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Short Recall of Two-Dimensional Conformal Field Theory *

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Abstract: An introduction to two-dimensional conformal field theory is given which compares it to conformal field theory in higher dimensions and which emphasizes the relationship between the Noether theorem and the Virasoro and Kac-Moody algebras. The standard models, free fermions and bosons, Liouville, Toda, and WZNW, are described.

1. Introduction.

As some of the participants at this conference are not working on two-dimensional conformal field theories and many of the talks will be in this field I have been asked to give an introductory lecture on the subject. In order to do this I will attempt to present the theory in terms which are familiar to all, namely, in terms of Noether's theorem [1]. This theorem is well-known and, perhaps surprisingly, it is sufficiently broad to describe the basic structures. By Noether's theorem I mean here, of course, not merely the conservation of the Noether charges, but the implementation of group transformations by means of those charges, and the fact that the charges represent the Lie algebra of the symmetry group on the Hilbert-space or phase-space of the theory. It will be seen, for example, that the well-known Virasoro and Kac-Moody algebras [2][3][4] of two-dimensional conformal field are just Noetherian representations of conformal and internal symmetry group transformations, respectively.

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Accordingly, we start with the general Noether theorem and then specialize to the case of space-time symmetries, for which the Noether charges are moments of the energy-momentum tensor. We then consider the restrictions that are introduced by adding scale invariance (and hence, for local Lagrangian theories, conformal invariance [5]) to the usual Poincare invariance. In order to emphasize the difference that exists between the two cases, we first consider the higher-dimensional cases $d = 3, 4$ and only afterwards specialize to the two-dimensional case $d = 2$. The big difference, of course, is that, in contrast to the higher-dimensional cases, the conformal group in two dimensions is infinite-dimensional and admits central extensions.

In order to illustrate the general principles, and because these models will undoubtedly be widely discussed at the conference we then present the better-known two-dimensional conformal models, namely the free-field models for both fermions and bosons [2], the Liouville and Toda models [6], which describe interacting scalar fields, and the Wess-Zumino-Novikov-Witten model [2] which describes group-valued scalar fields, and which have non-trivial interactions and topological terms if the group is non-abelian.

2. Recall of Noether Theorem.

As mentioned in the Introduction, the Noether theorem actually consists of three parts and we now recapitulate the three statements. Let $L(\phi(x), \partial\phi(x))$ be a Lagrangian density for any set of fields $\phi(x)$. Then the Lagrangian field equations may be written in the form

$$\partial_\mu \pi_\mu(x) = \frac{\partial L}{\partial \phi}, \quad \text{where} \quad \pi_\mu \equiv \frac{\delta L}{\delta \partial_\mu \phi}. \tag{1}$$

Now suppose that the fields vary with respect to some infinitesimal rigid (x-independent) group transformation $\phi(x) \to \phi(x) + \delta_g \phi(x)$ and define the current $j_\mu = \sum \phi \pi_\mu \delta_g \phi$. Then, as a consequence of the field equations it is easy to see that

$$\partial_\mu j_\mu = \delta_g L. \tag{2}$$

Accordingly, if the Lagrangian is group-invariant we have

$$\partial_\mu j_\mu = 0, \quad \text{and hence} \quad \partial_t Q = 0 \quad \text{where} \quad Q = \int j_o(x), \tag{3}$$

where the integral is over space only. Thus, if the Lagrangian is group- invariant the current j_μ and the charge Q are conserved. This is the first part of Noethers theorem.

Next let $[\pi_o(x), \phi(y)] = \delta(x-y)$ be the equal-time commutation or Poisson-bracket relation for the fields. Using these and the definition of Q one finds at once that

$$[Q, \phi(x)] = \delta_g \phi(x). \tag{4}$$

In other words one finds that the conserved charge Q *implements* the infinitesimal group transformation. This is the second part of Noether's theorem.

