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THE CHARACTERISTIC FUNCTIONS FOR THE SQUEEZED COHERENT CHAOTIC PHOTON STATE WITH APPLICATION TO THE JAYNES-CUMMINGS MODEL

T. Garavaglia

Institiúid Árd-Léighinn Bhaile Átha Cliath*

Baile Átha Cliath 4, Éire

The characteristic functions for the operators $\hat{x}, \hat{p}$, and $\hat{H}$ are derived for the squeezed coherent chaotic state of a single mode photon field. The photon probability distribution for this state is obtained from the characteristic function for $\hat{H}$, and it is applied to the Jaynes-Cummings model for the interaction of a single mode photon field with a two level system.

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* Present address: Superconducting Super Collider Laboratory, 2550 Beckleymeade Avenue, Dallas, TX 75237-3946.

I. INTRODUCTION

The statistical properties of a squeezed coherent state have recently been of interest in quantum optics applications to the Jaynes-Cummings model [1] and in the study of multiplicity distributions in high energy interactions [2,and 3]. In this paper, the characteristic functions for the operators, defined in terms of the boson operators a and $a^\dagger$, $\hat{x} = (a + a^\dagger)/\sqrt{2}, \hat{p} = (a - a^\dagger)/i\sqrt{2}$, and $\hat{H} = a^\dagger a + 1/2$ are given. The last one is used to obtain the probability distribution for the single mode squeezed coherent chaotic photon quantum state.

These states can be represented in terms of the thermal states [4,5, and 6]

$$S(\xi)D(\alpha)G^\dagger(\theta)|0) \otimes |\tilde{0}) \tag{1.1a}$$

$$D(\alpha)S(\xi)G^\dagger(\theta)|0) \otimes |\tilde{0}) \tag{1.1b}$$

where $S(\xi), D(\alpha)$, and $G(\theta)$ are respectively the squeezing[7, and 8], coherence[9], and thermal [10] operators defined for complex ξ, α, and real θ as

$$S(\xi) = e^{(\bar{\xi}a^2 - \xi a^{\dagger 2})/2} \tag{1.2a}$$

$$D(\alpha) = e^{\alpha a^\dagger - \bar{\alpha} a} \tag{1.2b}$$

$$G(\theta) = e^{\theta(a\tilde{a} - a^{\dagger}\tilde{a}^{\dagger})} \tag{1.2c}$$

where $(a, a^{\dagger})$ and $(\tilde{a}, \tilde{a}^{\dagger})$ form commuting sets of boson operators. The parameter θ is related to temperature T through the expressions

$$\bar{n}^{1/2}(\beta) = sinh(\theta)$$

$$\bar{n}(\beta) = (exp(\beta) - 1)^{-1} \tag{1.3}$$

$$\beta = \hbar\omega / K_B T.$$

For a single mode free boson field represented in the form

$$E = \hat{x} cos(\omega t) + \hat{p} sin(\omega t), \tag{1.4a}$$

one finds for the two states in (1.1) the matrix elements

$$(\xi\alpha\theta|E|\xi\alpha\theta) = e^{-\xi} x_c\ cos(\omega t) + e^{\xi} p_c\ sin(\omega t), \tag{1.4b}$$

and

$$(\alpha\xi\theta|E|\alpha\xi\theta) = x_c\ cos(\omega t) + p_c\ sin(\omega t) \tag{1.4c}$$

with

$$x_c = (\alpha + \bar{\alpha})/\sqrt{2}\ \ and\ \ p_c = (\alpha - \bar{\alpha})/i\sqrt{2}. \tag{1.4d}$$

The first state represents the physically interesting state of a squeezed coherent signal emitted from a thermal source. The second state represents the situation where coherence follows squeezing of a thermal signal. The results of the latter can be obtained from those of the former upon using the identity

$$S(\xi)D(\alpha)S^{\dagger}(\xi) = D(\tilde{\alpha}) \tag{1.5a}$$

where

$$\tilde{\alpha} = \alpha\ cosh|\xi| - \bar{\alpha}\ e^{i\phi}sinh|\xi| \tag{1.5b}$$

with $\xi = |\xi|e^{i\phi}$. The physical interpretation of states related to (1.1) which result when the operators $S(\xi), D(\alpha)$, and $G(\theta)$ are applied in different order has been discussed in Refs.[5 and 6].

The mean value of an operator $A(a, a^{\dagger})$ is defined as

$$\begin{aligned} Tr\rho_{\xi\alpha}(\beta) &= (\xi\alpha\theta|A(a,a^{\dagger})|\xi\alpha\theta) \\ &= (\theta|D^{\dagger}(\alpha)S^{\dagger}(\xi)A(a,a^{\dagger})S(\xi)D(\alpha)|\theta) \\ &= Tr\rho(\beta)D^{\dagger}(\alpha)S^{\dagger}(\xi)A(a,a^{\dagger})S(\xi)D(\alpha). \end{aligned} \tag{1.6}$$

This suggests for the state (1.1a) the definition of its density matrix as

$$\rho_{\xi\alpha}(\beta) = S(\xi)D(\alpha)\rho(\beta)D^{\dagger}(\alpha)S^{\dagger}(\xi) \tag{1.7}$$

where the density matrix for the Hamiltonian $\hat{H}$ is

$$\rho(\beta) = U(\beta)/Z(\beta) \tag{1.8}$$

with $U(\beta) = exp(-\beta\hat{H})$ and $Z(\beta) = Tr\rho(\beta)$.

