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37 NEW ORBIFOLD CONSTRUCTIONS OF THE MOONSHINE MODULE ?

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ABSTRACT

We propose 37 new constructions of the Moonshine Module $V^\natural$ of Frenkel, Lepowsky and Meurman, the bosonic CFT whose automorphism group is the Fischer-Griess Monster M. We consider the Leech lattice compactified bosonic string and construct a family of orbifolds based on 38 Leech lattice automorphisms of a specific type (including the original Z_2 reflection automorphism considered by FLM). All of these theories are shown to have no massless states and to share the same partition function as $V^\natural$. For orbifolds based on a prime ordered lattice automorphism we identify the orbifold automorphism group (where untwisted and twisted sectors are not mixed) with an appropriate centraliser of M. This explains the form observed for this centraliser by Conway and Norton. We also provide a new explicit formula for the orbifold Thompson series for elements of this centraliser generalising the formula found by FLM. Finally, the Thompson series is calculated for a distinguished orbifold automorphism which can be identified in each case with an element of M.

Introduction

The Moonshine Module constructed by Frenkel, Lepowsky and Meurman (FLM) [1,2] is historically the first example of an orbifold theory [3,4]. This meromorphic bosonic CFT [5] is constructed by means of a Z_2 reflection automorphism of the Euclidean bosonic string toroidally compactified [6] by the Leech lattice [7]. The automorphism group of the resulting orbifold Hilbert space $V^\natural$ is then the Fischer-Griess Monster group M, the largest sporadic finite simple group [8]. An essential feature of $V^\natural$ is the absence of any massless states so that a symmetrised form of the commutator of level two operators forms a closed algebra. This algebra is an affine version of the Griess algebra whose automorphism group is M [8].

The purpose of this paper is to investigate a suggestion we made in ref.[9] that there are a number of alternative constructions of the Moonshine Module. We consider a family of 38 Leech lattice automorphisms g (including the original Z_2 automorphism) which can be employed to construct a new meromorphic orbifold theory with Hilbert space $\mathcal{H}_{\mathrm{orb}}$ which contains no massless states. The partition function for $\mathcal{H}_{\mathrm{orb}}$ is shown to equal to that of $V^\natural$ in each case. It has been conjectured by FLM [2] that $V^\natural$ is the unique meromorphic bosonic theory without any massless states so that $\mathcal{H}_{\mathrm{orb}} \equiv V^\natural$. We provide evidence to support this conjecture by analysing the automorphism group of $\mathcal{H}_{\mathrm{orb}}$ where no mixing between the various twisted and untwisted sectors occurs. In particular we find the general form for the automorphism group C_n for the Hilbert space of untwisted states and states twisted by g with no mixing between the two sectors. In the case of an orbifold constructed from g of prime order p, we find that C_p can be identified as the centraliser in M of a particular Monster element $p-$. This centraliser was observed by Conway and Norton [10] to be related to the centraliser of g in $.0$, the Conway group of Leech lattice automorphisms. Our orbifold construction provides the first explanation for this relationship generalising the result of FLM for $p = 2$.

We also provide an explicit formula for the Thompson series for elements of C_p. If $\mathcal{H}_{\mathrm{orb}} \equiv V^\natural$, this gives a new way of calculating the Thompson series for these Monster group elements. We also calculate the Thompson series for a distinguished automorphism i_p of $\mathcal{H}_{\mathrm{orb}}$ for which C_p is the centraliser and show that i_p has the same Thompson series in $\mathcal{H}_{\mathrm{orb}}$ as $p-$ has in $V^\natural$. Thus i_p can be identified with $p-$. Lastly, we generalise this result for all of the 38 Leech lattice automorphisms to show that the corresponding distinguished automorphism i_n of $\mathcal{H}_{\mathrm{orb}}$ can be identified with an appropriate element of M.

The outline of the paper is as follows. Section 2 describes the orbifolding procedure for any Leech lattice automorphism g where g leaves no lattice vectors invariant. We discuss the structure of the g twisted vacuum sector in some detail and calculate the partition function for the resulting modular invariant theory. In section 3 we place a further constraint on g to ensure that the resulting orbifold theory $\mathcal{H}_{\rm orb}$ contains no massless states. The partition function is shown to be that of $V^\natural$ in each case. A table of all 38 Leech lattice automorphisms which give rise to this partition function is provided. We then discuss the automorphism group of $\mathcal{H}_{\rm orb}$ and find that the automorphism group C_p is isomorphic to the centraliser of $p- \in M$ for an automorphism of prime order p. In section 4 we discuss the explicit calculation of the Thompson series for elements of C_p in terms of Siegel modular functions. We calculate the Thompson series for the distinguished automorphism i_p and show that the result is the Thompson series for $p-$ in M. This last property is generalised using the Moonshine properties for certain Thompson series [10] to establish that the Thompson series for the distinguished orbifold automorphism i_n is the same as that of an appropriate element of M. Section 5 concludes with a few remarks concerning further results required to conclusively prove that $\mathcal{H}_{\rm orb} \equiv V^\natural$.

2. Orbifolds from Leech Lattice Automorphisms

In this section we will review the construction of a meromorphic orbifold [3,4] CFT based on an automorphism group of a Euclidean bosonic closed string compactified to a 24 dimensional torus T^{24} [6]. The torus T^{24} we choose is defined by quotienting R^{24} with the Leech lattice Λ, the unique even self-dual lattice in 24 dimensions without roots (vectors of length squared 2) [7,11]. The automorphism group we consider is the cyclic group generated by an automorphism of the Leech lattice which fixes no lattice elements. A further condition on the lattice automorphism will be imposed in section 3 to ensure that the resulting orbifold theory has no massless states as is the case for the FLM Moonshine Module [1,2]. We will discuss in this section the construction of the corresponding twisted Hilbert space with particular emphasis on the nature of the vacuum structure.

We begin with the usual left-moving bosonic string variables $x^i(z)$ which obey the closed string boundary condition $x^i(e^{2\pi i}z) = x^i(z) + 2\pi\beta$ for $\beta \in \Lambda$. The mode expansion for $x^i(z)$ is then

$$x^i(z) = q^i - ip^i lnz + i\sum_{m\neq 0}\frac{\alpha^i_m}{m}z^{-m} \tag{2.1}$$

with

$$
\begin{aligned}
[q^i, p^j] &= i\delta^{ij} \\
[\alpha^i_m, \alpha^j_n] &= m\delta^{ij}\delta_{m+n,0}
\end{aligned}
\tag{2.2}
$$

A similiar expression holds for the right-moving part of the string $x^i(\bar{z})$. Since Λ is even self-dual [11], the 1-loop partition function takes the factorised modular invariant form $Z(\tau)Z(\bar{\tau})$ where

$$
Z(\tau) = \mathrm{Tr}(q^{L_0}) = \frac{\Theta_\Lambda}{\eta^{24}} \tag{2.3a}
$$

$$
\Theta_\Lambda(\tau) = \sum_{\beta\in\Lambda} q^{\beta^2/2} \tag{2.3b}
$$

with $q = e^{2\pi i\tau}$ and where L_0 is the normal ordered Virasoro Hamiltonian operator

$$
L_0 = \frac{1}{2}p^2 + \sum_{m=1}^{\infty} \alpha^i_{-m}\alpha^i_m - 1 \tag{2.4}
$$

$\eta = q^{\frac{1}{24}}\prod_n(1-q^n)$ is the Dedekind eta function arising from the oscillator modes. Θ_Λ is the theta function associated with the Leech lattice Λ and is a modular form of weight 12 [12]. The normal ordering constant gives the usual bosonic tachyonic vacuum energy -1.

The Hilbert space of states for this theory also factorises into meromorphic/anti-meromorphic (in z) pieces. We may therefore consistently regard the left-moving string as a meromorphic CFT [13,5]. The Hilbert space $\mathcal{H}_0$ for this meromorphic CFT is generated as a Fock space by the action of the operators $\{\alpha^i_m\}$ on the highest weight states $\{|\beta>\}$ (which are annihilated for $m > 0$) where $p^i|\beta> = \beta^i|\beta>$ [11]. The trace in (2.3a) is over $\mathcal{H}_0$ which is graded by L_0 with integer levels.

The partition function $Z(\tau)$ for $\mathcal{H}_0$ is a meromorphic and modular invariant function of τ with a unique simple pole at $q = 0$ due to the tachyonic vacuum energy. $Z(\tau)$ may therefore be found in terms of the modular invariant function $J(\tau)$ as follows

$$
Z(\tau) = J(\tau) + 24 \tag{2.5a}
$$

$$
J(\tau) = \frac{E_2^3}{\eta^{24}} - 744 = \frac{1}{q} + 0 + 196884q + \ldots \tag{2.5b}
$$

where $E_2(\tau)$ is the Eisenstein modular form of weight 4 [12]. The constant 24 reflects the existence of only 24 level 1 massless states $\{\alpha^i_{-1}|0>\}$ since the Leech lattice contains no roots.

