing $M_- = M$. A natural question is what $sl(2)$ subalgebras arise in this way, or equivalently, given an arbitrary $sl(2)$ subalgebra, can the resulting $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra be obtained as the $\mathcal{W}$-algebra corresponding to the triple (Γ, M, H), for some integral grading operator H? Whether this occurs or not depends only on how the $sl(2)$ is embedded, and it is therefore a pure group-theoretic question. According to Section 3.4, the $sl(2)$ subalgebras having this property are the H-compatible ones. This appendix is devoted to establishing when a given $sl(2)$ embedding is H-compatible, and if so, what the corresponding H is.

The question of an $sl(2)$ being H-compatible is very much related to another one, which was mentioned at the end of Section 4.2. We noted that in some instances, a generalized Toda theory associated to an $sl(2)$ embedding could as well be regarded as a Toda theory associated to an integral grading operator H. This means that the effective action of the theory is a special case of both (4.12) and (4.3) at the same time. We have seen that this is the case when the corresponding halving is H-compatible, i.e., when the Lie algebra decomposition $\mathcal{G} = (\mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}}) + (\mathcal{Q}_{\frac{1}{2}} + \mathcal{G}_0 + \mathcal{Q}_{-\frac{1}{2}}) + (\mathcal{P}_{-\frac{1}{2}} + \mathcal{G}_{\leq -1})$ (subscripts are M_0-grades) can be nicely recasted into $\mathcal{G} = \mathcal{G}_{\geq 1}^H + \mathcal{G}_0^H + \mathcal{G}_{\leq -1}^H$. Our second problem, addressed at the end of the appendix, is to find the list of those $sl(2)$ subalgebras which allow for an H-compatible halving. Clearly, an $sl(2)$ subalgebra which possesses an H-compatible halving is also H-compatible in the above sense, but it will turn out that the converse is not true.

Let $\mathcal{S} = \{M_-, M_0, M_+\}$ be an $sl(2)$ subalgebra embedded in a maximally non-compact real simple Lie algebra $\mathcal{G}$. For the classical algebras A_l, B_l, C_l and D_l, these real forms are respectively $sl(l+1, R)$, $so(l, l+1, R)$, $sp(2l, R)$ and $so(l, l, R)$. (We do not consider the exceptional Lie algebras.) For $\mathcal{S}$ to be an H-compatible $sl(2)$, one should find an H in $\mathcal{G}$ with the following properties:

1. ad_H is diagonalizable with eigenvalues being integers,
2. $H - M_0$ must commute with the $\mathcal{S}$-triple,

3. $\dim \mathrm{Ker}(\mathrm{ad}_H) = \dim \mathrm{Ker}(\mathrm{ad}_{M_\pm})$.

We remark that here the equivalence of relations (B.1) and (B.5), proven in the previous appendix, has been taken into account. Under conditions 1-3, the decomposition

$$\Gamma^\perp = [M_-, \Gamma] + \mathrm{Ker}(\mathrm{ad}_{M_+}) \qquad (C.1)$$

holds, where $\Gamma = \mathcal{G}^H_{\geq 1}$ in the (Γ, M_-, H) setting, or $\Gamma = \mathcal{P}_{\frac{1}{2}} + \mathcal{G}^{M_0}_{\geq 1}$ in the $sl(2)$ setting, respectively. (For clarity, note that these two gauge algebras are in general not equal.) As a consequence, $J_{\mathrm{red}}(x) = M_- + j_{\mathrm{red}}(x)$ with $j_{\mathrm{red}}(x) \in \mathrm{Ker}(\mathrm{ad}_{M_+})$ is a DS gauge in both settings, and thus the $\mathcal{W}$-algebras are the same.

