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ON THE RENORMALIZATION GROUP APPROACH TO STRING EQUATIONS OF MOTION

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Abstract

We discuss the program of deriving the string field theory equations of motion for all (massless and massive) string modes as the renormalization group fixed point equations for the most general sigma model containing all possible ("non-renormalizable") interactions. We review the approach based on the Wilson RG equation and point out the problem of cutoff dependence of the interaction term in the corresponding "quadratic" beta function. The relation between the sigma model path integral and the string scattering amplitudes is clarified. We suggest a new approach to derivation of the generalized sigma model beta functions in which the central role is played by the condition of completeness of the set of interaction terms ("vertex operators") present in the sigma model action. The use of the completeness relation makes it possible to obtain closed expressions for the sigma model partition function and the beta functions. The resulting beta functions contain all higher powers of the couplings (fields).

1. Introduction

One of the remarkable properties of string theory is its connection with 2-d σ-models. First established at the level of perturbatively renormalizable σ-models /1-4/, corresponding to low energy dynamics of the massless string modes /5/, this connection played important role in search for vacuum configurations in string theory /6/. Since conformal or Weyl invariance is a central consistency requirement at the string tree level, it is natural to expect that a version of conformal invariance condition imposed on a most general 2-d σ-model (containing all possible "non-renormalizable" interactions corresponding to all higher order massive string modes) may be equivalent to the exact tree-level string equations of motion for all the infinite system of string local fields.

Below we shall review and elaborate on a number of suggestions concerning the program of deriving the complete system of string field equations from the σ-model (see in this connection refs./7-21/). We shall consider the two approaches. The first /10,17-19/ uses the Wilson RG equation /22-23/. The second is based on the partition function for the σ-model and on the "completeness" relation for the interaction terms in the σ-model action. The "completeness" relation is simply the expression of the fact that all possible local interactions are included in the action.

We start with general remarks on the Wilson's RG approach (sect.2) and then consider its application to string theory (sect. 3). Attempts to establish correspondence with string S-matrix are reviewed in sect.4. We clarify the issue of subtraction of the Möbius infinities and point out the existence of two different procedures for identifying string scattering amplitudes with some σ-model objects. One is based on the interpretation of the σ-model partition function as a generating functional for string S-matrix and the other starts with the β-function for an effective renormalizable σ-model and considers the solution of the $\beta=0$ equation.

In sect.5 we develop an alternative approach in which the central role is played by the completeness condition. We obtain the closed expressions for the β-functions which, in contrast to the β-functions in the Wilson's approach, are "less" cutoff dependent and contain terms of all orders in the fields. Some open problems are discussed in sect.6.

The present paper can be considered as a complement to our recent review of the σ-model approach to string theory /24/ where only the renormalizable σ-models were considered.

2. General remarks on the Wilson's renormalization group approach

Below (in sects.2 and 3), we shall discuss an attempt to interpret the Wilson's RG equation in a 2-d quantum field theory as a classical equation of motion of a string field theory. In view of the well-established relation between the renormalizable 2-d sigma models and the effective equations of motion for the massless string modes it is natural to expect that considering a most general 2-d sigma model on a curved background and imposing the condition of its Weyl invariance one should get the fundamental equations for all string modes*/. It was observed in ref./10/ that the

*/ Generalizations of the RG approach to include the massive string modes were considered, e.g., in /1,7-9/.

"bilinear" structure of the Wilson RG equation /21/ is reminiscent of the "cubic" interaction in a string field theory. The suggestion /10/ to consider this RG equation as a classical string field theory equation of motion was recently elaborated upon in refs. /17-19/.

While it is certainly essential to impose the Weyl (or a kind of BRST) invariance in order to consistently discribe the dynamics of all string modes, it is sufficient, as a first step, to consider the condition of the global scale invariance of a 2-d theory on a flat background.

We shall discuss and try to clarify some points of the program of ref./10/ and, in particular the relation to the string S-matrix /18/. Let x^μ denote a set of dimensionless fields defined on a 2-d plane*/. Consider the theory

$$I = I_0 + I_{int} \quad , \quad I_0 = \frac{1}{2}\int d^2\sigma \, x^\mu \Delta_0 x^\mu \quad , \quad \Delta_0 = -\Box \quad . \quad (2.1)$$

$G_0 = \Delta_0^{-1}$ is a regularized propagator depending on a fixed cutoff Λ_0. The basic object is the partition function

$$Z[\bar{x}] = z_0^{-1}\int \mathcal{D}x \, \exp\left(-\frac{1}{2} x G_0^{-1} x - I_{int}[x+\bar{x}]\right) , \quad (2.2)$$

$$Z = \langle \exp\left(-I_{int}[x+\bar{x}]\right) \rangle , \quad \langle 1 \rangle = 1 \, ,$$

where $\bar{x}$ is arbitrary. It is easy to prove that

$$Z[\bar{x}] = \exp\left(\frac{1}{2} G_0 \cdot \frac{\delta^2}{\delta \bar{x}^2}\right) \exp\left(-I_{int}[\bar{x}]\right) , \quad (2.3)$$

$$G_0 \cdot \frac{\delta^2}{\delta x^2} \equiv \int d^2\sigma_1 \, d^2\sigma_2 \, G_0(\sigma_1, \sigma_2) \frac{\delta^2}{\delta x^\mu(\sigma_1) \, \delta x^\mu(\sigma_2)} \quad . \quad (2.4)$$

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*/ In what follows we shall discuss only the 2-d theories corresponding to the closed Bose string theory expanded near a flat D-

In fact,

$$< F[x+\bar{x}]> = \int \mathcal{D}J\ \tilde{F}[J]\ < e^{iJx} > e^{iJ\bar{x}} =$$
$$= \int \mathcal{D}J\ \tilde{F}[J]\ \exp\left(-\tfrac{1}{2} J G_0 J\right) e^{iJ\bar{x}}$$
$$= \int \mathcal{D}J\ \tilde{F}[J]\ \exp\left(\tfrac{1}{2} G_0 \cdot \frac{\delta^2}{\delta \bar{x}^2}\right) e^{iJ\bar{x}} = e^{\frac{1}{2} G_0 \cdot \frac{\delta^2}{\delta \bar{x}^2}} F[\bar{x}] \quad (2.5)$$

Suppose now we split G_0 into the two pieces, introducing some "intermediate" cutoff Λ, $0<\Lambda<\Lambda_0<\infty$,

$$G_0(\Lambda_0) = G(\Lambda) + \bar{G}(\Lambda_0, \Lambda) \quad . \quad (2.6)$$

For example,

$$G_0(\Lambda_0) = \int_0^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{e^{ik\sigma_{12}}}{k^2} + \int_{\Lambda}^{\Lambda_0} \frac{d^2k}{(2\pi)^2} \frac{e^{ik\sigma_{12}}}{k^2} \quad , \quad \sigma_{12} = \sigma_1 - \sigma_2 \quad (2.7)$$

or

$$G_0(\Lambda_0) = -\frac{1}{4\pi} \ln\left(\sigma_{12}^2 + \Lambda_0^{-2}\right) \quad , \quad (2.8)$$
$$\bar{G}(\Lambda_0, \Lambda) = -\frac{1}{4\pi} \ln\left(\frac{\sigma_{12}^2 + \Lambda_0^{-2}}{\sigma_{12}^2 + \Lambda^{-2}}\right) \quad .$$

Then (2.3) implies that

$$Z[\bar{x}] = \exp\left(\tfrac{1}{2} G \cdot \frac{\delta^2}{\delta \bar{x}^2}\right) e^{-I_{eff}[\bar{x}]} = \quad (2.9)$$
$$= z^{-1} \int \mathcal{D}x\ \exp\left(-\tfrac{1}{2} x G^{-1} x - I_{eff}[x+\bar{x}]\right) \quad ,$$

$$e^{-I_{eff}[\bar{x}]} \equiv e^{\frac{1}{2}\bar{G} \cdot \frac{\delta^2}{\delta \bar{x}^2}} e^{-I_{int}[\bar{x}]} = \bar{z}^{-1} \int \mathcal{D}x\ e^{-\frac{1}{2} x \bar{G}^{-1} x - I_{int}[x+\bar{x}]} \quad , \quad (2.10)$$

$$z = \int \mathcal{D}x\ \exp\left(-\tfrac{1}{2} x G^{-1} x\right) , \quad \bar{z} = z_0 / z \quad .$$

Hence I_{eff} defines the effective theory corresponding to the result of integration over the modes with momenta $\Lambda < k < \Lambda_0$.

Let us consider the dependence of I_{eff} on Λ . Since Z in (2.2) does not depend on Λ

$$\frac{d\,Z[\bar{x}]}{d\Lambda} = 0 , \quad \frac{d}{d\Lambda}\left[z^{-1} \int \mathcal{D}x\ e^{-\frac{1}{2} x G^{-1} x - I_{eff}[x+\bar{x}]} \right] = 0 \quad (2.11)$$

Differentiating over Λ we find $(\Delta \equiv G^{-1})$

$$z^{-1}\dot{z} + \tfrac{1}{2} < x \dot{\Delta} x > + < \dot{I}_{eff} > = 0 \quad , \quad \dot{F} \equiv \Lambda \frac{\partial}{\partial \Lambda} F \quad .$$

It is easy to prove (using functional integration by parts) that

$$\langle x \dot{\Delta} x \rangle =$$

$$\int d^2\sigma_1 d^2\sigma_2 \; \dot{G}(\sigma_{12}) \left\langle \frac{\delta^2 I_{eff}}{\delta x^\mu(\sigma_1)\delta x^\mu(\sigma_2)} - \frac{\delta I_{eff}}{\delta x^\mu(\sigma_1)} \frac{\delta I_{eff}}{\delta x^\mu(\sigma_2)} \right\rangle + tr(\Delta^{-1}\dot{\Delta}) \quad (2.12)$$

Since $\zeta^{-1}\dot{\zeta} = -\frac{1}{2} tr(\Delta^{-1}\dot{\Delta})$ we thus get

$$0 = \left\langle \dot{I}_{eff}[x+\bar{x}] + \frac{1}{2}\int d^2\sigma_1 d^2\sigma_2 \; \dot{G}(\sigma_{12}) \left(\frac{\delta^2 I_{eff}}{\delta x^\mu(\sigma_1)\delta x^\mu(\sigma_2)} - \frac{\delta I_{eff}}{\delta x^\mu(\sigma_1)} \frac{\delta I_{eff}}{\delta x^\mu(\sigma_2)} \right) \right\rangle_{x+\bar{x}} \quad (2.13)$$

The Wilson's RG equation is obtained by dropping the expectation value brackets*/

$$\dot{I}_{eff} + \frac{1}{2}\int d^2\sigma_1 d^2\sigma_2 \; \dot{G}(\sigma_{12}) \left(\frac{\delta^2 I_{eff}}{\delta x^\mu(\sigma_1)\delta x^\mu(\sigma_2)} - \frac{\delta I_{eff}}{\delta x^\mu(\sigma_1)} \frac{\delta I_{eff}}{\delta x^\mu(\sigma_2)} \right) = 0 \quad (2.14)$$

We can check directly that I_{eff} defined by the functional integral in eq.(2.10),(2.9) satisfies the functional equation (2.14). In general, eq.(2.14) is satisfied by any I_{eff} of the form

$$I_{eff} = -\ln H \; , \quad H[x] = \exp\left(\frac{1}{2} K \cdot \frac{\delta^2}{\delta x^2}\right) F[x] \; , \quad (2.15)$$

where F is an arbitrary functional independent of Λ and

$$\dot{K}(\Lambda) = -\dot{G}(\Lambda) \; , \quad K(\Lambda) = -G(\Lambda) + c \; , \quad \dot{c} = 0 \; . \quad (2.16)$$

In fact,

$$I''_{eff} - (I'_{eff})^2 = -\frac{H''}{H} \; , \qquad (\;)' \equiv \frac{\delta}{\delta x}(\;) \; , \quad (2.17)$$

$$\dot{I}_{eff} = -\frac{\dot{H}}{H} = -\frac{1}{2}\dot{K}\,\frac{H''}{H} \; .$$

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*/ We thus replace the functional integral $\langle F[x+\bar{x}] \rangle = 0$ by the functional differential equation $F[x]=0$. There is, of course, the freedom of making simultaneous redefinition of the integration variable, $x' = x + \varepsilon\xi[x]$, $\int \mathcal{D}x \, e^{-I_0[x]} F[x+\bar{x}] = \int \mathcal{D}x' e^{-I_0[x']} F[x'+\bar{x}$

Eq.(2.10) corresponds to $C=G_0(\Lambda_0)$, $F=e^{-I_{int}}$.

