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Generalized Skein Relations from Chern-Simons Field Theory

Danny Birmingham

Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

Siddhartha Sen

School of Mathematics, Trinity College, Dublin 2, Ireland

Abstract

Using a variational approach based only on three dimensional properties of Chern-Simons theory, a skein relation for the expectation value of Wilson line operators in the adjoint representation of $SU(2)$ is derived, in the large k limit. The result agrees with that obtained from RCFT. The generalization to arbitrary representations is then straighforward, once an important phase factor present in our example is understood.

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1 Introduction

Chern-Simons quantum field theory provides a useful framework for understanding and generalizing knot and link invariants [1]. One can also show how certain $2d$ integrable lattice models arise naturally, together with the notion of quantum groups [2]. In explicit calculations of these invariants an important role is played by the skein relations. Witten has shown that when the gauge field is in the fundamental representation of $SU(N)$ the Chern-Simons field theory leads to the $HOMFLY$ polynomial [3], which is a two-variable generalization of the Jones polynomial [4]. Specifically, the skein relation associated with this polynomial was derived. However, in deriving this relation essential use was made of the intimate connection between Chern-Simons theory in three dimensions and rational conformal field theory (RCFT) in two dimensions. In fact recent work [5] has shown that generalized skein relations for arbitrary groups and representations can be obtained using results from RCFT, such as the dimensionality of physical Hilbert spaces and the known eigenvalues of the braiding matrix [6].

Subsequently, Cotta-Ramusino et al [7] derived this skein relation directly from the Chern-Simons theory, without making use of results from RCFT. The method is based on a variational approach [8] and the existence of a Fierz identity for the generators of $SU(N)$ in the fundamental representation. However, it should be emphasized that the coefficients in the skein relation are evaluated to first order in the large k limit, where k is the integer parameter multiplying the Chern-Simons action. As such the method can be regarded as a large k perturbative scheme. To this order, the results agree with those obtained from RCFT.

The main motivation for the present work is to point out that when applying this method to more general cases, one encounters a relative framing phase factor, which is not present in the original calculation. The correct interpretation of this phase factor is crucial for obtaining results which agree with those from the RCFT method. We illustrate this for the case when the gauge field is in the adjoint representation of $SU(2)$; indeed it is for this case that we also have a simple Fierz identity allowing us to proceed. The resulting skein relation corresponds to the Akutsu-Wadati polynomial [9]. However, having understood the origin of this phase in the simple case, the application of the method to general groups and representations can then, in principle, proceed.

The plan of this paper is as follows. In the next section we briefly review the RCFT approach to the derivation of skein relations. Following this we apply the variational method to the case of the adjoint representation of $SU(2)$, showing that the two methods coincide, to order $\frac{1}{k}$. We also present two simple consistency checks on the procedure. Section 4 contains our concluding remarks.

2 Skein Relations from RCFT

In this section we quickly review the derivation of skein relations using knowledge of RCFT [1,5,6]. To make the discussion concrete, and for comparison with results in the following section, we treat the case of the adjoint representation of $SU(2)$.

The basic idea is to consider an arbitrary link on S^3, and then cut the link on a two-sphere S^2, exposing a two-sphere with a certain number of marked points. The three-sphere with this two-sphere as boundary corresponds to a vector in the physical Hilbert space, which we denote by χ, the other half of the cut-surface is represented by a vector ψ. Since we are interested in deriving skein relations for the locally four-valent planar graphs associated with the link projection, the number of marked points will be four. We thus have a two-sphere with four charges, all in the adjoint representation of $SU(2)$. One now uses the fact the the physical Hilbert space with these four charges has dimension three [1]. This can be seen simply from the fact that for large k, the physical Hilbert space corresponds to the $SU(2)$-invariant subspace

$$\mathcal{H} = Inv(A \otimes A \otimes A \otimes A) \ . \tag{1}$$

Since $1 \otimes 1 = 2_s \oplus 1_a \oplus 0_s$, we see that there are three invariants. The subscripts s and a correspond to whether the representation occurs symmetrically or anti-symmetrically in the decomposition. This means that any four vectors in $\mathcal{H}$ obey a relation of linear dependence. This relation isprecisely the skein relation. The four vectors which we choose are represented pictorially as follows:

