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Conformal Reduction of WZNW Theories *

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Abstract: It is shown that Wess-Zumino-Novikov-Witten (WZNW) field theories can be reduced to other families of conformally-invariant integrable field theories, including the Toda field theories. The advantages of regarding these conformal theories as reduced WZNW theories are outlined, and include the natural appearance of two-dimensional gravity, the easy derivation of the general solutions from the WZNW solutions, and, for the Toda theories, an intuitive understanding and relatively simple construction of the W-algebras. An interesting formula connecting the WZNW Kac-Moody centres with the centres of the reduced theories is derived.

1. Introduction.

Perhaps the most celebrated class of conformally-invariant two-dimensional field theories is the Wess-Zumino-Novikov-Witten (WZNW) class [1] described in an earlier lecture and recently it has been shown that these theories can be reduced to other conformallly-invariant field theories, notably the Toda field theories [2], by putting linear constraints on the currents [3]. More generally, one can obtain a whole series of conformal integrable field theories, which interpolate between the WZNW and Toda theories [4], and there even exists a non-conformal version of the constraints that yields a set of non-conformal integrable systems, which includes the affine Toda systems.

In the present talk I should like to describe these reductions of WZNW theory and to outline the various features and advantages that emerge by regarding the reduced conformal theories as constrained WZNW theories. Perhaps the most

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remarkable feature that emerges is the appearance in all cases of a two-dimensional gravitational field, in non-trivial interaction with itself and with the other fields. The emergence of this gravitational field is in some sense the converse of the Polyakov's embedding [5] of the string-induced two-dimensional gravity in the WZNW group $SL(2,R)$, but it is present for all WZNW groups. One of the most immediate practical advantages that accrues from regarding the conformal theories as constrained WZNW theories is that their general solution can be obtained in a rather simple manner from the well-known general WZNW solution. For reasons of space the derivation will not be given in this talk but the general method will be indicated and the end-result, which is quite simple, presented. (The details can be found in ref.[4]).

One of the remarkable features of the Toda theories in particular is that they realize [6] the polynomial algebras (so-called W-algebras) defined [7] abstractly by Zamolochikov. Within the confines of Toda theories it is not immediately obvious why these algebras should exist, and one of the advantages of regarding Toda theory as a reduced WZNW theory is that in the broader WZNW context their existence becomes quite natural and understandable. Indeed in the reduced WZNW context the W-algebras have a simple intuitive interpretation as the algebras of gauge-invariant polynomials of the constrained currents (and their derivatives), the gauge group in question being the one generated by the constraints. This identification not only provides an intuitive understanding of W-algebras, but also provides a relatively simple algorithm for their computation. This is because of the existence of a gauge in which the gauge-invariant polynomials reduce to the currents themselves. In this gauge the W-algebras manifest themselves as the Dirac star algebras of the gauge-fixed constrained currents and, because all constraints are linear in the currents, the W-algebras can then be computed relatively easily from the Kac-Moody algebras of the associated WZNW theories.

Those not completely familiar with two-dimensional conformal field theory may refer to my opening lecture on that subject, which contains all the material needed to make the reduction understandable.

2. Recall of WZNW and Toda Theories and Definition of W-Algebras.

We first recall the WZNW Lagrangian, which takes the form

$$L_{WZ} = \frac{k}{2} \oint d^2x \mathrm{tr}(g^{-1}(x)\partial g(x))^2 + \frac{2k}{3} \oint d^3x \mathrm{tr}(g^{-1}(x)\partial g(x))^3, \quad (1)$$

where the three-dimensional integral is over a space whose boundary is the two-dimensional one of the first (kinetic) integral. As a result of the addition of the three-dimensional integral, whose variation is purely topological, the field equations of the theory take the form

$$\partial_w J(x) = 0 \quad \text{and} \quad \partial_z \bar{J}(x) = 0, \quad (2)$$

where

$$J(x) = (\partial_z g(x))g^{-1}(x) \quad \text{and} \quad \bar{J}(x) = g^{-1}(x)\partial_w g(x).$$

The field equations mean, of course, that the currents $J(x)$ and $\bar{J}(x)$ are functions only of z and w respectively and from the symmetry of L_{WZ} with respect to (rigid) left and right group multiplication ($g \to hg$ and $g \to gh$), and the Noether theorem, it follows that they satisfy Kac-Moody (KM) algebras with centres k. Thus $J(z)$, for example, satisfies the KM algebra

$$[J_a(z), J_b(z')] = f^c_{ab} J_c(z)\delta(z-z') + k\delta_{ab}\partial_z\delta(z-z'). \quad (3)$$