Finally, let us suppose that the infinitesimal group transformation is part of a Lie group of such transfomations i.e.

$$\delta_g \phi(x) = \epsilon \sigma_a \phi(x), \quad \text{where} \quad [\sigma_a, \sigma_b] = f_{ab}^c \sigma_c, \tag{5}$$

the fab^c being the structure constants. Then from (4) one sees by inspection that

$$[[Q_a, Q_b] - f_{ab}^c Q_c, \phi(x)] = 0, \qquad .(6)$$

from which it follows that

$$[Q_a, Q_b] = f_{ab}^c Q_c + C_{ab}, \quad \text{where} \quad [C_{ab}, \phi(x)] = 0. \tag{7}$$

Equation (7) shows that the conserved charges Q satisfy the Lie algebra of the original symmetry group, up to quantities C_{ab} that commute with the $\phi(x)$, hence with everything in sight, and are therefore called *central charges*. This is the third part of Noether's theorem. Note that the Qs are operators on the infinite-dimensional Hilbert (or phase) space of the field theory, whereas the σs are operators on the space of the Lie algebra, which is often only finite dimensional. For most of the traditional Lie symmetry groups (which are finite-dimensional and either compact or semi-simple, or both) the structure of the Lie algebra is such that central charges are not permitted, so up to recently they have not been very familiar objects. For two-dimensional conformal field theories, however, central charges not only exist but play a crucial role. To sum up, the three parts of Noether's theorem are (I) conservation (3) (II) implementation (4), and (III) commutation up to central charges (7).

3. Poincare Symmetries in Higher-Dimensions.

Conformal symmetries are space-time symmetries and hence to put the two-dimensional ones in perspective it is useful to recall first the space-time symmetries of Lagrangians in higher dimensions, in particular the Poincare and conformal symmetries in higher dimensions. In this section we consider the Poincare symmetries. As with all space-time symmetries the Noether charges are moments of the energy-momentum tensor $T_{\mu\nu}$ belonging to the Lagrangian in question. We recall that $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = \frac{\partial L}{\partial \phi_\mu}\frac{\partial l}{\partial \phi_\nu} - g_{\mu\nu}L, \tag{8}$$

where $\phi_\mu = \partial_\mu \phi$, and a similiar expression with spin corrections added for higher-spin fields.

The simplest Poincare transformations are, of course, the translations, $x_\mu \to x_\mu + \epsilon_\mu$, where the ϵ's are constant parameters, whose infinitesimal action on the fields are $\phi(x) \to \phi(x) + \epsilon_\mu \partial_\mu \phi(x)$. Using the rules of section 2 to compute the corresponding Noether currents one finds that

$$j_{(\mu)\nu} = T_{\mu\nu}, \quad \text{and hence} \quad P_\mu \equiv Q_{(\mu)} = \int T_{\mu o}, \tag{9}$$

where we have used the traditional symbol P for the momentum. Hence the three parts of Noethers theorem are in this case

$$(I) \quad \partial_\nu T_{\mu\nu} = 0 \quad \Rightarrow \quad \partial_t P_\mu = 0, \tag{10}$$

$$(II) \quad [P_\mu, \phi(x)] = \partial_\mu \phi(x), \tag{11}$$

and

$$(III) \quad [P_\mu, P_\nu] = 0. \tag{12}$$

The more complicated Poincare transformations are the Lorentz transformations $x_\mu \to x_\mu + \epsilon_{\alpha\beta}\sigma^{\alpha\beta}_{\mu\nu}x_\nu$, where the ϵs are constant parameters and the σ the infinitesimal generators. The Noether rules then give for the currents and charges

$$j_{(\alpha\beta)\mu} = x_\alpha T_{\beta\mu} - x_\beta T_{\alpha\mu}, \quad \text{and} \quad Q_{\alpha\beta} = \int [x_\alpha T_{\beta o} - x_\beta T_{\alpha o}]. \tag{13}$$