Let us consider next the construction of a meromorphic orbifold CFT based on an abelian automorphism group of $\mathcal{H}_0$. Let g be an automorphism of Λ i.e. $g : \Lambda \rightarrow \Lambda$ with $(g\alpha).(g\beta) = \alpha.\beta$ for all $\alpha, \beta \in \Lambda$. The full set of such automorphisms is .0, the Conway group. Let n be the order of g and define the projection operator

$\mathcal{P}_g = (1+g+g^2+...+g^{n-1})/n$. We will consider here a lattice automorphism g which leaves no lattice vector fixed so that $\mathcal{P}_g\Lambda = 0$. g generates an abelian cyclic group of order n (isomorphic to Z_n) which we denote by $< g >$. We may now construct the bosonic string theory with coordinates $x^i(z)$ on the orbifold $T^{24}/ < g >$. The states of this theory consist of $\mathcal{P}_g$ projections of the 'untwisted' Hilbert space $\mathcal{H}_0$ together with new 'twisted' Hilbert spaces defined below. Together these states then form a modular invariant theory [3,4].

The automorphism g induces an automorphism on $\mathcal{H}_0$ by $g : |\beta >\rightarrow |g\beta >$ with an obvious generalisation to the other Fock states. (As will be briefly explained in section 3, the Z_2 cocycles associated with the vertex construction of these states actually allow each g to be centrally extended by Z_2^{24} as an automorphism on $\mathcal{H}_0$ [1]. Here we will consider the trivial extension for which a twisted sector construction is explicitly known). With this induced automorphism on $\mathcal{H}_0$ we can consider the projected Hilbert space $\mathcal{H}_0^{(0)} = \mathcal{P}_g\mathcal{H}_0$ which has partition function $\mathrm{Tr}_{\mathcal{H}_0^{(0)}}(q^{L_0})$ which we also denote by $\mathcal{P}_g \underset{1}{\square}$ where the boundary conditions on the world-sheet torus are specified in the standard way. Note that the assumed absence of any fixed lattice vectors under g implies that $\mathcal{H}_0^{(0)}$ contains no level 1 states i.e. the 24 states $\{\alpha_{-1}^i|0>\}$ are projected away. To compute the corresponding trace it is useful to parameterise g according to its characteristic equation as follows

$$\det(x-g) = \prod_{k|n}(x^k - 1)^{g_k} \tag{2.6}$$

where $k|n$ denotes that k divides n and where each g_k is a not necessarily positive integer with

$$\sum_{k|n} kg_k = 24 \tag{2.7a}$$

$$\sum_{k|n} g_k = 0 \tag{2.7b}$$

(2.7b) follows from the assumption that g has no unit eigenvalues. Since the characteristic equation is invariant under conjugation by $h \in .0$ with $g \rightarrow h^{-1}gh$, the parameters $\{g_k\}$ depend only on the conjugacy class of g. For $n = p$ prime, these parameters are particularly simple with $g_p = -g_1 = 2d$ where $(p-1)2d = 24$ which implies $p = 2, 3, 5, 7, 13$ for $d = 12, 6, 3, 2, 1$.

One can perform the trace over $\mathcal{H}_0$ to obtain the following result

$$\mathrm{Tr}(gq^{L_0}) = g \underset{1}{\square} = \frac{1}{\eta_g(\tau)} \tag{2.8a}$$

$$\eta_g(\tau) = \prod_{k|n}\eta(k\tau)^{g_k} \tag{2.8b}$$

In general one finds for the other traces that $h\,\underset{1}{\square} = \Theta_h/\eta_h$ where $\Theta_h(\tau) = \sum_{h\alpha=\alpha} q^{\alpha^2/2}$ for $h = g^k$. The resulting partition function for $\mathcal{H}_0^{(0)}$ is not modular invariant requiring us to introduce twisted Hilbert spaces. Thus under $S : \tau \to -1/\tau$ the boundary conditions in (2.8a) are interchanged to obtain

$$1\,\underset{g}{\square} = D_g^{1/2} \prod_{k|n} \eta(\tau/k)^{-g_k} \tag{2.9}$$

where

$$D_g = \prod_{k|n} k^{g_k} = \det(1-g) \tag{2.10}$$

For $n = p$ prime, we find $D_g = p^{2d}$. In general we therefore expect the sector twisted by g to have a vacuum degeneracy of $D_g^{1/2}$ and vacuum energy

$$E_0^g = -\frac{1}{24}\sum_{k|n}\frac{g_k}{k} \tag{2.11}$$

given by expanding the η functions in q. This formula is generally valid even if (2.7b) is not satisfied.

We now discuss the construction of the g twisted sector [14,15,3,16,17]. The starting point is a mode expansion for $x^i(z)$ satisfying the twisted boundary condition $x^i(z) = g^{ij}x^j(e^{2\pi i}z) + \beta^i$ for $\beta \in \Lambda$. It is convenient to choose a diagonal basis for g so that $g = \mathrm{diag}(e^{2\pi i r_1/n}, \ldots e^{2\pi i r_{24}/n})$ for $0 < r_i < n$. The mode expansion is then given by

$$x^i(z) = \tilde{q}^i + i \sum_{m \in Z + r_i/n} \frac{\tilde{\alpha}^i_m}{m} z^{-m} \tag{2.12}$$

where $\{\tilde{\alpha}^i_m\}$ obey the usual commutation relations (2.1). The vector $\tilde{q}$ denotes any fixed point of T^{24} under g i.e. $\tilde{q} \in (1-g)^{-1}\Lambda$. The set of inequivalent fixed points is determined by the coset $L_g = \Lambda/(1-g)\Lambda$ which is a finite abelian group of dimension $D_g = \det(1-g)$ by (2.10).

Each fixed point $\tilde{q} \in L_g$ corresponds to a vacuum state of the full left and right moving theory (a similar expansion to (2.12) exists for $x^i(\bar{z})$ involving the same fixed point set L_g). We expect from (2.9) that we may associate $D_g^{1/2}$ of these states $\{|\sigma^a>\}, r = 1, \ldots D_g^{1/2}$ with the left-moving meromorphic sector. To understand this in more detail let us consider the construction of this twisted vacuum sector as a representation space for the cocycle factors appearing in the vertex construction of twisted states [14,15,2]. To ensure the associativity of the algebra of vertex operators for twisted states it is necessary, as in the untwisted case, to introduce into every vertex operator a cocycle factor $\tilde{c}(\alpha)$ for each $\alpha \in \Lambda$ where $\tilde{c}(\alpha)$ is an element of a central extension $\hat{\Lambda}$ of the lattice Λ. $\hat{\Lambda}$ is defined in the following way. Let $\omega = e^{2\pi i/n}$

where n is the order of g. The commutator for elements $a, b \in \hat{\Lambda}$ of the central extension of Λ by $<(-1)^n\omega>$ (the cyclic group generated by $(-1)^n\omega$) is defined as follows

$$aba^{-1}b^{-1} = S(\alpha, \beta) \tag{2.13}$$

where a and b are extensions of α and β and where $S : \Lambda \times \Lambda \rightarrow <(-1)^n\omega>$ is the bilinear commutator map given in general by [14]

$$S(\alpha, \beta) = S^{-1}(\beta, \alpha) = \prod_{k=0}^{n-1} (-\omega^k)^{(g^k\alpha).\beta} \tag{2.14}$$

This reduces to the familiar commutator $(-1)^{\alpha.\beta}$ both for the untwisted case and for the Z_2 reflection twist $r : \alpha \rightarrow -\alpha$. In the more general case where g has no unit eigenvalues one finds that (2.14) simplifies to [18]

$$S(\alpha, \beta) = e^{-2\pi i((1-g)^{-1}\alpha).\beta} \in <\omega> \tag{2.15}$$

With $\hat{\Lambda}$ so defined we can then choose a section $\tilde{c} : \Lambda \rightarrow \hat{\Lambda}$ which gives the cocycle factors appearing in the vertex construction i.e. $\hat{\Lambda} = \{\omega^k \tilde{c}(\alpha) | \alpha \in \Lambda, k \in Z_n\}$. Associated with each such section is a 2-cocycle $\epsilon(\alpha, \beta)$ given by

$$\tilde{c}(\alpha)\tilde{c}(\beta) = \epsilon(\alpha, \beta)\tilde{c}(\alpha + \beta) \tag{2.16}$$

obeying the cocycle conditions

$$\begin{aligned} \epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) &= \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma) \\ \epsilon(\alpha, \beta)\epsilon^{-1}(\beta, \alpha) &= S(\alpha, \beta) \end{aligned} \tag{2.17}$$

(2.15) similarly defines the commutator for a central extension $\hat{L}_g$ of dimension nD_g of the finite abelian group L_g by $<\omega>$. It is clear from (2.15) that if $S(\alpha, \beta) = 1$ for all $\beta \in \Lambda$ then $\alpha \in (1-g)\Lambda$. Therefore the center Cent $\hat{L}_g = <\omega> = [\hat{L}_g, \hat{L}_g]$, the commutator group given by $\{aba^{-1}b^{-1} | a, b \in \hat{L}_g\}$. In the case where $n = p$ prime, $L_g = (Z_p)^{2d}$ i.e. an elementary abelian p-group which is denoted by p^{2d} [10]. The central extension is called an 'extraspecial p-group' denoted p_+^{1+2d} since $\hat{L}_g$ obeys the defining properties of such a group (see e.g. ref.[2]) : (1) Cent $\hat{L}_g = <\omega> = [\hat{L}_g, \hat{L}_g]$ and (2) $\hat{L}_g / <\omega>$ is an elementary abelian p-group.

