DIAS-STP-89-08

Explicit Evaluation of The BRST Operator for Ashtekar's Chiral Gravity

B.P. Dolan

Dept. of Mathematical Physics
St. Patrick's College
Maynooth
Ireland

and

Dublin Institute for Advanced Studies
10 Burlington Road
Dublin 2
Ireland

ABSTRACT

The second order structure functions for the gauge algebra of Ashtekar's chiral canonical gravity are evaluated. The third order structure functions vanish. This allows the construction of the classical BRST field. The BRST extended Hamiltonian is also explicitly evaluated.

A tidy formulation of the canonical Hamiltonian approach to quantum gravity has recently been formulated which casts the constraints into the form of quadratic polynomials [1]. The construction involves identifying chiral connection one forms, ${}^{+}\omega^{mn}$, (m,n =0,1,2,3), with SU(2) connection one forms A^{i} (i=1,2,3), and extending the usual four constraints of canonical gravity (associated with general co-ordinate invariance) by an extra three constraints (associated with local, space-like, SO(3) rotations of an orthonormal basis).

We have explicitly evaluated the full constraint algebra and shown that all third order structure functions vanish. This involves an explicit evaluation of the classical BRST field, Ω [2,3] . The BRST extended Hamiltonian is also calculated.

First we briefly review Ashtekar's formalism [1] , as presented in [4], with the slight modification that all reference to spinor indices is eliminated. Then the full expressions for Ω and the extended Hamiltonian are presented.

As usual in canonical gravity, space-time is foliated into $\mathbb{R} \times \Sigma$, where Σ is a space-like surface and $\mathbb{R}$ represents a time variable, t (Σ is assumed compact to avoid surface terms).

The components of the four dimensional metric , $g_{\mu\nu}$, are replaced by lapse and shift functions $N^{\perp}$ and N^{a}, defined by (See e.g. [5])

$$g_{\mu\nu} = \begin{pmatrix} -(N^{\perp})^2 & g_{bc}N^c \\ g_{ac}N^c & g_{ab} \end{pmatrix}$$

where g_{ab} is a three dimensional space-like metric on Σ . (a,b, ... = 1,2,3 label co-ordinates on Σ).

In the tetrad formalism, we may take

$$e_{\mu}{}^{m} = \begin{pmatrix} N^{\perp} & e_c{}^{i} N^c \\ 0 & e_a{}^{i} \end{pmatrix}$$

where m,n, ... =0,1,2,3 are four dimensional orthonormal frame indices and i,j,k, ... = 1,2,3 are three dimensional orthonormal frame indices. Then $e_a{}^i$ is a triad for g_{ab} on Σ.

The inverse tetrad matrix is

$$e^{m\mu} = \begin{pmatrix} -1/N^{\perp} & N^a/N^{\perp} \\ 0 & e^{ia} \end{pmatrix} \qquad (1)$$

(The signature of g is (- + + +)).

Defining (anti) self-dual conection one forms

$$^{\pm}\omega^{mn} = \frac{1}{2}\left(\pm\omega^{mn} - i\,\frac{\varepsilon^{mn}}{2}{}_{pq}\,\omega^{pq}\right)$$

$(\varepsilon^{1230} = +1)$,

the curvature two forms split naturally

$$R(\omega)^{mn} = R(^{+}\omega)^{mn} - R(^{-}\omega)^{mn} .$$

The philosophy of Ashtekar's formalism is to use only one of these, say $R(^{+}\omega)^{mn}$, throwing away $R(^{-}\omega)^{mn}$. Then the Einstein-Hilbert action is modified to

$$S[e,{}^{+}\omega] = \int d^4x\, |e|\, e^{\mu m} e^{\nu n} R_{\mu\nu mn}(^{+}\omega) \qquad (2)$$

where $|e| = \det(e_\mu{}^m)$.

