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Non-Fractal Chaos in Two Dimensional Discrete Systems

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Abstract

Chaos in a two dimensional discrete system is analysed. The usual description of strange attractors in terms of the Lyapunov dimension was found to be inadequate to characterize the totality of chaotic behaviour observed in this system. An examination of the stability of the periodic orbits embedded in these attractors provides evidence to support this conclusion.

We consider the following two dimensional mapping described by Ushiki[1,2]:

$$X_{i+1} = (A - X_i - B_1 Y_i)\, X_i$$

$$Y_{i+1} = (A - Y_i - B_2 X_i)\, Y_i$$

We choose B_1=0.1, B_2=0.15 and A as the control parameter. For the range of values considered this map is bounded[1] and noninvertible. Other noninvertible systems[3,4] have been found to display chaotic behaviour similar to that described here and, experimentally, this chaotic behaviour is observed to be produced by coupled nonlinear oscillators[5,6].

In Figure 1(a) we show a plot of an attractor for the parameter value A=3.74572. An enlargement of one of the eight pieces is shown in Figure 1(b) for this period eight strange attractor. Apart from a slight distortion the other pieces have identical fractal structure. The Lyapunov exponents are λ_1=0.015 and λ_2=-0.062 and the dimension of the attractor using the Kaplan and York conjecture[7] is D_L=1.24. The correlation dimension D_2 computed from the Grassberger and Procaccia algorithm[8] is

$D_2=1.14\pm0.01$, which as expected is a lower bound on the Lyapunov dimension.

Also shown in Figure 1(b) are three unstable periodic orbits[9] of period 8,16 and 120. The period eight is non-attracting with two positive Lyapunov exponents. The period 16 orbit is such that it collides with the attractor as the control parameter A is reduced, leaving the vicinity of the fractal attractor by shooting out along its unstable manifold on to the attractor shown in Figure 2(a). The two period 120 orbits are the lowest unstable periodic orbits on this fractal attractor. The next consecutive orbits on the attractor are of period 128 and 136. All of these orbits were located using a Newton-Raphson iteration scheme. Apart from the period eight it was also possible to locate these orbits by scanning a time series for pairs of points separated by n time steps that are within a small preassigned spatial distance. For the case of this strange attractor it is the unstable and the stable manifolds of the periodic orbits that are responsible for the fractal (self-similar) structure.

In contrast, for A=3.745 we obtain the non-fractal attractor shown in Figure 2(a). In this case the Lyapunov exponents are $\lambda_1=0.26$ and $\lambda_2=0.06$. Since the sum of Lyapunov exponents is positive the Kaplan and York conjecture does not apply and hence no Lyapunov dimension can be defined. The correlation dimension D_2 was computed from the Grasberger and Procaccia algorithm. In Figure 3(a) we plot $\ln C_d(l)$ vs $\ln l$ for d=2,4,.....12. The time series consisted of 40000 points. The common slope in the scaling region is $D_2= 1.55\pm0.05$ as can be seen from Figure 3(b). In spite of the fact that this is a two dimensional system an extremely long time series was required because of the lack of spatially correlated data on the attractor. We note in passing that while we are unable to obtain a Lyapunov dimension, the correlation dimension still continues to be a useful measure.

This attractor is dense in periodic orbits, the majority of which have two positive Lyapunov exponents. It is these orbits that are responsible for the non-fractal structure. The most attracting orbits have the smallest Lyapunov exponents and it was found that these were the least difficult to locate when scanning a time series. Five of the most attracting period 8 orbits are shown in Figure 2(b) and, in comparison to Figure 2(a), it can be seen that they occupy the high density regions of the attractor.

In summary, we have studied fractal and non-fractal attractors in a two dimensional discrete system. The structure of these attractors has been shown to be related to the stability of the periodic orbits. The Lyapunov dimension was unobtainable for a wide range of parameters. We have reason to believe that the behaviour of this system is similar to the chaotic behaviour observed in infinite dimensional systems, both delay- differential and partial differential systems.