In order to answer the question of whether an $sl(2)$ embedding is H-compatible, it is useful to know what these embeddings actually are. For a classical *complex* Lie algebra $\mathcal{G}_c$, this question has been completely answered by Malcev (and Dynkin for the exceptional complex Lie algebras) [39]. The result can be nicely stated in terms of the way the fundamental vector representation reduces into irreducible representations of the $sl(2)$:

A_l : the $sl(2)$ reduction of the (l+1)-dimensional representation can be arbitrary,

B_l : the $(2l+1)$-dimensional representation of B_l reduces in such a way that the multiplicity of each $sl(2)$ *spinor* appearing in the reduction is even,

C_l : the $2l$-dimensional representation of C_l reduces in such a way that the multiplicity of each $sl(2)$ *tensor* appearing in the reduction is even,

D_l : same restriction as the B_l series: the spinors come in pairs.

The above conditions are necessary and sufficient, i.e., every possible $sl(2)$ content satisfying the above requirements actually occurs for some $sl(2)$ embedding. Moreover, for the classical complex Lie algebras, the way the fundamental reduces completely specifies the $sl(2)$ subalgebra, up to automorphisms of the embedding $\mathcal{G}_c$ [39].

The above description of the $sl(2)$ embeddings remains valid for the maximally non-compact classical real Lie algebras, except the last statement. First of all, this means that the above restrictions apply to the possible decompositions of the fundamental under the $sl(2)$ subalgebras in the real case as well. It is also obvious that those $sl(2)$ embeddings for which the content of the fundemantal is different are inequivalent. The converse however ceases to be true in the real case in general: inequivalent $sl(2)$ subalgebras can have the same multiplet content in the fundamental of $\mathcal{G}$. The answer to the problem of H-compatibility will in fact be provided by looking more closely at the decomposition of the fundamental of $\mathcal{G}$ under the $sl(2)$ subalgebra in question, as will be clear below.

As an immediate consequence of condition 2, $H - M_0$ is an $sl(2)$ invariant and can only depend on the value of the Casimir. If, in the reduction of the fundamental of $\mathcal{G}$, a spin j representation occurs with multiplicity m_j, the $sl(2)$ generators $\vec{M}$ and H can be written

$$\vec{M} = \sum_j \vec{M}^{(j)} \times I_{m_j}, \tag{C.2a}$$

$$H = M_0 + \sum_j I_{2j+1} \times D(j), \tag{C.2b}$$

where I_n denotes the unit $n \times n$ matrix, and the $D(j)$'s are $m_j \times m_j$ diagonal matrices. Hence, within each irreducible representation of $sl(2)$, H is equal to M_0 shifted by a constant. Obviously, this is also true in the adjoint representation and, in turn, this implies that ad_H takes the value zero at most once in each $sl(2)$ multiplet in the adjoint of $\mathcal{G}$. From condition 3, ad_H must take the value zero exactly once, i.e., each $sl(2)$ representation must intersect $\mathrm{Ker}(\mathrm{ad}_H)$ exactly once. In particular, the $sl(2)$ singlets must be ad_H-eigenvectors with zero eigenvalue.

The trivial solution $H = M_0$ exists whenever ad_{M_0} is diagonalizable on the integers, i.e., when the reduction of the fundamental of $\mathcal{G}$ is either purely tensorial or purely spinorial. From now on, we suppose that the reduction involves both kinds of $sl(2)$ representations.

1) A_l algebras.

The problem for the A_l series is simple to solve since, in this case, an H *always* exists. As a proof, we explicitly give an H which fulfills all the requirements. In (C.2b), we set

$$D(j) = \begin{cases} \lambda \cdot I_{m_j} & \text{if } j \in N, \\ (\lambda + \frac{1}{2}) \cdot I_{m_j} & \text{if } j \in N + \frac{1}{2}, \end{cases} \tag{C.3}$$

where λ is a constant that makes H traceless. In order to show that the H so defined has the required properties, we recall that for the A_l algebras, the adjoint representation is obtained by tensoring the fundamental with its contragredient. As a result, the roots are the differences of the weights of the fundamental (up to a singlet) and we have

$$\mathrm{ad}_H = \mathrm{ad}_{M_0} + [D(j_1) - D(j_2)], \qquad (C.4)$$

where j_1 and j_2 are the spins of the states in the fundamental representation from which a given state in the adjoint representation is formed. That the conditions 1-3 are satisfied is obvious from the fact that $\mathrm{ad}_H = \mathrm{ad}_{M_0}$ on tensors and $\mathrm{ad}_H = \mathrm{ad}_{M_0} \pm \frac{1}{2}$ on spinors, with $+\frac{1}{2}$ occurring as many times as $-\frac{1}{2}$.