The vacuum states of the twisted Hilbert space $\mathcal{H}_g$ are now provided by an irreducible representation of the group $\hat{L}_g$. In fact, there is a unique faithful irreducible representation $\pi(\hat{L}_g)$ which acts on a vector space T of (integer) dimension $D_g^{1/2}$ on which Cent $\hat{L}_g$ is faithfully represented by elements of $<\omega>$ [14,2,18]. The vacuum states $\{|\sigma^a>\}$ then form a basis for the vector space T. For $n = p$ prime, T is of dimension p^d. In the simplest case of the Z_2 reflection twist $r : \alpha \rightarrow -\alpha$, one can construct $\pi(\hat{L}_g)$ from a Clifford algebra basis [19,16].

The twisted Hilbert space $\mathcal{H}_g$ can now be constructed as the Fock space generated by the action of $\{\tilde{\alpha}^i_m\}$ on the vacuum states $\{|\sigma^a>\}$ which are annihilated for $m \geq r_i/n$. These states are graded by the Virasoro Hamiltonian $L_0 = \frac{1}{2}\sum \tilde{\alpha}^i_{-m}\tilde{\alpha}^i_m + E^g_0$ with (energy) level $E \in E^g_0 + Z/n$. E^g_0 is the normal ordering constant given by $E^g_0 = -1 + \sum_{i=1}^{24} r_i(n - r_i)/4n^2$ [4]. Using the parameterisation (2.6) this can be shown to give (2.11) as expected [9]. The twisting of an L_0 eigenstate $|E> \in \mathcal{H}_g$ is then expressed by

$$(g)e^{2\pi i L_0}|E>=|E> \tag{2.18}$$

assuming that no global phase anomaly arises [20] which would spoil the modular consistency of the theory. Therefore each state $|E>$ is an eigenstate of g with eigenvalue $\exp(-2\pi i E)$. The explicit action of g on a twisted state is described in section 3. The absence of any global phase anomaly leading to a modular consistent theory is guaranteed by the condition [4,20]

$$nE = nE^g_0 = 0 \bmod 1 \tag{2.19}$$

which follows from (2.18) by applying g n times or, equivalently, from the invariance of (2.9) under $T^n : \tau \to \tau + n$. Therefore the condition (2.19) is equivalent to the invariance of ${}^{g}\square_{1}$ under ST^nS. Assuming that this condition is similarly satisfied for each twisted Hilbert space $\mathcal{H}_{g^k}$ (where n is replaced by the order of g^k), the resulting orbifold theory will be a modular invariant and consistent meromorphic CFT with Hilbert space $\mathcal{H}_{\rm orb} = \mathcal{H}^{(0)}_0 \oplus \mathcal{H}^{(0)}_g \oplus \ldots \mathcal{H}^{(0)}_{g^{n-1}}$ where $\mathcal{H}^{(0)}_{g^k} = \mathcal{P}_g\mathcal{H}_{g^k}$. The resulting modular invariant partition function is meromorphic with a simple pole at $q = 0$ and is therefore given by

$$\begin{aligned} Z_{\rm orb}(\tau) &= \sum_{k=0}^{n-1} \mathcal{P}_g \, \square_{g^k} \\ &= J(\tau) + N_0 \end{aligned} \tag{2.20}$$

where N_0 is the number of level 1 massless states. Since g has no unit eigenvalues, these states can only arise in the twisted sectors $\mathcal{H}^{(0)}_{g^k}$. In the next section we will impose a further condition on g to ensure that these massless states are avoided in order to reproduce the properties of the Moonshine Module of FLM.

3. Constructing the Moonshine Module

The FLM Moonshine Module [1,2] $V^\natural$ is the meromorphic orbifold CFT constructed in the above way from the Z_2 reflection automorphism $r : \alpha \to -\alpha$ i.e. $V^\natural = \mathcal{H}_0^{(0)} \oplus \mathcal{H}_r^{(0)}$. The vacuum energy of $\mathcal{H}_r$ is $E_0^r = \frac{1}{2}$ using (2.11) with $g_2 = -g_1 = 24$ from (2.6). Hence the twisted sector introduces no massless states and the full partition function is $Z_{\rm orb}(\tau) = J(\tau)$. FLM show that the states of $V^\natural$ at each Virasoro level form a representation of M, the Fischer-Griess Monster group [8], which is the largest sporadic simple finite group. Their original motivation for studying such a theory was the observation of MacKay and Thompson [21] that the coefficients of $J(\tau)$ in (2.5b) are sums of dimensions of the irreducible representations of M. In particular, the coefficient 196884 of q is 1 greater than the dimension of the lowest dimensional representation 196883. FLM go on to show that the corresponding 196883 level 2 operators (together with L_0 which corresponds to a singlet of M) form a closed commutative non-associative algebra under a symmetrized form of the commutator of two operators. This algebra is an affine version of the Griess algebra whose automorphism group is M [8]. Since L_0 is a singlet, the states at each level of $V^\natural$ form a representation of M with those of level 2 forming the adjoint representation. A more complete description of this construction in the language of CFT can be found in ref.[13].

An important part of the FLM construction involves the identification of the automorphism group $C \subset M$ where no mixing between untwisted and twisted states occurs. If we define the 'fermion number' involution (order two) element i of M under which the states of $\mathcal{H}_0$ ($\mathcal{H}_r$) have eigenvalue $+1(-1)$. Then C is the centraliser of i in M i.e. $C = C(i) = \{g \in M | ig = gi\}$. FLM also identify a second involution σ, which mixes $\mathcal{H}_0$ and $\mathcal{H}_r$ according to a triality symmetry inherent in the construction [22]. Then, as was shown by Griess [8], $M =< C, \sigma >$ i.e. M is generated by C and σ.

It has been conjectured by FLM [2] that $V^\natural$ is the unique bosonic meromorphic CFT without massless states with light-cone central charge 24. This characterisation of the theory is analogous to that for the Leech lattice as the unique even self-dual lattice in 24 dimensions without roots. If this conjecture is correct, then any bosonic theory with partition function $Z_{\rm orb}(\tau) = J(\tau)$ must be isomorphic to $V^\natural$. In this section we will describe a number of such theories with the correct partition function constructed by the orbifold procedure from a Leech lattice automorphism g of a specified type of order n. We will also consider the automorphism group of the resulting orbifold theory $\mathcal{H}_{\rm orb}$ where no mixing between the untwisted and various twisted sectors occurs. This group is given by the centraliser $C(i_n) = \{g \in M_n | g i_n = i_n g\}$ where i_n is a distinguished automorphism of $\mathcal{H}_{\rm orb}$ of order n (which generalises the fermion number involution in the original Z_2 construction) under which all states of $\mathcal{H}_{g^k}$ are eigenvectors with eigenvalue $\omega^k = e^{2\pi i k/n}$. In the case of orbifolds constructed from an automorphism of prime order p we will be able to correctly identify $C(i_p)$ as an appropriate centraliser in M lending weight to the assertion that indeed $\mathcal{H}_{\rm orb} \equiv V^\natural$.

Let us now list the properties of a Leech lattice automorphism g that guarantee the absence of massless states so that $Z_{\rm orb} = J$. They are as follows:

(i) g has no unit eigenvalues as given by (2.7b).

(ii) The vacuum energy for each twisted sector obeys the constraint (2.19).

(iii) The vacuum energy E_0^g for the sector $\mathcal{H}_g$ is positive.

As already described in section 2, condition (i) implies that no untwisted massless states survive the $\mathcal{P}_g$ projection whereas condition (ii) ensures modular consistency. The last condition is equivalent to the absence of massless states in the twisted sector $\mathcal{H}_g$. It follows from this that no twisted massless states occur in $\mathcal{H}_{\rm orb}$ either. Let $|0> \in \mathcal{H}_{g^k}$ be some massless state. Then following (2.18) we have

$$(g^k)e^{2\pi i L_0}|0>=|0> \tag{3.1}$$

so that $|0>$ is an eigenstate of g^k with unit eigenvalue. If $|0>$ is a unit eigenstate of g also then it would obey (2.18) for a twisting by g contradicting assumption (iii). Therefore $\mathcal{P}_g|0>=0$ and hence $|0> \notin \mathcal{H}^{(0)}_{g^k}$. Thus no twisted massless states exist in $\mathcal{H}_{\rm orb}$ and so from (2.20) $Z_{\rm orb}(\tau) = J(\tau)$ as claimed.