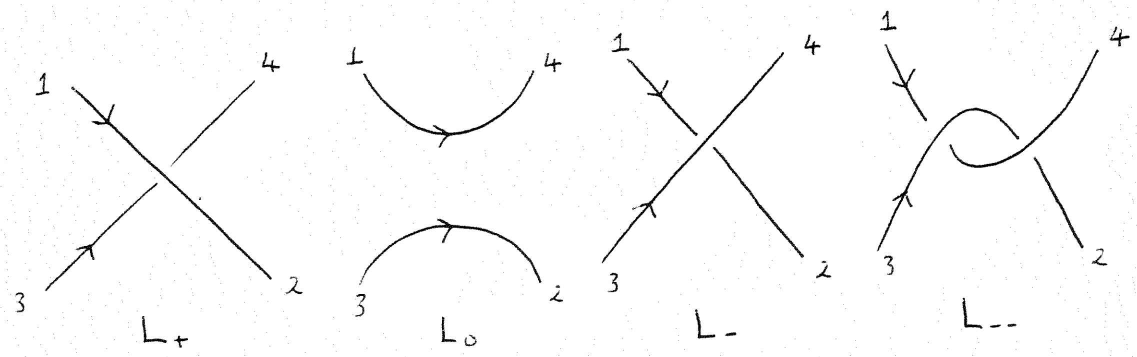

Fig. 1

Each configuration differs from the previous one by a diffeomorphism which braids two of the charges. This is called the 'half- monodromy' operation B [1,6]. The term proportional to the identity corresponds to L_+, i.e. a single over-crossing, while the remaining terms proportional to B, B^2, B^3 correspond to the diagrams L_0, L_-, L_{--}, respectively.. One now glues these manifolds back together giving the inner product relation

$$< \psi, [B^3 - B^2(\lambda_1+\lambda_2+\lambda_3) + B(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) - \lambda_1\lambda_2\lambda_3 I]\chi > \ , \quad (2)$$

where the inner product corresponds to the natural pairing of vectors in $\mathcal{H}$ and its dual. Equation (2) follows simply from the Cayley-Hamilton theorem, where B has been diagonalized with eigenvalues $\lambda_i, i = 1,2,3$. These eigenvalues are known [6] in terms of the conformal dimensions of the various fields which enter in the decomposition:

$$A \otimes A = \oplus_i E_i \ , \quad (3)$$

and are given by

$$\lambda_i = \pm e^{-\pi i(2h_A - h_{E_i})} \ , \tag{4}$$

where the $\pm$ sign depends on whether E_i occurs symmetrically or anti-symmetrically in the decomposition (3). Using the fact that the conformal dimension of a spin j field is $h_j = j(j+1)/(k+2)$, we find

$$\lambda_{E_1=2} = +e^{\frac{2\pi i}{k+2}} \ , \lambda_{E_2=1} = -e^{-\frac{2\pi i}{k+2}} \ , \lambda_{E_3=0} = +e^{-\frac{4\pi i}{k+2}} \ . \tag{5}$$

Inserting (5) in (2), taking the large k limit and multiplying through by $(1+\frac{2\pi i}{k})$ we arrive at the relation

$$\begin{aligned} (1-\frac{2\pi i}{k}) < W(L_+) > \ &- \ (1+\frac{2\pi i}{k}) < W(L_-) > \\ = (1-\frac{2\pi i}{k}) < W(L_0) > \ &- \ (1+\frac{2\pi i}{k}) < W(L_{--}) > \ , \end{aligned} \tag{6}$$

where $< W(L_+) >$ represents, in the notation of [7], a Wilson line expectation value with a single over-crossing, and so on. Equation (6) is the desired skein relation, and corresponds to the Akutsu-Wadati polynomial [9].

In the form (6) we have neglected to take into account the relative framing of diagrams. In the process of cutting, performing a diffeomorphism B^n, and gluing back together, one shifts the framing of the diagram by n units, relative to L_+. Equation (6) thus represents a regular isoptopy invariant, which is invariant under Reidemeister moves of type II and III only. To obtain an ambient isoptopy invariant, which is invariant under all three Reidemeister moves, ones simply reinserts the relative framing factors, see [1,5,7].