It should be pointed out that (C.3) is by no means the only solution. Since in the product $j_1 \times j_2$, the highest weights have an M_0-eigenvalue at least equal to $|j_1 - j_2|$, another solution is given by $D(j) = (\lambda + j) \cdot I_{m_j}$.

2) C_l algebras.

For the symplectic algebras, the adjoint representation is obtained from the *symmetric* product of the fundamental with itself and we therefore have

$$\mathrm{ad}_H = \mathrm{ad}_{M_0} + [D(j_1) + D(j_2)]. \qquad (C.5)$$

Since the symmetric product of a tensor with itself produces a singlet, which must belong to $\mathrm{Ker}(\mathrm{ad}_H)$, we have $2D(t) = 0$ for every integer $j = t$. Hence in the fundamental representation, $H = M_0$ on tensors. Similarly, the symmetric product of a spinor with itself always produces a triplet, one member of which must belong to $\mathrm{Ker}(\mathrm{ad}_H)$. This implies that the diagonal entries of $2D(s)$ are either 0 or ± 1, for every half-integer $j = s$. However $D(s)$ cannot have a zero on the diagonal, because ad_H would not be integral on the representations contained in $s \times t$. Therefore, in the fundamental, $H = M_0 \pm \frac{1}{2}$ on spinors.

Let us now look at the m_s spinor representations of spin s, say $s^1, s^2, \ldots, s^{m_s}$. The product $s^i \times s^j$ of any two of those contains a singlet, and that implies $D(s^i) + D(s^j) = 0$. This equality must hold for any pair of spin s representations, which is impossible unless $m_s \leq 2$.

Let us consider the restriction g_s of the symplectic form to the spin s representations. The restricted form is non-degenerate, because the original non-degenerate metric is block-diagonal with respect to the eigenvalues of the $sl(2)$ Casimir.

If $m_s = 1$, then the H given by $M_0 \pm \frac{1}{2} \cdot I$ on the unique spin s representation, should be in the symplectic algebra: $g_s H + H^t g_s = 0$. Since M_0 is already symplectic, we require that the identity be symplectic, which is impossible for a non-degenerate form. Hence m_s must be 2.

If $m_s = 2$, $H - M_0$ and g_s look like (in the basis where M_0 and H are diagonal)

$$H - M_0 = \pm \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad g_s = \begin{pmatrix} a & b \\ -b^t & c \end{pmatrix}, \tag{C.6}$$

where the blocks a and c are antisymmetric. $H - M_0$ being symplectic leads to $a = c = 0$.

To summarize, for an integral H to exist, the $sl(2)$ embedding must be such that: (i) the multiplicity of any spinor representation in the fundamental of $\mathcal{G}$ is 2, (ii) if (s, s') is such a pair of spinors, they must be the dual of each other with respect to the symplectic form. If these two conditions are met, then H is given in the fundamental by

$$H = \begin{cases} M_0 & \text{on tensors,} \\ M_0 +/- \frac{1}{2} & \text{on a pair of spinors } s/s'. \end{cases} \tag{C.7}$$

Conditions 1-3 are satisfied since (C.7) implies $\mathrm{ad}_H = \mathrm{ad}_{M_0}$ on singlets, $\mathrm{ad}_H = \mathrm{ad}_{M_0} \pm (1$ or $0)$ on tensors and $\mathrm{ad}_H = \mathrm{ad}_{M_0} \pm \frac{1}{2}$ on spinors.

3) B_l and D_l algebras.

The analysis here is similar to what has been done in 2), and we can therefore go through the proof quickly.