We see that eq.(2.14) has a little dynamical content: it knows nothing about $G_0(\Lambda_0)$ and I_{int}, i.e. the parameters of the "fundamental" theory. The information about the fundamental theory should be introduced by hand through the initial condition:

$$\left(I_{eff} \right)_{\Lambda=\Lambda_0} = I_{int} \qquad (2.18)$$

In this respect the Wilson RG equation is different from the Gell-Mann-Low equation $\dot{g}=\beta(g)$ in which the β-function "knows" about the dynamics of the theory.

Additional dynamical input is provided by the assumption that I_{eff} can be represented as a combination of a complete set of local "operators" V_i built out of x and all its derivatives

$$I_{eff} = \sum_i \varphi^i \cdot V_i \quad , \quad V_i = \Lambda^{\gamma_i} \bar{V}_i \qquad (2.19)$$

Here $\varphi^i(\Lambda)$ are the corresponding dimensionless couplings and γ_i are the dimensions of the operators $\bar{V}_i$. I_{eff} depends on the cutoff implicitly through $\varphi^i(\Lambda)$ and explicitly through Λ^{γ_i}. In view of the completeness of the set $\{V_i\}$ it should be possible to represent the functional derivative terms in (2.14) as

$$\frac{1}{2}\int \dot{G} \cdot \frac{\delta^2}{\delta x^2} V_i = -\sum_j \Delta^j{}_i \, V_j \quad , \qquad (2.20$$

$$\frac{1}{2}\int \dot{G} \cdot \frac{\delta V_i}{\delta x} \; \frac{\delta V_j}{\delta x} = \sum_k f^k_{ij} \, V_k \quad . \qquad (2.20$$

Substituting (2.19), (2.20), into (2.14) we get an equation of the form $\sum_k \psi^k V_k = 0$ or, in view of the completeness of V_i, $\psi^k=0$. Explicitly, we find

$$\beta^i \equiv \dot{\varphi}^i = -\gamma_i \varphi^i + \sum_j \Delta^i{}_j \varphi^j + \sum_{j,k} f^i_{jk} \varphi^j \varphi^k \qquad (2.2$$

Given a theory with the corresponding set of V_i we can in principle compute the basic matrices Δ^i_j and f^i_{jk} and hence determine the "β-function" (2.21). Recalling the derivation of eq.(2.14) we conclude that eq.(2.21) is essentially equivalent to the condition of Λ-independence of Z (2.9),

$$\Lambda \frac{\partial}{\partial \Lambda} Z + \beta^i \frac{\partial Z}{\partial \varphi^i} = 0 \quad . \tag{2.22}$$

The important difference of the "β-function" (2.21) from the ordinary Gell-Mann-Low function in renormalizable models is that (as it is clear from (2.20)) the coefficients f^i_{jk} in (2.21) in general depend on the cutoff Λ. Hence it is not possible to represent the solution of (2.21) in the usual form

$$\varphi^i(\Lambda) = \left(\frac{\Lambda}{\mu}\right)^{-\gamma_i} \left[\varphi^i(\mu) + \sum_{n=1}^{\infty} T^i_n(\varphi(\mu)) \left(\ln \frac{\Lambda}{\mu}\right)^n\right] , \tag{2.23}$$

where T_n do not explicitly depend on Λ or μ . Eq.(2.23) gives the relation between the bare and renormalized couplings in the renormalizable thories in which we take the limit $\Lambda \to \infty$ (i.e. drop the inverse powers of Λ). In contrast, in the Wilson approach we keep the cutoff Λ fixed. If (2.23) were true, we would get

$$\beta^i = \dot{\varphi}^i = -\gamma_i \varphi^i + T^i_1(\varphi) \quad . \tag{2.24}$$

This is clearly inconsistent with (2.21) if f^i_{jk} are Λ-dependent. It is only in the "continuum limit" that we can eliminate the nonrenormalizable ($\gamma_i < 0$) couplings from (2.21) obtaining non-linear but Λ-independent β-functions for the renormalizable ones.

3. Application of the Wilson RG equation to string theory

Let us try to apply the above considerations to the case of the closed Bose string theory. In a string theory context the coupling constants φ^i in (2.19) correspond to the components of a string field and the interaction vertices V_i ("vertex operators") are given by the 2-d integrals of all possible combinations of derivatives of coordinates x^μ ($\mu = 1, \dots, D$). The product $\varphi^i \cdot V_i$ can be represented either in the coordinate or in the momentum space as

$$\varphi^i \cdot V_i = \int d^D y \, \varphi^i(y) \, \tilde{V}_i(y) = \int d^D k \, \tilde{\varphi}^i(k) \, V_i(k) \,, \qquad (3.1)$$

$$\tilde{V}_i(y) = \int d^2\sigma \, \delta^{(D)}(y^\mu - x^\mu(\sigma)) \, v_i(\partial x)$$

$$V_i(k) = \int d^2\sigma \, e^{ikx(\sigma)} \, v_i(\partial x) \,, \quad v_i = \{ \partial^{n_1} x^{\mu_1} \dots \partial^{n_s} x^{\mu_s} \} \qquad (3.2)$$

The index "i" is thus a shorthand notation for a set of D-dimensional indices. We assume that a non-trivial v_i contains at least two $\partial^n x$ -factors. The dimension of V_i is thus $\gamma_i = 2 - 2N_i$, $2N_i = \sum_k n_k$ total number of derivatives in v_i. γ_i is always even since the 2-d indices in v_i are contracted either by δ_{ab} or by ε_{ab}. Note that there can be several operators with the same γ_i. It is convenient to include the integration over y^μ or k^μ in the convention of summation over i. Thus the interaction term in the string action I_{int} or I_{eff} (2.19) takes the form

$$I_{eff} = \Lambda^2 \int d^2\sigma \, \Phi(x(\sigma)) + \int d^2\sigma \, \partial_a x^\mu \partial^a x^\nu H_{\mu\nu}(x(\sigma)) \qquad (3.3)$$
$$+ \dots + \Lambda^{-4} \int d^2\sigma \, \partial^a x^\mu \partial^b x^\nu \partial_a \partial_b x^\lambda \partial^2 x^\rho B_{\mu\nu\lambda\rho}(x(\sigma)) + \dots$$

We choose $2\pi\alpha' = 1$ and $[x] = cm^0$, $[\Lambda] = cm^{-1}$ $[\sigma] = cm$, so that all the space-time fields φ^i are dimensionless.

The basic dynamical principle of the RG approach is to identify the scale invariance or RG "fixed point" conditions $\beta^i=0$ with the equations of motion for string fields φ^i. Using eq.(2.21) for the "β-functions" we thus get

$$\omega^i_j \varphi^j + f^i_{jk} \varphi^j \varphi^k = 0 \ , \quad \omega^i_j \equiv \Delta^i_j - \gamma_i \delta^i_j \qquad (3.4)$$

(we shall always imply summation over the repeated indices i,j,k,... except in the case when γ_i is present).

Eq.(3.4) looks like an equation of motion in a field theory with a cubic interaction. The problem with this equation (which was not sufficiently appreciated in the previous discussions) is that while (3.4) expresses the condition of the Λ-independence of φ^i the "coupling matrix" f^i_{jk} in (3.4) depends non-trivially on the cutoff Λ. Hence it appears that the solution of (3.4) should also depend on Λ, in contradiction with the original assumption $\dot{\varphi}^i=0$. A possible clue to a resolution of this paradox is provided by the observation that being dimensionless f^i_{jk} may depend on Λ only through the dimensionless combination ΛR of Λ and an IR cutoff R, $|\Delta\sigma| \leq R$. Hence it may be that eq.(3.4) is consistent only if ΛR has some fixed value, i.e. if $R \sim \Lambda^{-1}$ so that f^i_{jk} is Λ-independent. In any case, the role of the IR cutoff needs to be clarified.

The above mentioned difficulty may be suggesting that while the "initial" I_{int} in (2.2) may be given by the local expression (3.3) we should not try to solve the Wilson equation (2.14) using the local ansatz (3.3) for I_{eff}. In fact, we know already that (2.14) is solved by the path integral (2.10), which produces a com licated non-local expression for $I_{eff}[x]$ parametrized by Λ - independent couplings φ^i appearing in I_{int}. The problem is then what is the scale invariance condition for non-local I_{eff}. Should it be $\dot{I}_{eff}=0$? But this is then contradicting our previous suggestion $\dot{\varphi}^i=0$ in the case of $I_{eff}=\varphi^i V_i$ since V_i are supposed to depend on Λ in order for φ^i to be dimensionless. Note also that in view of the completeness property of $\{V_i\}$ any non-local I_{eff} can be formally rewritten as a local one (2.19), at the expense of introducing the explicit dependence on an IR cutoff. A possible resolution of this puzzle is to formulate the scale invariance condition as[22] $\dot{I}_{eff}=-\int d^2\sigma\, \sigma^a \partial_a x^\mu \frac{\delta I_{eff}}{\delta x^\mu(\sigma)}$. In fact, if $I_{eff}=I_1+\dots$, $I_1=\Lambda^2\int d^2\sigma\, \Phi(x)$, we get: $\dot{I}_1=2I_1$, $\int d^2\sigma\, \sigma^a\partial_a x^\mu \frac{\delta I_1}{\delta x^\mu}$ $=\Lambda^2\int d^2\sigma\, \sigma^a\partial_a x^\mu \partial_\mu \Phi = \Lambda^2\int d^2\sigma\, \sigma^a \partial_a \Phi = -2I_1$.

Hence

$$\dot{I}_{eff} = -\int d^2\sigma\, \sigma^a \partial_a x^\mu \frac{\delta I_{eff}}{\delta x^\mu(\sigma)} \tag{3.5}$$

is a natural scale invariance condition which applies in the general case of a non-local I_{eff} and reduces to $\beta^i=\dot{\varphi}^i=0$ in the case of the local ansatz (2.19). Eq.(3.5) expresses the vanishing of the "anomalous dimensions" of the interaction terms in I_{eff}, generalizing the condition $\beta^i=0$. The term in the r.h.s. of (3.5) simply counts the number of derivatives in a given term in I_{eff}. We shall use a non-local ansatz for I_{eff} discussing the correspondence with the string S-matrix.