Next we recall the Toda Lagrangian, which takes the form

$$L_{Toda} = \int d^2x [C_{\alpha\beta}\partial\phi^\alpha(x)\partial\phi^\beta(x) + \sum_\beta \exp(K_{\beta\alpha}\phi^\alpha(x)\phi^\beta(x))], \quad (4)$$

where $C_{\alpha\beta} = \frac{4(\alpha.\beta)}{\alpha^2.\beta^2}$ and $K_{\alpha\beta} = \frac{2(\alpha.\beta)}{\beta^2}$ are the Coxeter and Cartan-Killing matrices for any semi-simple Lie group with fundamental roots α, β. From this it follows, of course, that to every Dynkin diagram there is associated a Toda field theory, and conversely.

Finally we consider the definition of the W-algebras. According to Zamolodchikov, who first introduced them [7], a W-algebra is an extension of a Virasoro algebra by primary fields, such that the Poisson (or commutator) bracket of any two primary fields is a polynomial in the fields and their derivatives (both primary and Virasoro), the order of the polynomial being less than the combined order of the two primary fields. In other words, a W-algebra consists of the Virasoro algebra, the conformal transformation law of the fields and a set of Poisson bracket (or commutation) relations of the form

$$[\phi^{(a)}(z), \phi^{(b)}(z')] = P^{(a,b)}(\phi(z), L(z), \delta(z - z')), \tag{5}$$

where $P^{(a,b)}$ is polynomial of lower order than (a+b) in $L(z)$, $\phi(z)$, $\delta(z-z')$ and their derivatives. In counting the order the delta function and the derivative are each given unit weight. Recently it has been shown that every Toda field theory admits a W-algebra, the W's being coefficients in an equation called the Gelfand-Dickey equation [8]. This equation is a linear differential (or pseudo-differential) equation of the same order as the dimension of the defining representation of G, and which is satisfied by certain left-and right-moving functionals of the Toda fields. Its role in our discussion will be to help identify the W-algebras.

The reduction of the WZNW theories requires setting some of the WZNW currents equal to non-zero constants and since these currents, being space-time vectors, have conformal weights (1,0) or (0,1) with respect to the usual conformal group, the problem is how to set them equal to constants without breaking conformal invariance. By the usual conformal group is meant here the group generated by the Noether currents $L(z)$ and $\bar{L}(w)$ belonging to the energy-momentum tensor of the WZNW theory and the way that is used to circumvent this difficulty is to note that the this conformal group is not unique. In fact, there is a two-parameter family of conformal groups equivalent to it and the procedure will be to choose a member of this family with respect to which some of the currents are no longer vectors but scalars i.e. have conformal weights (0,0). However, to make the appropriate choice of member requires some Lie-algebraic technicalities and these will be discussed in the next section.

3. Lie Algebraic Technicalities.

For the reduction of WZNW theories there are some algebraic technicalities the Lie algebraic technicalities that are needed are as follows: First, the simple WZNW groups G that are used for the reduction are be the (maximally non-compact) ones generated by the *real* linear span of the Cartan generators i.e. by the generators $[H_i, E_\alpha]$ in conventional notation. For the A and D algebras, for example, these are the groups SL(n,r) and SO(n,n).

Now within the Cartan algebra of any semi-simple Lie group there exists an element H such that each of the simple roots E_{α_i} is an eigenvector of H with eigenvalue unity or zero.

$$[H, E_{\alpha_i}] = hE_{\alpha_i} \quad \text{where} \quad h = 0, 1, \quad i = 1, 2 \ldots l, \tag{6}$$

and l is the rank. (To see this note that H can be written as $\mathbf{w}.\mathbf{H}$, where $\mathbf{w}$ is a sum over any subset of the l fundamental coweights). Then H provides an integer grading of the whole Lie algebra,

