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One-Dimensional Hard-Rod Caricature of Hydrodynamics: Navier-Stokes Correction

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Abstract . One-dimensional system of hard-rod particles of length a is studied in the hydrodynamical limit. The Navier-Stokes correction to Euler's equation is found for an initial locally-equilibrium family of states of constant density $\bar{\rho} \in [0, a^{-1})$. The correction is given, at $t \sim 0$, by the non-linear second-order differential operator

$$(Bf)(q,v) = \frac{a^2}{2}\frac{\partial}{\partial q}\Big[\int dw\,|v-w|\,f(q,w)\,\frac{\partial}{\partial q}\,f(q,v) -$$

$$- f(q,v)\int dw\,|v-w|\,\frac{\partial}{\partial q}\,f(q,w)\,\Big]\,(1-\bar{\rho}a)^{-1}$$

where $f(q,v)$ is the (hydrodynamical) density at a point $q \in \mathbf{R}^1$ of the species of particles with velocity $v \in \mathbf{R}^1$.

Key Words: hard-rod time-dynamics, hydrodynamical limit, Euler's equation, locally-equilibrium family of states, Navier-Stokes correction

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1. Introduction

One-dimensional hard-rod time-dynamics corresponds to a simple law of motion: particles move on the real line $\mathbf{R}^1$ freely between the epochs of collisions (which are happen when the distance between particles becomes equal to a , the length of a hard-rod) and at the time of collision they exchange their velocities (we suppose that the mass of every particle is equal to one). Such a simple character of motion makes the hard-rod model exactly solvable (in a certain sense) and that is why this model attracts a lot of attention.

Of a particular interest is the case of an infinite hard-rod system: this is a good model to check various conceptions of non-equilibrium Statistical Mechanics. Among a vast literature devoted to this model we mention here the following rigorous papers which are related to the theme of the present work: (i) the papers of Sinai (1972) [12] and of Aizenman, Goldstein and Lebowitz (1975) [1] where ergodic properties of the hard-rod dynamical system with an equilibrium measure have been established, (ii) the paper of Boldrighini, Dobrushin and Suhov (1979) [2] where the problem of convergence to an equilibrium state for hard rods has been studied in a general set-up, (iii) the paper of Boldrighini, Dobrushin and Suhov (1983) [4] where the hydrodynamical limit for hard-rods was performed and a non-linear "Euler's equation" has been derived (earlier, this equation appeared in a paper of Percus (1969) [11]), and finally, (iv) the papers of Spohn (1982 b) [14] and of Boldrighini and Wick (1988) [5] and (1990) [6] where equilibrium fluctuations around Euler's regime were studied (a starting point here was an earlier result of Lebowitz, Percus and Sykes (1968) [9]).

This paper is devoted to studying hydrodynamical limit for the hard-rod system. The general problem is to derive equations oh hydrodynamics (of primary interest are the Euler and Navier-Stokes equations) from Hamiltonian equations of motion, and the hydrodynamical limit is a kind of procedure to establish a connection between the "macroscopic" description of a system (hydrodynamics) and its "microscopic" description (motion of particles). In a general situation, the hydrodynamical equations are written for densities of "canonical" first integrals of the motion: mass, momentum (or velocity) and energy (sometimes angular momentum is considered, too). For details we refer the reader to a review paper by Dobrushin, Sinai and Suhov (1985) [8].

The hard-rod model is degenerate in the sense that the portion of particles with a prescribed velocity is preserved in time, and hence, hydrodynamical equations for this model should connect densities of various species of a "fluid" labeled by the value of velocity $v \in \mathbf{R}^1$. The hydrodynamical limit is related to scaling both the (microscopic) time and length by multiplying them by the factor ϵ where $\epsilon \to 0$. We arrive therefore to the idea to consider a family $\{ P_\epsilon \}$ of initial states (probability measures on the phase space of the infinite hard-rod system, or, in the probabilistic terminology, of random marked point processes with marks from $\mathbf{R}^1$) such that a structure of a state P_ϵ is changed "very little" on distances $\sim o(\epsilon^{-1})$, but such a change becomes noticeable on distances $\sim O(\epsilon^{-1})$ or more.

The quantity under investigation is the moment function $\rho_{T_{\epsilon^{-1}t}P_\epsilon}(\epsilon^{-1}q, v)$ (see below) giving the density of particles with velocity v at a (micro-) point $\epsilon^{-1}q$ and at a (micro-) time $\epsilon^{-1}t$ (more precisely, in the state $T_{\epsilon^{-1}t}P_\epsilon$ obtained from P_ϵ in the course of hard-rod dynamics up to time $\epsilon^{-1}t$). The result of Boldrighini, Dobrushin and Suhov (1983) [4] is that under certain general conditions on initial states P_ϵ there exists the (weak) limit

$$f_t(q,v) \;=\; \lim_{\epsilon \to 0} \rho_{T_{\epsilon^{-1}t}P_\epsilon}(\epsilon^{-1}q, v) \tag{1.1}$$

which gives a (unique) solution of the equation

$$\frac{\partial}{\partial t} f_t(q,v) \;=\; (\, A\, f_t\,)(q,v) \tag{1.2}$$

where A is a non-linear first-order differential operator

$$(A\, f_t)\,(q,v) \;=\; -\,\frac{\partial}{\partial q}\,[\, f_t(q,v)\,(v + \,a \int_{\mathbf{R}^1} dw\,(v-w)\,f_t(q,w)\,(1 - a \int_{\mathbf{R}^1} d\bar{w}\; f_t(q,\bar{w}))^{-1})\,]\,. \tag{1.3}$$

Equation (1.2), (1.3) is interpreted as Euler's equation for hard-rod fluid.

Relation (1.1) may be strengthened: for any $t > 0$

$$\lim_{\epsilon \to 0} \sup_{|s| \le \epsilon^{-1}t} \;|\, f_{\epsilon s}(q,v) \;-\; \rho_{T_s P_\epsilon}(\epsilon^{-1}q, v)\;| = 0\,. \tag{1.4}$$

For a couple of years the problem of derivation of a Navier-Stokes equation for hard-rod fluid is actively discussed. A popular approach is to treat a Navier-Stoles equation as giving a "correction" to Euler's equation, either up to the terms of order ϵ in the RHS of (1.1), or up to convergence for (macro-) times $t \sim \epsilon^{-1}$ in (1.4) (it is not clear, whether the first version will be appropriate for the second purpose, or vice versa). Some controversial variants of a possible Navier-Stokes equation were proposed by Boldrighini, Dobrushin and Suhov (1980) [3], by Spohn (1982 a) [13] and in Dobrushin (1989) [7]. Those equations were written, basing on different assumptions (explicit or implicit) on an initial family $\{ P_\epsilon \}$, which reflected various features of "local equilibrium", a notion which is widely exploited in the physical literature. However, so far there was no rigorous derivation of any of those equations. One of difficulties was to give a reasonable definition of locally-equilibrium initial states P_ϵ (it may be hardly believed that a Navier-Stokes equation can be derived without some kind of locally-equilibrium assumptions).

In this paper we give a derivation of a hard-rod Navier-Stokes equation treated as a correction to Euler's equation (1.2) up to the $O(\epsilon)-$ terms in the RHS of (1.1). The equation is established in a local sense: we prove that for appropriately defined locally-equilibrium initial states P_ϵ there exists (in a weak sense) the quantity

$$\frac{\partial}{\partial t} \lim_{\epsilon \to 0} \epsilon^{-1} [\rho_{T_{\epsilon^{-1}t} P_\epsilon} (\epsilon^{-1} q, v) - f_t(q,v)] |_{t=0} = (B f_0)(q,v). \tag{1.5}$$

Here f_t is the solution of Euler's equation (1.2) and B is a non-linear second-order differential operator

$$(B f_0)(q,v) = \frac{a^2}{2} \frac{\partial}{\partial q} [\int dw \, |v-w| \, f_0(q,w) \, (1-\bar{\rho} a)^{-1} \frac{\partial}{\partial q} f_0(q,v) - \\ - f_0(q,v) \frac{\partial}{\partial q} \int dw \, |v-w| \, f_0(q,w) \, (1-\bar{\rho} a)^{-1}]. \tag{1.6}$$

A value $\bar{\rho} \in [0, a^{-1})$ is the particle density in the initial state P_ϵ (one of the conditions onto the family $\{ P_\epsilon \}$ is that $\bar{\rho}$ is constant).

This result suggests the following version of a "short-time" Navier-Stokes equation which "corrects" Euler's equation for hard-rod fluid with locally-equilibrium states P_ϵ :

$$\frac{\partial}{\partial t} f_t^{(\epsilon)} = A f_t^{(\epsilon)} + \epsilon B f_t^{(\epsilon)}, \tag{1.7}$$

a form predicted by Spohn (1982 a) [13].