Given that the energy-momentum tensor is divergence-free i.e. satisfies (10) on account of translational invariance, the first part of Noethers theorem reduces in this case to

$$(I) \quad \partial_\mu (x_\alpha T_{\beta\mu} - x_\beta T_{\alpha\mu}) = T_{\beta\alpha} - T_{\alpha\beta} = 0, \tag{14}$$

and therefore requires that the energy-momentum tensor be symmetric. (It sometimes happens that the energy-momentum tensor that is computed by some canonical procedure, is not symmetric, but terms (so-called Belinfante terms) can always be added so that it becomes symmetric and remains divergence-free). The second part of Noether's theorem takes the form

$$(II) \quad [Q_{\alpha\beta}, \phi(x)] = [x_\alpha \partial_\beta - x_\beta \partial_\alpha + \Sigma_{\alpha\beta}]\phi(x), \tag{15}$$

where Σ denotes the standard spin part of the transformation, and the third part of the theorem is just the statement that the $Q_{\alpha\beta}$ satisfy the usual commutation relations for the generators of the lorentz group.

A simpler example, perhaps, is the three-dimensional rotation group in Quantum Mechanics, for which the position X_i and momentum P_i for $i = 1, 2, 3$ transform as vectors i.e. $X_i \rightarrow X_i + \epsilon_{aij} X_j$ and similiarly for P_i, and the charges Q are just the angular momentum operators $Q_a = \epsilon_{aij} X_j P_k$. These charges actually satisfy the commutation relations $[Q_a, Q_b] = \epsilon_{abc} Q_c$ of the rotation group generators (Part (III) of Noether's theorem) and implement the infinitesimal rotations $[Q_a, X_i] = \epsilon_{aij} X_j$, and similiarly for the Ps (Part (II) of the theorem), whether or not they are conserved, but they satisfy the conservation law (Part(I) of the theorem) if, and only if, the potential is rotationally invariant.

4. Scale and Conformal Invariance in Higher Dimensions.

We now wish to consider the case when the Lagrangian is scale-invariant as well as Poincare-invariant, that is to say, is invariant with respect to the coordinate transformations $x_\mu \rightarrow \lambda x_\mu$ and the field transformations $\phi(x) \rightarrow \lambda^s \phi(\lambda x)$, where λ is a rigid (x-independent) scale parameter and s is an exponent depending on the spin of the field. The Lagrangian is typically scale-invariant when it containsno dimensional parametrs, in particular contains no mass-terms. In the scale-invariant case the Noether current and charge turn out to be

$$j_\mu = x_\nu T_{\nu\mu} \quad \text{and} \quad Q = \int x_\nu T_{\nu o}, \tag{16}$$

respectively, and the Noether conservation law (I) reduces to

$$\partial_\mu (x_\nu T_{\nu\mu}) = T_{\mu\mu} = 0. \tag{17}$$

That is to say, Noether's theorem requires in this case that the energy-momentum tensor be traceless. Again it sometimes happens that when computed by some canonical procedure the energy-momentum tensor does not turn out to be traceless, but terms can always be added to make it traceless in such a way that it remains divergence free and symmetric. The traceless symmetric energy-momentum tensor obtained in this way is sometimes called the improved energy-momentum tensor. (It is worth remarking, perhaps, that if the Lagrangian is written in a manifestly covariant manner, with metric tensor $g_{\mu\nu}$ then the energy-momentum tensor defined as $T_{\mu\nu} = \frac{\partial L}{\partial g_{\mu\nu}}$ will be automatically divergence-free, symmetric and traceless).