Let us now proceed to the analysis of the scale invariance equation (3.4) in the case of the local ansatz for I_{eff}. Our first task is to compute the matrices Δ^i_j and f^i_{jk} defined in eq. (2.20). Let us choose the regularized propagator as in eq.(2.8). Then

$$\dot{G}(\sigma_{12}) = \frac{1}{2\pi} \frac{\Lambda^{-2}}{\sigma_{12}^2 + \Lambda^{-2}} , \quad \dot{G}(0) = \frac{1}{2\pi} . \tag{3.6}$$

Using the momentum space representation for the vertex operator (3.2) we note that the $\frac{\delta^2}{\delta x^2}$ -operator in (2.20a) may act either on $\mathcal{V}_i(\partial x)$ or on e^{ikx}. If both functional derivatives act on e^{ikx} we get

$$\int d^2\sigma\, d^2\sigma_1\, d^2\sigma_2 \; \dot{G}(\sigma_{12}) (-k^2)\, \delta^{(2)}(\sigma_{12})\, \delta^{(2)}(\sigma_2 - \sigma) \times$$
$$\times \left(\mathcal{V}_i(\partial x) e^{ikx}\right)(\sigma) = -\frac{1}{2\pi} k^2 \, \overline{V}_i(k) . \tag{3.7}$$

If one of the $\frac{\delta}{\delta x}$ acts on $\mathcal{V}_i$ and the other on e^{ikx} we obtain, e.g.,

$$\int d^2\sigma\, d^2\sigma_1\, d^2\sigma_2\, \dot{G}(\sigma_{12}) \; ik^\mu\, \delta^{(2)}(\sigma - \sigma_1)\, \partial^{n_1} \delta^{(2)}(\sigma - \sigma_2)\, e^{ikx(\sigma)} \times$$
$$\times \left(\partial^{n_2} x^\lambda \ldots \partial^{n_k} x^\rho\right)(\sigma) . \tag{3.8}$$

Observing that $\partial_\sigma^{n_1} \delta^{(2)}(\sigma - \sigma_2) = (-\partial_{\sigma_2})^{n_1} \delta^{(2)}(\sigma - \sigma_2)$ we can integrate by parts to get: $\partial_{\sigma_2}^{n_1} \dot{G}(\sigma_{12})\big|_{\sigma_1 = \sigma_2} \sim \Lambda^{n_1}$ times an operator with a number of derivatives less by n_1. Similar result is found in the case when both $\frac{\delta}{\delta x(\sigma_1)}$ and $\frac{\delta}{\delta x(\sigma_2)}$ act on $\mathcal{V}_i(\partial x)$. Thus Δ^i_j in (2.20a) has the following "triangle" structure:

$$\Delta^i_j = \begin{cases} \frac{1}{4\pi} k^2 , & i = j \\ 0 , & N_j < N_i \\ O(k^\mu, \delta_{\mu\nu}) \neq 0 , & N_j > N_i \end{cases} , \quad \gamma_i = 2 - 2N_i \tag{3.9}$$

This implies that the linear term in the β -function for some field $\varphi^{\mu_1 \ldots \mu_n}$ corresponding to the level N_i is (in momentum representation)

$$\beta^{\mu_1 \ldots \mu_n} = \left(\frac{1}{4\pi} k^2 - 2 + 2N_i\right) \varphi^{\mu_1 \ldots \mu_n} +$$
$$+ (k_\lambda \text{ or } \delta_{\lambda\rho} \quad \text{contracted with the fields } \varphi^{\mu_1 \ldots \mu_n \lambda} \text{ or } \varphi^{\mu_1 \ldots \mu_n \lambda\rho} \tag{3.10}$$

We thus see that fields of different masses (levels) mix in the "kinetic term" in (3.4). The additional "non-diagonal" terms in (3.10) are absent if we assume that all φ^i are traceless and transverse. Then the linearized equations (3.4) are simply

$$(k^2+m_i^2)\varphi^i=0 \quad , \quad m_i^2 = 4\pi\,(2N_i-2) \qquad (3.11)$$

($m^2=-8\pi$ corresponds to the tachyon, $m^2=0$ - to the massless modes, etc.).

Turning to the analysis of the interaction term in (2.21) let us first consider the case when both $\frac{\delta}{\delta x}$ in (2.20b) act on e^{ikx} in the vertex operators. Then

$$\int d^D k_1\, d^D k_2 \int d^2\sigma_1\, d^2\sigma_2\, d^2\sigma_1'\, d^2\sigma_2'\, \dot{G}(\sigma_{12})(-k_1k_2) \times$$
$$\times\, \delta^{(2)}(\sigma_1-\sigma_1')\,\delta^{(2)}(\sigma_2-\sigma_2')\,(V_i e^{ik_1x})(\sigma_1')\,(V_j e^{ik_2x})(\sigma_2')$$
$$= \int d^D k_1\, d^D k_2\, (-k_1k_2)\int d^2\sigma_1\, d^2\sigma_2\, \dot{G}(\sigma_{12}) \times \qquad (3.12)$$
$$\times\, (V_i e^{ik_1x})(\sigma_1)\,(V_j e^{ik_2x})(\sigma_2) \quad .$$

In order to apply the completeness property of the operators $V_i(\partial x)$, let us first formally expand the integrand near σ_2, $\sigma_1=\sigma_2+u$, and then integrate over u. We get

$$\int d^D k_1\, d^D k_2\, (-k_1k_2)\Big[a_0\Lambda^{\gamma_i+\gamma_j-2}\int d^2\sigma\, V_i V_j\, e^{i(k_1+k_2)x} +$$
$$+ \sum_{\ell} a_\ell \Lambda^{\gamma_\ell}\int d^2\sigma\, V_\ell(\partial x)\, e^{i(k_1+k_2)x}\Big] , \qquad (3.13)$$

$$a_0 = \Lambda^2\int d^2u\,\dot{G}(u) \;, \quad a_\ell = \Lambda^{2n_\ell+2}\int d^2u\,\dot{G}(u)\,u^{2n_\ell} , \qquad (3.14)$$

$$\gamma_\ell = 2-2N_\ell \,, \quad N_\ell = N_i + N_j + n_\ell \quad ,$$

V_ℓ may contain powers of k_1, k_2 and we have formally used the translation invariance on the infinite plane to rewrite $\int d^2\sigma_1\, d^2\sigma_2$

as $\int d^2\sigma_2\, d^2u$. While we have thus succeeded to represent the product $\int \dot{G}\, \frac{\delta V_i}{\delta x}\frac{\delta V_j}{\delta x}$ as a linear combination of V_k this was achieved at the expense of introducing the IR divergent coefficients a_ℓ . Assuming that we have already integrated over the phase of u we have (see (3.6))

$$a_\ell \sim \frac{1}{2}\int_0^{R^2} dy\, \frac{\Lambda^{2n_\ell}}{y+\Lambda^{-2}}\, y^{n_\ell} = \qquad (3.15)$$
$$= \frac{1}{2}(-1)^{n_\ell}\Big\{ \ln(\Lambda^2R^2+1) + \sum_{m=1}^{n_\ell}\frac{1}{m}\binom{n_\ell}{m}\big[(\Lambda^2R^2+1)^m - 1\big]\Big\} \quad , y=|u|$$

We have introduced the explicit IR cutoff R in the integral over u Alternatively, we could assume that an IR cutoff is **present** in the Green function G. Thus $a_\ell = f_\ell(\Lambda R)$ and so the operators in (3.13) in which $\mathcal{V}_k$ contains $2N_k$ derivatives are naturally multiplied by Λ^{γ_k} , $\gamma_k = 2-2N_k$. This, of course, is clear from the dimensional considerations only. What is important is that the coefficients in (3.13) are dimensionless functions of ΛR .

The case when one or both $\frac{\delta}{\delta x}$ act on $\mathcal{V}_i$ or $\mathcal{V}_j$ is treated similarly. We again use $\partial^n_{\sigma_1}\, \delta^{(2)}(\sigma_1-\sigma_1') = (-\partial_{\sigma_1'})^n\, \delta^{(2)}(\sigma_1-\sigma_1')$ and integrate by parts to put the derivatives on G. This operation reduces the level number of the resulting operator by n units. Expandi[ng] in $u=\sigma_{12}$ and integrating over it we again get an expression similar to (3.13). It is easy to understand that V_k appearing in the r.h.s. of (2.20b) may have any $N_k \geqslant 2$. For example, V_k with $N_k=2$ may appear in the "product" (2.20b) for arbitrary N_i and N_j i[f] $\mathcal{V}_i \sim \partial^{2N_i-1}x\,\partial x$, $\mathcal{V}_j \sim \partial^{2N_j-1}x\,\partial x$, etc. Thus, e.g., the "β-function" (2.21) for the graviton may contain in its interaction term fields from arbitrarily high levels.

Let us now consider more explicitly the interaction term in the β-function or eq.(3.4) for the tachyon field ($N_i=0$). The linear term was already found in eq.(3.10)

$$\beta^{\Phi}_{linear} = \left(\frac{k^2}{4\pi} - 2\right)\Phi(k) + \text{(traces of 2-nd rank tensor fields from all higher levels)} \qquad (3.16)$$

The structure of the quadratic term is clear from (3.12), (3.13): to obtain the tachyonic ($N_k=0$) operator in the "product" (2.20b) is possible only if we take both V_i and V_j to have $N_i=N_j=0$ and leave only the first term in the sum (3.13) (i.e. the first term in the expansion in powers of u). We get

$$\beta^{\Phi}_{quadratic} = -\frac{1}{2} a_0 \int d^D k' \; k' \cdot (k-k') \, \Phi(k') \, \Phi(k-k') , \qquad (3.17)$$

$$a_0 = \Lambda^2 \int d^2 u \, \dot{G}(u) = \frac{1}{2} \ln (\Lambda^2 R^2 + 1) \simeq \ln \Lambda R \qquad (3.18)$$

If we formally ignore the non-tachyonic terms in (3.16) the equation of motion for the tachyon takes the following form in the coordinate representation ($2\pi\alpha' = 1$)

$$\left(-\frac{1}{4\pi}\Box - 2\right)\Phi + a_0(\Lambda R)\, \partial_\mu \Phi \, \partial^\mu \Phi = 0 . \qquad (3.19)$$

The interaction term in (3.19) looks strange in several respects. As we have already mentioned above, the Λ -dependence of the coefficient a_0 implies an inconsistency since (3.4) and hence (3.19) is simply the condition of the Λ -independence of Φ (differentiating (3.19) over Λ and using $\dot{\Phi}$ =0 we get $\partial_\mu \Phi \partial^\mu \Phi = 0$ i.e. Φ =const and hence, from (3.19), Φ =0). Also, the interaction term $\partial_\mu \Phi \partial^\mu \Phi$ is different from the standard Φ^2 -interaction one expects to find from a string field theory action. It easy to check that eq.(3.19) cannot be derived from a local cubic action which is second order in derivatives of Φ . Moreover, solving (3.19) we do not get the usual 3-tachyon amplitude as the coefficient of the $(\Phi_{in})^2$-term in the classical solution.