Equation (1.7) gives the following expression for the "viscosity" of a hard-rod fluid which is determined as the coefficient in front of $\frac{\partial^2}{\partial q^2} f_0(q,v)$:

$$\kappa(q,v) = \epsilon \frac{a^2}{2} \int dw \, |v-w| \, f_0(q,w) \, (1-\bar{\rho} a)^{-1}. \tag{1.8}$$

In fact, all the variants of a Navier-Stokes equation which we mentioned before give the same viscosity term. Our result in this paper does not mean that (1.7) is the only one possible version of a correction to Euler's equation (1.2). It is clear that the situation depends crucially on what is taken as a local equilibrium at time zero. We suppose to return to this question in one of forthcoming papers.

We should notice that we do not prove in this paper any theorem on existence or uniqueness of a solution to equation (1.7) (such a solution should be related to a non-linear infinity-dimensional diffusion process), although we establish a formula for the limit

$$\lim_{\epsilon \to 0} \epsilon^{-1} [\rho_{T_{\epsilon^{-1}t} P_\epsilon} (\epsilon^{-1} q, v) - f_t(q,v)]$$

for any $t \in \mathbf{R}^1$ which provides some information about this limit. One of the difficulties in dealing with (1.7) is that local equilibrium in our definition is not preserved in time. On the other hand, however, this fact may be a base for explanation for appearence of irreversibility in equation (1.7).

The paper is organized as follows. In Section 2 we introduce basic notions related to an infinite hard-rod particle system. Section 3 deals with Euler's equation for hard-rod fluid, and in Section 4 we formulate and prove our result on the Navier-Stokes correction. Throughout the paper we use mainly the physical terminology for probabilistic notions (state, moment measure, moment function, fugacity, etc), but the paper is self-contained and does not require any use of physical literature. On the other hand, the probabilistic background may be provided, e.g., by the book of Matthes, Kerstan and Mecke (1978) [10].

2. Dynamics of hard-rod particles

The configuration space $\mathcal{X}^{(a)}$ of a system of hard-rod particles of length $a > 0$ is defined as the collection of subsets $X \subset \mathbf{R}^1$ such that (a) $|x - x'| \geq a$ for any pair $x, x' \in X$ of distinct points and (b) $\sum_{\langle x,x' \rangle}{}^{(+)}(|x-x'|-a) = \sum_{\langle x,x' \rangle}{}^{(-)}(|x-x'|-a) = \infty$ where $\langle x, x' \rangle$ denotes a pair of "neighbour" points from X and the sum $\sum^{(+)}$ ($\sum^{(-)}$) is taken over positive (respectively, negative) $x, x' \in X$ (if a point $x \in X$ does not have a neighbour from one side, we make the agreement that the neighbouring point is $\pm\infty$ so that the corresponding sum becomes infinite). The phase space $\mathcal{M}^{(a)}$ of a hard-rod particle system is the collection of subsets $Y \in \mathbf{R}^1 \times \mathbf{R}^1$ such that $|x - x'| \geq a$ for any pair (x, v), $(x', v') \in Y$ with $x \neq x'$ and $v \leq v'$ for any pair (x, v), $(x', v') \in Y$ with $x' = x + a$ and the condition (b) above is fulfilled (meaning that neighbours are considered in the sense of particle positions x). Both $\mathcal{X}^{(a)}$ and $\mathcal{M}^{(a)}$ are equipped with standard vague topologies; the corresponding Borel $\sigma-$ algebras are denoted by $\Im^{(a)}$ and $\Re^{(a)}$, respectively. A configuration state of a hard-rod particle system (briefly, a configuration hard-rod state) is a probability measure (PM) on ($\mathcal{X}^{(a)}, \Im^{(a)}$). A full hard-rod state is a PM on ($\mathcal{M}^{(a)}, \Re^{(a)}$). Given a full state P , we can construct a configuration state Q as the image of P under the projection $\mathcal{M}^{(a)} \to \mathcal{X}^{(a)}$ which is induced by the map $(x, v) \mapsto x$ (we shall call the state Q the configuration projection of a full state P). Conversely, given a configuration state Q , we can build up a full state P by indicating a family of conditional PM's which we denote as $P(\,\cdot \mid X\,)$, $X \in \mathcal{X}^{(a)}$; formally, one deals with the conditional distribution wrt the $\sigma-$ subalgebra $\Re^{(a)}[\mathcal{X}^{(a)}] \subset \Re^{(a)}$ which is generated by the foregoing projection and is isomorphic to $\Im^{(a)}$. Physically speaking, a PM $P(\,\cdot \mid X\,)$ describes the conditional distribution of velocities of particles provided that their positions form a configuration X . Besides hard-rods, we shall deal with point particles. The configuration space $\mathcal{X}$ (phase space $\mathcal{M}$) of a system of point particles is defined as the collection of non-negative integer-valued Borel measures on $\mathbf{R}^1$ (respectively, on $\mathbf{R}^1 \times \mathbf{R}^1$) which are $\sigma-$ finite (respectively, are projected to $\sigma-$ finite measures by the map $(x, v) \mapsto x$). As before, both spaces $\mathcal{X}$ and $\mathcal{M}$ are provided with vague topologies and we denote their Borel $\sigma-$algebras by $\Im$ and $\Re$, respectively. A configuration state Q and a full state P of a point-particle system are PM's on $(\mathcal{X}, \Im)$ and $(\mathcal{M}, \Re)$, respectively; they are related to each other in the same way as before.

We shall deal mostly with so-called simple states which are supported on the set of integer-valued measures X or Y whish assign a value at most one to any one-point set $\{ x \}$ or $\{ (x, v) \}$. Those measures are identified with their supports which are subsets of $\mathbf{R}^1$ (respectively, of $\mathbf{R}^1 \times \mathbf{R}^1$) with a finite number of points in any compact $C \subset \mathbf{R}^1$ (respectively, in any set $C \times \mathbf{R}^1$ where $C \subset \mathbf{R}^1$ is a compact). It will be convenient for us to use both measure-theoretical and set-theoretical notions and notations: $\int X(dx)$, $y \in X$, etc. For example, the spaces $\mathcal{X}^{(a)}$ and $\mathcal{M}^{(a)}$ may be regarded as Borel subsets of $\mathcal{X}$ and $\mathcal{M}$, respectively, and hard-rod states may be treated merely as pont-particle states concentrated on those subsets.

To write down formulas for hard-rod dynamics we need to define dilation and contraction transformations acting on $\mathcal{X}$ and $\mathcal{M}$. For definiteness, we consider the case of the phase space $\mathcal{M}$; to pass to the configuration space $\mathcal{X}$ it is enough to use the projection. Given $Y \in \mathcal{M}$ and $(x, v) \in Y$, we denote by $\mathbf{D}_{(x,v)}\, Y$ the element of $\mathcal{M}^{(a)}$ obtained as follows. Let us label points $(\tilde{x}, \tilde{v}) \in Y$ by integers $n \in \mathbf{Z}^1$, according to the lexicographic order on $\mathbf{R}^1 \times \mathbf{R}^1$, and giving the number zero to (x, v) . Then $\mathbf{D}_{(x,v)}\, Y$ is formed by the points $(x_n + na, v_n)$, $n \in \mathbf{Z}^1$. Conversely, given $Y \in \mathcal{M}^{(a)}$ and $(x, v) \in Y$, we denote by $\mathbf{C}_{(x,v)}\, Y$ the element of $\mathcal{M}$ which is formed by the points $(x_n - na, v_n)$, $n \in \mathbf{Z}^1$, with the same rule of labeling points $(\tilde{x}, \tilde{v}) \in Y$ as before.

The free dynamics of a point (x, v) is defined by $(x, v) \mapsto (x + \tau v, v)$, $\tau \in \mathbf{R}^1$ being the time variable. Given $Y \in \mathcal{M}$ and $(x, v) \in Y$, we denote by $M(x, v; \tau, Y)$ the algebraic number of intersections for the trajectory $\{(x + \tilde{\tau} v, v)\}$ of a point (x, v) during the time between zero and τ :

$$M(x, v; \tau, Y) = \text{Card}\, \{ (\tilde{x}, \tilde{v}) \in Y \ : \tilde{x} > x,\ \tilde{x} + \tau\tilde{v} \leq x + \tau v \} -$$

$$- \text{Card}\, \{ (\tilde{x}, \tilde{v}) \in Y \ : \tilde{x} < x,\ \tilde{x} + \tau\tilde{v} \geq x + \tau v \}. \tag{2.1}$$

Given $Y \in \mathcal{M}^{(a)}$, we now define the hard-rod dynamics by

$$T_\tau\, Y = \{ (x + \tau v + aM(x, v; \tau, \mathbf{C}_{(x,v)}\, Y), v) \ : \ (x, v) \in Y \}, \quad \tau \in \mathbf{R}^1. \tag{2.2}$$

Our aim is to study the evolution of a (full) hard-rod state P under the hard-rod dynamics:

$$T_\tau P = P(\overset{H}{T}_{-\tau}\,\cdot\,), \quad \tau \in \mathbf{R}^1. \tag{2.3}$$

The rest of this section is devoted to introducing basic tools to be used in the sequel.