$$[H, E_\alpha] = hE_\alpha \quad \text{where} \quad h \in Z. \tag{7}$$

In particular the elements of the algebra of the little group of H, which we shall call B, will have zero grade. It is not difficult to see that the set of little groups B for all possible choices of H are just the non-compact versions of the set of little groups in the adjoint representation of the compact form of G. In particular for $\mathbf{w} = \mathbf{s}$, where $\mathbf{s}$ is the sum over *all* the simple coweights (=half the sum of the positive coroots), the little algebra is the generic one, namely the Cartan algebra itself. (It will be seen later that this case corresponds to the Toda reduction). Finally we note that G admits a local Gauss decomposition $G = ABC$, where B is the little group and A and C are the (nilpotent) groups generated by the root vectors E_α with weights which are strictly positive and negative with respect to H, respectively. (This decomposition may not be global, but the parameter space may be divided into a finite number of patches on each of which the decomposition is valid up to left- or right-multiplication with a constant group element).

At the KM level we have, correspondingly,

$$[H(z), J^B(z')] = 0 \quad \text{except} \quad [H(z), H(z')] = k\partial_z\delta(z - z')\text{tr}H^2, \tag{8}$$

and

$$[H(z), J^\alpha_h(z')] = hJ^\alpha_h(z)\delta(z - z'). \tag{9}$$

4. Preservation of Conformal Invariance in the Reduction.

We come now to the crucial point. Let $L(z)$ denote the Virasoro operator which is the component $T_{zz}(z)$ of the energy-momentum tensor of the WZNW theory, and with respect to which all the KM currents $J(z)$ are have conformal weights unity, or, more precisely, (1,0). Then we replace $L(z)$ by $\Lambda(z)$, where

$$\Lambda(z) = L(z) + \partial_z H(z), \tag{10}$$

It is to be noted that $\Lambda(z)$ is again a Virasoro operator i.e. satisfies the standard Virasoro algebra. The only difference is that the centre c changes to $c - 12k\mathrm{tr}H^2$. It will turn out that $\Lambda(z)$ is actually the improved (i.e.traceless) energy-momentum tensor of the reduced theory.

Once the crucial change (10) has been made the rest is almost automatic. With respect to the conformal group generated by $\Lambda(z)$ the KM currents $J(z)$ are no longer vectors of conformal weight (1,0) but have the following transformation properties:

(i) Except for $H(z)$ the currents $J^B(z)$ belonging to the little group B are still vectors i.e. have conformal weights (1,0).

(ii) the field $H(z)$ now transforms not as a spin-one vector but as a spin-one *connection.*

(iii) The currents $J^\alpha(z)$ transform as conformal tensors (primary fields) of conformal weight $(1+h)$.

Thus, in particular, *the currents of grade* $h = -1$ *transform as conformal scalars.*

With this information in hand we are ready to impose the constraints, namely,

$$J^\alpha_{-1}(z) = J^\alpha_{-1}(0) \neq 0, \quad \text{and} \quad J^\alpha_h(z) = 0, \quad h < -1. \tag{11}$$

Here the set of constraints with non-zero right-hand-side do not break the conformal invariance generated by the new Virasoro operator $\Lambda(z)$ since they are scalars with respect to this operator, and the set of constraints with zero right-hand-side are added so that the complete system of constraints is first class. For the right-handed currents $\bar{J}(w)$ similiar constraints are imposed, but with $h < 0$ replaced by $h > 0$. In order to obtain an intuitive feeling for the meaning of the constraints (11) let us consider the case of G=SL(n,R), in which case the constrained current $J(z)$ takes the form

$$J^{constr.}(z) = \begin{pmatrix} J_{11}(z) & J_{12}(z) & J_{13}(z) & \cdots\cdots & J_{1n}(z) \\ J_{21}(0) & J_{22}(z) & J_{23}(z) & \cdots\cdots & J_{2n}(z) \\ 0 & J_{23}(0) & J_{33}(z) & \cdots\cdots & J_{3n}(z) \\ 0 & 0 & J_{34}(0) & \cdots\cdots & J_{4n}(z) \\ 0 & 0 & 0 & \cdots\cdots & J_{5n}(z) \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ 0 & 0 & 0 & J_{nn-1}(0) & J_{nn}(z) \end{pmatrix},$$

where the $J_{ab}(z)$ denote submatrices of currents which in general are not single entry or even square. Note that the constraints can also be expressed as

$$J^{\alpha}_{neg} = M \quad \text{and} \quad \bar{J}^{\alpha}_{pos} = N, \tag{12}$$