All this seems to suggest that eq.(3.19) and hence the whole approach based on combining the Wilson equation with the local ansatz for I_{eff} in terms of the complete set of local operators (2.19 and imposing the scale invariance condition (3.5) may have nothing to do with string theory.*/ A possible resolution of this puzzle may be that the above approach corresponds to an unusual choice of the parametrization of the string field variables. In fact, eq.(3.16) implies that the tachyon mixes with the traces of the 2-nd rank tensors $\varphi^{(n)}_{\mu\nu}\delta^{\mu\nu}$. Redefining Φ to eliminate this mixing we get the $\Phi\,\varphi^{(n)}_{\mu\mu}$ and $\varphi^{(n)}_{\mu\mu}\varphi^{(m)}_{\lambda\lambda}$ interaction terms in (3.19). In order to compare with the tachyonic string S-matrix we are then first to solve the equations for $\varphi^{(n)}_{\mu\nu}$. To check the correspondence with the string S-matrix in a straightforward way thus appears to be quite difficult in the approach based on the local representation for I_{eff}. Below we are going to show (following the idea of ref./18/) that this check can be carried out easily if we start with a non-local ansatz for the effective 2-d string action.

ddendum

4. Correspondence with string S-matrix

Suppose we were given a classical string field theory action $S[\hat{\Phi}]$ for a string functional $\hat{\Phi}$. We could, in principle, solv the corresponding equations of motion for the local component fields, expressing them in terms of their "in" values, $\varphi_{cl} = \varphi_{cl}(\varphi_{in})$

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*/ In principle, the Λ-dependence of the interaction term in (3.19) may not by itself imply a contradiction with the string theory since, e.g. the regularized 3-tachyon amplitude is known to contain the $O(\ln(\Lambda R))$ term /25,26/.

In general, it is easy to prove that the coefficients in the classical solution $\varphi^i_{cl}(\varphi_{in})$ are proportional to the scattering amplitudes with N+1 legs : $\varphi^i_{cl} = \sum_N^\infty \frac{A^i_{N+1}}{k^2+m_i^2}(\varphi_{in})^N$ (see, e.g., /18,27/). This follows, e.g., from the expression for the generating functional for the tree S-matrix: $\mathcal{S}[\varphi_{in}] = S[\varphi_{cl}] - \int \varphi_{in}\Delta\varphi_{cl} = \sum_N A_N \varphi^N_{in}$ (one should compare the powers of φ_{in} in both expressions). Hence in order to compare with the string S-matrix it is not necessary to know the action; **it is sufficient only to look a** coefficients in the expansion of the classical solution in powers of the "in"-field.

Let us now suppose that only the tachyon has a non-vanishing "in"-value Φ_{in} . Then all the component string fields which solve the corresponding equations of motion are functions of Φ (and hence of Φ_{in}). Substituting them into the tachyon equation of motion we obtain the effective non-linear equation for Φ . We can also substitute $\varphi^i_{cl}(\Phi_{in})$ directly into the local representation for I_{eff} (eq.(2.19)) $\left((k^2+8\pi)\Phi_{in}=0\right)$

$$I_{eff} = \varphi^i_{cl}(\Phi_{in})V_i \quad , \quad \text{or} \tag{4.1}$$

$$I_{eff} = \sum_{N=0}^\infty \frac{1}{N!}\prod_{n=1}^N \left[\Lambda^2\int d^2\sigma_n\, d^Dk_n\, e^{ik_n x(\sigma_n)}\Phi_{in}(k_n)\right] f_N(k_1,\dots,k_N,\sigma_1,\dots\sigma_N,\Lambda) \tag{4.2}$$

Here we used that $\varphi^i_{cl} = \sum_N^\infty \frac{A^i_{N+1}}{k^2+m_i^2}(\Phi_{in})^N$, where A^i_{N+1} is proportional to the scattering amplitudes with one φ^i-leg and N tachyonic legs. We have noted that I_{eff} in (4.1) should contain terms of all powers in Φ_{in} which can be represented in a non-local form with f_N being appropriate dimensionless functions. Such a representation is suggestive since in string theory we expect A_{N+1} to be given by the integrals over the Koba-Nielsen variables. Expanding the integrand near one point (σ_1) we can rewrite (4.2) back in the local form (4.1), in which the operators with all higher derivatives are present. We thus expect that (4.2) may correspond to a string classical solution in which all modes are expressed in terms of Φ_{in}. We could, in principle, introduce a more general non-local ansatz for I_{eff}, including dependence on the "in" values of other string modes and hence with $e^{ik_n X}$ in (4.2) replaced by $\partial^{n_1}X \ldots \partial^{n_i}X\, e^{ik_n X}$. This would correspond to a general perturbative classical string field solution. Let us note also that it is even natural to consider a non-local representation for I_{eff} since it is an <u>effective action</u>, i.e. a result of integrating out some 2-d degrees of freedom.

I_{eff} in (4.2) first should satisfy the Wilson's equation (2.14). We know already the solution of (2.14) (see (2.10),(2.15),(2.16))

$$I_{eff} = -\ln\left[\exp\left(\tfrac{1}{2}\left[G_0(\Lambda_0) - G(\Lambda)\right]\cdot\frac{\delta^2}{\delta x^2}\right) e^{-I_{int}[x]}\right] \tag{4.3}$$

(we assume that the initial condition is $\left(I_{eff}\right)_{\Lambda=\Lambda_0} = I_{int}$). As for the scale invariance condition (3.5) it would obviously be satisfied if f_N were independent of Λ. However, this does not follow from (4.3). In fact, to discuss the issue of the scale invariance we need to specify some procedure of removal of the cutoff. This point is related to the problem of the cutoff dependence of the interaction term in the "$\beta=0$"-equation (3.4) we confronted in sect.2.

Since we consider only the tachyonic "in"-background, let us choose I_{int} to have the following simple form

$$I_{int} = I_1 \equiv \Lambda_0^2 \int d^2\sigma\, d^D k\, e^{ikx(\sigma)}\, \Phi_{in}(k) \quad . \tag{4.4}$$

Then the "initial" theory (2.2) corresponds to the partition function

$$Z[\bar{x}] = Z^{-1} \int \mathcal{D}x \exp\left(-\tfrac{1}{2}x G_0^{-1} x - I_1[x+\bar{x}]\right) \tag{4.5}$$

and

$$e^{-I_{eff}[\bar{x}]} = \bar{Z}^{-1}\int \mathcal{D}x \, \exp\left(-\tfrac{1}{2} x \bar{G}^{-1} x - I_1[x+\bar{x}]\right) \quad . \tag{4.6}$$

Expanding (4.6) in powers of Φ_{in} we obtain I_{eff} in the form (4.2) with f_N being related to the correlators (I_{eff} contains only the "connected" parts of the correlators)

$$M_N = \left< \prod_{n=1}^{N} e^{ik_n x(\sigma_n)} \right> , \quad \langle \ldots \rangle = \bar{Z}^{-1}\int \mathcal{D}x \, e^{-\frac{1}{2}x\bar{G}^{-1}x} \ldots , \quad \langle 1 \rangle = 1 \quad . \tag{4.7}$$

They are computed using the propagator $\bar{G}$, regular both in the ultraviolet and the infrared (see (2.8))

$$\bar{G}(0) = -\frac{1}{4\pi} \ln \frac{\Lambda^2}{\Lambda_0^2} \quad , \quad \bar{G}\Big|_{\sigma_{12}\to\infty} \longrightarrow 0 \quad . \tag{4.8}$$

The basic observation is that if we expand the arguments in (4.7) near one point, integrate over the points and separate the leading singularity, then the residue will be the regularized expression for the Virasoro-Shapiro amplitude. The prescription of removing the cutoff is the following: we take $\Lambda_0 \to \infty$ keeping Λ fixed. Consider, for example, the two leading terms in I_{eff} (4.6)

$$I_{eff}[\bar{x}] = \langle I_1 \rangle - \frac{1}{2}\left(\langle I_1^2 \rangle - (\langle I_1 \rangle)^2 \right) + \ldots , \tag{4.9}$$

$$\begin{aligned} \langle I_1 \rangle &= \Lambda_0^2 \int d^2\sigma \, d^D k \, \Phi_{in}(k) \, e^{ik\bar{x}} \langle e^{ikx} \rangle \\ &= \Lambda_0^2 \int d^2\sigma \, d^D k \, \Phi_{in}(k) \, e^{ik\bar{x}} \, e^{-\frac{1}{2}k^2 \bar{G}(0)} \\ &= \Lambda^2 \int d^2\sigma \, d^D k \, \Phi_{in}(k) \, e^{ik\bar{x}} = I_1 (\Lambda_0 \to \Lambda) , \end{aligned} \tag{4.10}$$

$$\begin{aligned} \langle I_1^2 \rangle = \Lambda_0^4 \int & d^D k_1 \, d^D k_2 \, d^2\sigma_1 \, d^2\sigma_2 \, \Phi_{in}(k_1) \, \Phi_{in}(k_2) \, \times \\ & \times \exp\left(ik_1 \bar{x}(\sigma_1) + ik_2 \bar{x}(\sigma_2)\right) \exp\Big[\frac{1}{8\pi}(k_1^2 + k_2^2) \ln \frac{\Lambda^2}{\Lambda_0^2} + \\ & + \frac{1}{4\pi} k_1 \cdot k_2 \ln \frac{\sigma_{12}^2 + \Lambda_0^{-2}}{\sigma_{12}^2 + \Lambda^2} \Big] = \end{aligned} \tag{4.11}$$

$$= \Lambda^4 \int d^2\sigma_1\, d^2\sigma_2\, d^D k_1\, d^D k_2\; \Phi_{in}(k_1)\, \Phi_{in}(k_2)\, \exp\left(ik_1 \bar{x}(\sigma_1) + ik_2 \bar{x}(\sigma_2)\right) \times$$

$$\times \left[\frac{|u|^2 + \Lambda_0^{-2}}{|u|^2 + \Lambda^{-2}} \right]^{\ell} , \quad \ell = \frac{1}{8\pi}(k_1+k_2)^2 - 2 \quad , \quad u = \sigma_{12} \quad .$$

Taking $\Lambda_0 \to \infty$ with Λ =fixed, expanding in u and integrating over u we get the leading singularity from the $|u| \to 0$ region

$$\int_{|u|\simeq 0} d^2 u\, |u|^{2\ell} \sim \frac{1}{\frac{1}{8\pi}(k_1+k_2)^2 - 1} \quad . \tag{4.12}$$

This is the tachyonic pole(in the total momentum)so that

$$\langle I_1^2 \rangle = \Lambda^2 \int d^2\sigma\, d^D k\; e^{ikx} \left[\tilde{f}_2(k) + O(\partial x) \right] \quad ,$$

$$\tilde{f}_2 \sim \frac{1}{\frac{1}{8\pi}k^2 - 1} \int d^D q\; \Phi_{in}(q)\, \Phi_{in}(k-q) \quad , \quad k^2 \approx 8\pi \quad . \tag{4.13}$$

For large but fixed Λ_0 the pole is replaced by the $\ln \frac{\Lambda_0}{\Lambda}$ -term $\tilde{f}_2$ is indeed just the second order term in the classical solution of the standard tachyonic equation of motion with the Φ^2-interaction corresponding to the well known 3-tachyon amplitude.