The action of the space-translation group $\{ S_y,\ y \in \mathbf{R}^1 \}$ on each spase introduced is defined in the standard way; a given configuration or full state G is called translation-invariant if $G(S_y A) = G(A)$ for any $y \in \mathbf{R}^1$ and any event A. The moment measure (MM) R_Q (R_P) of a configuration state Q (respectively, of a full state P) is a Borel measure on $\mathbf{R}^1$ (respectively, on $\mathbf{R}^1 \times \mathbf{R}^1$) given by

$$R_Q(B) = \mathbf{E}_Q\, N_B\,, B \subseteq \mathbf{R}^1 \quad \text{(respectively,} \quad R_P(B) = \mathbf{E}_P\, N_B\,, B \subseteq \mathbf{R}^1 \times \mathbf{R}^1).$$

Here (and below) $\mathbf{E}_G$ denotes the expectation wrt a PM G and

$$N_B(X) = \int X(dx)\, \chi_B\,(x) \quad \text{(respectively,} \quad N_B(Y) = \int Y(dx \times dv)\, \chi_B\,(x,v))\,;$$

$N_B(X)$ (respectively, $N_B(Y)$) is merely the number of points $x \in X$ (respectively, $(x,v) \in Y$) which are in B. The Radon-Nicodym derivative $\rho_Q(x) = \frac{d\,R_Q}{d\,\lambda}\,(x)$, $x \in \mathbf{R}^1$, (respectively, $\rho_P(x,v) = \frac{d\,R_P}{d\,(\lambda\times\nu)}\,(x,v)$, $x, v \in \mathbf{R}^1$, where λ is the Lebesgue measure on $\mathbf{R}^1$ and ν is a $\sigma-$ finite non-negative Borel measure on $\mathbf{R}^1$, is called the moment function (MF) of a configuration state Q (respectively, the $\nu-$ moment function ($\nu-$ MF) of a full state P (in the case where ν is fixed, simply the MF of P). For a translation-invariant configuration state Q the MF $\rho_Q(x)$ is a constant, $\bar{\rho}_Q$, interpreted as the particle density in the state Q. For a translation-invariant full state P the MM is of the form $R_P = \bar{\rho}_P\,(\lambda \times \mu)$ where μ is a PM on $\mathbf{R}^1$; a constant $\bar{\rho}_P$ gives the particle density in the state P.

We shall need the notion of the Palm distribution associated with a configuration state Q or with a full state P. For the sake of brevity, let us restrict ourselves to the case where our state is simple (see above). We shall give the definition for the case of a full state; the reader can easily cover the configuration case by taking the projection $\mathcal{M} \to \mathcal{X}$. Consider a non-negative measure π on $(\mathbf{R}^1 \times \mathbf{R}^1) \times \mathcal{M}$ (more precisely, on the set $\bar{\mathcal{M}} = \{ ((x,v),Y) \,:\, (x,v) \in Y \}$) given by

$$\int \pi((dx \times dv) \times dY)\, g((x,v),Y) = \mathbf{E}_P\, N_{(g)}$$

where

$$N_{(g)}\,(Y) = \int Y\,(dx \times dv)\, g((x,v),Y)$$

and g is a (non-negative) measurable function on $\bar{\mathcal{M}}$. The image of the measure π under the projection $((x,v),Y) \mapsto (x,v)$ is precisely the moment measure R_P . Hence, by Fubini's Theorem,

$$\int \pi((dx \times dv) \times dY)\, g((x,v),Y) = \int R_P(dx \times dv) \int \hat{P}_{(x,v)}\,(dY)\, g((x,v),Y)$$

where $\{ \hat{P}_{(x,v)}\,,\ (x,v) \in \mathbf{R}^1 \times \mathbf{R}^1 \}$ is a family of PM's on $\mathcal{M}$ (more precisely, on $\mathcal{M}_{(x,v)} = \{ Y\,; Y \ni (x,v) \}$) which are defined for R_P- aa $(x,v) \in \mathbf{R}^1 \times \mathbf{R}^1$. This family is called the Palm family of the state P , and a single PM $\hat{P}_{(x,v)}$ is called the Palm distribution (state) associated with P at a point (x,v).

We shall not give here the definitions of a configuration Poissonian state and of full Possonian state with IID velocities: they are well-known and of a common use. We notice that a configuration Poissonian state is determined by its fugacity (or particle density) $\bar{\rho}_0$ and a full Poissonian state by fugacity $\bar{\rho}_0$ and velocity distribution μ . We notice also that the Palm state $\hat{P}_{(x,v)}$ associated with a full Poissonian state P at a point (x,v) is defined by

$$\hat{P}_{(x,v)}(A) = P(A^{(x,v)}), \quad A \subseteq \mathcal{M}_{(x,v)}\,,$$

where $A^{(x,v)}$ is the image of the event A under the map $\mathcal{M}_{(x,v)} \to \mathcal{M}$ given by $Y \in \mathcal{M}_{(x,v)} \mapsto Y \setminus \{(x,v)\}$. The similar assertion is valid for configuration Poissonian states, too.

A configuration state Q is called a hard-rod Gibbs state with fugacity $\bar{\rho}_0$ if its Palm state $\hat{Q}_x$ is the image, under the dilation $\mathbf{D}_x$, of the configuration Poissonian state $\hat{Q}^0_x$ associated with the configuration Poissonian state Q^0 of fugacity $\bar{\rho}_0$:

$$\hat{Q}_x = \mathbf{D}_x \hat{Q}^0_x, \quad \text{or} \quad \hat{Q}^0_x = \mathbf{C}_x \hat{Q}_x, \quad x \in \mathbf{R}^1. \tag{2.4}$$

This condition determines the state Q uniquely; Q is translation-invariant and has the density $\bar{\rho} = \bar{\rho}_0(1-\rho_0 a)^{-1} \in [0, a^{-1})$ (we can call the state Q a hard-rod Gibbs state with particle density $\bar{\rho}$).

A full state P is called an equilibrium hard-rod state with fugacity $\bar{\rho}_0$ (or particle density $\bar{\rho} = \bar{\rho}_0(1+\bar{\rho}_0 a)^{-1}$) and velocity distribution μ if its Palm state $\hat{P}_{(x,v)}$ is the image, under the dilation $\mathbf{D}_{(x,v)}$, of the Palm state $\hat{P}^0_{(x,v)}$ associated with the full Poissonian state P^0 of fugacity $\bar{\rho}_0$ and velocity distribution μ :

$$\hat{P}_x = \mathbf{D}_{(x,v)} \hat{P}^0_{(x,)}, \quad \text{or} \quad \hat{P}^0_{(x,v)} = \mathbf{C}_{(x,v)} \hat{P}_{(x,v)}, \quad (x,v) \in \mathbf{R}^1 \times \mathbf{R}^1. \tag{2.5}$$

The equivalent condition is that the configuration projection of P is the hard-rod Gibbs state and the conditional PM $P(\cdot \mid X)$ corresponds to IID velocities with the marginal distribution μ.

A similar construction relating translation-invariant hard-rod and point-particle states may be performed in a general situation as well. The point-particle state is called contracted and the hard-rod state dilated; their densities $\bar{\rho}_0$ and $\bar{\rho}$ are connected, as before, by the pair of equalities $\bar{\rho}_0 = \bar{\rho}(1-\bar{\rho}a)^{-1}$, $\bar{\rho} = \bar{\rho}_0(1+\bar{\rho}a)^{-1}$.

We conclude this section with a summary of some known results which are related to hard-rod dynamics and provide an insight into concepts and arguments which follow. For details and proofs see Sinai (1972) [12], Aizenman, Goldstein and Lebowitz (1975) [1] and Boldrighini, Dobrushin and Suhov (1979) [2].

Proposition 2.1. *(i) An equilibrium hard-rod state is invariant under hard-rod dynamics:*

$$T_\tau P \equiv P, \quad \tau \in \mathbf{R}^1.$$

(ii) Let P be an equilibrium hard-rod state with velocity distribution μ such that $\mathbf{E}_\mu |v| < \infty$ and μ has no atom at $v_0 = \mathbf{E}_\mu v$. Then $(\mathcal{M}, T_\tau, P)$ is a $K-$ system.

(iii) Let P be as before, and $\mu([v_0-\delta, v_0+\delta]) = 0$ for some $\delta > 0$. Then $(\mathcal{M}, T_\tau, P)$ is a Bernoulli system.

(iv) Let G be a translation-invariant hard-rod state with the MM $R_G = \bar{\rho}\,(\lambda \times \mu)$ where $\mathbf{E}_\mu |v| < \infty$, and G^0 be the corresponding contracted state. Let P denote the equilibrium hard-rod state with density $\bar{\rho}$ and velocity distribution μ and P^0 be the corresponding Poissonian state. Then the state $T_\tau G$ weakly converges, as $\tau \to \pm\infty$, to P iff the state $T^0_\tau G^0$ obtained from G^0 in the course of free dynamics converges to P^0.