Scale-invariance has the remarkable property that for most local, Lagrangian, field theories in more than two dimensions it implies also conformal invariance [3][5][7]. The exact conditions under which this holds are known, but it is known to hold in almost all cases of interest and hence we will regard it as a folk theorem, namely: Poincare invariance + locality + scale-invariance = conformal invariance. Here conformal invariance means invariance with respect to the (non-linear) space-time transformations $x_\mu \rightarrow x_\mu/x^2$. The classic example of a conformally-invariant field theory is the pure Maxwell or Yang-Mills Lagrangian in four dimensions i.e. $\int d^4x \mathrm{tr} F^{\mu\nu}F_{\mu\nu}$. The conformal transformations evidently add n new parameters to the space-time symmetry group in n dimensions, so in three and four dimensions, for example, the conformal group, (consisting of the Poincare, scale and pure conformal transformations) is $6+1+3=10$ and $10+1+4=15$ dimensional, respectively. Thus it is a finite-dimensional group. Furthermore, it is not difficult to see that it is a simple group, namely, $SO(n-1,2)$ in n dimensions, and thus admits no central extensions. Although the conformal group is only finite-dimensional the requirement of conformal invariance places very strong restrictions on local renormalizable field theories in three and four dimensions. The general structure of conformally invariant field theories for $n = 3,4$ is described in the books of Coleman [5] and Todorov et al.[7].

5. Conformal Invariance in Two Dimensions.

Against this higher-dimensional background let us now consider the case of conformal invariance in two dimensions. In this case the energy-momentum tensor $T_{\mu\nu}$ has only four components, $(T_{11}, T_{22}, T_{12}, T_{21})$, of which, in a Poincare-invariant theory, only three are independent, since $T_{\mu\nu}$ must be symmetric. In a scale-independent theory $T_{\mu\nu}$ must be traceless as well, so that there are actually only *two* independent components in this case. In order to express the symmetry and tracelessness of $T_{\mu\nu}$ in a compact manner it is convenient to introduce the so-called left- and right-moving coordinates $z = x+y,\ w = x-y$ or ($z = x+iy\ w = \bar{z} = x-iy$ in the Euclidean version). Then $T_{\mu\nu}$ can be wriiten as $(T_{zz}, T_{ww}, T_{zw}, T_{wz})$ and the symmetry and tracelessness consists in the statement that $T_{zw} = T_{wz} = 0$. Thus only the components T_{zz} and T_{ww} survive. But then, the Noether conservation laws corresponding to translational invariance become

$$\partial_z T_{zw} + \partial_w T_{zz} = \partial_w T_{zz} = 0, \tag{18}$$

and

$$\partial_z T_{ww} + \partial_w T_{wz} = \partial_z T_{ww} = 0. \tag{19}$$

Thus they amount to the statement the components T_{zz} and T_{ww} depend only on z and w respectively. Furthermore the equations (18) (19) are invariant with respect to the transformations $z \to f(z)$ and $w \to g(w)$ for arbitrary smooth functions $f(z)$ and $g(z)$ so the theory is invariant with respect to these transformations. But these are just the transformations of the conformal group in two dimensions, which is just the conformal group of the theory of one complex variable. Thus the folk-theorem that scale-invariance implies conformal-invariance is realized explicitly in two dimensions. two dimensions What distinguishes the two-dimensional case, however, is the structure of the conformal group. First it is a direct product of two identical groups, namely, the left- and right-handed groups $z \to f(z)$ and $w \to g(w)$. Second, each of these two groups is infinite-dimensional since it consists of the set of all analytic functions $f(z)$ or $g(w)$. Finally, each side of the conformal group in two dimensions is known to admit one central extens1lion. The transformation law of the energy-momentum tensor with respect to the conformal group tensor is

$$T_{zz}(z) \to (f'(z))^2 T_{zz}(f(z)) + c\frac{2f'''f' - 3(f'')^2}{24(f')^2}, \tag{20}$$

where c is a constant (and its coefficient is called the Schwarzian derivative of $f(z)$). As long as c is not zero, this transformation is inhomogeneous., but it is easy to see that the Jacobi identity for succesive conformal transformations does not require c to be zero. This is why the conformal group admits central extensions. It can be shown that it actually admits only two central extensions (one for each side) and so the constant c, which depends on the detailed structure of the theory, and its partner from the w part of the group, are the parameters that characterize the central extensions.