For the orthogonal algebras, the adjoint is got from the *antisymmetric* product of the fundamental with itself and we still have

$$\mathrm{ad}_H = \mathrm{ad}_{M_0} + [D(j_1) + D(j_2)]. \tag{C.8}$$

The antisymmetric product of a tensor (spinor) with itself produces a triplet (singlet), so that with respect to the symplectic algebras, the situation is reversed in the sense that the tensors and the spinors have their roles interchanged: $H = M_0 \pm \frac{1}{2}$ on tensors, $H = M_0$ on spinors and $m_t \leq 2$ for any tensor representation of spin t.

If as in 2), we look at the restriction g_t of the orthogonal metric to the spin t tensors, we have $m_t = 2$ on account of the non-degeneracy of g_t. From this, we get at once that there can be no solution for the B_l algebras. Indeed, the fundamental being odd-dimensional, at least one tensor representation must come on its own.

On the $2(2t+1)$-dimensional subspace made up by the two spin t tensors, $H - M_0$ and g_t take the form

$$H - M_0 = \pm \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad g_s = \begin{pmatrix} a & b \\ b^t & c \end{pmatrix}, \tag{C.9}$$

where a and c are now symmetric. Requiring that $H - M_0$ be orthogonal, we again obtain $a = c = 0$.

Therefore, for the orthogonal algebras, we get the following conclusions. There is no solution for the B_l series if the $sl(2)$ embedding is not integral. As to the D_l series, the $sl(2)$ embedding must be such that: (i) every tensor in the fundamental of $\mathcal{G}$ has a multiplicity equal to 2, (ii) if (t, t') is such a pair of tensors, they must be the dual of each other with respect to the orthogonal metric. In this case, H is given in the fundamental by

$$H = \begin{cases} M_0 + / - \frac{1}{2} & \text{on a pair of tensors } t/t', \\ M_0 & \text{on spinors.} \end{cases} \tag{C.10}$$

Summarizing the analysis, the H-compatible $sl(2)$ embeddings are the following ones:

A_l : any $sl(2)$ subalgebra,

B_l : only the integral $sl(2)$'s,

C_l : those for which each *spinor* occurs in the fundamental of C_l with a multiplicity 0 or 2, the pairs of spinors being symplectically dual,

D_l : those for which each *tensor* occurs in the fundamental of D_l with a multiplicity 0 or 2, the pairs of tensors being orthogonally dual.

The reader may wish to check that the above results are consistent with the isomorphisms $B_2 \sim C_2$ and $A_3 \sim D_3$.

We now come to the second question alluded to at the beginning of this appendix, namely the problem of H-compatible halvings. From the definition, an $sl(2)$ subalgebra allows for an H-compatible halving if in addition to conditions 1-3 one also has

4. $\mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1} = \mathcal{G}^H_{\geq 1}$, and $\mathcal{P}_{-\frac{1}{2}} + \mathcal{G}_{\leq -1} = \mathcal{G}^H_{\leq -1}$.

In particular, this fourth condition implies $\mathcal{G}_0^{M_0} \subset \mathcal{G}_0^{H}$. So we readily obtain that H and M_0 must satisfy

$$\mathrm{ad}_H = \mathrm{ad}_{M_0}, \qquad \text{on} \quad \text{tensors}, \tag{C.11}$$

since we know, from the previous analysis, that $\mathrm{ad}_H - \mathrm{ad}_{M_0}$ is a constant in every representation (condition 2). Therefore, we can simply look at those solutions of the first problem which satisfy (C.11) and check if condition 4 is fully satisfied or not. We get that the $sl(2)$ embeddings allowing for an H-compatible halving are as follows:

- A_l : any $sl(2)$ subalgebra. There are only two solutions for H given by setting in (C.2b): $D(j) = (\lambda \pm \epsilon(j)) \cdot I_{m_j}$ with $\epsilon(j) = 0/\frac{1}{2}$ for a tensor/spinor,
- B_l : only the integral $sl(2)$'s with $H = M_0$,
- C_l : only the integral $sl(2)$'s,
- D_l : the integral $sl(2)$'s, and those for which the fundamental of D_l reduces into spinors and two singlets, with H given by (C.10).

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