where M and N are constants matrices of grade minus one and plus one respectively, and *neg* and *pos* refer to the sign of h. The constraints (11) are not invariant with respect to general KM transformations, $J(z) \to U(z)^{-1}J(z)U(z) + U(z)^{-1}\partial_z U(z)$ but there exists a residual group of KM transformations with respect to which they are invariant. These are the KM transformations for which $U(z)$ lies in the group A of the Gauss decomposition which is generated by the root vectors with negative grade (E^{α}_{h}, for $h < 0$). Thus they are just the transformations that would be generated by the constraints themselves. The idea is to regard these residual KM transformations as gauge transformations and regard only those functions, or functionals, of $J(z)$ which are invariant with respect to this gauge group as physical. Thus finally one has (dimG-dimB)/2 constraints and (dimG-dimB)/2 gauge degrees of freedom, leaving just dimB physical fields altogether. It is possible to choose the gauge (at least locally) so that the physical fields are just the currents $J^B(z)$ belonging to the little group B.

5. Field Equations.

It is easy to see that the constraints (11) are consistent with the WZNW field equations (2), indeed are special solutions of some of them, and hence the WZNW field equations can be reduced to field equations for the unconstrained components of the current $J(z)$ If one now makes the Gauss decomposition $g(x) = a(x)b(x)c(x)$, defines the currents $J^A(x) = a^{-1}(x)\partial_z a(x)$, $J^B(x) = b^{-1}(x)\partial_z b(x)$, $\bar{J}^A(x) = a(x)\partial_w a^{-1}(x)$ etc., and imposes the constraints (12), the WZNW equations (2) reduce to

$$\partial_w J^B(x) = [b(x)Nb^{-1}(x), M], \tag{13}$$

and

$$\begin{aligned} J^A(x) &\equiv a^{-1}(x)\partial_z a(x) = b(x)Mb^{-1}(x), \\ \bar{J}^C(x) &\equiv (\partial_w c(x))c(x)^{-1} = b^{-1}(x)Nb(x) \end{aligned} \tag{14}$$

Note that, in contrast to the WZNW currents, the currents $J^A(x), J^B(x), \bar{J}^C(x)$ and $\bar{J}^B(x)$ are not functions of z and w alone.

The most interesting feature of (13) is that the equations for $J^B(z)$ do not involve the fields $J_h(z)$ for $h \neq 0$ and thus are self contained. Furthermore, it is easy to verify that they can be derived from the effective Lagrangian

$$L_{\text{eff}}(b(x)) = L_{WZNW}(b(x)) \quad + \quad \int \text{tr}(b(x)Mb^{-1}(x)N), \quad \text{where} \quad b(x) \in B. \tag{15}$$

This Lagrangian can be interpreted in two ways. First, it can be regarded as the generalization of the WZNW Lagrangian for fields belonging to the group B, but where, because B is reducible, there are interactions between the simple and abelian parts of B. Note, however, that since the constant matrices M and N have grades ± 1 there is a non-zero interaction only between the components of B which differ by one grade (nearest neighbour components). Second, by noting that in the special case when B is abelian i.e. when B is the Cartan subgroup ($b(x) = \exp(H_\alpha \phi_\alpha(x))$) and M and N are linear combinations of the positive and negative simple root-vectors respectively, the Lagrangian (15) reduces to the Toda Lagrangian, one sees that (15) can be regarded as a generalization of the Toda Lagrangian to the case in which the nearest-neighbour interacting fields are no longer abelian fields. Thus (15) may be regarded as describing either interacting WZNW fields or generalized Toda fields.

6. Two-Dimensional Gravity.

As mentioned in the Introduction, the Lagrangian (15) incorporates also a two-dimensional gravitational field. This comes about as follows: Since B is defined as the little group of H it follows that the one-parameter abelian group $R(1)$ generated by H is in the centre of B. Hence, locally at least, B may be written as the direct product $R(1) \otimes B_o$, where B_o denotes the rest of B. If we let $h(x)$ be the WZNW field belonging to $R(1)$ then the Lagrangian (15) can be re-written in the form

$$L_{\text{eff}}(b(x)) = L_{WZNW}(b_o(x)) \quad + \quad \oint (\partial h(x))^2 \quad + \quad \oint h(x)\text{tr}\left(b_o(x)Mb_o^{-1}(x)N\right). \tag{16}$$