6. Noethers Theorem for the Two-Dimensional Conformal Group.

Let us now consider Noether's theorem for the conformal group in two dimensions. First, in contrast to the higher-dimensional cases, in which the Noether charges are only lower-order moments of the energy-momentum tensor, in two dimensions they consist of *all* the moments of the energy- momentum tensor. That is to say, the charges that are conserved according to part (I) of Noether's theorem turn out to be

$$Q_n = \int z^n : T_{zz} : \quad \text{and} \quad \bar{Q}_n = \int w^n : T_{ww} :, \tag{21}$$

where the colons denote normal ordering in the quantized case. It is evident that these charges are conserved since they are independent of space and time. In practice it is usual to deform the space-integrals so that they become integrals around the unit circles $|z| = 1$ and $|w| = 1$.

Now, with respect to the conformal group, tensor fields $\phi(z, w)$ (which are called *primary* fields) have the transformation laws

$$\phi(z,w) \rightarrow (\frac{\partial z}{\partial z'})^j (\frac{\partial w}{\partial w'})^{\bar{j}} \phi(z', w') \tag{22}$$

where j and $\bar{j}$ are called the conformal weights, and, according to Part (II) of Noether's theorem, the Q's and $\bar{Q}$'s should implement the infinitesimal version of these transformations. This they do by means of the commutation (or Poisson bracket) relations

$$[Q_n, \phi(z,w)] = z^n[nj + z\partial_z]\phi(z,w) \quad \text{and} \quad [\bar{Q}_n, \phi(z,w)] = w^n[n\bar{j} + w\partial_w]\phi(z,w), \tag{23}$$

respectively.

Finally, according to Part (III) of Noether's theorem, the Q's and $\bar{Q}$'s should satisfy the Lie algebra of the conformal group (up to central extensions) i.e. should satisfy the well-known Virasoro algebra. By taking the Fourier transform of (20) one finds that,

$$[Q_n, Q_m] = (n-m)Q_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,o}, \tag{24}$$

which is indeed the Virasoro algebra, and similiarly for $\bar{Q}$. One may also verify that (24) follows, up to the central term, from the Jacobi relations for (23).

6. Free-Field Examples.

Let us now consider as examples the case of free fermion and boson fields. Since free fermion fields in two-dimensions scale like $(d)^{-\frac{1}{2}}$, where d is the dimension, the simplest scale-invariant Lagrangian for free fermion fields is evidently

$$L(\psi) = \sum_{a=o}^{a=N} \int d^2x(\bar{\psi}_a(x)\gamma^\mu \partial_\mu \psi_a(x)), \tag{25}$$

and the equal-time canonical commutation relations or Poisson brackets are

$$\{\psi_a(x), \psi_b(y)\} = \delta_{ab}\delta(x-y), \tag{26}$$

where $x = (z, w)$. The Lagrangian (25) is, of course, guaranteed to be conformally invariant as well, and if we define Q_n according to (21), and use the field equations, which simply state that $\psi(x) = \psi(z) + \psi(w)$, we find that

$$Q_n \equiv \oint \frac{dz}{z} z^n T_{zz} = \oint \frac{dz}{z} \sum_a \psi_a(z)^\dagger \psi_a(z), \tag{27}$$

with or without colons according as to whether the theory is quantized or not. On using (26) we find that, as required by Noether's theorem, the Q's implement the infinitesimal conformal transformations of the fields, $\psi(z) \to \frac{\partial z}{\partial z'}\psi(z')$, and satisfy the Virasoro algebra. However, in the classical case one finds that the central parameter $c = 0$ is zero, whereas in the quantized case, due to the normal-ordering, one finds that $c = N/2$.