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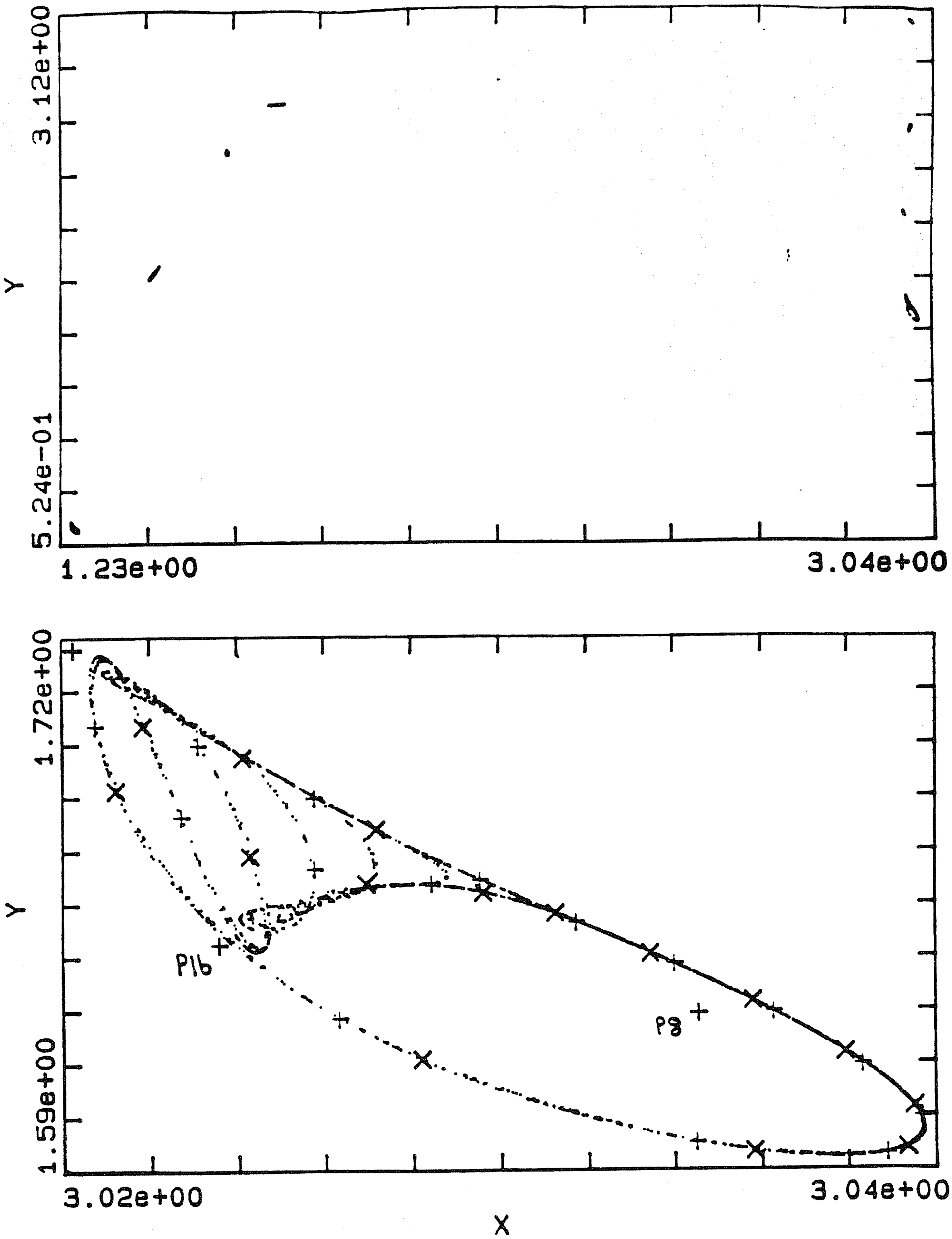

Figure 1. (a) Fractal attractor of the Ushiki mapping for A=3.74572. (b) Enlargement of one of the eight pieces shown in (a) and also shown are a period 8, 16 and two 120 periodic orbits.