To understand the appearence of the VS amplitudes (in their standard Möbius gauge fixed form) in the present context, let us make a digression on the two possible approaches to establishing the correspondence between the σ-model path integral (4.5) and string scattering amplitudes. One is the "β-function approach" originally proposed by Lovelace /1/ and discussed, e.g., in /3,7, 11,13,14,28,29/. The other is the "string partition function approach" /2,30,31,26/. Below we shall review these approaches on the example of the tachyonic coupling. Let us start with the second approach. It is based on the observation /2/ that the σ-model partition function is the generating functional for the correlators of the vertex operators

$$Z = \int \mathcal{D}x \, e^{-I_0[x] - I_{int}[x]} = \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \langle V_1 \dots V_N \rangle \varphi^N \qquad (4.14)$$

$$I_0 = \frac{1}{2} \int d^2\sigma \, \partial_a x^\mu \partial^a x^\mu , \quad I_{int} = \varphi^i V_i , \quad V_i = \Lambda^{\gamma_i} \int d^2\sigma \, \gamma_i(\partial x) e^{ikx} , \quad \sum_{i=1}^{N} k_i$$

In order to relate the UV regularized correlators $\langle V_1 \dots V_N \rangle$ to the string scattering amplitudes we should go on the mass shell taking $\varphi = \varphi_{in}$ and also to make some subtraction corresponding to the elimination of the Möbius infinites in the unregularized string amplitudes. Since the regularized closed string Möbius volume is logarithmically divergent /25,26/ it is necessary to subtract the "overall" $\ln \Lambda$ -infinity in $\langle V_1 \dots V_N \rangle$ /26/. For example,

$$\langle V_1 V_2 V_3 \rangle = c_1 (\Lambda R)^2 + c_2 (\Lambda R)^2 \ln \Lambda R + c_3 \ln \Lambda R + c_4 \qquad (4.15)$$

where it is c_3 that is the ordinary Möbius gauge fixed 3-point amplitude. In general, the correlator $\langle V_1 \dots V_N \rangle = \int d^2\sigma_1 \dots d^2\sigma_N \, M_N(\sigma_1, \dots$ contains the overall <u>momentum-independent logarithmic</u> infinity coming from the region of integration where N-1 points σ_k are clos to each other /26/. There are of course logarithmic subdivergences originating from the regions of integration where $k < N-1$ point are close to each other, but they are momentum dependent (and correspond to the physical poles in the absence of the cutoff). This is easy to understand qualitatively from the factorization property of the correlators*/. The limit in which all N points are close to each other corresponds to the factorization of the correlator into the product of the N+1 point amplitude, a zero momentum propagator and a tadpole on the sphere and hence gives the overall <u>quadratic</u> (tachyonic) infinity ($\Lambda^2 \ln \Lambda$ comes from the massless state contribution). In the limit when some N-1 points are close we get the product of the N-point amplitude, the propagator tak

*/ This property is , in fact, a consequence of the path integral representation.

for on-shell momentum and the 2-point correlator on the sphere. The latter contains a finite piece and hence

$$\left(\langle V_1 \ldots V_N \rangle\right)_{z_2 \approx \ldots \approx z_{N-1}} \sim \ln\Lambda \, A_N + \ldots \quad , \tag{4.16}$$

where A_N is the SV amplitude in its standard Koba-Nielsen Möbius gauge fixed form. This qualitative analysis can be confirmed explicitly by making the change of the variables $\{z_n\} \to \{z_1, z_2, u, w_k\}$ $z_3 = z_2 + u$, $z_k = z_2 + u w_k$, $k = 4, \ldots, N$ in the integral representation e.g. for the tachyonic correlator $\langle V_1 \ldots V_N \rangle$ /26/. Thus $\ln\Lambda$ in (4.16) can be interpreted as the Möbius group volume infinity. If V_n correspond to the massless particles and $\langle V_1 \ldots V_N \rangle$ are computed by expanding in powers of momenta (i.e. by expanding near the mass shell) the correct prescription of subtraction of the Möbius infinity is to take $\partial/\partial \ln\Lambda$ of the correlator $\langle V_1 \ldots V_N \rangle$ in which all power infinities are omitted /26/.

Let us now consider the β-function approach. To compute the counterterms we need to introduce some background field $\bar{x}$

$$Z[\bar{x}] = \int \mathcal{D}x \, e^{-I_0 - I_{int}[x + \bar{x}]} \quad . \tag{4.17}$$

Above in eq.(4.14) we assumed that $\bar{x}=0$ (and hence the integration over the constant part of x gave the δ^D-function, implying the conservation of the total momentum). If $\bar{x} \neq 0$ we can say that there is a non-vanishing flow of momenta outside. Suppose I_{int} in (4.17) contains only the tachyonic term

$$I_{int} = I_1 \ , \quad I_1 = \Lambda^2 \int d^2\sigma \, d^D k \, e^{ikx} \, \Phi_0(k) \tag{4.18}$$

where Φ_0 is the bare field. The resulting theory is renormalizable near the tachyonic mass shell. We are thus considering the effective theory for tachyons only. Then

$$I_{eff}[\bar{x}] = -\ln Z[\bar{x}] = \int d^2\sigma \; \{ \Lambda^2 F[\bar{x}] + O(\partial\bar{x}) \} \quad + \text{non-local}, \tag{4.19}$$

$$F[\bar{x}] = \Phi_0(\bar{x}) + O(\ln\Lambda) \quad .$$

Assuming that the <u>renormalized</u> field is on-shell (i.e. is equal to Φ_{in}), we get

$$\Phi_0 = \Phi_{in} + \sum_N C_N (\Phi_{in})^N \ln\Lambda + O(\ln^2\Lambda) \quad . \tag{4.20}$$

In order to find the expression for the β -function coefficients C_N we are to consider the leading (overall) divergence in I_{eff}. We have

$$I_{eff} = \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \prod_{n=1}^{N} \left[\int d^2\sigma_n \, d^D k_n \, e^{ik_n\bar{x}(\sigma_n)} \Lambda^2 \Phi_0(k_n) \right] \times f_N(\sigma_1,\dots,\sigma_N, k_1,\dots,k_N, \Lambda) \quad , \tag{4.21}$$

$$f_N = (< V_1 \dots V_N >)_{connected} \quad . \tag{4.22}$$

To isolate this leading $\ln\Lambda$-term in (4.21) we may replace Φ_0 by Φ_{in} putting all the momenta on shell. We are thus to analyze the integral

$$X_N = \Lambda^{2N} \int d^2\sigma_1 \dots d^2\sigma_N \; \exp\left(ik_1\bar{x}(\sigma_1) + \dots + ik_N\bar{x}(\sigma_N)\right) \times < \prod_{n=1}^{N} e^{ik_n x(\sigma_n)} > \quad , \quad k_n^2 = 8\pi \quad . \tag{4.23}$$

The overall divergence comes from the region where <u>all</u> <u>N points</u> are close to each other (other limits correspond to subleading divergences that are subtracted in the renormalization process and do not influence the β -function). Changing the variables: $\sigma_2 = \sigma_1 + u$, $\sigma_k = \sigma_1 + u w_k$, $k = 3,\dots,N$ and expanding in powers of $u \to 0$ we get (Λ^2-factors in (4.23) cancell out since k^2 are on shell)

$$X_N = \int d^2\sigma_1 e^{iq\bar{x}(\sigma_1)} \Big[\int d^2 u \, |u|^{2(N-2)} \prod_{k=3}^{N} d^2 w_k \exp\Big(-\sum_{n<m} k_n k_m G(\sigma_{nm})\Big)$$

$$+ O(\partial\bar{x}) \Big] \sim \int d^2\sigma_1 \, e^{iq\bar{x}(\sigma_1)} \Big[\ln\Lambda \, A_{N+1}(k_1, \dots, k_N, -q) \; +$$

$$+ O(\partial\bar{x}) \Big] \; , \qquad q = \sum_{i=1}^{N} k_i \quad . \tag{4.24}$$

Here $\ln\Lambda$ comes from the integration over u. The integral over u takes the form: $\int d|u| \, |u|^{\frac{q^2}{4\pi} - 3} + \dots$. We have assumed that $q^2 \approx 8\pi$, isolated the leading tachyonic pole and replaced $(\frac{1}{8\pi} q^2 - 1)^{-1}$ by $\ln\Lambda$. One can check directly that A_{N+1} is the standard VS amplitude computed in the Möbius gauge $w_1=0$, $w_2=1$, $w_{N+1}=\infty$. Thus the coefficient C_N in the bare field (4.20) is indeed proportional to the N+1-point VS amplitude.*/ Note that in much similarity with what we have in (4.16), $\ln\Lambda$ in (4.24) can be interpreted as the Möbius infinity. Computing β we thus effectively fix a Möbius gauge by separating one point ($\int d^2\sigma_1 \, e^{iq\bar{x}(\sigma_1)}$) and integrating over one point ($\int d^2 u \, |u|^{\frac{q^2}{4\pi} - 4}$) to produce $\ln\Lambda$ (the total momentum corresponds to z_{N+1} which is also fixed being simply absent). Eq.(4.20) (with $\ln\Lambda$ replaced by the tachyonic propagators) may be considered also as giving the expression for the classical solution of the β =0 equation,

$$\beta(\Phi_{in}) \sim \sum_{N=2}^{\infty} (A_{N+1})_{subtr.} \; \Phi_{in}^{N} \quad . \tag{4.25}$$

Let us now discuss the relation between the two approaches. Consider the partition function (4.14) or (4.17) with $\bar{x}$=0. The renormalizability implies the relation

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*/ More exactly, we need also to subtract all other tachyonic pole in A_{N+1} in order to identify it with a coefficient in the β -func-

$$\Lambda \frac{\partial Z}{\partial \Lambda} + \beta^{\Phi} \frac{\partial Z}{\partial \Phi} = 0 \qquad (4.26)$$

According to the partition function approach (4.14), (4.16) (we disregard the dependence on powers of Λ)

$$Z[\Phi_{in}] = \ldots + \sum_{N=3}^{\infty} A_N \, \Phi_{in}^N \ln \Lambda + \ldots ,$$
$$\frac{\partial Z}{\partial \ln \Lambda} = \sum_N A_N \, \Phi_{in}^N + \ldots , \quad \frac{\partial Z}{\partial \Phi_{in}} = 1 + c\,\Phi_{in} + \ldots . \qquad (4.27)$$

Then from eq.(4.26) we get

$$\beta(\Phi_{in}) = \sum_N C_N \, \Phi_{in}^N , \quad C_N \sim A_{N+1} + \ldots \qquad (4.28)$$

This, in fact, is what we have found in the β-function approach. The validity of (4.28) depends crucially on the presence of the finite Φ^2-term in Z. Note that (4.28) is consistent with the expectation that the equation β =0 should be equivalent to the effective equation of motion $\partial S / \partial \Phi = 0$ with $S = \sum_N (A_N)_{subtr.} \, \Phi^N$ being the effective action (since we differentiate once over Φ, C_N in (4.26) should be $\sim (A_{N+1})_{subtr.}$). We thus have a clear qualitative understanding of why a subtraction of the overall $\ln \Lambda$ corresponds to a subtraction of the Möbius infinity.

Returning to our discussion of I_{eff} in eq.(4.6) it is thus not too surprizing that representing I_{eff} formally in the local form (by expanding the integrand near one point) and isolating the leading tachyonic pole (or the leading logarithmic singularity) we indeed get the ordinary VS amplitude A_{N+1} as the coefficient of the Φ_{in}^N-term. Assuming $\Lambda_0 \to \infty$ and defining the amplitudes by the analytic continuation we satisfy the condition of scale invariance. This was the argument of ref./18/ which we recognize as being simply an application of the β-function approach of refs. /1,7,11,13,14/. An attempt to relate a solution of the Wilson's renormalization group equation (combined with scale invariance condition) was also made in ref./17/. As was already pointed out in ref./18/ this attempt was erroneous. The authors of ref./17/ assumed that $\ln Z$ is the generating functional for string S-matrix and hence ignored the problem of subtraction of the logarithmic Möbius infinity: they considered the zero momentum tachyonic term in I_{eff}, expanded it in Φ_{in} and claimed that the coefficients are proportional to N-point string amplitudes A_N. However, as we have seen above, these coefficients are, in fact, the N+1-point amplitudes A_{N+1} (times a logarithmic divergence).