Since bosonic fields are scale invariant in two dimensions the simplest scale-invariant free-field Lagrangian is

$$L(\phi) = \sum_{a=o}^{a=N} \int d^2x(\partial_\mu \phi^a(x) \partial_\mu \phi^a(x)), \tag{28}$$

and the equal-time commutation or Poisson-bracket relations are

$$[\phi_a(x), \partial_t \phi_b(y)] = i\delta_{ab}\delta(x-y). \tag{29}$$

As before, the Lagrangian is guaranteed to be scale-invariant, and if we define the Noether charges

$$Q_n \equiv \oint \frac{dz}{z} z^n T_{zz}(z) = \oint \frac{dz}{z} z^n \sum_a \partial_z \phi_a(x) \partial_z \phi_a(x), \tag{30}$$

with or without colons according as the theory is quantum or classical, we find that, as required by Noether's theorem, the Q's implement the conformal transformations $\phi(z,w) \to \phi(z',w')$ of the fields and satisfy the Virasoro algebra. For the centre parameter c, one finds that $c = 0$ and $c = N$ in the classical and quantized cases respectively. The results for the free-field centres in the quantized cases just mentioned are actually part of a general [8] free-field formula

$$c = \epsilon[N + 6\sum_a j_a(j_a - 1)], \tag{31}$$

where $\epsilon = \pm$ for bosons and fermions respectively, j^a are the conformal weights and N is the number of fields.

7. Liouville and Toda Lagrangians.

As mentioned earlier, conformal invariance severely limits the form of the Lagrangian in any dimension, and in two dimensions the conditions for a single scalar field are so strong that the only interacting Lagrangian density is one with an exponential potential i.e.

$$L(\phi) = \frac{1}{2}(\partial\phi)^2 + \mu e^{k\phi}, \tag{32}$$

whereμ and k are constants. This Lagrangian density is called the Liouville one, and it has the propert that not only is it conformally invariant but the field equations derived from it are integrable. It surfaces in many problems in physics, in particular it is the Lagrangian that describes the two-dimensional gravity theory that is induced by string theory in less than the critical number of dimensions [9].

For a number of scalar fields the situation is a little more complicated. The most general conformal-invariant Lagrangian density is still one in which the potential is exponential, namely,

$$L(\phi) = \sum_{ab} \frac{1}{2} C_{ab}\partial\phi^a\partial\phi^b + \mu_a e^{K_{ab}\phi^a\phi^b}, \tag{33}$$

where the C_{ab} and K_{ab} are constant matrices for $a, b = 1...N$, where N is the number of fields. This Lagrangian is conformally invariant, but for for general choices of the matrices C and K the field equations are not integrable. The interesting point is that the field equations become integrable for a special class of C and K matrices, namely when C and K are the Coxeter and Cartan-Killing matrices of a simple Lie group G of rank $l = N$, i.e. when

$$C_{ab} = \frac{4(\alpha_a.\alpha_b)}{\alpha_a^2.\alpha_b^2}, \quad \text{and} \quad K_{ab} = \frac{2(\alpha_a.\alpha_b)}{\alpha_b^2}, \tag{34}$$

where the α_a's and α_b's are the l simple roots of G i.e. the l roots from which all the positive roots can be obtained by addition. In this (integrable) case the Lagrangian density (33) is called the *Toda* Lagrangian density [6]. Since the Coxeter and Cartan-Killing matrices are known [10] to have non-zero elements only for neighbouring values of i and j for the classical groups (and also next to neighbouring values for the exceptional groups) one sees that the Toda interaction is actually a nearest-neighbour type interaction.