General conditions ensuring convergence of the state $T^0_\tau F$ to a Poissonian state are formulated in the paper of Boldrighini, Dobrushin and Suhov (1979) [2]. The main condition is space-mixing:

$$\lim_{s\to\infty} \sup_{y \in \mathbf{R}^1} \sup \mid F(A_1 \cup A_2) - F(A_1)\, F(A_2) \mid = 0;$$

the internal supremum is taken here over the pairs of events $A_1 \in \Re((-\infty, y))$, $A_2 \in \Re((y+s, \infty))$, where $\Re(I)$, $I \subseteq \mathbf{R}^1$, denotes the $\sigma-$ subalgebra of $\Re$ generated by the RV's N_B with $B \subseteq I \times \mathbf{R}^1$.

3. Dynamics of hard-rod fluid: Euler's equation

Henceforth, we shall use the term "state" for a full state of our system (either of hard rods, or point particles). Let us suppose that a family of initial hard-rod states P_ϵ , $\epsilon > 0$, be given which satisfies the following conditions I and II.

I. The MM R_{P_ϵ} is absolutely continuous wrt a measure $\lambda \times \mu$ where μ is a $\sigma-$ finite measure on $\mathbf{R}^1$ with $\mu(C) \leq c\lambda(E(C))$ for any compact $C \subset \mathbf{R}^1$ where $E(C) = C \cup \{x \in C^c : \operatorname{dist}(x, C) \leq 1\}$, and $c > 0$ is a constant and the $\mu-$ MF $\rho_{P_\epsilon}(x, v)$ satisfies the bound

$$\rho_{P_\epsilon}(x, v) \leq \phi(v) , \quad x, v \in \mathbf{R}^1 , \tag{3.1}$$

where

$$\int \mu(dv)\, \phi(v) < a^{-1} , \tag{3.2}$$

and

$$\int \mu(dv)\, \phi(v)|v| < \infty , \tag{3.3}$$

and the following relation holds

$$\lim_{\epsilon \to 0} \rho_{P_\epsilon}(\epsilon^{-1} q, v) = f(q, v) , \quad x, v \in \mathbf{R}^1 , \tag{3.4}$$

where f is a function of class C^1 for which the bound (3.1) is fulfilled.

To formulate our condition II, we need some definitions. Given q, v and t , we set

$$r(q, v; t, f) = \int_{-\infty}^{v} \mu(dw) \int_0^{t(v-w)} ds\, f_q^0(q + s, w; f) -$$

$$- \int_v^{\infty} \mu(dw) \int_{t(v-w)}^{0} ds\, f_q^0(q + s, w; f) . \tag{3.5}$$

Here, a function $f_q^0(\cdot\, , \cdot\, ; f)$ is the image of f under a continuous analog of contraction $\mathbf{C}_{(\cdot\, , \cdot)}$:

$$f_q^0(q + s, w; f) = f(q + s^*, w)\, (1 - a \int_{\mathbf{R}^1} \mu(d\bar{w})\, f(q + s^*, \bar{w}))^{-1} \tag{3.6}$$

where $s^* = s^*(q, s; f)$ is determined from the equations

$$s^* - a \int_q^{q+s^*} d\bar{s} \int_{\mathbf{R}^1} \mu(d\bar{w})\, f(\bar{s}, \bar{w}) = s , \quad \text{for } s > 0 , \tag{3.7a}$$

and

$$s^* + a \int_{q+s^*}^{q} d\bar{s} \int_{\mathbf{R}^1} \mu(d\bar{w})\, f(\bar{s}, \bar{w}) = s , \quad \text{for } s < 0 . \tag{3.7b}$$

We notice, for the further use, that formula (3.5) becomes particularly simple provided that f is of the form

$$f(q, v) = \bar{\rho} h(q, v)$$

where $\bar{\rho} \in [0, a^{-1})$ is a constant and $\int_{\mathbf{R}^1} \mu(dw)\, h(q, w) = 1$ for any $q \in \mathbf{R}^1$. In this case

$$r(q, v; t, f) = \bar{\rho}_0 \left[\int_{-\infty}^{v} \mu(dw) \int_0^{t(v-w)} ds\, h(q + s(1 + \bar{\rho}_0 a), w) - \right.$$

$$\left. - \int_v^{\infty} \mu(dw) \int_{t(v-w)}^{0} ds\, h(q + s(1 + \bar{\rho}_0 a), w) \right] , \tag{3.8}$$

where $\bar{\rho}_0 = \bar{\rho}(1-\rho a)^{-1}$.

Condition II reads now as follows:

II. For any $\delta > 0$, $t \in \mathbf{R}^1$ and any bounded $C \subset \mathbf{R}^1$ the following relation holds uniformly in $(q,v) \in \mathbf{R}^1 \times C$

$$\lim_{\epsilon \to 0} \mathbf{C}_{(\epsilon^{-1}q,v)} \left((\hat{P}_\epsilon)_{(\epsilon^{-1}q,v)} \right) \left(\{ Y : | \epsilon M(\epsilon^{-1}q, v; \epsilon^{-1}t, Y) - \right.$$

$$\left. - r(q,v;t,f) \mid > \delta \} \right) = 0 \tag{3.9}$$

(a law of large numbers for $M(\epsilon^{-1}q, v; \epsilon^{-1}t, \cdot))$. Here $(\hat{P}_\epsilon)_{(\epsilon^{-1}q,v)}$ is the Palm state associated with P_ϵ at the point $(\epsilon^{-1}q, v)$ and $\mathbf{C}_{(\epsilon_{-1}q,v)}((\hat{P}_\epsilon^{-1})_{(\epsilon^{-1}q,v)})$ is its contraction.

Examples of families $\{ P_\epsilon \}$ which obey I and II are given in Sect. 5 of Boldrighini, Dobrushin and Suhov (1983) [4].

The following assertion was proved by Boldrighini, Dobrushin and Suhov (1983) [4]:

Proposition 3.1. *Under the conditions I and II, for any* $t \in \mathbf{R}^1$ *the rescaled MM* $R_t^{(\epsilon)}$ *given by*

$$\int R_t^{(\epsilon)} (dq \times dv)\, g(q,v) = \epsilon \int R_{T_{\epsilon^{-1}t} P_\epsilon} (dx \times dv)\, g(\epsilon x, v) ,$$

converges, in the vague topology, as $\epsilon \to 0$ *, to a measure* $\bar{R}_t$ *which is absolutely continuous wrt* $\lambda \times \mu$ *with the Radon - Nicodym derivative* f_t *which gives a (unique) solution to the equation (1.2) (with the measures* $\mu(dw)$ *and* $\mu(d\bar{w})$ *instead of* dw *and* $d\bar{w}$ *):*

$$\frac{\partial}{\partial t} f_t(q,v) = -v \frac{\partial}{\partial q} f_t(q,v) - \frac{\partial}{\partial q} \Big[\int_{\mathbf{R}^1} \mu(dw)\, (v-w)\, f_t(q,w) \times$$

$$\times \Big(1 - a \int_{\mathbf{R}^1} \mu(d\bar{w})\, f_t(q,\bar{w})\Big)^{-1} f_t(q,v) \Big] , \quad q,v,t \in \mathbf{R}^1 , \tag{3.10}$$

with the initial date

$$f_0(q,v) = f(q,v), \quad q,v \in \mathbf{R}^1 . \tag{3.11}$$

An important formula for the solution of the Cauchy problem (3.10), (3.11) is

$$\int dq\, f_t(q,v)\, g(q,v) = \int dq\, f(q,v)\, g(q + tv + ar(q,v;t,f), v) , \quad t \in \mathbf{R}^1 , \tag{3.12}$$

where g is an arbitrary measurable bounded function $\mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1$ with a compact support and $r(q,v;t,f)$ is given by (3.5).

4. Dynamics of hard-rod fluid: Navier-Stokes' correction

The aim of this section is to establish the first-order correction to equation (1.2) (or (3.10)) in the sense discussed in Introduction. First of all, we need a notion of a locally-equilibrium hard-rod state, or, more precisely, of a locally-equilibrium family of hard-rod states.

Let us fix a Borel measure μ on $\mathbf{R}^1$ which has the properties listed in condition I of the preceding Section. Furthermore, we fix a non-negative function $h = h(q,v)$, $q, v \in \mathbf{R}^1$, which is of class C^4 in q for any given $v \in \mathbf{R}^1$. Suppose that

$$\int_{\mathbf{R}^1} \mu(dv)\, h(q,v) \equiv 1, \quad q \in \mathbf{R}^1, \tag{4.1}$$

and

$$h(q,v), \mid \frac{\partial^i}{\partial q^i} h(q,v) \mid \leq \varphi(v), \quad q, v \in \mathbf{R}^1, \quad i = 1,2,3,4 \tag{4.2}$$

where

$$\int_{\mathbf{R}^1} \mu(dv) |v|^i < \infty, \quad i = 1,2 \tag{4.3}$$

(cf. (3.2), (3.3)). Finally, we choose a value $\bar{\rho} \geq 0$ such that

$$\bar{\rho},\ \bar{\rho} \int_{\mathbf{R}^1} \mu(dv) \varphi(v) < a^{-1}. \tag{4.4}$$

We set then

$$f(q,v) = \bar{\rho}\, h(q,v), \quad q, v \in \mathbf{R}^1. \tag{4.5}$$

Definition. A family of hard-rod states $\{ P_\epsilon,\ \epsilon > 0 \}$ is called *locally – equilibrium* hard-rod family with particle density $\bar{\rho}$ and velocity distributions $\{ h(q,v)\mu(dv),\ q \in \mathbf{R}^1 \}$ if the following two conditions are fulfilled.