Now define SU(2) connection one-forms

$$A^i = \left(\pm\omega^{0i} + \frac{i}{2}\varepsilon^{ijk}\omega^{jk}\right).$$

Then

$$F^i = dA^i + \frac{i}{2}\varepsilon^{ijk}A^j \wedge A^k$$
$$= 2\,R({}^+\omega)^{0i} = i\,\varepsilon^{ijk}\,R({}^+\omega)^{jk} \qquad (3)$$

are the curvature two forms of a (four dimensional) SU(2) gauge field. Using (1) and (3) in (2), the action S reduces to

$$S[e, {}^+\omega] = \int dt \int_{\Sigma_t} d^3x \left\{ \tilde{e}^{ai}\dot{A}^i_a + A^i_0 \mathcal{G}_i - N^a \mathcal{H}_a - \underset{\sim}{N}^{\perp}\mathcal{H}_{\perp} \right\} \qquad (4)$$

where the following definitions have been used

$$\cdot \equiv \frac{\partial}{\partial t} = \partial_0 \quad , \tilde{e}^{ai} \equiv \det(e_b{}^j)\, e^{ai} \quad , \underset{\sim}{N}^{\perp} \equiv \frac{1}{\det(e_b{}^j)}\, N^{\perp}$$
$$\mathcal{H}_a \equiv F^i_{ab}\,\tilde{e}^{bi} \qquad (5)$$
$$\mathcal{H}_{\perp} \equiv \frac{i}{2} F^i_{ab}\;\tilde{e}^{aj}\tilde{e}^{bk}\,\varepsilon^{ijk}$$
$$\mathcal{G}_i \equiv (\mathcal{D}_a \tilde{e}^a)^i$$

with the SU(2) co-variant derivative

$$\mathcal{D}^{ij}_a \equiv \delta^{ij}\partial_a + i\,\varepsilon^{ikj}A^k_a \,.$$

In arriving at (4), an integration by parts has been carried out on the F^i_{0a} term of (2).

Thus, treating $\tilde{e}^{ai}$, N^a, $\underset{\sim}{N}^{\perp}$ and A^i_μ as canonical variables, (4) gives the constraints

$$\mathcal{H}_a \approx 0 \;, \quad \mathcal{G}_i \approx 0 \;, \quad \mathcal{H}_{\perp} \approx 0 \qquad (6)$$

together with the canonical equations

$$\dot{A}^i_a(x) = \frac{\delta H(t)}{\delta \tilde{e}^{ai}(x)} \qquad\qquad \dot{\tilde{e}}^{ai}(x) = -\frac{\delta H(t)}{\delta A^i_a(x)}$$

where the Hamiltonian is

$$H(t) = \int_{\Sigma_t} d^3x \left\{ N^a \mathcal{H}_a + \underset{\sim}{N}^{\perp}\mathcal{H}_{\perp} - A^i_0\, \mathcal{G}_i \right\}. \qquad (7)$$

Thus A_a^i and $\tilde{e}^{ai}$ are naturally canonically conjugate variables, with equal time Poisson brackets [4]

$$\{ A_a^i(x), \tilde{e}^{bj}(y) \} = \delta_a^b \, \delta^{ij} \, \delta(x-y).$$

The fact that A_a^i are complex and $\tilde{e}^{ai}$ are real is not a problem. As explained in [4], the extra components of A_a^i merely serve to reproduce the Bianchi identities $R_{mn} \wedge e^m \wedge e^n \wedge e^p = 0$ of the four dimensional curvature two forms. They are not, therefore, true dynamical degrees of freedom.

Now the Poisson bracket of any two quantities $B(A, \tilde{e})$ and $C(A, \tilde{e})$ can be defined as

$$\{B(x), C(y)\} = \int d^3z \left[\frac{\delta B(x)}{\delta A_a^j(z)} \frac{\delta C(y)}{\delta \tilde{e}^{aj}(z)} - \frac{\delta B(x)}{\delta \tilde{e}^{aj}(z)} \frac{\delta C(y)}{\delta A_a^j(z)} \right].$$

In particular, the constraints, (6) are all first class [6], as a little work shows. Denoting the constraints by ϕ_α, $\alpha = 1, \ldots, 7$, their equal time Poisson brackets are

$$\{ \phi_\alpha(x), \phi_\beta(y) \} = \int d^3z \, C_{\alpha\beta}{}^\gamma(x,y;z) \, \phi_\gamma(z)$$

where the structure functions are

$$C_{ab}{}^i(x,y;z) = -F^i_{ab} \, \delta(x-z) \, \delta(y-z)$$

$$C_{ab}{}^c(x,y;z) = \delta_a^c \, \delta(y-z) \, \partial_b^x \delta(x-z) - \delta_b^c \, \delta(x-z) \, \partial_a^y \delta(y-z)$$

$$C_{\perp a}{}^i(x,y;z) = i \, \varepsilon^{ijk} F^j_{ab} \, \tilde{e}^{bk} \, \delta(x-z) \, \delta(y-z)$$

$$C_{\perp a}{}^\perp(x,y;z) = \delta(y-z) \, \partial_a^x \delta(x-z) - \delta(x-z) \, \partial_a^y \delta(y-z)$$