8. WZNW Lagrangians.

The last example which we shall consider (and which will be treated in much more detail in the next lecture) is the Wess-Zumino-Novikov-Witten (WZNW) Lagrangian. This is a Lagrangian in which the individual scalar fields $\phi(x)$ are replaced by *group-valued* scalar fields $g(x)$ i.e. by fields $g(x)$ which take their values in the defining representation of a simple Lie group$G$, i.e. $g(x) \in G$. The Lagrangian in this case takes the form

$$L(g) = \frac{\kappa}{2} \int d^2x \mathrm{tr}(j_\mu j_\mu) + \frac{\kappa}{3} \int d^3x \epsilon_{abc} \mathrm{tr}(j_a j_b j_c), \tag{35}$$

where $j_\mu \equiv g(x)^{-1} \partial g(x)$, and the three-dimensional integral, whose variation is purely topological, is over a volume whose boundary is the two-dimensional space of the kinetic term. Since the g's are conformal scalars this Lagrangian is manifestly conformally-invariant, and the topological term is added so that the field equations reduce to

$$\partial_w J(x) = 0 \quad \text{and} \quad \partial_z \bar{J}(x) = 0, \tag{36}$$

where the currents J and $\bar{J}$ are defined as

$$J(x) = (\partial_z g(x)) g(x)^{-1} \quad \text{and} \quad \bar{J}(x) = g(x)^{-1} \partial_w g(x), \tag{37}$$

respectively i.e. so that they reduce to the statement that $J(x)$ and $\bar{J}(x)$ depend only on z and w respectively, $J(x) = J(z)$ and $\bar{J}(x) = \bar{J}(w)$. The canonical commutation relations for these currents, corresponding to the space-derivative of the free-field bosonic commutation relations (29) take the form

$$[J^a(z), J^b(z')] = f_c^{ab} J^c \delta(z - z') + g^{ab} \kappa \partial_z \delta(z - z'), \quad \text{where} \quad J^a(z) = \mathrm{tr}\sigma^a J(z), \tag{38}$$

the σ^a being the generators of G and κ the overall factor in the Lagrangian, and similiarly for $J(w)$. Those familiar with Kac-Moody (KM) algebras will recognize that (38) is a KM algebra of centre K. Note that that this algebra can be regarded as the non-abelian version of the free-field commutation algebra (29), to which it reduces for $f_c^{ab} = 0$. This is not surprising because the WZNW theory with G replaced by an abelian group is just the free-field bosonic theory. The importance of this observation, however, is that it means that, just as in the free-field theory, the canonical commutation relations can be represented on a Fock-space with vacuum $|o>$ defined as $J_n|o>= 0$ for $n > 0$, where the J_n are the components in the Fourier transform of the $J(z)$ (on the unit circle). Such a Fock-space representation is also called a highest-weight representation.

The energy momentum density for the WZNW Lagrangian (35) is

$$T_{zz} =: J^a(z)J^a(z) :, \tag{39}$$

and similiarly for the $\bar{J}(w)$, where the normal ordering is with respect to the Fock-space vacuum $|o>$. Hence the Noether charges corresponding to the conformal symmetry in this case are

$$Q_n = \oint \frac{dz}{z} z^n : J^a(z)J^a(z) : \tag{40}$$

where again the integral is taken around the unit circle in z-space (and similiarly for the $\bar{J}(w)$). Since the Q_n do not depend on space-time they are trivially conserved, as required by Part (I) of Noether's theorem and, using the KM algebra (37), one can verify that they implement the conformal transformations of the fields i.e. that

$$[Q_n, g(x)] = z^n g(x), \tag{41}$$

as required by Part (II). Finally, by using the KM algebra again, one finds that the Q's satisfy a Virasoro, algebra,as required by Part (III) of the theorem, and that the central parameter c is

$$c = \frac{K}{K+g}\text{dimG}, \tag{42}$$

where g, defined as $lg = \sum_{\alpha>0}(\alpha)^2$, where l is the rank, is the Coxeter number of the Group G. Note that when G is abelian, ($g = 0$), this formula agrees with the free-field formula (31) since for scalar fields the weights j^a are zero.

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