A. The configuration projection of the state P_ϵ is the hard-rod DLR state Q with fugacity $\bar{\rho}_0 = \bar{\rho}(1 - \bar{\rho} a)^{-1}$.

B. The conditional PM $P_\epsilon(\ \cdot\ |X)$ corresponds to independent velocities, and the marginal velocity distribution for a particle with position at $x \in X$ is $h(\epsilon x, v)\, \mu(dv)$.

We have to make several remarks related to this definition.

1. The $\mu-$ MF of the state P_ϵ is given by

$$\rho_{P_\epsilon}(x,v) = f(\epsilon x, v), \quad x, v \in \mathbf{R}^1. \tag{4.6}$$

2. The Palm state $(\hat{P}_\epsilon)_{(x,v)}$ associated with P_ϵ at a point (x,v) may be described as the image, under the dilation $\mathbf{D}_{(x,v)}$, of a point particle state $P^0_{\epsilon,(x,v)}$ which is determined by the following two properties.

A$'$. The configuration projecion of $P^0_{\epsilon,(x,v)}$ is the Palm state $\hat{Q}^0_{(x,v)}$ associated with the Poissonian state Q^0 of density $\bar{\rho}_0$ at the point (x,v).

B$'$. The conditional PM $(\hat{P}_\epsilon)_{(x,v)}\ (\ \cdot\ |X)$ corresponds to independent velocities for particles with positions at $\tilde{x} \in X \backslash \{x\}$, and the marginal velocity distribution for a paricle with position at $\tilde{x} \in X \cap (x, \infty)$ is

$$h(\epsilon(\tilde{x} + aN_{(x,\tilde{x})}(X) + a), \tilde{v})\, \mu(d\tilde{v}), \quad \tilde{v} \in \mathbf{R}^1, \tag{4.7a}$$

and for a paricle with position at $\tilde{x} \in X \cap (-\infty, x)$

$$h(\epsilon(\tilde{x} - aN_{(\tilde{x},x)}(X) - a), \tilde{v})\, \mu(d\tilde{v}), \quad \tilde{v} \in \mathbf{R}^1. \tag{4.7b}$$

In other words,

$$\mathbf{C}_{(x,v)}(\, (\hat{P}_\epsilon)_{(x,v)}\,) = P^0_{\epsilon,(x,v)} \tag{4.8}$$

where $P^0_{\epsilon,(x,v)}$ is determined by A$'$ and B$'$. Notice that we do not claim that $\{ P^0_{\epsilon,(x,v)} \}$ is the Palm family of a certain (full) state: this is not true in general.

3. It is possible to prove that the locally-equilibrium hard-rod family $\{ P_\epsilon \}$ satisfies both conditions I and II of the preceding section with $f(q,v)$ given by (4.5). For the proof see Sect. 5 of Boldrighini, Dobrushin and Suhov (1983) [4].

The aim of this section (and of the whole paper) is to prove the following assertion:

Theorem. *Let* $\{ P_\epsilon,\ \epsilon > 0\}$ *be a locally-equilibrium hard-rod family with particle density* $\bar\rho \in [0, a^{-1})$ *and velocity distributions* $\{ h(q,v)\,\mu(dv),\ q \in \mathbf{R}^1 \}$ *where* μ *and* h *satisfy the aforementioned conditions. Then, for any test-function* $g\ :\ \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1$ *of class* C^4 *and with a compact support, the following relation holds:*

$$\lim_{t\to 0} \frac{\partial}{\partial t} \lim_{\epsilon\to 0} \epsilon^{-1} [\int dq\,\mu(dv)\,\rho_{T_{\epsilon^{-1}t}P_\epsilon}(\epsilon^{-1}q,v)\,g(q,v) - \int dq\,\mu(dv)\,f_t(q,v)\,g(q,v)] =$$

$$= \lim_{t\to 0} \frac{1}{t} \lim_{\epsilon\to 0} \epsilon^{-1} [\int dq\,\mu(dv)\,\rho_{T_{\epsilon^{-1}t}P_\epsilon}(\epsilon^{-1}q,v)\,g(q,v) -- \int dq\,\mu(dv)\,f_t(q,v)\,g(q,v)] =$$

$$= \int dq\,\mu(dv)\,(\ Bf\ )\,(q,v)\,g(q,v). \tag{4.9}$$

Here $\rho_{T_{\epsilon^{-1}t}P_\epsilon}$ *is the* $\mu-$ *MF of the time-shifted hard-rod state* $T_{\epsilon^{-1}t}P_\epsilon$, f_t *is the solution of the Cauchy problem (3.5), (3.6) for the hard-rod Euler equation with the initial date* $f(q,v)$ *given by (4.5) and* B *is the non-linear second-order differential operator (cf. (1.5)):*

$$(\ Bf\ )\,(q,v) = \frac{a^2}{2} (\frac{\partial}{\partial q} [\int \mu(dw)\,|v-w|\,f(q,w)\,(1-a\bar\rho)^{-1} \frac{\partial}{\partial q} f(q,v)] -$$

$$- \frac{\partial}{\partial q} [\,f(q,v)\,\frac{\partial}{\partial q} \int \mu(dw)\,|v-w|\,f(q,w)\,(1-a\bar\rho)^{-1}\,]). \tag{4.10}$$

Proof. As follows from Remark 1 (see (4.3)) and from the definitions of the Palm state and of hard-rod dynamics (see (2.2)),

$$\int dq\,\mu(dv)\,\rho_{T_{\epsilon^{-1}t}P_\epsilon}(\epsilon^{-1}q,v)\,g(q,v) =$$

$$= \int dq\,\mu(dv)\,f(q,v)\,\mathbf{E}_{(\hat P_\epsilon)_{(\epsilon^{-1}q,v)}}\, g(q+tv+\epsilon a\,M(\epsilon^{-1}q,v;\epsilon^{-1}t;\ \mathbf{C}_{(\epsilon^{-1}q,v)}\,\cdot\,),v). \tag{4.11}$$

By using Remark 2 (see (4.8)), the RHS of(4.11) may be written as

$$\int dq\,\mu(dv)\,f(q,v)\,\mathbf{E}_{\mathbf{C}_{(\epsilon^{-1}q,v)}((\hat P_\epsilon)_{(\epsilon^{-1}q,v)})}\, g(q+tv+\epsilon a\,M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,),v)\ ; \tag{4.12}$$

this is the basic expression we shall work with.

First of all, we put the expectation into the argument of our function g . To simplify the notations we write henceforth $\mathbf{E}$ instead of $\mathbf{E}_{\mathbf{C}_{(\epsilon^{-1}q,v)}((\hat P_\epsilon)_{(\epsilon^{-1}q,v)})}$. The derivatives g', g'' , etc, (and also h', h'' , etc, below) are understood as the partial derivatives wrt the first argument. Finally, given a RV γ , we denote by $\tilde\gamma$ its centered version: $\tilde\gamma = \gamma - \mathbf{E}\gamma$. Using Taylor's expansion yields

$$\text{RHS of } (4.12) = \int dq\,\mu(dv)\,f(q,v)\,[\,g(q+tv+\epsilon a\,\mathbf{E}\,M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,),v) =$$

$$+\,\epsilon a\,g'(q+tv+\epsilon a\,\mathbf{E}\,M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,),v)\,\mathbf{E}\tilde M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,) +$$

$$+ \frac{(\epsilon a)^2}{2}\,g''(q+tv+\epsilon a\,\mathbf{E}\,M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,),v)\,\mathbf{E}\,(\tilde M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,))^2 +$$

$$+ \frac{(\epsilon a)^3}{3!}\,g'''(q+tv+\epsilon a\,\mathbf{E}\,M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,),v)\,\mathbf{E}\,(\tilde M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,))^3 +$$

$$+ \frac{(\epsilon a)^4}{4!}\,\mathbf{E}\,(g^{\mathrm{IV}}\,(q+tv+\Theta_1,v)\,(\tilde M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,))^4)\,]\,. \tag{4.13}$$

Here Θ_1 is an RV between $\epsilon a\,M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,)$ and the expectation value $\epsilon a\,\mathbf{E}\,M(\epsilon^{-1}q,v;\epsilon^{-1}t;\,\cdot\,)$. The term with g' gives, of course, zero contribution. The terms with g''' and g^{IV} may be neglected as shown by the following Lemma 4.1:

Lemma 4.1. *Under the conditions of the Theorem, for any* $t \in \mathbf{R}^1$

$$\lim_{\epsilon\to 0} \mid \int dq\, \mu(dv)\, h(q,v)\, [\, \frac{\epsilon^2 a^3}{3!}\, g'''(q+tv+\epsilon a\, \mathbf{E}\, M(\epsilon^{-1}q, v; \epsilon^{-1}t;\ \cdot\ ), v)\, \mathbf{E}\, (\tilde{M}(\epsilon^{-1}q, v; \epsilon^{-1}t;\ \cdot\ ))^3 +$$

$$+ \frac{\epsilon^3 a^4}{4!}\, \mathbf{E}\, (\, g^{\mathrm{IV}}(q+tv+\Theta_1, v)\, (\tilde{M}(\epsilon^{-1}q, v; \epsilon^{-1}t;\ \cdot\ ))^4)\,]\, \mid = 0. \tag{4.14}$$

In order not to interrupt the proof of our Theorem, we postpone the proof of Lemma 4.1 until the end of this section (the same is about other Lemmas which follow). The next step is to analyze the two remaining terms in the RHS of (4.13). We start with the equality

$$\epsilon\, \mathbf{E}\, M(\epsilon^{-1}q, v; \epsilon^{-1}t;\ \cdot\ ) = \int_{-\infty}^{v} \mu(dw) \int_0^{t(v-w)} ds\, \bar{\rho}_0\, \mathbf{E}\, h(q+s+\epsilon a\, (N_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))}+1), w) -$$

$$- \int_v^{\infty} \mu(dv) \int_{t(v-w)}^{0} ds\, \bar{\rho}_0\, \mathbf{E}\, h(q+s-\epsilon a\, (N_{(\epsilon^{-1}(q+s), \epsilon^{-1}q)}+1), w) \tag{4.15}$$

which follows immediately from properties A$'$ and B$'$ of the state $\mathbf{C}_{(\epsilon^{-1}q,v)}\ ((\hat{P}_\epsilon)_{(\epsilon^{-1}q,v)})$ and from (2.1). Using the same sort of ideas as before we write for $s > 0$

$$\mathbf{E}\, h(q+s+\epsilon a\, (N_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))}+1), w) = h(q+s+\epsilon a\, (\mathbf{E}\, N_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))}+1), w) +$$

$$+ \epsilon a\, h'(q+s+\epsilon a\, (\mathbf{E}\, N_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))}+1), w)\, \mathbf{E}\, \tilde{N}_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))} +$$

$$+ \frac{(\epsilon a)^2}{2}\, h''(q+s+\epsilon a\, (\mathbf{E}\, N_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))}+1), w)\, \mathbf{E}\, (\tilde{N}_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))})^2 +$$

$$+ \frac{(\epsilon a)^3}{3!}\, h'''(q+s+\epsilon a\, (\mathbf{E}\, N_{(\epsilon^{-1}q, \epsilon^{-1}q(q+s))}+1), w))\, \mathbf{E}\, (\tilde{N}_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))})^3 +$$

$$+ \frac{(\epsilon a)^4}{4!}\, \mathbf{E}\, (h^{\mathrm{IV}}(q+s+\Theta_2^{(+)})\, (\tilde{N}_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))})^4) \tag{4.16}$$

where $\Theta_2^{(+)}$ is an RV between $\epsilon a N_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))}$ and the expectation value $\epsilon a\, \mathbf{E}\, N_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))}$. The similar formula holds for $s < 0$. The terms with h' give, as before, zero contribution. The terms with h''' and h^{IV} will be estimated below and neglected. The non-trivial contribution will come from the remaining two terms.

Using again property A$'$ yields for $s > 0$

$$\mathbf{E}\, N_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))} = \mathbf{E}\, (\tilde{N}_{(\epsilon^{-1}q, \epsilon^{-1}(q+s))})^2 = \epsilon^{-1}\bar{\rho}_0\, s\, ; \tag{4.17}$$

the similar equalities take place for $s < 0$, too. We can write for $s > 0$

$$h(q+s+\epsilon a(\, \mathbf{E}\, N_{(\epsilon^{-1}q, \epsilon(q+s))}+1), w) = h(q+s(1+\bar{\rho}_0 a), w) +$$

$$+ \epsilon a h'(q+s(1+\bar{\rho}_0 a), w) + \frac{(\epsilon a)^2}{2} h''(q+s(1+\bar{\rho}_0 a)+\vartheta_1^{(+)}, w)\, ; \tag{4.18}$$

again the similar expansion holds for $s < 0$ as well. Here $\vartheta_1^{(+)}$ is an RV between zero and ϵa .

Combining (4.13) – (4.18) and using once more Taylor's expansion, we get

$$\epsilon\, \mathbf{E}\, M(\epsilon^{-1}q, v; \epsilon^{-1}t;\ \cdot\ ) = \int_{-\infty}^{v} \mu(dw) \int_0^{t(v-w)} ds\, \bar{\rho}_0\, [\, h(q+s(1+\bar{\rho}_0 a), w) +$$

$$+ \frac{\epsilon a^2}{2}\, \bar{\rho}_0\, h''(q+s(1+\bar{\rho}_0 a), w)\, s + \epsilon a h'(q+s(1+\bar{\rho}_0 a), w) +$$

$$+ \frac{(\epsilon a)^2}{2} h''(q + s(1+\bar\rho_0 a) + \vartheta_1^{(+)}, w)] -$$

$$- \int_v^\infty \mu(dw) \int_{t(v-w)}^0 ds\ \bar\rho_0\ [\ h(q + s(1+\bar\rho_0 a), w) -$$

$$- \frac{\epsilon a^2}{2} \bar\rho_0 h''(q + s(1+\bar\rho_0 a), w)\ s\ -\ \epsilon a h'(q + s(1+\bar\rho_0 a), w) +$$

$$+ \frac{(\epsilon a)^2}{2} h''(q + s(1+\bar\rho_0 a) + \vartheta_1^{(-)}, w)]\ . \tag{4.19}$$

Taking into account formula (3.8), we rewrite the sum of the zero-order and first-order parts of the RHS of (4.19) as

$$r(q,v;t,f)\ +\ \epsilon a \bar r_1(q,v;t,f)\ +\ \epsilon a r_1(q,v;t,f) \tag{4.20}$$

where

$$r_1(q,v;t,f)\ =\ (\ \int_{-\infty}^v \mu(dw) \int_0^{t(v-w)} ds\ +$$

$$+ \int_v^\infty \mu(dw) \int_{t(v-w)}^0 (ds)\ )\ f(q + s(1+\bar\rho_0 a), w)\ (1-\bar\rho a)^{-1} \tag{4.21a}$$

and

$$\bar r_1(q,v;t,f)\ =\ \frac{a}{2}\ (\ \int_{-\infty}^v \mu(dw) \int_0^{t(v-w)} ds\ +$$

$$+ \int_v^\infty \mu(dw) \int_{t(v-w)}^0 ds\ )\ f''(q + s(1+\bar\rho_0 a), w)\ s\ \bar\rho\ (1-\bar\rho a)^{-2}\ . \tag{4.21b}$$

Now, going back to (4.13), we have

$$g(q + tv + \epsilon a\ \mathbf{E}\ M(\epsilon^{-1}q, v; \epsilon^{-1}t;\ \cdot\ ), v)\ =\ g(q + tv + ar(q,v;t,f), v) +$$

$$+ \epsilon a\ g'(q + tv + ar(q,v;t,f), v)\ [\ r_1(q,v;t,f)\ +\ \bar r_1(q,v;t,f)\ ]\ +\ \epsilon^2\ g_1(q,v;t,\epsilon)\ . \tag{4.22}$$

The contribution of the ϵ^2- term may be neglected: this is the direct corollary of Lemma 4.2:

Lemma 4.2. *Under the conditions of the Theorem, for any* $t \in \mathbf{R}^1$

$$\lim_{\epsilon\to 0}\ |\ \int dq\ \mu(dv)\ h(q,v)\ \epsilon\ g_1(q,v;t,\epsilon)\ |= 0\ . \tag{4.23}$$

The assertion of Lemma 4.2 completes the analysis of the first term in the RHS of (4.13).
We pass now to the $g''-$ term in the RHS of (4.13).