II. CHARACTERISTIC FUNCTIONS

The characteristic function for the operator $A(a, a^\dagger)$ is defined as

$$C(\lambda, \xi, \alpha, \beta)_A = Tr\rho_{\xi\alpha}(\beta)e^{i\lambda A(a,a^\dagger)} \tag{2.1}$$

The characteristic functions for the operators $\hat{x}, \hat{p}$, and $\hat{H}$ can be evaluated when (2.1) is expressed in the x-representation. For the state (1.1a),the characteristic function of the operator $\hat{H}$ is

$$C(\lambda, \xi, \alpha)_H = TrS(\xi)D(\alpha)\rho(\beta)D^\dagger(\alpha)S^\dagger(\xi)U(-i\lambda), \tag{2.2a}$$

and it can be expressed in the x-representation as

$$C(\lambda, \xi, \alpha)_H = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x|\rho_{\xi\alpha}(\beta)|x')(x'|U(-i\lambda)|x)dxdx'. \tag{2.2b}$$

The density matrix in the x-representation,$(x|\rho(\beta)|x')$, is found from the identity

$$\int_{-\infty}^{\infty}(x|U(\beta)\hat{x}U(-\beta)|x'')(x''|\rho(\beta)|x')dx'' = x'(x|\rho(\beta)|x') \tag{2.3}$$

Since

$$U(\beta)\hat{x}U(-\beta) = \hat{x}\ \ cosh\beta + i\hat{p}\ \ sinh\beta, \tag{2.4}$$

one finds the differential equation

$$(cosh\beta \;\; x + sinh\beta \;\; d/dx)(x|\rho(\beta)|x') = x'(x|\rho(\beta)|x'). \qquad (2.5)$$

The solution to this equation satisfying the normalization condition $Tr\rho(\beta) = 1$ is

$$(x|\rho(\beta)|x') = \frac{1}{\sqrt{2\pi}\;\sigma(\beta)} exp - [\frac{(x+x')^2}{8\sigma^2(\beta)} + \frac{\sigma^2(\beta)}{2}(x-x')^2] \qquad (2.6)$$

where

$$\sigma(\beta) = \sqrt{\frac{1}{2} coth(\beta/2)} \qquad (2.7)$$

Representing the operators $S(\xi)$, for real ξ, and $D(\alpha)$ in terms of $\hat{x}$ and $\hat{p}$, one finds for L^2 test functions $\Psi(x) = (x|\Psi)$

$$S(\xi)\Psi(x) = e^{\xi/2}(e^{\xi}x|\Psi) \qquad (2.8a)$$

$$D(\alpha)\Psi(x) = e^{-ip_c x_c/2} e^{ip_c x}\Psi(x - x_c) \qquad (2.8b)$$

which implies

$$S^{\dagger}(\xi)|x) = e^{\xi/2}|e^{\xi}x) \qquad (2.9a)$$

$$D^{\dagger}(\alpha)|x) = e^{i(\frac{p_c x_c}{2} - p_c x)}|x - x_c) \qquad (2.9b)$$

Using these results along with $\beta \rightarrow -i\lambda$ in (1.8) and (2.6), the characteristic function (2.2b) becomes

$$C(\lambda, \xi, \alpha)_H = \frac{Z(-i\lambda)}{2\pi e^{-\xi}\sigma(\beta)\sigma(-i\lambda)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dx'\; e^{ip_c e^{\xi}(x-x')}$$
$$\times\, exp - [(x + x' - 2e^{-\xi}x_c)^2/(8e^{-2\xi}\sigma^2(\beta)) + e^{2\xi}\sigma^2(\beta)(x - x')^2/2]$$
$$\times\, exp - [(x + x')^2/(8\sigma^2(-i\lambda)) + (\sigma^2(-i\lambda)/2)(x - x')^2] \qquad (2.10)$$

With a change of variables $y = x + x'$ and $z = x - x'$, this characteristic function is expressed in terms of a product of integrals as

$$C(\lambda, \xi, \alpha = \frac{Z(-i\lambda)}{\pi e^{-\xi}\sigma(\beta)\sigma(-i\lambda)} I_1(\xi, \beta, \lambda) \times I_2(\xi, \beta, \lambda) \qquad (2.11)$$

with

$$I_1 = \int_{-\infty}^{\infty} dy\; e^{-(y/2 - e^{-\xi}x_c)^2/(2e^{-2\xi}\sigma^2(\beta)) - (y/2)^2/2\sigma^2(-i\lambda)} \qquad (2.12a)$$

$$I_2 = \int_{-\infty}^{\infty} dz\; e^{ip_c e^{\xi} z}\;\; e^{-(1/2)(e^{2\xi}\sigma^2(\beta) + \sigma^2(-i\lambda))z^2} \qquad (2.12b)$$