But we have already seen that, unlike the rest of the components of the current which transform like primary fields, the components in the direction H transform like spin-one connections, and it is not difficult to deduce from this that the field $h(x)$ transforms like $\sqrt{g}$ where $g_{\mu\nu}$ is a two-dimensional metric. Accordingly, if one *defines* a metric as $g_{\mu\nu} = h(x)\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is any flat (constant) non-singular metric, introduces general coordinate transformations, and extends the tensor properties of the currents to be the same with respect to general coordinate transformations, one finds that the Lagrangian (16) may be written as

$$L_{\text{eff}}(b) = L_{WZNW}(b_o) \quad + \oint R\triangle^{-1}R \quad + \quad \oint \sqrt{(g)}\text{tr}\,(b_o M b_o^{-1} N)\,, \qquad (17)$$

where $R(x)$ is the Gauss curvature and $\triangle$ the two-dimensional d'Alembertian operator. It is clear that this Lagrangian describes a theory in which a two-dimensional gravitational field $h(x)$ and the WZNW fields $b_o(x)$ interact with themselves and with each other. The purely gravitational part of the interaction (which is obtained by setting $b_o(x) = 1$) is just the Liouville gravitational interaction which is induced by string theory in less than 26 dimensions [9]. This Liouville theory was embedded in an SL(2,R) Kac-Moody theory by Polyakov [5] in order to facilitate its quantization, so our procedure may be regarded as the converse of Polykov's for SL(2,R), and a generalization of the converse for the other groups.

7. Solutions of the Field equations.

The general solution of the field equations (13)(14) for the fields $b(x)$ are obtained from the general solution $g_s(z,w) = g(z)g(w)$, where $g(z)$ and $g(w)$ are arbitrary smooth matrices (not necessarily the same), of the WZNW equations as follows: First one makes a Gauss-decomposition of the general WZNW solution,

$$g_s(z,w) = g(z)g(w) = a(z)b(z)c(z)a(w)b(w)c(w), \qquad (18)$$

and notes that the field-equations (14) are then equivalent to the equations

$$\partial_z c(z) = [b(z)Mb(z)^{-1}]c(z) \quad \text{and} \quad \partial_w a(w) = a(w)[b(w)Nb^{-1}(w)]. \qquad (19)$$

This leaves $a(z), c(w)$ and $B(z), b(w)$ arbitrary. But $a(z)$ and $c(w)$ are pure gauge. So the solution is the B-part in the Gauss-decomposition of $b(z)c(z)a(w)b(w)$, where $b(z)$ and $b(w)$ are arbitrary, and $c(z)$ and $a(w)$ are given uniquely in terms of the b's by (9) (with initial values $a(0) = c(0) = 1$). Furthermore, it is easy to see that the B-part of the Gauss-decomposition of $b(z)c(z)a(w)b(w)$ is just $b(z)D(z,w)b(w)$, where $D(z,w)$ is the B-part in the Gauss-decomposition of $c(z)a(w)$. So we arrive at the following algorithm for constructing the general solution of the reduced system: choose any arbitrary (smooth) matrices $b(z), b(w)$ (not necessarily the same b in each case), solve (14) for $a(w)$ and $c(z)$, extract the B-part, $D(z,w)$ in the Gauss-decomposition of $c(z)a(w)$ and then the general solution is $b(z)D(z,w)b(w)$. The second step in the algorithm is, of course, purely algebraic, but the first step requires the solution of the differential equations (14), and so it might be thought that in this sense the algorithm is not complete. However, because of the nilpotency of the groups A and C the equations (14) can be solved by simple iteration. Indeed, if $c(z)$, for example, is decomposed into its H grades $c_h(z)$ then the solution is

$$c_h(z) = \int_o^z dz' c_{h-1}(z')(b(z')Mb^{-1}(z')), \qquad c_o(z) = 1. \tag{20}$$

Note the resemblance between the general solution (20) and the general solution $b(z)\bar{b}(w)$ for non-interacting WZNW fields belonging to the little group B. Indeed (20) reduces to this solution in the non-interacting case, for which $M = N = 0$ and hence, from (14), $D(z,w) = 1$.