In Table 1 we give an exhaustive list of 38 automorphisms $\{g\}$ of the Leech lattice that obey the conditions (i)-(iii) given above. The notation employed follows that of ref.[10]. Each conjugacy class of the full group of lattice automorphisms $.0$ with representative g can be labelled according to the parameters $\{g_k\}$ of (2.6). A complete list of conjugacy classes can be found in ref.[23] but the restricted set obeying (i)-(iii) already appears in ref.[10]. The first column of Table 1 gives a short-hand notation (the Frame shape) for η_g of (2.8b) where we write $p^{g_p}q^{g_q}.../r^{g_r}s^{g_s}...$ for $\eta(p\tau)^{g_p}\eta(q\tau)^{g_q}.../\eta(r\tau)^{g_r}\eta(s\tau)^{g_s}...$ where $g_p, g_q, ... > 0$ and $g_r, g_s, ... < 0$. Thus $\eta_{2-} = [\eta(2\tau)/\eta(\tau)]^{24} \equiv 2^{24}/1^{24}$. In the second column we provide an alternative labelling given in ref.[10] which describes the modular invariance group of η_g. This labelling does not in general uniquely specify the elements of $.0$ but does do so for g obeying (i) and (iii) above. If the modular invariance group of η_g is $\Gamma_0(n)$ alone then the the order n conjugacy class is labelled $n-$. The remaining order n conjugacy classes labelled $n+e_1, e_2, ...$ have a corresponding η_g which is invariant under $\Gamma_0(n)+e_1, e_2, ...$, the group generated by $\Gamma_0(n)$ and the Atkin Lehner transformations $W_{e_1}, W_{e_2}, ...$ [10] where

$$\Gamma_0(n) = \{\begin{pmatrix} a & b \\ cn & d \end{pmatrix}\}, \quad \det = 1 \tag{3.2a}$$

$$W_e = \{\begin{pmatrix} ae & b \\ cn & de \end{pmatrix}\}, \quad \det = e, \quad e|n, \quad (e, \frac{n}{e}) = 1 \tag{3.2b}$$

with $a, b, c, d \in Z$ and where $(e, n/e)$ denotes the greatest common divisor of e and n/e. The Atkin-Lehner transformations close in the following sense : $W_{e_1}W_{e_2} = W_{e_3} \bmod \Gamma_0(n)$ where $e_3 = e_1e_2/(e_1, e_2)^2$ which also implies that $W_e^2 = 1 \bmod \Gamma_0(n)$. W_e is also in the normaliser of $\Gamma_0(n)$ in SL$(2, R)$ i.e. $\Gamma_0(n) = W_e\Gamma_0(n)W_e^{-1}$. Finally, we note that Table 1 consists of all possible modular group labels in ref.[10] of the form $n+e_1, e_2, ..\beta.$ with $e_i \neq n$.

We can now confirm that the constraints (i)-(iii) are obeyed by the automorphisms listed. The first constraint (2.7b) can be seen to be obeyed by inspection. Constraint (iii) follows from the inversion of η_g under the Atkin-Lehner transformation $W_n : \tau \to -1/n\tau$ (the Fricke involution). This can be seen by observing that in each case the characteristic equation parameters obey the symmetry condition

$$g_k = -g_{n/k} \tag{3.3}$$

so that from (2.8) and (2.9b) we find that

$$\eta_g(-1/n\tau) = D_g^{1/2}\eta_g^{-1}(\tau) \tag{3.4}$$

Thus the vacuum energy is $E_0^g = 1/n > 0$ from (2.9b). Alternatively, applying (3.3) to (2.11) and using (2.7a) we obtain the same result. We note from ref.[10] that a number of different classes of .0 obeying (i) may share the same modular group label $n + e_1, e_2, \ldots$ in some cases. This occurs when the respective η functions are the same up to an overall additive constant. However, the constraint (iii) singles out a unique class of .0 in each such case.

Lastly, constraint (ii) requires us to check that (2.19) is obeyed for each g^k of order $n' = n/(n,k)$ or, equivalently, that $g^k \underset{1}{\square}$ is $ST^{n'}S$ invariant. If $(n,k) = 1$ this is automatically the case since g and g^k then have the same characteristic equation (2.6). Then applying (3.4), we know that $\eta_g = \eta_{g^k}$ is ST^nS invariant. For $(n,k) \neq 1$ we may invoke the 'Power Law Map' formula from ref.[10]. This states that the k^{th} power of a class with modular group label $n + e_1, e_2, \ldots$ is a class with modular group label $n' + e'_1, e'_2, \ldots$ where $e'_1, e'_2, \ldots$ denotes all elements of $e_1, e_2, \ldots$ that divide n'. Thus, for example, the 10^{th} power of $g = 2.6.10.30/1.3.5.15$ with modular group label $30+3,5,15$ is $g^{10} = 1^6 3^6$ with modular group label $3+3$. Strictly speaking, this formula applies to Thompson series formed from traces over the Moonshine Module but may be checked explicitly for the automorphisms given in Table 1 using the invariance modular group properties listed in refs. [10] and [24]. We therefore find that $g^k \underset{1}{\square}$ is $\Gamma_0(n')$ invariant and in particular, is $ST^{n'}S$ invariant. The second constraint is thus satisfied.

Let us now consider the automorphism group M_n of the orbifold theory $\mathcal{H}_{\mathrm{orb}}$ constructed from a lattice automorphism g of order n. We will offer evidence to suggest that $M_n \equiv M$, the Monster group, as expected from the FLM conjecture that $\mathcal{H}_{\mathrm{orb}} \equiv V^\natural$. We will concentrate on the automorphism group C_n under which no mixing between the untwisted and g twisted sector occurs. In the cases where $n = p$ prime, we will identify C_p as an appropriate centraliser of M. The analysis below generalises that of FLM for the original Z_2 construction. Where possible, we label the various automorphism groups that arise in an analogous way to the notation found in ref.[2].

We first consider the untwisted Hilbert space $\mathcal{H}_0$. The automorphism group of $\mathcal{H}_0$ is a central extension of .0, $2^{24}(.0)$, which is the automorphism group of the Z_2 central extension $\hat{\Lambda}_0$ of Λ with commutator map $S(\alpha,\beta) = (-1)^{\alpha\cdot\beta}$ which appears in the untwisted vertex construction [1.2]. Each automorphism $g \in .0$ is centrally extended to an automorphism $\hat{g}$ of $\hat{\Lambda}_0$ where choosing a section of cocycle factors $\{c(\alpha)\} \in \hat{\Lambda}_0$ we have

$$\hat{g} : c(\alpha) \rightarrow e^{2\pi i \mu_g . \alpha} c(g\alpha) \tag{3.5}$$

where $\mu_g \in \Lambda/2$ so that the phase is ± 1. The associated cocycle conditions (2.17) are preserved by the map (3.5). The set of inequivalent extensions is then determined by the coset $\Lambda/2\Lambda$ of dimension 2^{24}. The action of $\hat{g}$ on a highest weight state is $\hat{g}|\beta >= e^{2\pi i \mu_g . \beta}|g\beta>$ since $|\beta >= \hat{c}(\beta)|0>$ with $\hat{c}(\beta) = e^{i\beta . q}c(\beta)$ [1,2,11]. A general Fock state similarly transforms under $\hat{g}$ where, in the diagonal basis, each creation operator α^i_{-m} becomes $g^i_j \alpha^j_{-m} = \omega^{r_i} \alpha^i_{-m}$.

We can similarly describe the automorphism group C_1 for the projected untwisted Hilbert space $\mathcal{H}_0^{(0)} = \mathcal{P}_g \mathcal{H}_0$ where g is one of the automorphisms of Table 1. Here we choose the trivial Z_2 trivial extension of g corresponding to $\mu_g = 0$ above which we denote by θ_0. Define the centraliser subgroup of g in $.0$ by $n.G_n = \{h \in .0 | hg = gh\}$. The notation follows that of ref.[10] where $A.B$ denotes the property that a group of type A is a normal subgroup of $A.B$ with $B = A.B/A$ and where n is shorthand for the cyclic group $< g >$. The group G_n is then the automorphism group of $\Lambda/ < g >$.

Consider now a state of $\mathcal{H}_0^{(0)}$ as follows

$$\mathcal{P}_g|\mathbf{m}, \beta >= \frac{1}{p} \alpha^{i_1}_{-m_1} ....\alpha^{i_k}_{-m_k}[\hat{c}(\beta) + \omega^N \hat{c}(g\beta) + ...\omega^{N(n-1)}\hat{c}(g^{n-1}\beta)]|0> \tag{3.6}$$

where $N = r_{i_1} + ... + r_{i_k}$ and $\mathbf{m} = (m_1, ...m_k)$. Clearly $\mathcal{P}_g|\mathbf{m}, g\beta >= \omega^{-N}\mathcal{P}_g|\mathbf{m}, \beta>$ so that the independent highest weight states for $\mathcal{H}_0^{(0)}$ are determined by the elements of $\Lambda/ < g >$ up to a phase factor in $< \omega >$. Let $h \in n.G_n$ and assume that it is also diagonal in the basis chosen already for g with $h = \mathrm{diag}(e^{2\pi i u_1}, ..., e^{2\pi i u_{24}})$. h then acts on α^i_{-m} with eigenvalue $e^{2\pi i u_i}$. We can define a central extension $\hat{h}$ of h such that each cocycle factor transforms according to $\hat{h} : c(\alpha) \rightarrow \exp(2\pi i f_h(\alpha))c(h\alpha)$ where $f_h(\alpha + \beta) = f_h(\alpha) + f_h(\beta)$ which guarantees that the cocycle conditions (2.17) are preserved. Let $\{e^{(r)}\}$ be a basis for Λ and $\{e_{(r)}\}$ a dual basis $(e^{(r)}.e_{(s)} = \delta^r_s)$. Then define $\mu_h = f_h(e_{(r)})e^{(r)}$ so that $f_h(\alpha) = \mu_h.\alpha$ implying that each central extension of h is determined by a vector μ_h as in (3.5).