3 The Variational Method

We now come to the main object of the paper, that is, to derive the skein relation (6) directly from the Chern-Simons action, without making use of results from RCFT. Following [7], we begin with the Chern-Simons action in the form

$$S = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} Tr(A_\mu \partial_\nu A_\rho + i\frac{2}{3} A_\mu A_\nu A_\rho) \ . \tag{7}$$

Here $A_\mu = A^a_\mu T^a$ is the gauge field, and the Hermitian generators T^a are normalized in the fundamental representation as $Tr(T^a T^b) = \frac{1}{2}\delta^{ab}$. This ensures that once k is chosen to take integer values, the action S is invariant under all gauge transformations, both large and small.

The fundamental property of the the action S is that

$$F^a_{\mu\nu}(x) = \frac{4\pi}{k}\epsilon^{\mu\nu\rho}\frac{\delta S}{\delta A^a_\rho(x)} \quad . \tag{8}$$

It should be pointed out that since the Chern-Simons action is gauge invariant, eqn. (8) will be supplemented with gauge fixing and ghost terms. However, as shown in [10], and discussed at the end of this section, these terms do not affect the analysis. If we now consider a Wilson line operator

$$U(x_1, x_2) = Pexp \; i\int_{x_1}^{x_2} dx^\mu A^a_\mu R^a \quad , \tag{9}$$

then an infinitesimal variation of the path produces a $F_{\mu\nu}$ insertion, i.e.

$$U(x_1, x_2) \rightarrow U(x_1, x) i \Sigma^{\mu\nu} F^a_{\mu\nu} R^a U(x, x_2) \quad , \tag{10}$$

where $\Sigma^{\mu\nu} = dx^\mu dx^\nu$ is the area element, and there is no summation over μ, ν.

In order to evaluate the effect of the $F_{\mu\nu}$ insertion on $< W(L) >$ we make use of the identity (8). This yields, upon integration by parts, the following relation

$$< F^a_{\mu\nu}(x) O_1 \cdots O_N > = Z^{-1}\int dA \; e^{iS}\,\frac{4\pi i}{k}\epsilon_{\mu\nu\rho}\frac{\delta}{\delta A^a_\rho(x)}(O_1 \cdots O_N) \quad . \tag{11}$$

where $O_1, ..., O_N$ are gauge invariant observables, and Z is the partition function.

We can now proceed to derive the skein relation [7,8]. Let us consider the following identity:

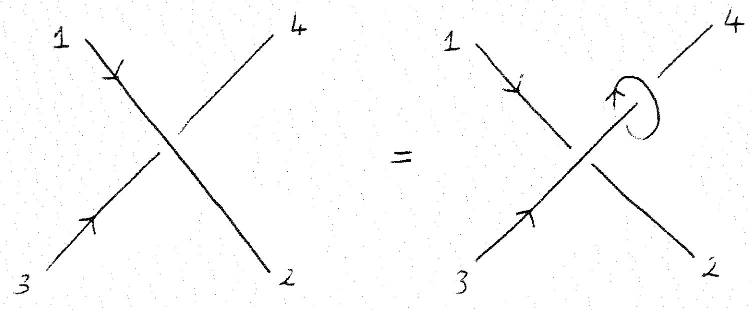

Fig. 2

where the circle attachment is to be regarded as a perturbation in the background of $< W(L_-) >$. This allows us to relate $< W(L_+) >$ and $< W(L_-) >$ by

$$< W(L_+) > = < W(L_-) > + < ...U(1,x) i \Sigma^{\mu\nu} F^a_{\mu\nu}(x) R^a U(x,2) U(3,4)... > \quad . \tag{12}$$

Using (11) in (12), and noting that the functional derivative acts on both paths $1 \to 2$ and $3 \to 4$, we obtain the relation

$$\begin{aligned} < W(L_+) > &= < W(L_-) > \\ &- \frac{4\pi i}{k} \sum_a < ...U_{ci}(1,x) R^a_{ij} U_{jb}(x,2) U_{nk}(3,x) R'_{kl} U_{lm}(x,4)... > \quad , \end{aligned} \tag{13}$$

where R and R' are the representations carried by the paths $1 \to 2$ and $3 \to 4$. It is important to point out that when the derivative acts on the Wilson lines, a differential line element dx^μ is produced. It is then necessary to determine whether this belongs to the plane defined by $\Sigma^{\mu\nu}$, or points outwards from this plane, see [7].