8. Two-Block example.

To illustrate how the algorithm works in practice let us consider the case of a 2-block reduction of the SL(n,R) WZNW theory. Such a reduction is obtained by choosing as the H matrix of section 3 the matrix $H = \text{diag}(pI_q, qI_p)$ where I_q, I_p denote the unit matrices in q and p dimensions respectively, and $p+q=n$. Then

$$B(z) = \begin{pmatrix} B_{11}(z) & 0 \\ 0 & B_{22}(z) \end{pmatrix}, \quad C(z) = \begin{pmatrix} 1 & 0 \\ \gamma(z) & 1 \end{pmatrix}, \quad \text{and} \quad M = \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix}, \tag{21}$$

and similiarly for the right-handed fields and currents. Solving the equation for $C(z)$ in (19) one obtains at once

$$\gamma(z) = \oint_o^z dz' B_{11}(z') m B_{22}^{-1}(z'), \tag{22}$$

and a similiar expression for the corresponding upper-right-handed matrix entry $\alpha(w)$ in $A(w)$. On the other hand the B-part of the Gauss-decomposition of the matrix

$$C(z)A(w) = \begin{pmatrix} 1 & 0 \\ \gamma(z) & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha(w) \\ 0 & 1 \end{pmatrix}, \tag{23}$$

is easily seen to be

$$\begin{pmatrix} 1 - \alpha(w)\frac{1}{1+\gamma(z)\alpha(w)}\alpha(w) & 0 \\ 0 & 1+\gamma(z)\alpha(w) \end{pmatrix}. \tag{24}$$

Hence, according to the algorithm the general solution for a two-block reduction of the SL(n,R) WZNW theory is

$$B_{11}(z,w) = B_{11}(z)[1 - \alpha(w)\frac{1}{(1+\gamma(z)\alpha(w))}\gamma(z)]B_{11}(w) \tag{25}$$

and

$$B_{22}(z,w) = B_{22}(z)[1 + \gamma(z)\alpha(w)]B_{22}(w). \tag{26}$$

It is interesting to compare this result with the well-known solution of the Liouville ($SL(2,R)$-Toda) system,

$$e^{\phi/2} = \sqrt{f'}\frac{1}{1+fg}\sqrt{g'} = \sqrt{F}\frac{1}{(1+\int F\int G)}\sqrt{G}, \tag{27}$$

and

$$e^{-\phi/2} = \frac{1}{\sqrt{F}}[1 + \int F \int G]\frac{1}{\sqrt{G}}, \tag{28}$$

to which it reduces for $G = SL(2,R)$ i.e. when the entries in the above matrices are just numbers and not submatrices.

9. The W-Algebras of Toda Theory.

In this section we show that the W-algebras that have emerged in the Toda theory become much more understandable and tractable in the reduced WZNW context. First we identify them by means of the equation $\partial_z g(z) = J(z)G(z)$ connecting the WZNW fields $g(z)$ with their currents $J(z)$. These equations can be regarded as first-order differential equations for $g(z)$, given $J(z)$, and, it turns out that, in the constrained case, they can easily be reduced to higher order differential (or pseudo-differential) equations for those components of $g(z)$ that are gauge-invariant with respect to the residual gauge group discussed earlier. Since the coefficients of the powers of ∂_z in these higher-order equations are gauge-invariant with respect to the residual gauge group by construction, and are polynomials in the constrained currents and their derivatives because the group A is nilpotent, we see that they are gauge-invariant polynomials in the constrained currents and their derivatives. The crucial point now is that the higher-order equations obtained in this way are just the Gelfand-Dicke equations. Since the coefficients of the latter equations are just the base elements of the W-algebra of the Toda theory this immediately gives us an identification of the W-algebra as the algebra of local gauge-invariant polynomials in the constrained currents.

Although the identification of the W-algebra of Toda theory as the algebra of gauge-invariant polynomials of the constrained WZNW theory is very natural and intuitive it is not of great help for practical computations in arbitrary gauges.