We next describe the set of allowed inequivalent vectors μ_h. By acting on $\mathcal{P}_g|\mathbf{m}, \beta>$, we see from (3.6) that the transformation $\hat{h}$ will be an automorphism of $\mathcal{H}_0^{(0)}$ provided $c(\beta)$ and $c(g\beta)$ transform with the same phase. Thus $\mu_h.\beta - g\mu_h.\beta \in Z$ for all β which implies that $\mu_h \in (1-g)^{-1}\Lambda$ because Λ is self-dual. Using the identity $(1-g)^{-1} = -(g + 2g^2 + ...(n-1)g^{n-1})/n$ we find that $(1-g)^{-1}\Lambda \subset \Lambda/n$ and hence $\exp(2\pi i \mu_h.\beta) \in < \omega >$. By the self-duality of Λ again we also see that two vectors μ_h and μ'_h determine equivalent extensions if and only if $\mu_h - \mu'_h \in \Lambda$. Thus the set of inequivalent choices for μ_h is given by the abelian group $L_g = \Lambda/(1-g)\Lambda$ already introduced in section 2. To summarise, each $h \in n.G_n$ is centrally extended to an automorphism $\hat{h}$ of $\mathcal{H}_0^{(0)}$ where

$$\hat{h} : c(\alpha) \rightarrow e^{2\pi i \mu_h \cdot \alpha} c(h\alpha), \qquad \mu_h \in (1-g)^{-1}\Lambda/\Lambda \equiv L_g \tag{3.7}$$

This central extension of $n.G_n$ by the set of phases $\exp(2\pi i \mu_h.\beta) \in <\omega>$ (isomorphic to L_g) is denoted by $C_0 = L_g(n.G_n)$. It is clear from (3.6) that not all such automorphisms are independent since θ_0 (the trivial Z_2 extension of g used in defining $\mathcal{P}_g$) acts on $\mathcal{H}_0^{(0)}$ with unit eigenvalue. In general, the action of an extension of hg^k is equivalent to some extension of h for each k. Therefore the independent automorphisms of $\mathcal{H}_0^{(0)}$ are given by $C_1 = L_g(G_n)$. For $n = p$ prime, $C_1 = p^{2d}(G_p)$ since in this case $L_g \equiv p^{2d}$.

Let us now describe the corresponding automorphism group for the twisted sector $\mathcal{H}_g^{(0)}$. As was discussed in section 2, the vacuum states $\{|\sigma^a>\}$ form a basis for the vector space T which is acted on by the unique irreducible representation $\pi(\hat{L}_g)$. We now generalise the arguments given in refs. [1,2] to show that every $\hat{h} \in C_0$ gives rise to an induced linear transformation h_T on T as follows. Recall the general twisted central extension $\hat{\Lambda}$ of Λ by $<\omega>$ defined by (2.13) and (2.15). The automorphism group of $\hat{\Lambda}$ is then given by C_0 where choosing some section $\{\tilde{c}(\alpha)\}$, each $\tilde{c}(\alpha)$ tranforms as in (3.7) (noting that $S(h\alpha, h\beta) = S(\alpha, \beta)$ for all $h \in n.G_n$). Next choose a section of $\hat{\Lambda}$ such that the subset $K = \{\tilde{c}((1-g)\alpha) | \alpha \in \Lambda\}$ closes under multiplication. It is easy to show that K is a central subgroup isomorphic to $(1-g)\Lambda$ and that $\hat{\Lambda}/K \equiv \hat{L}_g$, the central extension of L_g defined in section 2. In addition, K is invariant under C_0 and therefore each $\hat{h} \in C_0$ induces an automorphism on $\hat{L}_g$. Finally, the representation $\pi(\hat{L}_g)$ is unique and irreducible and hence the automorphism induced acts on the elements of $\pi(\hat{L}_g)$ by conjugation with some matrix h_T i.e $\pi(x) \rightarrow h_T^{-1}\pi(x)h_T$ for $x \in \hat{L}_g$. Thus each $\hat{h} \in C_0$ induces a linear transformation h_T on T which we note is only specified up to an overall phase at this stage.

We may again identify a distinguished element $\theta_0 \in C_0$ of order n given by the following extension of g :

$$\theta_0 : \tilde{c}(\alpha) \rightarrow \epsilon(\alpha, (1-g)\alpha)\tilde{c}(g\alpha) \tag{3.8}$$

where the section $\{\tilde{c}(\alpha)\}$ is chosen as above in defining K. One can check that $\theta_0 \in C_0$ and that θ_0 acts as the identity on $\hat{L}_g$. We may identify θ_0 with the automorphism of the untwisted space $\mathcal{H}_0$ used in forming $\mathcal{P}_g$. The remaining extensions of g are then equivalent as automorphisms on $\hat{L}_g$ to the other nontrivial extensions of the identity element of $n.G_n$. In general the induced action of an extension on $\hat{L}_g$ of $hg^k \in n.G_n$ is equivalent to that of an extension of h . Thus the set of independent automorphisms induced by C_0 on $\hat{L}_g$ is again $C_1 = L_g(G_n)$.

Let us now consider the set of linear maps $\{h_T : T \rightarrow T\}$ induced by C_1 on $\pi(\hat{L}_g)$. As noted earlier, each $\hat{h} \in C_1$ induces a map h_T which is specified only up to an overall phase. We will now consider a particular set C_T of maps induced from C_1 where these phases are given. The representation space $\pi(\hat{L}_g)$ is itself a subset of $\{h_T\}$ since it may be considered as the set of linear maps induced by the inner automorphisms of $\hat{L}_g$ as follows. For $x \in \hat{L}_g$ we may define the inner automorphism

$x : y \to xyx^{-1} = S(\alpha, \beta)y$ from (2.14) for all $y \in \hat{L}_g$ where α, β are representatives of L_g which are centrally extended to x, y respectively. (The full group of such inner automorphisms of $\hat{L}_g$ is therefore isomorphic to L_g corresponding to the extensions of the identity element of C_1). We therefore find that $\pi(\hat{L}_g) \subset \{h_T\}$ as claimed.

Recall from section 2 that the representation $\pi(\hat{L}_g)$ is faithful [2] where a central element $\pi(\omega^k.K)$ acts as ω^k on T. We therefore consider the set of induced representations C_T on $\pi(\hat{L}_g)$ where each $\hat{h} \in C_1$ is covered by n elements of $\{h_T\}$ differing only by phases in $< \omega >$. Thus we define $C_T =< \omega > (C_1) = \hat{L}_g(G_n)$ which is the group of induced automorphisms on T forming a minimal covering of C_1 which contains the representation $\pi(\hat{L}_g)$ itself.

We can now turn to the twisted Hilbert space $\mathcal{H}_g$ formed as a Fock space from the vacuum states $\{|\sigma^a>\}$ by acting with the operators $\{\tilde{\alpha}^i_{-m}\}$. We form a composite group $\hat{C}$ from the groups C_0 and C_T as follows

$$\hat{C} = \{(\hat{h}, h_T) \in C_0 \odot C_T\} \tag{3.9}$$

where h_T is induced by the action of $\hat{h} \in C_0$ on $\pi(\hat{L}_g)$. The groups C_0 and C_T are both cosets of $\hat{C}$ with $C_0 = \hat{C}/ < (1, \omega) >$ and $C_T = \hat{C}/ < (\theta_0, 1) >$. Each $(\hat{h}, h_T) \in \hat{C}$ is an automorphism of $\mathcal{H}_g$ where $\tilde{\alpha}^i_{-m} \to (h)^{ij}\tilde{\alpha}^j_{-m}$ and $|\sigma^a > \to (h_T)^a_b|\sigma^b>$. In particular, the twisting of a state of $\mathcal{H}_g$ by g as expressed by (2.18) can now be understood as follows. g acts on the creation operators according to $\tilde{\alpha}^i_{-m} \to \omega^{r_i}\tilde{\alpha}^i_{-m}$, in a diagonal basis for g. The action on $|\sigma^a>$ is a particular element $\theta_T \in C_T$ induced from the distinguished central extension $\theta_0 \in C_0$ of g given in (3.8). θ_0 acts as the identity on $\hat{L}_g$ and therefore extends to an element of $< \omega >$ in its induced action on T. We therefore choose $\theta_T \in C_T$ induced by θ_0 such that θ_T acts as ω^{-1} on T which is the appropriate phase for the vacuum energy $E^g_0 = 1/n$. The automorphism describing the twisting in (2.18) is therefore given by $(\theta_0, \omega^{-1}) \in \hat{C}$. We may then define the corresponding projection operator, denoted as before by $\mathcal{P}_g$, to form $\mathcal{H}^{(0)}_g = \mathcal{P}_g\mathcal{H}_g$.