Equation (13) is the basic relation that we need. In our specific example we will choose $R = R' = A$, where A is the adjoint of $SU(2)$. In this case we have the explicit representation of the generators

$$R^a_{ij} = -i\epsilon_{aij} \quad ,[R^a, R^b] = i\epsilon^{abc}R^c \quad . \tag{14}$$

Futhermore, in this representation we have the following Fierz identity

$$R^a_{ij}R^a_{kl} = R^a_{kj}R^a_{il} - \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} \quad . \tag{15}$$

Inserting (15) in (13) we can now interpret the various terms pictorially, leading to

$$\begin{aligned} < W(L_+) > &= (1 + \frac{4\pi i}{k}) < W(L_-) > - \frac{4\pi i}{k} < W(L_0) > \\ &- \frac{4\pi i}{k} < ...U_{ci}(1,x)R^a_{il}U_{lm}(x,4)U_{nk}(3,x)R^a_{kj}U_{jb}(x,2)... > \quad , \end{aligned} \tag{16}$$

where it is important to remember that all terms of $O(\frac{1}{k})$ are defined in the background of $< W(L_-) >$.

At this point it remains only to interpret the final term in (16). To this effect we consider the following identity which represents a perturbation about $< W(L_{--}) >$:

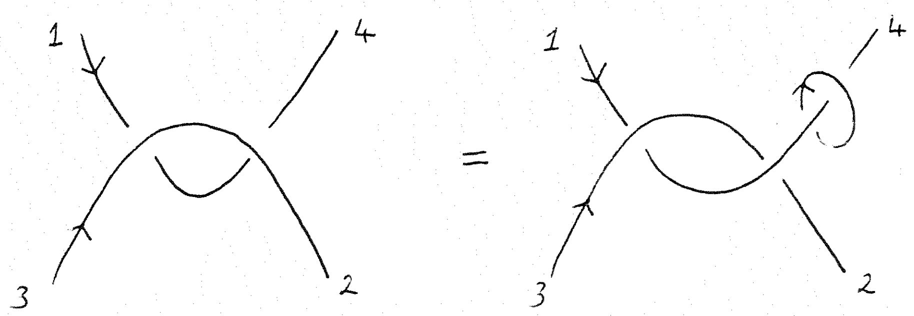

Fig. 3

In this case (13) leads to

$$
\begin{aligned}
< W(L_0) > \quad &- \quad < W(L_{--}) >= \qquad\qquad (17)\\
&- \quad \frac{4\pi i}{k} < ...U_{ci}(1,x)R^a_{il}U_{lm}(x,4)U_{nk}(3,x)R^a_{kj}U_{jb}(x,2)... > \quad .
\end{aligned}
$$

However, it is at this point that we must address the relative framing phase factor mentioned in the introduction. The final term in (16) is defined in the background of $< W(L_-) >$, while in (17) we have expressed it in the background of $< W(L_{--}) >$. Since these two diagrams differ by a twist of one unit, or in other words a single application of the braiding operator B, it is easy to see that the correct interpretation of the final term in (16) is in fact equation (17) multiplied on the left-hand-side by a factor of $(1 + \frac{4\pi i}{k})$. This value of this framing phase can also be obtained from the variational method [7], and equals $e^{\frac{2\pi i}{k+2}c_2(R)}$, where $c_2(R) = 2$ for the adjoint representation. Inserting (17) with the phase correction into (16) we get

$$
< W(L_+) >= (1+\frac{4\pi i}{k}) < W(L_-) > + < W(L_0) > -(1+\frac{4\pi i}{k}) < W(L_{--}) > \quad . \qquad (18)
$$

To $O(\frac{1}{k})$ we can rewrite this as

$$
\begin{aligned}
(1 - \frac{2\pi i}{k}) < W(L_+) > \quad &- \quad (1 + \frac{2\pi i}{k}) < W(L_-) >= \\
(1 - \frac{2\pi i}{k}) < W(L_0) > \quad &- \quad (1 + \frac{2\pi i}{k}) < W(L_{--}) > \ , \qquad (19)
\end{aligned}
$$

which agrees with the result obtained from RCFT. This skein relation coresponds to the Akutsu-Wadati polynomial [9].