$$C_{\perp\perp}{}^a(x,y;z) = (\tilde{e}^{ai} \tilde{e}^{bi})_y \, \delta(y-z) \, \partial_b^x \delta(x-z) - (\tilde{e}^{ai} \tilde{e}^{bi})_x \, \delta(x-z) \, \partial_b^y \delta(y-z)$$

$$C_{ij}{}^k(x,y;z) = i \, \varepsilon^{ijk} \, \delta(x-z) \, \delta(y-z).$$

The second order structure functions are defined via the Jacobi identity (the notation is that of [3])

$$\{ \phi_{[\alpha}(x), \{ \phi_\beta(y), \phi_{\gamma]}(z) \} \} = 0$$

$$\Rightarrow \int d^3u \, \{ \phi_{[\alpha}(x), C_{\beta\gamma]}{}^\delta \} \, \phi_\delta(u) + \int d^3u \, d^3v \, C_{[\beta\gamma}{}^\delta(y,z;u) \, C_{\alpha]\delta}{}^\varepsilon(x,u;v) \, \phi_\varepsilon(v)$$

$$\equiv 2 \int D_{\alpha\beta\gamma}{}^\delta \, \phi_\delta \, d^3u .$$

It is shown in [3] that this implies

$$D_{\alpha\beta\gamma}{}^{\delta}(x,y,z;u) = \frac{1}{2}\int \overset{(2)}{U}_{\alpha\beta\gamma}{}^{\delta\varepsilon}(x,y,z;u,v)\ \phi_{\varepsilon}(v)\, d^3v$$

where $\overset{(2)}{U}_{\alpha\beta\gamma}{}^{\delta\varepsilon}$ are the second order structure functions, completely anti-symmetric on $[\alpha\beta\gamma]$ and $[\delta\varepsilon]$.
Straightforward, but tedious, calculation shows that only two of the $\overset{(2)}{U}$ are non-zero.

$$\overset{(2)}{U}_{ab\perp}{}^{ij}(x,y,z;u,v) = -\frac{i}{3!}\,\varepsilon^{ijk}F^{k}_{ab}\,\delta(x-v)\,\delta(y-v)\,\delta(z-v)\,\delta(u-v)$$

$$\overset{(2)}{U}_{a\perp\perp}{}^{ib}(x,y,z;u,v) = \frac{1}{2.3!}\left(\tilde{e}^{ci}\delta_a{}^b + \tilde{e}^{bi}\delta_a{}^c\right)_v$$
$$\times\left[\delta(y-v)\,\partial_c^z\,\delta(z-v) - \delta(z-v)\,\partial_c^y\,\delta(y-v)\right]\delta(x-v)\,\delta(u-v).$$

The third order structure functions vanish identically. The BRST field, Ω ,for the gauge algebra of the classical theory is obtained, [3], by introducing anti-commuting ghost fields $\eta^{\perp}(x)$, $\eta^{a}(x)$ and $\lambda^{i}(x)$, one for each of the constraints (6), together with conjugate momenta $\mathcal{P}_{\perp}(x)$, $\mathcal{P}_{a}(x)$ and $\Pi_{i}(x)$ satisfying anti-commutator Poisson brackets,

$$\{\mathcal{P}_a(x),\eta^b(y)\}_+ = \{\eta^b(x),\mathcal{P}_a(x)\}_+ = -\delta_a{}^b\,\delta(x-y)$$

etc.

Then

$$\Omega(t) = \int_{\Sigma_t} d^3x\,\eta^{\alpha}\phi_{\alpha} - \frac{1}{2}\int_{\Sigma_t} d^3x\,d^3y\,d^3z\,\eta^{\beta}(y)\eta^{\alpha}(x)\,C_{\alpha\beta}{}^{\gamma}(x,y;z)\,\mathcal{P}_{\gamma}(z)$$
$$+\int_{\Sigma_t} d^3x\,d^3y\,d^3z\,d^3u\,d^3v\,\eta^{\gamma}(z)\eta^{\beta}(y)\eta^{\alpha}(x)\,\overset{(2)}{U}_{\alpha\beta\gamma}{}^{\delta\varepsilon}(x,y,z;u,v)\,\mathcal{P}_{\varepsilon}(v)\,\mathcal{P}_{\delta}(u)$$

where η^{α} are the seven ghosts and $\mathcal{P}_{\alpha}$ the seven ghost momenta. This construction ensures that $\{\Omega,\Omega\}_+ = 0$ automatically [3]. The BRST field is explicitly