However, there exists a set of gauges in which it is very useful and practical, and in which we obtain an alternative interpretation of the W-algebras as Dirac star algebras. These are the (DS) gauges introduced [10] by Drinfeld and Sokolow. In these gauges the local gauge-invariant polynomials in the constrained currents reduce to the currents themselves,

$$P(J_{(i)}(z), \partial_z^n J_{(k)}(z)) = J_{(i)}^{DS}(z), \tag{29}$$

where the $J_{(i)}(z)$, of which there are l, form a basis for the W-algebra. The gauge-fixing is complete in these gauges and the system of constraints obtained by combining the original constraints and the gauge fixing form a second class system of constraints. Hence their Poisson-bracket algebra (which, from (23), is just the W-Poisson-bracket algebra) is not their normal Kac-Moody algebra but the corresponding Dirac star algebra,

$$[P_{(i)}, P_{(k)}] = [J_{(i)}^{DS}, J_{(k)}^{DS}]^* = [J_{(i)}^{DS}, J_{(k)}^{DS}] - [J_{(i)}^{DS}, C_\alpha][C_\alpha, C_\beta]^{-1}[C_\beta, J_{(k)}^{DS}]. \tag{30}$$

We thus obtain an alternative identification of the W-algebra as the Dirac star algebra of the constrained currents in the DS gauge. This identification is very useful for practical purposes because in this gauge the gauge-fixing as well as the original constraints impose *linear* conditions on the currents. This means that the the constraints C_α in (30) can be replaced by components J_α of the currents themselves, in which case the right-hand-side of (30) can be expressed completely in terms of KM commutators. Furthermore, because of the nilpotency of the gauge group it turns out that the inverse constraint matrix $[J_\alpha^{DS}, J_\beta^{DS}]^{-1}$ is easy to compute and is a polynomial in the currents. Again we shall not give the details of the computation here but refer to the literature [3] in which, as examples, the W-algebras for the groups $G = A_2, B_2$ and G_2 are computed. It is well-known that that the W-algebra for G_2, which involves the Poisson bracket of two sixth-order polynomials, is very difficult to compute by direct methods. Indeed, as far as we know it has not yet been computed this way.

10. Kac-Moody and Virasoro Centres.

Finally, as a first step towards the quantization of the theory we derive a formula connecting the KM and Virasoro centres which is valid for any highest weight

(Fock) representation of the algebra, and should therefore be valid for any quantized version. The range of each centre seperately is not determined, but presumably other physical conditions for the quantization, such as unitarity, will restrict these ranges. The starting point is the well-known formula [8]

$$c = \frac{\dim G}{1 + g/k}, \tag{26}$$

where g is the Coxeter number of G, which connects the KM and Virasoro centres for unconstrained KM theories. When the Virasoro operator is modified according to (12) this formula acquires an extra term from the $\partial_z H(z)$ part and becomes

$$c = \frac{\dim G}{1 + g/k} - 12k\mathrm{tr}H^2. \tag{27}$$

What we now have to add is the contribution from the BRST ghosts due to the constraints. For this we observe that the current components to be constrained are primary fields with weights h_α for negative α. We must introduce a ghost pair for each of these constraints and from the usual formula [11] for the contribution of primary fields of these weights for highest weight representations we see that the ghost contribution is just

$$c_{ghost} = \sum_{\alpha>0} [12h_\alpha(h_\alpha - 1) - 2]. \tag{28}$$

Using the Lie algebraic formulae

$$\sum_{\alpha>0} h_\alpha = 2\bar{\mathrm{s}}.\mathrm{s} \quad \text{and} \quad \sum_{\alpha>0} h_\alpha^2 = \frac{1}{2}\mathrm{Tr}H^2, = g\mathrm{tr}H^2 \tag{29}$$

where $\bar{\mathrm{s}}$ denotes half the sum of the positive roots, and Tr denotes trace in the adjoint representation, we see that the ghost contribution to the centre can be written as

$$c_{ghost} = l - dimG - 12g\mathrm{tr}H^2 + 24\mathrm{s}.\bar{\mathrm{s}}. \tag{30}$$

By adding the ghost contribution to (27) and using the Freudenthal-de Vries formula $gD = 12\mathrm{s}^2$ one obtains finally

$$c = l - 12(\sqrt{(k+g)}\bar{\mathrm{s}} - (\sqrt{(k+g)})^{-1}\mathrm{s})^2, \tag{31}$$

as the required expression for the centre. This expression resembles the expressions in rational conformal field theories, but it is to be noted that what is being squared in (31) is a vector, not a number. For simply-laced groups, for which $s = \bar{s}$, equation (31) simplifies to

$$c = l - \frac{gD}{k+g}(k+g-1)^2 = l[1 - g(g+1)\frac{(r-s)^2}{rs}], \tag{32}$$

where $s/r = k+g$. The last expression resembles the formula in rational conformal field theory even more closely, but it must be rememkered that, without further information, the quantity r/s cannot be assumed to be rational.

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