We have thus shown, in this simple example, how the method of [7,8] can indeed be used to derive generalized skein relations directly from the Chern-Simons theory, without using facts from RCFT. The coefficients in the skein relation are determined in the large k limit, and it is a straightforward exercise to obtain the $O(\frac{1}{k})$ correction. As we have seen, the basic problem is to interpret pictorially the various terms that arise in the Fierz identity. But it is clear that for general groups and representations the only subtle point will be in interpreting analogues of the relative framing phase which we have encountered above. Thus, having understood this point in the simple example, one can now proceed to derive more general skein relations with this method.

Before ending this section, it is useful to point out two simple consistency checks on the above procedure. If we connect the points 3 and 2 in fig. 1 we find from (19) that

$$(1-\frac{2\pi i}{k}) < W(\hat{L}_+) > -(1+\frac{2\pi i}{k}) < W(\hat{L}_-) >= -\frac{4\pi i}{k} < W(\hat{L}_0) >< W(C_0) > \ , \tag{20}$$

where the hat notation is used to distinguish these Wilson line operators from their previous counterparts in fig. 1, and C_0 denotes an unknotted knot. Since the term on the right-hand- side is already of $O(\frac{1}{k})$ we can replace $< W(C_0) >$ by 3. This follows from the fact that [1]

$$< W(C_0) >= \frac{S_{0,A}}{S_{0,0}} = q + 1 + q^{-1} \ , \tag{21}$$

where $S_{i,j}$ is the matrix which generates the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ among the characters $\chi_i(\tau)$ of the affine Lie algebra $\hat{G}$ at level k. Equation (20) is then seen to be consistent with the known framing conventions [7,1]

$$< W(\hat{L}_\pm) >= (1 \mp \frac{4\pi i}{k}) < W(\hat{L}_0) > \ . \tag{22}$$

A second interesting check is to use this skein relation to evaluate the expectation value of two linked, but unknotted Wilson lines. This is achieved by also connecting the points 1 and 4 in fig. 1, leading to the relation

$$< W(L(R_1, R_1)) >= (1 - \frac{4\pi i}{k}) < W(C_0) >^2 + \frac{4\pi i}{k} < W(C_0) > \ , \tag{23}$$

where $L(R_1, R_1)$ denotes two linked Wilson lines in the adjoint representation. In the large k limit we find its value to be 9, which is in agreement with the result of Witten [1], namely $< W(L(R_1, R_1)) >= \frac{S_{2,2}}{S_{0,0}}$.

It is worth pointing out the basis dependence of the Fierz identity which we have used, namely (15). Since the generators in the adjoint representation are proportional to the ϵ symbol, this identity is well known. However, one could just as well choose to derive a Fierz identity in the Cartan basis, for example. While the two identities will differ, the final results, namely the skein relations, will coincide.

To close this section we address the important issue of gauge fixing in the application of the variational method. The basic equation used is (8). However, beacuse of the gauge invariance of the Chern-Simons action, one must include the necessary gauge fixing and ghost terms. This means that eqn (8) gets modified to

$$F^a_{\mu\nu}(x) = \frac{4\pi}{k}\epsilon_{\mu\nu\rho}\left(\frac{\delta S_q}{\delta A^a_\rho(x)} + f^{abc}\partial^\rho \bar{c}^b(x)c^c(x) - \frac{k}{4\pi}\partial^\rho B^a(x)\right) \ , \qquad (24)$$

where S_q denotes the complete quantum action, including the gauge fixing and ghost terms; $c, \bar{c}, B$ denote the ghost, antighost, and multiplier fields, respectively. However, as shown in [7], these extra terms do not effect the important relations (11-13). This follows simply from the BRST invariance of the vacuum and of the observables O_i, and from the gauge covariant properties of the Wilson line operators $U(x,y)$.

4 Conclusion

We have shown that the variational method does indeed allow one to derive generalized skein relations, without using knowledge from RCFT. The principal object of the present work is to point out that in applying this method to more general situations, one will encounter relative framing phase factors, which must be accounted for and interpreted in the correct way, in order to obtain results in agreement with the RCFT method. However, it should be clear that having understood the interpretation of this phase in our simple illustrative example, no more complexity will be encountered in the more general case.

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