$$\Omega = \overset{(0)}{\Omega} + \overset{(1)}{\Omega} + \overset{(2)}{\Omega}$$

where

$$\overset{(0)}{\Omega} = \int d^3x \left\{ \lambda \cdot (\mathcal{D}_a \tilde{e}^a) + \eta^a (F_{ab} \cdot \tilde{e}^b) + \frac{i\eta^\perp}{2} F_{ab} \cdot (\tilde{e}^a \times \tilde{e}^b) \right\}$$

$$\begin{aligned} \overset{(1)}{\Omega} = \frac{1}{2} \int d^3x \{ & \eta^b (\partial_b \eta^a) \mathcal{P}_a + i \eta^\perp \eta^b F_{bc} \cdot (\tilde{e}^c \times \pi) \\ & + \eta^b \eta^a (F_{ab} \cdot \pi) - (\partial_b \eta^b) \eta^\perp \mathcal{P}_\perp - \eta^b (\partial_b \eta^\perp) \mathcal{P}_\perp \\ & + 2\eta^\perp \partial_b \eta^\perp (\tilde{e}^a \cdot \tilde{e}^b) \mathcal{P}_a + i (\lambda \times \lambda) \cdot \pi \} \end{aligned} \qquad (8)$$

$$\begin{aligned} \overset{(2)}{\Omega} = \frac{1}{3!} \int d^3x \{ & i \eta^\perp \eta^b \eta^a F_{ab} \cdot (\pi \times \pi) + \eta^\perp (\partial_a \eta^\perp) \eta^a \mathcal{P}_b (\tilde{e}^c \cdot \pi) \\ & + \eta^\perp (\partial_c \eta^\perp) \eta^a \mathcal{P}_a (\tilde{e}^c \cdot \pi) \} . \end{aligned}$$

All dot and cross products refer to SU(2) indices, $i, j, k, \ldots$.

With these results, the BRST extended Hamiltonian is not difficult to compute. It is a ghost number zero field which satisfies, [3] ,

$$\{ H_{BRST}(t), \Omega(t) \} = 0$$

and has an expansion

$$H_{BRST} = \overset{(0)}{H} + \int d^3x d^3y \, \overset{(1)}{H}_\alpha{}^\beta (x; y) \eta^\alpha(x) \mathcal{P}_\beta(y) + \ldots$$

where $\overset{(0)}{H}$ is the Hamiltonian, (7).
It is given by

$$H_{BRST}(t) = \int_{\Sigma_t} d^3x \left\{ N^a \frac{\overrightarrow{\partial}}{\partial \eta^a} + \underset{\sim}{N}^\perp \frac{\overrightarrow{\partial}}{\partial \eta^\perp} - A_0^i \frac{\overrightarrow{\partial}}{\partial \lambda^i} \right\} \Omega . \qquad (9)$$

This clearly has the correct form for $\overset{(0)}{H}$, so we need only prove that it commutes with Ω . This is immediate, since

$$0 = \frac{\overrightarrow{\partial}}{\partial \eta^\alpha} \{ \Omega, \Omega \}_+ = \left\{ \frac{\overrightarrow{\partial}}{\partial \eta^\alpha} \Omega, \Omega \right\} - \left\{ \Omega, \frac{\overrightarrow{\partial} \Omega}{\partial \eta^\alpha} \right\} = 2 \left\{ \frac{\partial \Omega}{\partial \eta^\alpha}, \Omega \right\} .$$

Hence (9) is the required Hamiltonian.

In conclusion, we have shown that the third order structure functions of Ashtekar's canonical gravity vanish, and we have evaluated the classical BRST field, (8), and the BRST extended Hamiltonian, (9), explicitly.

REFERENCES

1 A. Ashtekar, Phys.Rev.Lett,57, 2244 (1986);
Phys.Rev. D36, 1587, (1987)

2 E.S. Fradkin and M.A. Vasiliev, Phys.Lett. 72B, (1977), 70
E.S. Fradkin and T.E. Fradkina, Phys.Lett. 72B, (1978),343.

3 M. Henneaux, Phys.Rep. 126, (1985), 1.

4 T. Jacobson and L. Smolin, Class.Quantum grav. 5 (1988) 583

5 C.W. Misner, K.S. Thorne and J.A. Wheeler, "Gravitation", Freeman (1973).

6 P.A.M. Dirac "Lectures on Quantum Mechanics" Academic Press (1987).