Lemma 4.3. *Under the conditions of the Theorem, the following relation holds true*

$$\lim_{\epsilon\to 0}\ \epsilon\ \mathbf{E}\ (\tilde M(\epsilon^{-1}q, v; \epsilon^{-1}t,\ \cdot\ ))^2\ =\ r_2(q,v;t,f)\ +\ \bar r_2(q,v;t,f) \tag{4.24}$$

where

$$r_2(q,v;t,f)\ =\ (\ \int_{-\infty}^v \mu(dw) \int_0^{t(v-w)} ds\ +$$

$$+ \int_v^\infty \mu(dw) \int_{t(v-w)}^0 ds\ )\ f(q + s(1+\bar\rho_0 a), w)\ (1-\bar\rho a)^{-1}\ , \tag{4.25a}$$

and

$$\bar r_2(q,v;t,f)\ =\ (\int_{-\infty}^0 \mu(dw_1) \int_0^{t(v-w_1)} ds_1\ -\ \int_v^\infty \mu(dw_1) \int_{t(v-w_1)}^0 ds_1\ ) \times$$

$$\times \left(\int_{-\infty}^{v} \mu(dw_2) \int_0^{t(v-w_2)} ds_2 - \int_v^{\infty} \mu(dw_2) \int_{t(v-w_2)}^{0} ds_2\right)\times$$

$$[\, f''(q + s_1(1+\bar\rho_0 a), w_1)\, s_1\, f(q+s_2(1+\bar\rho_0 a), w_2) +$$

$$f(q+s_1(1+\bar\rho_0 a), w_1)\, f''(q+s_2(1+\bar\rho_0), w_2)\, s_2 +$$

$$+ f'(q+s_1(1+\bar\rho_0 a), w_1)\, f'(q+s_2(1+\bar\rho_0 a), w_2) \chi_{\mathbf{R}^1_+}(s_1 \cdot s_2)\,(s_1 \wedge s_2)\,]\,(1-\bar\rho a)^{-2}\,, \qquad (4.25b)$$

where

$$s_1 \wedge s_2 = \min[s_1, s_2], \quad \text{if} \quad s_1,\, s_2 \geq 0\,,$$

$$s_1 \wedge s_2 = \max[s_1, s_2], \quad \text{if} \quad s_1,\, s_2 < 0\,.$$

For the sake of brevity we make the convention to write henceforth

$$\int_{\mathbf{R}^1} \mu(dw) \int_0^{|t(v-w)|} ds \quad \text{instead of} \quad \left(\int_{-\infty}^{0} \mu(dw) \int_0^{t(v-w)} ds + \int_v^{\infty} \mu(dw) \int_{t(v-w)}^{0} ds\right)$$

and

$$\int_{\mathbf{R}^1} \mu(dw) \int_0^{t(v-w)} ds \quad \text{instead of} \quad \left(\int_{-\infty}^{0} \mu(dw) \int_0^{t(v-w)} ds - \int_v^{\infty} \mu(dw) \int_{t(v-w)}^{0} ds\right).$$

From Lemma 4.3 we derive that

$$\frac{(\epsilon a)^2}{2} g''(q + tv + \epsilon a\, \mathbf{E}\, M(\epsilon^{-1}q, v; \epsilon^{-1}t;\ \cdot\ ), v)\, \mathbf{E}\,(\tilde M(\epsilon^{-1}q, v; \epsilon^{-1}t,\ \cdot\ ))^2 =$$

$$= \frac{\epsilon a^2}{2} g''(q + tv + ar(q, v; t, f), v)\,[\, r_2(q, v; t, f) + \bar r_2(q, v; t, f)\,] + \epsilon^2 g_2(q, v, t, \epsilon)\,. \qquad (4.26)$$

It turns out that the ϵ^2 -term is again negligible:

Lemma 4.4. *Under the conditions of the Theorem, for any* $t \in \mathbf{R}^1$

$$\lim_{\epsilon \to 0} \left| \int dq\, \mu(dv)\, h(q, v)\, \epsilon\, g_2(q, v, t, \epsilon) \right| = 0\,. \qquad (4.27)$$

Combining the calculations done so far, we can write the "essential" part of the RHS of (4.13) as

$$\int dq\, \mu(dv)\, f(q, v)\,[\, g(q + tv + ar(q, v; t, f), v) +$$

$$+ \epsilon a\,(\, r_1(q, v; t, f) + \bar r_1(q, v; t, f)\,)\, g'(q + tv + ar(q, v; t, f), v) +$$

$$+ \frac{\epsilon a}{2}\,(\, r_2(q, v; t, f) + \bar r_2(q, v; t, f)\,)\, g''(q + tv + ar(q, v; t, f), v)\,]\,. \qquad (4.28)$$

According to (3.12), the first term in parenthesis in (4.28) gives just the integral

$$\int dq\, \mu(dv)\, f_t(q, v)\, g(q, v)\,,$$

and after subtracting and dividing by ϵ , we arrive to the expression

$$\frac{a}{2} \int dq\, \mu(dv)\, f(q, v)\,[\, 2\,(\, r_1(q, v; t, f) + \bar r_1(q, v; t, f)\,)\, g'(q + tv + ar(q, v; t, f), v) +$$

$$+ (\, r_2(q, v; t, f) + \bar r_2(q, v; t, f)\,)\, g''(q + tv + ar(q, v; t, f), v)\,]\,. \qquad (4.29)$$

Now we take the derivative $\frac{\partial}{\partial t}$ (or divide by t) and pass to the limit $t \to 0$. Reviewing formulas (3.5), (4.21a), (4.21b), (4.25a) and (4.25b), we conclude that the limiting expression is

$$\frac{a}{2} \int dq\, \mu(dv)\, f(q,v)\, (2 \frac{\partial}{\partial t} [r_1(q,v;t,f) + \bar{r}_1(q,v;t,f)] \,|_{t=0})\, g'(q,v) +$$

$$+ (\frac{\partial}{\partial t} [r_2(q,v;t,f) + \bar{r}_2(q,v;t,f)] \,|_{t=0})\, g''(q,v)) , \tag{4.30}$$

and furthermore,

$$\frac{\partial}{\partial t} r_1(q,v;t,f) \,|_{t=0} = \int_{\mathbf{R}^1} \mu(dw)\, |v-w|\, \frac{\partial}{\partial q} f(q,w)\, (1-\bar{\rho}a)^{-1} , \tag{4.31a}$$

$$\frac{\partial}{\partial t} \bar{r}_1(q,v;t,f) \,|_{t=0} = 0 , \tag{4.31b}$$

$$\frac{\partial}{\partial t} r_2(q,v;t,f) \,|_{t=0} = \int_{\mathbf{R}^1} \mu(dw)\, |v-w|\, f(q,v)\, (1-\bar{\rho}a)^{-1} \tag{4.31c}$$

and

$$\frac{\partial}{\partial t} r_2(q,v;t,f) \,|_{t=0} = 0 . \tag{4.31d}$$

After integrating by parts we arrive to the integral

$$\frac{a}{2} \int dq\, \mu(dv)\, g(q,v) \,[-2 \frac{\partial}{\partial q} (f(q,v) \int_{\mathbf{R}^1} \mu(dw)\, |v-w|\, \frac{\partial}{\partial q} f(q,w))\, (1-\bar{\rho}a)^{-1} +$$

$$+ \frac{\partial^2}{\partial q^2} (f(q,v) \int_{\mathbf{R}^1} \mu(dw)\, |v-w|\, f(q,v))\, (1-\bar{\rho}a)^{-1}] =$$

$$= \frac{a}{2} \int dq\mu(dv)\, g(q,v)\, [\frac{\partial}{\partial q} (\int_{\mathbf{R}^1} \mu(dw)\, |v-w|\, f(q,w)\, (1-\bar{\rho}a)^{-1} \frac{\partial}{\partial q} f(q,v)) -$$

$$- (\frac{\partial}{\partial q} (f(q,v) \frac{\partial}{\partial q} \int_{\mathbf{R}^1} \mu(dw)\, |v-w|\, f(q,w)\, (1-\bar{\rho}a)^{-1})] . \quad \text{Q.E.D.}$$

Proof of Lemma 4.1. The two terms which arise in (4.14) are investigated in a similar way, and to avoid a repetition we proceed with one of them which contains g^{IV} . First of all, we have

$$\epsilon^3 \,|\, \int dq\, \mu(dv)\, h(q,v)\, \mathbf{E}\, (g^{\mathrm{IV}}(q+tv+\Theta_1,v)\, (\tilde{M}(\epsilon^{-1}q,v;\epsilon^{-1}t,\,\cdot\,))^4) \,| \leq$$

$$\leq const\ \epsilon^3 \int dq\, \mu(dv)\, h(q,v)\, \mathbf{E}\, (\tilde{M}(\epsilon^{-1}q,v;\epsilon^{-1}t,\,\cdot\,))^4 . \tag{4.32}$$

We write then formulas which are used, in slightly different versions, in various arguments below. For simplicity, we write $\tilde{M}$ and M instead of $\tilde{M}(\epsilon^{-1}q,v;\epsilon^{-1}t,\,\cdot\,)$ and $M(\epsilon^{-1}q,v;\epsilon^{-1}t,\,\cdot\,)$ and $N(s)$ instead of $N_{(\epsilon^{-1}q,\epsilon^{-1}(q+s))}$, $s > 0$, or of $N_{(\epsilon^{-1}(q+s),\epsilon^{-1}q)}$, $s < 0$. We have

$$\mathbf{E}\, (\, \tilde{M}\,)^4 = \mathbf{E}\, M^4 - 4\, \mathbf{E}\, M^3\, \mathbf{E}\, M + 6\, \mathbf{E}\, M^2\, (\, \mathbf{E}\, M\,)^2 - 3\, (\, \mathbf{E}\, M\,)^4 . \tag{4.33}$$