Consider an element of $\mathcal{H}^{(0)}_g$ as follows

$$\mathcal{P}_g|\mathbf{m}, \sigma^a> = |\mathbf{m}, \sigma^a> = \tilde{\alpha}^{i_1}_{-m_1} \ldots \tilde{\alpha}^{i_k}_{-m_k}|\sigma^a> \tag{3.10}$$

where $N = r_{i_1} + \ldots + r_{i_k} = 1 \bmod n$. Notice that this condition implies that the states of $\mathcal{H}^{(0)}_g$ have integral L_0 levels as expected. The states of $\mathcal{H}^{(0)}_g$ are by definition unit eigenstates of (θ_0, ω^{-1}) and so the group of independent automorphisms is given by

$$C_n = \hat{C}/ < (\theta_0, \omega^{-1}) > \equiv \hat{L}_g(G_n) \tag{3.11}$$

The group $\hat{C}$ clearly also has a natural action on the original untwisted Hilbert space $\mathcal{H}^{(0)}_0$ where again θ_0 acts as the identity. Thus C_n forms an automorphism group of $\mathcal{H}^{(0)}_0$ (which corresponds to n copies of C_1). If we then consider the combined Hilbert space $\mathcal{H}^{(0)}_0 \oplus \mathcal{H}^{(0)}_g$, we find that automorphism group of this space is $C_n = \hat{L}_g(G_n)$

where no mixing between the two sectors occurs. For $n = p$ prime, $C_p = p_+^{1+2d}(G_p)$ since $\hat{L}_g$ is then extraspecial.

We can similarly see that C_n is also the automorphism group without mixing for the Hilbert space $\mathcal{H}_0^{(0)} \oplus \mathcal{H}_{g^k}^{(0)}$ where $(k,n) = 1$ since then $L_g = L_{g^k}$ and the analysis given above can be repeated to show that $\mathcal{H}_g$ and $\mathcal{H}_{g^k}$ are isomorphic. In the cases where $n = p$ prime, all sectors are isomorphic implying that the full automorphism group C_p has been obtained where no mixing between sectors occurs. For n not prime and $(k,n) \neq 1$, the g^k twisted sector is complicated by the appearance of a different fixed point space L_{g^k} which labels the vacuum states. In addition, there is the possibility of a further complication due to momentum (highest weight) states labelled by the elements of $\mathcal{P}_{g^k}\Lambda$ [16,9].

Let us therefore consider the cases where $n = p$ prime. We have found that the automorphism group for the full orbifold theory $\mathcal{H}_{\rm orb}$ where no mixing between sectors occurs is $C_p = p_+^{1+2d}(G_p)$. As described earlier, this implies that $p_+^{1+2d}(G_p)$ is the centraliser of i_p (which acts as ω^k on $\mathcal{H}_{g^k}$) in M_p, the full group of automorphisms of $\mathcal{H}_{\rm orb}$. It has been observed by Conway and Norton [10] that the centraliser of a prime ordered element $p-$ of the Monster group is precisely given by $p_+^{1+2d}(G_p)$ (where $p.G_p$ is the centraliser of g in $.0$). Thus the general orbifold construction presented here explains the structure of this centraliser for the first time. The defining characteristic of the Monster group element labelled $p-$ is its Thompson series which is defined in the next section. We will now demonstrate that $i_p \in M_p$ has precisely this Thompson series strengthening the conjecture that indeed $\mathcal{H}_{\rm orb} \equiv V^\natural$ and $M_p \equiv M$.

4. Calculating the Thompson Series

In this section we will discuss the explicit calculation of the Thompson series for $h \in C_p$ where we perform the trace over $\mathcal{H}_{\rm orb}$. We will give a formula for each such series generalising the expression found by FLM [1,2] for the original Z_2 orbifold. We will then demonstrate that the Thompson series for the distinguished 'fermion number' automorphism i_p is precisely that for $p- \in M$, the Monster group. We will also show that the Thompson series for i_n arising from the orbifold construction for any g in Table 1, is given by the Thompson series for $n + e_1, e_2, \ldots \in M$ in general.

Let us begin with the definition of the Thompson series $T_h(\tau)$ for any automorphism h of the orbifold $\mathcal{H}_{\rm orb}$ built from any of the Leech lattice automorphisms of Table 1. T_h is given by

$$T_h(\tau) = \mathrm{Tr}_{\mathcal{H}_{\rm orb}}(hq^{L_0}) = \frac{1}{q} + 0 + \ldots \tag{4.1}$$

where the leading terms are due to the unique tachyon and the absence of massless states. For $h \in C(\imath_n)$, the centraliser of $\imath_n$, h maps each sector into itself and so we may expand (4.1) to give

$$T_h(\tau) = \sum_{k=1}^{n} h\mathcal{P}_g \underset{g^k}{\square} \tag{4.2}$$

Here h is a shorthand notation for the appropriate action on each sector.

Let us now consider the explicit computation of these traces. We begin with the untwisted contribution which is given by

$$h\mathcal{P}_g \underset{1}{\square} = \frac{1}{n}\left[\frac{\Theta_{\hat{h}}(\tau)}{\eta_h(\tau)} + \frac{\Theta_{\hat{h}g}(\tau)}{\eta_{hg}(\tau)} + \ldots + \frac{\Theta_{\hat{h}g^{n-1}}(\tau)}{\eta_{hg^{n-1}}(\tau)}\right] \tag{4.3}$$

where $\hat{h}$ acts as a central extension of $h \in G_n$ on $\mathcal{H}_0$ according to (3.7) so that the trace over highest weight states $\{|\beta>\}$ contributes the generalised theta function

$$\Theta_{\hat{h}g^k}(\tau) = \sum_{\beta = hg^k\beta} q^{\beta^2/2} e^{2\pi i \mu_h . \beta} \tag{4.4}$$

The trace over the oscillator modes then gives rise to the η function contributions.

We turn next to the g twisted sector. The automorphism $h \in C(\imath_n)$ appearing in (4.2) is shorthand for the action of $(\hat{h}, h_T)$ on $\mathcal{H}_g$ as described in section 3. We choose a simultaneously diagonal basis for h and g with $g = \mathrm{diag}(e^{2\pi i r_i/n})$ and $h = \mathrm{diag}(e^{2\pi i u_i})$ for $i = 1, \ldots 24$. Acting with $(\hat{h}, h_T)$ on a state of $\mathcal{H}_g$ we then find

$$(\hat{h}, h_T)\tilde{\alpha}^{i_1}_{-m_1} \ldots \tilde{\alpha}^{i_k}_{-m_k}|\sigma^a> = e^{2\pi i(u_{i_1} + \ldots u_{i_k})}\tilde{\alpha}^{i_1}_{-m_1} \ldots \tilde{\alpha}^{i_k}_{-m_k}(h_T)^a_b|\sigma^b> \tag{4.5}$$

Tracing over all such states we find the following result for $\mathrm{Tr}_{\mathcal{H}_g}(hq^{L_0})$

$$h \underset{g}{\square} = \mathrm{Tr}(h_T) q^{E_0^g} \prod_{i=1}^{24} \prod_{m=1}^{\infty} (1 - e^{2\pi i u_i} q^{m - r_i/n})^{-1} \tag{4.6}$$

where the remaining trace is over the finite dimensional representation space T and E_0^g is the twisted vacuum energy which is given by $1/n$ for all g in Table 1. The infinite product can be expressed more concisely as follows

$$h \underset{g}{\square} = \mathrm{Tr}(h_T) \prod_{i=1}^{24} g_o(u_i, \frac{r_i}{n};\ \tau)^{-1/2} \tag{4.7}$$

where $g_0(u, v;\ \tau)$ is the (normalised) Siegel modular function [25,20]

$$g_0(u,v;\ \tau) = q^{(v^2-v+\frac{1}{6})/2}\prod_{m=1}^{\infty}(1-e^{2\pi i u}q^{m-v})(1-e^{-2\pi i u}q^{m+v-1}) \tag{4.8}$$

Notice that we have absorbed the vacuum energy factor into each Siegel function using the original formula $E_0^g = -\frac{1}{4}\sum_i(r_i(r_i-n)/n^2+1/6)$. One can also check that for $g=1, h=g$ or $h=1$ one recovers the expressions given in (2.8) and (2.9). The usual Siegel modular function is $g(u,v;\ \tau) = e^{-i\pi u(v-1)}g_0(u,v;\ \tau)$ which transforms under a general modular transformation $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,Z)$ according to [25]

$$g(u,v;\ \gamma(\tau)) = \epsilon(\gamma)g(\gamma^{-1}(u,v);\ \tau) \tag{4.9}$$

where $\epsilon(\gamma)$ is a phase independent of u,v which is a twelfth root of unity. Thus γ acts by changing the boundary conditions with $h\,\underset{g}{\square} \rightarrow h^d g^{-b}\,\underset{h^{-c}g^a}{\square}$ as usual [3,20].