We conclude our discussion of the approach based on Wilson's RG equation with the following remarks. It is not at all obvious that the basic equation of this approach can be interpreted as an equation of a string field theory. The basic problem is that of the cutoff dependence. In general, we might expect that a string field theory vertex is well-defined only if a cutoff is also introduced in it. However, to check the correspondence with the standard dual S-matrix we need to remove a cutoff. The procedure used in /18/ to demonstrate the correspondence with the string S-matrix was, in fact, a version of the argument used to prove that the β-functions in the "effective" (near mass-shell) tachyonic sigma-model are equivalent to the effective string equations of motion. It would be desirable to have a more direct argument for establishing a connection with the string S-matrix which is not based on the effective integrating out all the string modes except the tachyon.

5. Approach based on the completeness property of "vertex operators"

One of the most characteristic properties of string theory is that the corresponding sigma-model contains all possible local ope rators V_i constructed out of x^{μ} and its derivatives with coefficients related to the modes of the string spectra. It is important to stress that the completeness of the system $\{V_i\}$ is an additional input in the Wilson RG approach. We may try to avoid using the Wilson RG equation, directly employing the completeness property of $\{V_i\}$ in the analysis of the basic partition function

$$Z[\bar{x}] = z^{-1} \int \mathcal{D}x \, e^{-I_0 - I_{int}[x+\bar{x}]} \tag{5.1}$$

Here I_0 is a free action with a cutoff and I_{int} is given by the sum of all possible V_i's multiplied by the "bare" fields

$$I_0 = \frac{1}{2} \, x \, G^{-1}(\Lambda) \, x \, , \quad I_{int} = \varphi^i \cdot V_i \tag{5.2}$$

V_i obviously satisfy the local completeness relation

$$V_i = \Lambda^{\gamma_i} \int d^2\sigma \, \mathcal{V}_i(k,\sigma) \, , \quad \mathcal{V}_i(k,\sigma) = e^{ikx} \mathcal{V}_i(\partial x) \, , \tag{5.3}$$

$$\mathcal{V}_i(k,\sigma) \, \mathcal{V}_j(k',\sigma') = C^{\ell}_{ij}(\sigma-\sigma', k, k') \, \mathcal{V}_\ell(k+k', \sigma) \tag{5.4}$$

To prove (5.4) we note simply that expanding z' near z we get a series of terms which are again proportional to some $\mathcal{V}_i$. Eq.(5.4) can be rewritten also in a more symmetric form

$$\mathcal{V}_i(\sigma) \, \mathcal{V}_j(\sigma') = \tilde{C}^{\ell}_{ij} \, (\sigma-\sigma') \, \mathcal{V}_\ell \left(\frac{\sigma+\sigma'}{2}\right) . \tag{5.5}$$

From now on we shall not indicate explicitly the momentum argument of $\mathcal{V}_i$, including it in the index i. We shall assume that C^{ℓ}_{ij} always contains the δ-function $\delta^D(k_i+k_j-k_\ell)$ (and that the sum over ℓ includes the integration over k_ℓ). C and $\tilde{C}$ in (5.4) and (5.5) can be related by reexpanding $\mathcal{V}_\ell(\sigma)$ in the r.h.s. of eq.(5.4) near $\frac{1}{2}(\sigma+\sigma')$. Integrating (5.4) or (5.5) over σ and σ' we get the integrated form of the completeness relation

$$V_i \, V_j = \mathbb{C}^{\ell}_{ij} \, V_\ell \tag{5.6}$$

We use the translational invariance on the plane in order to be able to rewrite $\int d^2\sigma_1 d^2\sigma_2$ as $\int d^2u\, d^2\sigma$, $u=\sigma_1-\sigma_2$. Then

$$\mathbb{C}^{\ell}_{ij} = \int d^2u \; C^{\ell}_{ij}(u) \tag{5.7}$$

Since C^k_{ij} is a power series in u , the integral in (5.7) needs some large distance (IR) cutoff, $|u| \leq$ R. This is a weak point of this approach: the necessity of introducing R is a price to be paid for the possibility to use eq.(5.6) in order to rewrite the non-local expressions (like $V_i V_j$) in a local form.

It may be useful to stress that the completeness relations (5.4)-(5.6) are purely "classical" (not operator product) relations. They are true exactly, irrespective of any averaging procedure. As in (2.19) we assume that V_i are made dimensionless with a help of the appropriate powers of the cutoff Λ^{γ_i} , γ_i=2-2N_i, 2N_i=number of derivatives in γ_i . As is clear from (5.4),

$$C^{\ell}_{ij} = 0 \quad , \qquad N_i + N_j > N_\ell \quad , \tag{5.8}$$

$$\mathbb{C}^{\ell}_{ij} \sim (R\Lambda)^{\gamma_i+\gamma_j-\gamma_\ell} \; \delta^{(D)}(k_i+k_j-k_\ell) \quad ,$$

Let us now formally consider the system of objects V_i which form the commutative associative algebra (5.6) with

$$\mathbb{C}^{\ell}_{ij} = \mathbb{C}^{\ell}_{ji} \quad , \quad \mathbb{C}^{\ell}_{ij}\, \mathbb{C}^{n}_{\ell k} = \mathbb{C}^{\ell}_{jk}\, \mathbb{C}^{n}_{\ell i} \tag{5.9}$$

and use (5.9) to compute the partition function (5.1). Let us define the matrix

$$B^n_{\ m} = \mathbb{C}^n_{\ m\ell}\, \varphi^\ell \tag{5.10}$$

where the summation over ℓ includes the integration over momentum. Then

$$\frac{\partial B^n_{\ m}}{\partial \varphi^\ell} = \mathbb{C}^n_{\ m\ell} \quad , \quad B^n_{\ m} \frac{\partial B^m_{\ k}}{\partial \varphi^\ell} = \frac{\partial B^n_{\ m}}{\partial \varphi^\ell} B^m_{\ k} \quad , \tag{5.11}$$

$$\frac{\partial}{\partial\varphi^{\ell}}\left(f(B)\right)^{n}{}_{k} = \left(\frac{\partial f}{\partial B}\right)^{n}{}_{m}\frac{\partial B^{m}{}_{k}}{\partial\varphi^{\ell}} ,$$

i.e. the associativity implies the usual rule of differentiation of functions of B. Consider

$$\langle V_i[x+\bar{x}]\rangle = -\frac{\partial Z[\bar{x}]}{\partial\varphi^i} = Z^{-1}\int\mathcal{D}x\, e^{-I_0-I_{int}[x+\bar{x}]}\, V_i[x+\bar{x}] \tag{5.12}$$

$$\frac{\partial}{\partial\varphi^j}\langle V_i\rangle = -\langle V_i V_j\rangle = -\mathbb{C}^{k}_{ij}\langle V_k\rangle . \tag{5.13}$$

Since $\mathbb{C}^{k}_{ij}$ =const we can easily solve the equation (5.13)

$$\langle V_j\rangle = \left(e^{-B}\right)^{i}{}_{j}\,\langle V_i\rangle_0 \tag{5.14}$$

Here B was defined in (5.11) and

$$\langle\ldots\rangle_0 = Z_0^{-1}\int\mathcal{D}x\, e^{-I_0}\ldots \quad , \quad \langle 1\rangle_0 = 1 , \tag{5.15}$$

is the free expectation value. Eq.(5.14) gives the explicit dependence of the "tadpole" $\langle V_i\rangle$ on φ^j . Now we can solve eq.(5.12 for Z. Alternatively, we can find the dependence of Z on φ by representing it as a power series in φ

$$Z = 1 - \varphi^i\langle V_i\rangle_0 + \frac{1}{2}\varphi^i\varphi^j\langle V_i V_j\rangle_0 + \ldots \tag{5.16}$$

and using the completeness relation (5.6). The result is

$$Z = 1 - \langle V_i\rangle_0\,\Omega^{i}{}_{j}(\varphi)\,\varphi^j , \tag{5.17}$$

$$\Omega = \sum_{p=0}^{\infty}\frac{(-1)^p}{(p+1)!}B^p = \frac{1-e^{-B}}{B} . \tag{5.18}$$

Differentiating (5.17) with respect to φ^i we get back to eq. (5.14). Thus the completeness property makes it possible to expres all the correlators $\langle V_{n_1}\ldots V_{n_i}\rangle$ in terms of $\mathbb{C}$, φ^i and $\langle V_i\rangle$

The basic object is thus

$$\langle V_i\rangle_0 = Z_0^{-1}\int\mathcal{D}x\, e^{-I_0[x]}\, V_i[x+\bar{x}] \tag{5.19}$$

We have */

$$< V_i >_0 = V_j [\bar{x}]\, t^j{}_i \,, \quad t^j{}_i = (\Lambda a)^{-\frac{K_i^2}{4\pi}} P^j_i\, \delta^{(D)}(k_i - k_j)\,, \tag{5.20}$$
$$P^i_j = 0 \quad \text{if } N_i > N_j$$

where a is an IR cutoff in the Green function (e.g. $G = -\frac{1}{4\pi} \ln\left(\sigma_{12}^2 + \Lambda^{-2}\right)/a^2$) and the finite triangular matrix P_j may depend only on the momentum but not on the cutoffs. Here the summation over j again includes the integration over the momentum which is thus the same in both sides of the equation (5.20).