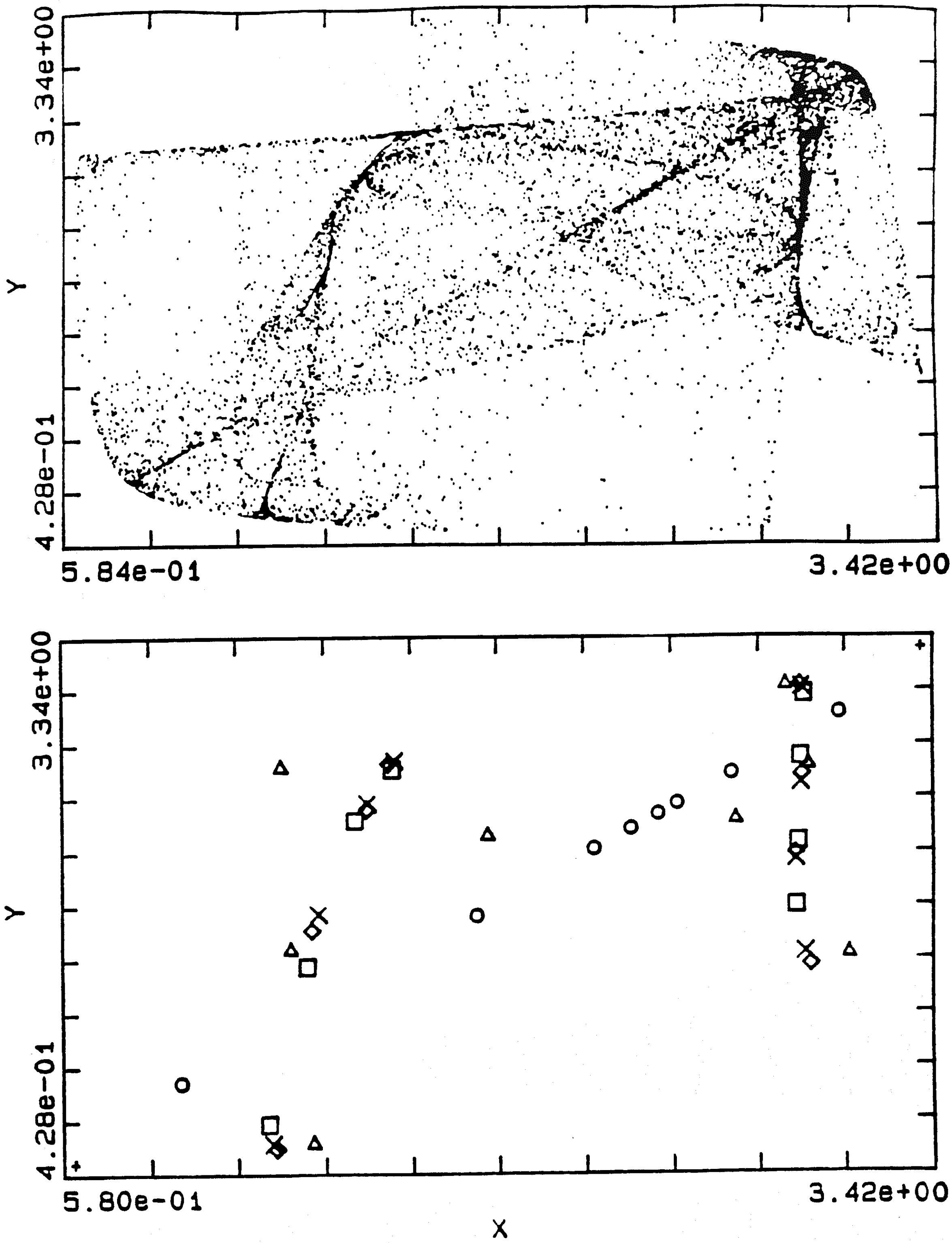

Figure 2. (a) Non-fractal attractor for A=3.745. (b) Unstable periodic orbits of period 8 for A=3.745.