The absence of the explicit phase factor for g_0 in (4.8) is compensated for by the contribution from the representation space T. As explained in section 3, the induced action θ_T of the twisting automorphism θ_0 on T is given by ω^{-1} corresponding to the vacuum energy $1/n$ in (2.18). Thus under the modular transformation T : $\tau \rightarrow \tau+1$ we find $h\,\underset{g}{\square} \rightarrow hg^{-1}\,\underset{g}{\square}$ where $\mathrm{Tr}(h_T\theta_T^{-1}) = \omega\mathrm{Tr}(h_T)$ provides the appropriate phase. Therefore the total contribution arising from the g twisted sector is $h\mathcal{P}_g\,\underset{g}{\square}(\tau) = \frac{1}{n}(1+T+\ldots+T^{n-1})\,h\,\underset{g}{\square}(\tau)$.

The remaining twisted sector traces $h\mathcal{P}_g\,\underset{g^k}{\square}$ can similarly be calculated for $(n,k)=1$ giving exactly the result of (4.7) with g replaced by g^k. For $(n,k)\neq 1$ the same Siegel function contributions arise. However, the representation space T is different and there may also be additional contributions from highest weight states labelled by the elements of $\mathcal{P}_{g^k}\Lambda$ [16,9]. We will therefore consider the simplest situation with $n=p$, prime, for which each trace can be given. We may then provide the explicit formula for T_h with $(\hat{h}, h_T) \in C_p \equiv p_+^{1+2d}(G_p)$ as follows

$$T_h(\tau) = h\mathcal{P}_g\,\underset{1}{\square} + \frac{1}{p}(1+T+\ldots+T^{p-1})\sum_{k=1}^{p-1} h\,\underset{g^k}{\square} \tag{4.10}$$

where $h\mathcal{P}_g\,\underset{1}{\square}$ and $h\,\underset{g^k}{\square}$ are given by (4.3) and (4.7) respectively with $n=p$. If $\mathcal{H}_{\mathrm{orb}} \equiv V^\natural$, as is strongly suggested by the evidence given in section 3, then (4.10) provides a new way of explicitly computing the Thompson series for elements of the Monster subgroup C_p.

The simplest example of such a Thompson series is that for the element $i_p \in C_p$ under which each state of $\mathcal{H}_{g^k}$ has eigenvalue ω^k. We then find that

$$
\begin{aligned}
T_{i_p}(\tau) =&\frac{1}{p}\{\frac{\Theta_\Lambda(\tau)}{\eta^{24}(\tau)}+(p-1)[\frac{\eta(\tau)}{\eta(p\tau)}]^{2d}\\
&+(\omega+...+\omega^{p-1})p^d(1+T+...+T^{p-1})[\frac{\eta(\tau)}{\eta(\tau/p)}]^{2d}\} \qquad (4.11)\\
=&[\frac{\eta(\tau)}{\eta(p\tau)}]^{2d}+2d\equiv \eta_g^{-1}(\tau)+2d
\end{aligned}
$$

using the identity $(1+S+TS+...+T^{p-1}S)[\eta(\tau)/\eta(p\tau)]^{2d}=J(\tau)-2d$ which follows from $T_1(\tau)=J(\tau)$. We therefore find, as was claimed earlier, that i_p has the same Thompson series as the elements of class $p-$ in M [10] where the constant $2d$ ensures that the massless contribution is zero as given in (4.1).

We may also check that this particular property generalises for any g in Table 1 where

$$
T_{i_n}(\tau)=\eta_g^{-1}(\tau)-g_1 \qquad (4.12)
$$

with g_1 the parameter of (2.6) corresponding to $k=1$ so that T_{i_n} obeys (4.1). Thus i_n has precisely the same Thompson series as $n+e_1,e_2,... \in M$ in the notation of ref.[10]. In fact all elements $n+e_1,e_2,... \in M$ with $e_i \neq n$ have a Thompson series of this form corresponding to $g=n+e_1,e_2,...$ in Table 1.

To prove (4.12) we will make use of the 'hauptmodul' property of the modular function $\eta_g^{-1}(\tau)$ [10,9]. This is an example of the basic Monstrous Moonshine property for Thompson series. The hauptmodul property states that $g\,\underset{1}{\square}=\eta_g^{-1}(\tau)=1/q+g_1+...$ is the unique (up to an additive constant) meromorphic function with a simple pole at $q=0$ which is invariant under the modular group $\Gamma_g=\Gamma_0(n)+e_1,e_2,...$. This uniqueness is equivalent to the statement that the compactification of the fundamental region H/Γ_g is the Riemann sphere of genus zero where H is the upper half plane, the domain of τ. Such a unique modular invariant function is referred to as a hauptmodul. Thus the basic function $J(\tau)$ of (2.5) is the hauptmodul for the full modular group.

Let us now quote from the analysis of ref.[9] where we showed that $g\,\underset{1}{\square}$ is a hauptmodul for $\Gamma_0(n)+e_1,e_2,...$ if and only if :

(i) All g^k twisted sectors have non-negative vacuum energy unless $k=cf$ where $f=n/e$ and $(c,e)=1$ where $e\in\{e_1,e_2,...\}$ in which case there is a unique tachyonic vacuum state with energy $-1/e$.

(ii) If the sectors twisted by g^{f_1} and g^{f_2} are tachyonic (with energies $-1/e_1,-1/e_2$) then the g^{f_3} sector is tachyonic with energy $-1/e_3$ where $e_3=e_1e_2/(e_1,e_2)^2$.

The condition (i) ensures that $g\,\underset{1}{\square}$ has the correct singularity structure whereas condition (ii) ensures closure under the composition of two Atkin-Lehner transformations (3.2b). We will now apply these conditions to the present case where

$g\,\underset{1}{\square} = \eta_g^{-1}$ to show that T_{i_n} is also a hauptmodul for $\Gamma_0(n)+e_1,e_2,\ldots$ and hence (4.12) follows.

We begin by showing that $T_{i_n}(\tau)$ is $\Gamma_0(n)$ invariant. (From ref.[9] we expect this to be the case anyway, once T_{i_n} is itself identified as a contribution to the partition function for a new orbifold created from $\mathcal{H}_{\mathrm{orb}}$ by means of the automorphism i_n). Let us express T_{i_n} in the following form

$$T_{i_n} = \sum_{k=1}^{n} \omega^k\, \mathcal{P}_g \underset{g^k}{\square} = \sum_k{}' \Big(\sum_{(k',n)=(k,n)} \omega^{k'} \Big) \mathcal{P}_g \underset{g^k}{\square} \tag{4.13}$$

where $\sum_k'$ denotes a sum over the distinct elements of $\{(j,n), j=1,\ldots n\}$. We have also used the isomorphism of $\mathcal{H}_{g^k}$ and $\mathcal{H}_{g^{k'}}$ for any k' with $(k',n)=(k,n)$ to give a representative trace for each k. We can next consider the action of $\gamma = \begin{pmatrix} a & b \\ cn & d \end{pmatrix} \in \Gamma_0(n)$ on (4.13). Under γ each trace contribution $g^l\,\underset{g^k}{\square}$ becomes $g^{dl-bk}\,\underset{g^{ak}}{\square}$. However, since det $\gamma = 1$ we have $(a,n)=1$ and so $(ak,n)=(k,n)$. Therefore the representative traces in (4.13) are individually preserved by γ and hence T_{i_n} is $\Gamma_0(n)$ invariant.