Since V_i is proportional to Λ^{γ_i} we have

$$\frac{\partial}{\partial\lambda} < V_i >_0 = - V_j [\bar{x}]\, \mathcal{B}^j{}_n\, t^n{}_i \,, \qquad \lambda = \ln\Lambda\,, \tag{5.21}$$
$$\mathcal{B}^j{}_n = \left(\tfrac{1}{4\pi} k_j^2 - \gamma_j\right) \delta^{(D)}(k_j - k_n)\,.$$

Consider now the renormalization group equation for Z, expressing its cutoff independence

$$\frac{\partial Z}{\partial\lambda} + \beta^i \frac{\partial Z}{\partial\varphi^i} = 0 \,, \tag{5.22}$$

$$\beta^i = \dot{\varphi}^i \,, \qquad \dot{\varphi}^i = \frac{\partial\varphi^i}{\partial\lambda} \,.$$

Let us solve this equation for β^i. Substituting the expressions for Z (5.17) and $\frac{\partial Z}{\partial\varphi}$ (5.12), (5.14) we get (see (5.8))

$$\frac{\partial Z}{\partial\lambda} = - \frac{\partial}{\partial\lambda} < V_i >_0 \Omega^i{}_j \varphi^j - < V_i >_0 \dot{\Omega}^i{}_j \varphi^j \,, \tag{5.23}$$

where

$$\dot{\Omega}^i_j \equiv \frac{\partial\Omega^i{}_j}{\partial\lambda} = \left(\frac{\partial\Omega}{\partial B}\right)^i_k \dot{\mathbb{C}}^k{}_{je}\, \varphi^e \,, \tag{5.24}$$

$$\dot{\mathbb{C}}^k{}_{je} = \left(\gamma_j + \gamma_e - \gamma_k\right) \mathbb{C}^k{}_{je} \,.$$

Hence

$$\dot{Z}[\bar{x}] = V_i [\bar{x}]\, \mathcal{B}^i{}_j\, t^j{}_e\, \Omega^e{}_s\, \varphi^s - \tag{5.25}$$

*/ The cutoff dependent factor in t^i_j is not essential: it will disappear from the final result for the β-function.

$$- V_i[\bar{x}]\, t^i_{\ j}\, \Omega'^{j}_{\ \ell}\, \dot{\mathbb{C}}^{\ell}_{\ ks}\, \varphi^k \varphi^s \quad ,$$

$$\Omega' = \frac{\partial \Omega}{\partial B} = \sum_{p=1}^{\infty} \frac{(-1)^p p}{(p+1)!} B^{p-1} \quad , \tag{5.26}$$

$$\beta^i \frac{\partial Z}{\partial \varphi^i} = -\beta^i (e^{-B})^j_{\ i}\, V_k[\bar{x}]\, t^k_{\ j} \quad . \tag{5.27}$$

Eq.(5.22) thus takes the following form

$$V_i[\bar{x}]\; E^i = 0 \quad . \tag{5.28}$$

Using again the completeness of V_m we finish with $E^i = 0$, i.e.

$$t^i_{\ j} (e^{-B})^j_{\ k}\, \beta^k = \omega^i_{\ j}\, t^j_{\ k}\, \Omega^k_{\ s}\, \varphi^s - \tag{5.29}$$
$$- t^i_{\ j}\, \Omega'^j_{\ k}\, \dot{\mathbb{C}}^k_{\ zs}\, \varphi^z \varphi^s \quad .$$

This equation gives the expression for β containing all higher powers of φ ,

$$\beta^i = (e^B)^i_{\ j} \left[\hat{\omega}^j_{\ k}\, \Omega^k_{\ s}\, \varphi^s - \Omega'^j_{\ k}\, \dot{\mathbb{C}}^k_{\ zs}\, \varphi^z \varphi^s \right] \quad , \tag{5.30}$$
$$\hat{\omega} \equiv t^{-1} \omega\, t = P^{-1} \omega\, P \quad .$$

It is remarkable that the use of the completeness relation makes it possible to obtain the closed expressions for Z and the β -function. The β -function in (5.30) depends on the cutoff only through $\mathbb{C}^i_{jk}$ (the cutoff dependence of $t^i_{\ j}$ cancels out in (5.30)). Eq.(5.30) is certainly different from what was found in the Wilson's RG approach (cf. eq.(2.21)). The expansion of β in powers of φ reads

$$\beta^i = \hat{\omega}^i_{\ j}\, \varphi^j + B^i_{\ k}\, \hat{\omega}^k_{\ j}\, \varphi^j - \tfrac{1}{2} \hat{\omega}^i_{\ k}\, B^k_{\ j}\, \varphi^j - \tag{5.31}$$
$$- \tfrac{1}{2}\, \dot{\mathbb{C}}^i_{\ jk}\, \varphi^j \varphi^k + O(\varphi^3) \quad =$$

$$= \hat{\omega}^i_{\ j}\varphi^j + \Big[C^i_{\ jk}\hat{\omega}^j_{\ s} - \frac{1}{2}\hat{\omega}^i_{\ j} C^j_{\ ks} - \qquad (5.32)$$

$$- \frac{1}{2}(\gamma_k + \gamma_s - \gamma_i) C^i_{\ ks}\Big]\varphi^k\varphi^s + O(\varphi^3) \; .$$

$\hat{\omega}$, in general, is not diagonal and hence "mixes" different fields.

Let us study the structure of β in more detail on the example of the tachyonic β -function. We shall concentrate on the dependence of β^Φ on the tachyon field Φ itself. According to the definition of $C^i_{\ jk}$ and $t^i_{\ j}$ (see (5.8) and (5.20))

$$C^\Phi_{\ ij} = C^\Phi_{\ \Phi_1\Phi_2}\,\delta_{i\Phi_1}\delta_{j\Phi_2} \;, \qquad C^\Phi_{\ \Phi_1\Phi_2} = (\Lambda R)^2\,\delta^{(D)}(k_1 + k_2 - k) \;, \qquad (5.33)$$

$$t^i_{\ \Phi} = \delta_{i\Phi'}\,t^{\Phi'}_{\ \Phi} \;, \qquad t^{\Phi'}_{\ \Phi} = (\Lambda a)^{-k^2/4\pi}\,\delta^{(D)}(k - k') \;,$$

$$\hat{\omega}^i_{\ j} = \sum_k t^{-1\,i}_{\ \ k}\,\omega_k\,t^k_{\ j} \;, \qquad (5.34)$$

$$\hat{\omega}^\Phi_{\ \Phi'} = \Big(\frac{1}{4\pi}k^2 - 2\Big)\,\delta^{(D)}(k - k') \equiv \omega_\Phi\,\delta^{(D)}(k - k') \, .$$

Hence we find

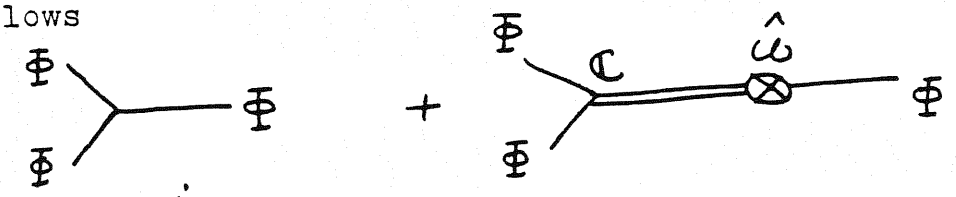
$$\beta^\Phi = \Big(\frac{1}{4\pi}k^2 - 2\Big)\Phi + \Big[C^\Phi_{\ \Phi_1\Phi_2}\omega_{\Phi_1} - \frac{1}{2}\hat{\omega}^\Phi_{\ i}\,C^i_{\ \Phi_1\Phi_2} - \qquad (5.35)$$

$$- C^\Phi_{\ \Phi_1\Phi_2}\Big]\Phi_1\Phi_2 + O(\varphi \neq \Phi, \Phi^3) = \omega_\Phi \Phi +$$

$$+ \Big[C^\Phi_{\ \Phi_1\Phi_2}\Big(\omega_{\Phi_1} - \frac{1}{2}\omega_\Phi - 1\Big) - \frac{1}{2}\sum_{i\neq\Phi}\hat{\omega}^\Phi_{\ i}\,C^i_{\ \Phi_1\Phi_2}\Big]\Phi_1\Phi_2 + \qquad (5.36)$$

$$+ O(\varphi \neq \Phi, \Phi^3) \qquad (5.37)$$

The interaction terms here can be represented graphically as follows



where $C^i_{\ \Phi\Phi'}$ corresponds to the transition of the two tachyons into some state which afterwards is transformed again into the tachyon with the help of the matrix $\hat{\omega}$. Note the important role of the matrix $t^i_{\ j}$ which makes $\hat{\omega}$ <u>non-diagonal</u> in field space. Since

$\hat{w}$ depends only on momenta (and not on the cutoff) β^{Φ} depends on the cutoff through $\mathbb{C}$. $\mathbb{C}^i_{jk}$ determines the interaction terms, which thus contain factors of powers of the cutoff.

To check the correspondence with the string S-matrix we are to solve the equation β^i =0, for example, eliminating all the fields in favour of the tachyon (as we did in the previous section). But as in the Wilson approach we need to understand what to do with the cutoff dependence in the interaction vertices. One possibility could be to make a redefinition of the tachyon ($\Phi \rightarrow (\Lambda R)^{-2}\Phi$) but this does not help in the case of the second interaction term in (5.37) (which contains the infinite number of possible powers of Λ).

Since t^i_j may depend on the momenta, the second interaction term in (5.37) has a complicated form: the interaction vertex depends essentially on all powers of momenta, i.e. is effectively non local. Similar results are found for other β-functions, e.g. the graviton one.

The scale invariance condition β^i=0, i.e. $\frac{\partial Z}{\partial \lambda}$ =0, can be represented as follows (cf.(5.25))

$$w^i_j\, t^j_k\, \Omega^k_s\, \varphi^s = t^i_j\, \Omega'^j_k\, \dot{\mathbb{C}}^k_{st}\, \varphi^s \varphi^t \quad , \tag{5.38}$$

or

$$\hat{w}^i_j\, \Omega^j_s\, \varphi^s = \Omega'^i_j\, \dot{\mathbb{C}}^j_{st}\, \varphi^s \varphi^t \tag{5.39}$$

This equation would look like as an equation of motion in a φ^3-field theory if not for the dependence of Ω and Ω' on φ. Observing that Ω =1+O(φ) we have

$$\begin{aligned}\hat{w}^i_j \varphi^j &= \tfrac{1}{2}\left(\hat{w}^i_j\, \mathbb{C}^j_{rs} - \dot{\mathbb{C}}^i_{rs}\right)\varphi^r\varphi^s + O(\varphi^3) \\ &= \tfrac{1}{2}\left[\hat{w}^i_j - \delta^i_j(\gamma_r+\gamma_s-\gamma_i)\right]\mathbb{C}^j_{rs}\,\varphi^r\varphi^s + O(\varphi^3)\ .\end{aligned} \tag{5.40}$$

As in the case of the Wilson's RG approach, the problem with this equation is that $\mathbb{C}$ explicitly depends on Λ , $\mathbb{C}^i_{jk} = (\Lambda R)^{\gamma_j+\gamma_k-\gamma_i}\ \widetilde{\mathbb{C}}^i_{jk}$. Hence the solution of (5.40) will, in general, depend on Λ (in apparent contradiction with $\dot{\varphi}^i = \beta^i = 0$).