Assuming for definiteness that $t > 0$, one can write

$$\mathbf{E}\, M^4 = \epsilon^{-1} \int_{\mathbf{R}^1} \mu(dw) \int_0^{|t(v-w)|} ds\, \bar{\rho}_0\, \mathbf{E}\, h(q+s \pm \epsilon a\,(1+N(s)), w) +$$

$$+ 4\, \epsilon^{-2} \prod_{j=1}^{2} (\int_{\mathbf{R}^1} \mu(dw_j) \int_0^{t(v-w_j)} ds_j\,)\, (\bar{\rho}_0)^2\, \mathbf{E} \prod_{k=1}^{2} h(q+s \pm \epsilon a\,(1+N(s_k)), w_k) +$$

$$+ 6\,\epsilon^{-2} \prod_{j=1}^{2} (\int_{\mathbf{R}^1} \mu(dw_j) \int_0^{|t(v-w)|} ds_j \,) \, (\bar\rho_0)^2 \, \mathbf{E} \prod_{k=1}^{2} h(q + s_k \pm \epsilon a \, (1 + N(s_k)), w_k) +$$

$$+ 6\,\epsilon^{-3} \prod_{j=1}^{2} (\int_{\mathbf{R}^1} \mu(dw_j) \int_0^{t(v-w_j)} ds_j \,) \int_{\mathbf{R}^1} \mu(dw_3) \int_0^{|t(v-w_3)|} (\bar\rho_0)^3 \times$$

$$\times \, \mathbf{E} \prod_{k=1}^{3} h(q + s_k \pm \epsilon a \, (1 + N(s_k)), w_k) +$$

$$+ \epsilon^{-4} \prod_{j=1}^{4} (\int_{\mathbf{R}^1} \mu(dw_j) \int_0^{t(v-w_j)} ds_j \,) \, (\bar\rho_0)^4 \, \mathbf{E} \prod_{k=1}^{4} h(q + s_k \pm \epsilon a \, (1 + N(s_k)), w_k); \qquad (4.34)$$

similar formulas hold for $\mathbf{E}\, M^3$ and $\mathbf{E}\, M^2$. The choice of sign in front of ϵ depends on whether s_k (more presisely, $t(v - w_k)$) is positive or negative.

We must worry about the terms of order ϵ^{-4} and ϵ^{-3} which arise in the RHS of (4.33). Let us consider the case of ϵ^{-4} which is sligtly more complicated. Notice that in the limit $\epsilon \to 0$ the coefficient on this term in the RHS of (4.34) is

$$\prod_{j=1}^{4} \int_{\mathbf{R}^1} \mu(dw_j) \int_0^{t(v-w_k)} ds_j \; h(q + s_j(1 + \bar\rho_0 a) \pm \epsilon a, w_j); \qquad (4.35)$$

this follows immediately from the law of large numbers and Lebesgue dominated convergence theorem. The same is true for ϵ^{-4} - terms which come from the other addends in the RHS of (4.33). The sum of all the coefficients gives zero.

It remains to study deviation of the ϵ^{-4} - coefficient in the RHS of (4.34) from the limiting value (4.35): this may create a term of order ϵ and, after multiplication by ϵ^{-4} , a term of order ϵ^{-3} . The similar problem arises with the other ϵ^{-4} - terms figuring in the RHS of (4.33).

Comparing the quantities

$$\mathbf{E} \prod_{k=1}^{4} h(q + s_k \pm \epsilon a \, (1 + N(s_k)), w_k) \quad \text{and} \quad \prod_{j=1}^{4} h(q + s_k(1 + \bar\rho_0) \pm \epsilon a, w_j),$$

we use again Taylor's expansion formula and formulas for moments of the RV $\tilde N(s)$ in the configuration Poissonian state. Then we arrive to the conclusion that the first derivative gives zero contribution and the second derivative creates ϵ^2 in front which is negligible. Similar arguments work for the other ϵ^{-4} - terms from the RHS of (4.33). This finishes the proof of Lemma 4.1.

Proof of Lemma 4.2. In many details the proof of Lemma 4.2 repeats that of Lemma 4.1 (this is also true for Lemmas 4.3 and 4.4) and we shall proceed in a more concise way. To analyze the term $\epsilon^2\, g(q, v, t, \epsilon)$, we notice that it accumulates various contributions which may be "suspected" of being of order ϵ^2 . Those contributions can be "labeled" by pairs (i, i') , $i = 1, 2$, $i' = 1, 2, 3$, and are associated with products of derivatives $\frac{\partial^i}{\partial q^i}\, g$ and $\frac{\partial^{i'}}{\partial q^{i'}}\, h$. Besides, there are contributions which are suspected to be of higher order: they correspond to multiple products. Since the arguments in analyzing all these items are similar, we restrict our attention to one of them, namely, to the term corresponding to the pair (1,3). This term is given by

$$\frac{(\epsilon a)^3}{3!}\, g'(q + tv + a r_1(q, v; t, f), v) \, [\int_{-\infty}^{v} \mu(dw) \int_0^{t(v-w)} ds \, \mathbf{E}\, (h'''(q + s + \epsilon a\, \Theta_2^{(+)}, w)\, \tilde N(s)^3) +$$

$$+ \int_v^{\infty} \mu(dv) \int_{t(v-w)}^{0} ds \, \mathbf{E}\, (h'''(q + s - \epsilon a\, \Theta_2^{(-)}, w)\, \tilde N(s)^3) \,] \qquad (4.36)$$

where $\Theta_2^{(\pm)}$ is a RV between the value $N(s)$ and its expectation $\mathbf{E}\, N(s)$. Using our assumption (4.2), (4.3) and the equality $\mathbf{E}\, \tilde{N}(s)^3 = \epsilon^{-1}\, |s|\, \bar{\rho}_0$, we conclude that the "true" order of (4.36) is ϵ^2 . Q.E.D.

Proof of Lemma 4.3. We perform the following straigtforward calculation (cf. (4.34)):

$$\epsilon\, \mathbf{E}\, M^2 = \int_{bfR^1} \mu(dw) \int_0^{|t(v-w)|} ds\, \bar{\rho}_0\, \mathbf{E}\, h(q \pm s \pm \epsilon a\, (1 + N(s)), w) +$$

$$+ \epsilon^{-1} \prod_{j=1}^{2} (\int_{\mathbf{R}^1} \mu(dw_j) \int_0^{t(v-w)} ds_j\,)\, (\bar{\rho}_0)^2\, \mathbf{E} \prod_{k=1}^{2} h(q + s_k \pm \epsilon a\, (1 + N(s_k)), w_k)\, ; \qquad (4.37)$$

as before, the choice of plus or minus is determined by $\mathrm{sign}\,(t(v-w))$ and $\mathrm{sign}\,(t(v-w_k))$, $i = 1, 2$. On the other hand,

$$(\, \mathbf{E}\, M\,)^2 = \epsilon^{-1} \prod_{j=1}^{2} (\int_{\mathbf{R}^1} \mu(dw_j) \int_0^{t(v-w_j)} ds_j\, \bar{\rho}_0\, h(q + s_j(1 + \bar{\rho}_0 a) \pm \epsilon a, w_j)). \qquad (4.38)$$

We have to look for the limiting non-vanishing terms from the $\epsilon\, \mathbf{E}\, (\tilde{M})^2 = \epsilon\, (\, \mathbf{E}\, M^2 - (\mathbf{E}\, M)^2\,)$. By virtue of Taylor's expansion and formulas for expectations $\mathbf{E}\, (\, \tilde{N}(s)\,)^i$, $i = 1, 2, 3,$ and $\mathbf{E}\, N(s_1) N(s_2)$ one finds that those terms are:

$$\int_{\mathbf{R}^1} \mu(dw) \int_0^{|t(v-w)|} ds\, \bar{\rho})_0\, h(q + s(1 + \bar{\rho}_0 a), w)$$

(this is the limit of the first addend from the RHS of (4.37)) and

$$\prod_{j=1}^{2} (\int_{\mathbf{R}^1} \mu(dw_j) \int_0^{t(v-w_j)} ds_j\,)\, (\bar{\rho}_0)^3\, [\, h''(q + s_1(1 + \bar{\rho}_0 a), w_1)\, h(q + s_2(1 + \bar{\rho}_0 a), w_2)\, s_1 +$$

$$+ h(q + s_1(1 + \bar{\rho}_0 a), w_1)\, h''(q + s_2(1 + \bar{\rho}_0 a), w_2)\, s_2 +$$

$$+ h'(q + s_1(1 + \bar{\rho}_0 a), w_1)\, h'(q + s_2(1 + \bar{\rho}_0 a), w_2)\, \chi_{\mathbf{R}^1_+}(s_1 \cdot s_2) \min_{i=1,2} |s_i|\,]$$

(which is the difference of the second addend from the RHS of (4.37) and the RHS of (4.38)). This completes the proof of Lemma 4.3.

Proof of Lemma 4.4 . After the investigation of the term $\epsilon\, \mathbf{E}\, (\, \tilde{M}\,)^2$ which was done in the proof of Lemma 4.3, all what remains is to study the difference

$$g''(q + tv + \epsilon a\, \mathbf{E}\, M, v) - g''(q + tv + ar(q, v; t, f), v)\, .$$

This is straightforward in view of the formula (4.15).

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