Let us next demonstrate that T_{i_n} is also W_e invariant for $e \in \{e_1,e_2,\ldots\}$. Employing the decomposition of $W_e = \begin{pmatrix} ae & b \\ cn & de \end{pmatrix} = \begin{pmatrix} a & b \\ cf & de \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$ with $n=ef$, we find the following action for W_e [9]

$$W_e : g^l\,\underset{g^k}{\square}(\tau) \to g^{del-bk}\,\underset{g^{-cfl+ak}}{\square}(e\tau) \tag{4.14}$$

Thus for $l=1, k=0$, invariance under W_e gives the unique twisted sector tachyonic vacuum energy $-1/e$ for the g^{cf} sector, where det $W_e = e$ implies that $(c,e)=1$. This is the origin of condition (i) above. Let us choose, for simplicity, the representative W_e with $c=d=1$. This we can always do since $e \neq n$ for any of the automorphisms of Table 1 we are considering. The action of W_e on T_{i_n} in (4.13) is then given by

$$T_{i_n}(W_e(\tau)) = \frac{1}{n}\sum_{l,k=1}^{n} \omega^{lf+ke}\, g^l\,\underset{g^k}{\square}(e\tau) \tag{4.15}$$

Let us consider the singularities of this expression due to the tachyonic poles in the sector twisted by g^k with $k=cf$, $(c,e)=1$. The contribution to (4.15) from this sector is

$$\frac{1}{n}\sum_{l=1}^{n} \rho^l\, g^l\,\underset{g^{cf}}{\square}(e\tau) \tag{4.16}$$

where $\rho = \omega^f$ which is the phase appropriate for the order e twisting g^{cf}. The vacuum state for this sector is unique by (i) and therefore the corresponding representation space T is one dimensional (where T is a generalised version of the vector space for $\pi(\hat{L}_g)$, discussed in section 2, where now Λ has an invariant sublattice under g^{cf} [14]). The induced action g_T on T then acts as some phase in $< \rho >$ which we note is consistent with the W_e invariance of $g\,\underset{1}{\square}$. The uniqueness of the vacuum also tells us that $1\,\underset{g^{cf}}{\square}(\tau) = q^{-1/e} + ...$ and by applying $T : \tau \rightarrow \tau + 1$ we obtain $g_T^{cf} = \rho$. Let us now consider c as the unique solution to $cf = -1 \bmod e$ with $(c, e) = 1$ and $0 < c < e$. We therefore find that $g_T = \rho^{-1}$ and $g^l\,\underset{g^{cf}}{\square}(\tau) = \rho^{-l} q^{-1/e} + ...$. The leading behaviour of (4.16) is therefore given by the the singular term $1/q$ because the phases appearing in (4.16) are exactly compensated for. On the other hand, it is easy to see that for any other choice of c, complete cancellation takes place between the phases in the leading term so that (4.16) is not singular at $q = 0$. Likewise, one can show that similar cancellations occur for the other tachyonic contributions including the untwisted sector. Therefore (4.15) has the leading behaviour $T_{i_n}(W_e(\tau)) = 1/q + ...$ and hence $\tau = W_e(\infty)$ is a (cusp) singularity for $T_{i_n}(\tau)$.

We may similarly identify all of the remaining independent cusp singularities of $T_{i_n}(\tau)$ as the set $\{\infty, W_{e_1}(\infty), W_{e_2}(\infty), ...\}$ (up to $\Gamma_0(n)$ transformations). We may next repeat the arguments of ref.[9] to prove (4.12). We form the modular function $\phi(\tau) = T_{i_n}(\tau) - T_{i_n}(W_e(\tau))$ which is invariant under $\Gamma_0(n)$ since $W_e\Gamma_0(n) = \Gamma_0(n)W_e$. From our discussion above, $\phi(\tau)$ is non-singular at $\tau = \infty$. However by condition (ii) above we see that $\phi(\tau)$ is also non-singular at $\tau = W_{e'}(\infty)$ for any $e' \in \{e_1, e_2, ...\}$. Thus $\phi(\tau)$ is a holomorphic function on the Riemann surface given by the compactification of the fundamental region $H/\Gamma_0(n)$ and therefore ϕ must be constant (since all holomorphic functions on a compact Riemann surface are constant). But from the definition of ϕ we have $W_e\phi = -\phi$ which implies that $\phi = 0$ and so T_{i_n} is W_e invariant. Therefore applying this argument for all $e \in \{e_1, e_2, ...\}$ we find that $T_{i_n}(\tau)$ is invariant under $\Gamma_g = \Gamma_0(n) + e_1, e_2, ...$ and has a unique simple pole at $q = 0$ ($\tau = \infty$) on the fundamental region H/Γ_g and is therefore a hauptmodul for this group. Therefore T_{i_n} is equal to η_g^{-1} up to a constant which is fixed by (4.1) to give the result (4.12).

5. Concluding remarks

Let us now summarise our results and consider some of the questions that still remain open. We have considered an element g of a special list of 38 Leech lattice automorphisms and have used g to construct an orbifold theory with partition function $J(\tau)$. FLM have conjectured that the bosonic theory with this partition function is unique and so $\mathcal{H}_{\mathbf{orb}} \equiv V^\natural$ with $M_n \equiv M$ where the automorphism group of $\mathcal{H}_{\mathrm{orb}}$ is M_n.

For g of prime order p we have shown that the centraliser group C_p of i_p in M_p is isomorphic to the centraliser $p_+^{1+2d}(G_p)$ of $p-$ in M. In addition, we have shown that the Thompson series for i_p in $\mathcal{H}_{\mathrm{orb}}$ is precisely that of $p-$ in $V^\natural$. These results support the conjecture that $\mathcal{H}_{\mathrm{orb}} \equiv V^\natural$ and $M_p \equiv M$, at least in these cases. To complete the proof of this, it is necessary to identify another set of automorphisms Σ_p in M_p which mix the various twisted and untwisted sectors and then to show that $M =< C_p, \Sigma_p >$. Σ_p generalises the triality symmetry involution σ in the original FLM formulation. The origin of this triality symmetry lies in the relationship between the construction of both the Leech lattice and the A_1^{24} lattice from the Golay code [1,2,22]. The actual method of construction of Σ_p, which relies on some generalisation of this triality symmetry, remains an open question.

For g of non-prime order n we have only found the automorphism group C_n for the Hilbert space $\mathcal{H}_0^{(0)} \oplus \mathcal{H}_g^{(0)}$ where no mixing between sectors occurs. The full centraliser $C(i_n)$ of i_n has not been calculated because of the complications due to non-isomorphic twisted sectors. We have however shown that the Thompson series of i_n is precisely the Thompson series for $n + e_1, e_2, ... \in M$. We likewise propose that there exists some generalisation of the triality symmetry, Σ_n, which mixes the various sectors. We then expect that $M =< C(i_n), \Sigma_n >$ in general.

It is interesting to note that we may perform a further orbifolding on $\mathcal{H}_{\mathrm{orb}}$ with the automorphism i_n which then returns us to the original untwisted Leech theory. Given that $\mathcal{H}_{\mathrm{orb}} \equiv V^\natural$, we have therefore shown that orbifolding $V^\natural$ with $n + e_1, e_2, ... \in M$, where $e_i \neq n$, results in the original untwisted Leech theory $\mathcal{H}_0$. This proves part of the conjecture stated in ref.[9] wherein we suggested that orbifolding $V^\natural$ with respect to elements of M either reproduces $V^\natural$ or returns us to $\mathcal{H}_0$. Given that Table 1 provides an exhaustive list, we therefore expect that orbifolding $V^\natural$ with the remaining elements of M of the form $n + e_1, e_2, ...$ with $e_i = n$ for some e_i, will reproduce $V^\natural$ again. These matters will be expanded upon elsewhere [26].

In conclusion, let us point out that we have only provided a complete list of orbifolds with partition function $J(\tau)$ based on a cyclic automorphism subgroup $< g >$. We can certainly expect that there exists orbifold constructions based on other subgroups (possibly non-abelian) which also share the same partition function $J(\tau)$.

η_g	Modular Group	η_g	Modular Group
$2^{24}/1^{24}$	$2-$	$2.16^2/1^2 8$	$16-$
$3^{12}/1^{12}$	$3-$	$2.3.18^2/1^2 6.9$	$18-$
$4^8/1^8$	$4-$	$9.18/1.2$	$18+2$
$5^6/1^6$	$5-$	$2^3 3^2 18^3/1^3 6^2 9^3$	$18+9$
$2.6^5/1^5 3$	$6-$	$2^2 5^2 20^2/1^2 4^2 10^2$	$20+4$
$3^4 6^4/1^4 2^4$	$6+2$	$7.21/1.3$	$21+3$
$2^6 6^6/1^6 3^6$	$6+3$	$2^2 22^2/1^2 11^2$	$22+11$
$7^4/1^4$	$7-$	$2.3^2 4.24^2/1^2 6.8^2 12$	$24+8$
$2^2 8^4/1^4 4^2$	$8-$	$4.28/1.7$	$28+7$
$9^3/1^3$	$9-$	$2^2 3.5.30^2/1^2 6.10.15^2$	$30+15$
$2.10^3/1^3 5$	$10-$	$2^3 3^3 5^3 30^3/1^3 6^3 10^3 15^3$	$30+6,10,15$
$5^2 10^2/1^2 2^2$	$10+2$	$2.6.10.30/1.3.5.15$	$30+3,5,15$
$2^4 10^4/1^4 5^4$	$10+5$	$3.33/1.11$	$33+11$
$2^2 3.12^3/1^3 4.6^2$	$12-$	$2.9.36/1.4.18$	$36+4$
$4^2 12^2/1^2 3^2$	$12+3$	$2^2 3^2 7^2 42^2/1^2 6^2 14^2 21^2$	$42+6,14,21$
$2^4 3^4 12^4/1^4 4^4 6^4$	$12+4$	$2.46/1.23$	$46+23$
$13^2/1^2$	$13-$	$3.4.5.60/1.12.15.20$	$60+12,15,20$
$2^3 14^3/1^3 7^3$	$14+7$	$2.5.7.70/1.10.14.35$	$70+10,14,35$
$3^2 15^2/1^2 5^2$	$15+5$	$2.3.13.78/1.6.26.39$	$78+6,26,39$

Table 1

A list of the 38 Leech lattice automorphisms that obey the constraints (i)-(iii).

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