It is of some interest to consider the partition function Z (5.1) with $\bar{x}$=0. Z[$\bar{x}$=0] is the generating functional for the correlators of the vertex operators and hence, in view of our discussion in sect.4, $\dot{Z}[\bar{x}=0]$ may have some relation to the string S-matrix generating functional. When $\bar{x}$=0 (cf.(5.20),(5.21))

$$\langle V_i \rangle = V_\Phi[0]\, t^\Phi_{\ i} \equiv \hat{t}^\Phi_{\ i} ,$$

$$\hat{t}^\Phi_{\ i} = (\Lambda R)^2 (\Lambda a)^{-\frac{k^2}{4\pi}}\ \tau_i , \qquad \tau_i = \tau_i(k) , \tag{5.41}$$

$$\frac{\partial}{\partial\lambda}\langle V_i \rangle = -\,\omega_\Phi\, \hat{t}^\Phi_{\ i} , \qquad \omega_\Phi = \frac{1}{4\pi}k^2 - 2 . \tag{5.42}$$

Then eq.(5.23) implies that

$$S = \frac{\partial Z}{\partial\lambda} = \omega_\Phi\, \hat{t}^\Phi_{\ i}\, \Omega^i_{\ j}\, \varphi^j - \hat{t}^\Phi_{\ i}\, \Omega'^i_{\ j}\, \dot{\mathbb{C}}^j_{rs}\, \varphi^r\varphi^s = \tag{5.43}$$

$$= \omega_\Phi\, \hat{t}^\Phi_{\ i}\, \varphi^i - \tfrac{1}{2}\,\omega_\Phi\, \hat{t}^\Phi_{\ i}\, \mathbb{C}^i_{rs}\, \varphi^r\varphi^s + \tag{5.44}$$

$$+ \tfrac{1}{2}\, \hat{t}^\Phi_{\ i}\, \dot{\mathbb{C}}^i_{rs}\, \varphi^r\varphi^s + O(\varphi^3) .$$

This does not look like as a reasonable action functional. For example, the leading tachyon-dependent terms in S (5.44) are

$$S = \omega_\Phi\, \hat{t}^\Phi_{\ \Phi}\, \Phi - \tfrac{1}{2}\, \hat{t}^\Phi_{\ i}\, (\omega_\Phi - 2)\, \mathbb{C}^i_{\Phi\Phi'}\, \Phi\Phi' + \tag{5.45}$$

$$+ O(\varphi \neq \Phi, \Phi^3) .$$

It would be interesting to prove (though we consider this to be unlikely) that the linear in φ term in S is always a total derivative and that

$$\frac{\partial S}{\partial \varphi^i} = \mathscr{X}_{ij}\, \beta^j \qquad , \tag{5.46}$$

where $\mathscr{X}_{ij}$ is some non-degenerate matrix. The basic objects of our approach based on the completeness relation are the partition function Z and the "structure coefficients" $\mathbb{C}^i_{jk}$. The latter we can formally interpret as a connection (which is flat because of the associativity property (5.9) and the φ -independence of $\mathbb{C}^i_{jk}$). Since

$$\frac{\partial^2 Z}{\partial \varphi^i \partial \varphi^j} = \langle V_i V_j \rangle = \mathbb{C}^k_{ij} \langle V_k \rangle = -\mathbb{C}^k_{ij} \frac{\partial Z}{\partial \varphi^k} \tag{5.47}$$

we have

$$\nabla_i \nabla_j Z = \frac{\partial^2 Z}{\partial \varphi^i \partial \varphi^j} - \Gamma^k_{ij} \frac{\partial Z}{\partial \varphi^k} = 0$$

$$\Gamma^k_{ij} = -\mathbb{C}^k_{ij} \tag{5.48}$$

Thus the only "natural" "metric" $\mathscr{X}_{ij} = \nabla_i \nabla_j Z$ which transforms as a tensor under the redefinitions of φ^i is, in fact, equal to zero.

6. Concluding remarks

As should be clear from the above discussion we are lacking a clear demonstration of a connection between scale invariance RG equations for a most general 2-d sigma-model and string field theory equations of motion. Still there are many indications that some connection exists, so both the approach based on the Wilson's RG equation and the approach based on the completeness of the system of vertex operators are worth further study. The "completeness" approach looks particularly appealing since it is just a natural generalization of the usual RG approach for the renormalizable sigma-model to the case when the σ -model action includes all possible local interactions. While the β -function of the completeness approach contains all powers of the fields this may not be a drawback since the "quadratic" form of the β -function in the Wilson's approach is achieved at the expence of introducing the logarithmic dependence of the interaction vertex on the cutoff.

The dependence of the β -functions on the cutoff is a problem in both approaches. It may indicate that it is not quite correct to directly impose the condition of the vanishing of the β -function of a general σ -model in attempt to interpret the resulting equations as string equations of motion. In any case, trying to solve $\dot{\varphi} = \beta(\varphi, \Lambda) = 0$ for φ with β non-trivially depending on Λ does not make sense. What may be of interest is to try to formulate the string theory with a built-in 2-d cutoff Λ and establish a connection with the σ -model for a finite Λ .

Both the "Wilson" and the "completeness" approaches are based on separating the free piece in the σ -model action. Hence they both are perturbative in nature, with the flat space time metric playing a distinguished role. We thus distinguish one particular operator ($\partial X \partial X$) among all other vertex operators present in the σ -model action. This, of course, implies that we study the theory in a "perturbative phase" in which the particles propagate in the usual way. This seems to be a disadvantage as compared to the case of the ordinary renormalizable σ -model in which we can develop the generally covariant loop expansion without specifying a particular D-dimensional metric. To develop a background-independent renormalization group approach for the generalized σ -model thus remains an important problem for future.

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Addendum to Sect. 3

Let us finish this section with the following re~~ma~~rks. We used the interaction vertices V_i which are <u>not</u> normal ordered with respect to the propagator G (see Eqs.(2.9),(2.19)). This seems natural within the path integral approach and makes possible to consider the products of V_i without worring about normal ordering (i.e. simply as classical, not operator, products; see, e.g., Eqs.(5.4), (5.5) of Sec.5). Still, assuming that $v_i(\sigma)$,

$$V_i \equiv \Lambda^{\gamma_i} \int d^2\sigma \; v_i(\sigma) \;, \quad v_i(\sigma) = e^{ik_i x}\, v_i(\partial x) \;, \qquad (3.20)$$

form a complete set of local operators, we are free to trade $\{v_i\}$ for the basis of the normal ordered operators (see (2.3)-(2.9); we take $G(0|\Lambda) = \frac{1}{2\pi} \ln \Lambda a$)

$$:v_i: \; = v_j\, Q^j{}_i \;, \qquad Q^i{}_j = (\Lambda a)^{k_j^2/4\pi}\, \bar{Q}^i{}_j\, \delta^D(k_i - k_j) \qquad (3.2$$

$$:\mathcal{O}[x]: \; = \exp\left(-\frac{1}{2} G \cdot \frac{\delta^2}{\delta x^2}\right) \mathcal{O}[x] \;, \qquad (3.2$$

$$\langle :\mathcal{O}[x]: \rangle_0 = \mathcal{O}[x]\Big|_{x=0} \;, \quad \langle F[x] \rangle_0 = \left(e^{\frac{1}{2} G \cdot \frac{\delta^2}{\delta x^2}} F[x]\right)\Big|_{x=0} \qquad (3.2$$

The matrix $\bar{Q}^i{}_j$ depends on k (but not on Λ) and has "triangular" structure (note that $\partial_a x^\mu \partial^a x^\nu = :\partial_a x^\mu \partial^a x^\nu: + \frac{1}{\pi} \Lambda^2 \delta^{\mu\nu}$, et

$$v_i = (\Lambda a)^{k_i^2/4\pi} :v_i: \; + \text{ operators at lower levels } N_j < N_i \qquad (3.2$$

Comparing (3.21),(3.22) with (5.20),(5.19) we conclude that

$$Q^i{}_j = t^{-1\,i}{}_j \;, \qquad \bar{Q}^i{}_j = P^{-1\,i}{}_j \;. \qquad (3.2$$

We can thus rewrite Eqs.(2.19)-(2.21) in terms of the normal ordered operators and the corresponding set of fields

$$I_{eff} = \varphi^i \cdot V_i = \hat{\varphi}^i \cdot \hat{V}_i \;, \quad \hat{V}_i = \Lambda^{\gamma_i} \int d^2\sigma \; :v_i(\sigma): \;, \qquad (3.2$$

$$\varphi^i = Q^i{}_j\, \hat{\varphi}^j = \bar{Q}^i{}_j\, \bar{\varphi}^j \;, \quad \bar{\varphi}^i = \varphi^i + \ldots \qquad (3.2$$

An advantage of the use of the normal ordered basis is that the corresponding matrix $\bar{f}^i_{ja}$ in (2.20b),(2.21) can be expressed in terms of the operator product coefficients[22,17]. Taking the $\Lambda\frac{\partial}{\partial\Lambda}$ - derivative of the relation (cf.(2.5))

$$:\gamma_i(\sigma)::\gamma_j(\sigma'): = :\gamma_i(\sigma)\exp\left[\int d^2\sigma_1 d^2\sigma_2\, G(\sigma_1,\sigma_2)\frac{\overleftarrow{\delta}}{\delta x(\sigma_1)}\frac{\overrightarrow{\delta}}{\delta x(\sigma_2)}\right] \times \gamma_j(\sigma'): \tag{3.28}$$

we get

$$\int \dot{G}\cdot\frac{\delta}{\delta x}:\gamma_i(\sigma):\frac{\delta}{\delta x}:\gamma_j(\sigma'): = \frac{\partial}{\partial \ln\Lambda}\left(:\gamma_i(\sigma)::\gamma_j(\sigma'):\right) \tag{3.29}$$

Hence (see (2.20b))

$$\bar{f}^k_{ij} = \frac{1}{2}\Lambda^{\gamma_{ijk}}\frac{\partial}{\partial\ln\Lambda}\int d^2u\, \hat{C}^k_{ij}(u), \qquad \gamma_{ijk} \equiv \gamma_i+\gamma_j-\gamma_k \tag{3.30}$$

where we have used the OPE relation implied by the completeness of $:\gamma_i:$

$$:\gamma_i(\sigma)::\gamma_j(\sigma'): = \hat{C}^k_{ij}(\sigma-\sigma'):\gamma_k\left(\frac{\sigma+\sigma'}{2}\right): \tag{3.31}$$

$\bar{f}^k_{ij}$ are the coefficients in the β -function (2.21) expressed in terms of $\bar{\varphi} = \bar{Q}^{-1}\varphi$ (cf.(5.30))

$$\bar{\beta}^i = \dot{\bar{\varphi}}^i = \bar{u}^i_j\bar{\varphi}^j + \bar{f}^i_{jk}\bar{\varphi}^j\bar{\varphi}^k, \tag{3.32}$$

$$\bar{u} = \bar{Q}^{-1}u\bar{Q}, \qquad \bar{f} = \bar{Q}^{-1}f\bar{Q}\bar{Q}, \quad u^i_j = \Delta^i_j - \gamma_i\delta$$

$\bar{\beta}^i$ =0 is equivalent to the equation derived in[17] (up to the non-diagonal terms in $\bar{u}$ absent in[17]). Note that $\hat{C}$ and $\bar{f}$ depend non-trivially on Λ through their dependence on G (being dimensionless $\bar{f}$ depends on ΛR, where R is an IR cutoff).

It is possible to get rid of the dependence of $\bar{f}$ on Λ by using a special prescription (cf. Ref.22) of how to introduce a cutoff (which, however, can hardly be considered as a consistent one in the present context). Let us assume that (3.31) is defined so that $\hat{C}^{k}_{ij}$ is Λ-independent (this is true e.g. if we take $\Lambda \to \infty$ in G before $\vec{\sigma} \to \vec{\sigma}'$), inserting instead the UV cut-off $|u| > \Lambda^{-1}$ in the integral in (3.30). Then on dimensional grounds

$$\hat{C}^{k}_{ij}(u) = \bar{C}^{k}_{ij} \cdot |u|^{\gamma_{ijk}-2} \qquad , \tag{3.33}$$

where $\bar{C}^{k}_{ij}$ is dimensionless and hence (cf. Ref.22)

$$\bar{f}^{k}_{ij} = \pi \Lambda^{\gamma_{ijk}} \bar{C}^{k}_{ij} \frac{\partial}{\partial \ln \Lambda} \left(\int_{\Lambda^{-1}} d|u| \, |u|^{\gamma_{ijk}-1} \right) = -\pi \bar{C}^{k}_{ij} \tag{3.34}$$

This coincides with the standard expression for the coefficient of the φ^2-term in the β-function computed using the perturbation theory near a conformal point[22]. However, in the latter case the β-function in general contains terms of all <u>higher</u> orders in φ (see also Sec.5). We conclude that the existence of a consistent scheme in which the β-function is quadratic in φ and does not depend explicitly on Λ remains an open question.