Upon integration, one finds the result

$$C(\lambda, \xi, \alpha)_H = \sqrt{\frac{\omega}{ab}}\;\; e^{-(Ab+Ba)/ab}\;\; \frac{e^{x_a y_a/(y_a - 1)}}{\sqrt{1 - y_a}}\;\; \frac{e^{x_b y_b/(y_b - 1)}}{\sqrt{1 - y_b}} \qquad (2.13)$$

with the introduction of the notations

$$w = e^{i\lambda},$$

$$y_a = \tilde{a} w, \quad y_b = \tilde{b} w,$$

$$\tilde{a} = (a-1)/a, \quad \tilde{b} = (b-1)/b,$$

$$x_a = A/a(1-a), \quad x_b = B/b(1-b),$$

$$A = (x_c^2/2) e^{-2\xi}, \quad B = (p_c^2/2) e^{2\xi}, \tag{2.14}$$

$$a = e^{-2\xi}(\bar{n}(\beta) + 1/2) + 1/2,$$

and

$$b = e^{2\xi}(\bar{n}(\beta) + 1/2) + 1/2.$$

Introducing the generating function for the Laguerrer polynomials[11], (2.13) may be written as

$$C(\lambda, \xi, \alpha)_H = \frac{e^{-(Ab+Ba)/ab}}{\sqrt{ab}} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \tilde{a}^l \tilde{b}^m L_l^{-1/2}(x_a) L_m^{-1/2}(x_b) w^{(l+m+1/2)}. \tag{2.15}$$

Following a similar method, one can obtain the characteristic functions for the operators $\hat{x}$ and $\hat{p}$ as follows:

$$C(\lambda, \xi, \alpha)_x = \int_{-\infty}^{\infty} dx (x|\rho_{\xi\alpha}(\beta)|x) e^{i\lambda x} = exp[i\lambda e^{-\xi} x_c - (e^{-\xi}\sigma(\beta))^2 \lambda^2/2] \tag{2.16}$$

$$
\begin{aligned}
C(\lambda,\xi,\alpha)_p &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(p|x)(x|\rho(\beta)_{\xi\alpha}|x^{'})(x^{'}|p)e^{i\lambda p}dpdxdx^{'} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdx^{'}(x|\rho_{\xi\alpha}(\beta)|x^{'})\delta(x^{'}-x+\lambda) \\
&= exp[i\lambda e^{\xi}p_c - (e^{\xi}\sigma(\beta))^2\lambda^2/2]
\end{aligned}
\tag{2.17}
$$

where

$$
(x|p) = \frac{1}{\sqrt{2\pi}}e^{ipx}
$$

and

$$
\delta(x^{'}-x+\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x^{'}-x+\lambda)}dp \tag{2.18}
$$

have been used.

III. PROBABILITY DISTRIBUTIONS

The above characteristic functions can be used upon differentiation with respect to λ to find the moments of their associated operators, and they can be used to find probability distributions. The photon number probability distribution for the squeezed coherent chaotic state (1.1a) is found from the Fourier transform of $C(\lambda,\xi,\alpha)$ in (2.15) as

$$
\begin{aligned}
P_n(\beta)_{\xi\alpha} &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i\lambda(n+1/2)}C(\lambda,\xi,\alpha)_H d\lambda \\
&= \frac{e^{-(Ab+Ba)/ab}}{\sqrt{ab}}\sum_{l=0}^{n}\tilde{a}^l\tilde{b}^{n-l}L_l^{-1/2}(x_a)L_{n-l}^{-1/2}(x_b).
\end{aligned}
\tag{3.1}
$$

Various special cases and known results for some of these cases can be found from (3.1). An approximation to this formula has appeared in [12]. The results for the state in (1.1b) are found using the replacement $\alpha \rightarrow \alpha e^{\xi}$. For the case with no squeezing ,$\xi = 0$, one finds from (2.14)

$$a = b = \bar{n}(\beta) + 1; \quad \tilde{a} = \tilde{b} = \bar{n}(\beta)/(\bar{n}(\beta) + 1); \tag{3.2}$$

$$x_a = \frac{-x_c^2}{2\bar{n}(\beta)(\bar{n}(\beta)+1)}; \quad x_b = \frac{-p_c^2}{2\bar{n}(\beta)(\bar{n}(\beta)+1)}$$

which yields the Glauber-Lachs distribution [13]

$$P_n(\beta)_{0\alpha} = \frac{e^{\frac{-|\alpha|^2}{(\bar{n}(\beta)+1)}}}{\sqrt{\bar{n}(\beta)(\bar{n}(\beta)+1)}} e^{-\beta(n+1/2)} L_n\left(\frac{-|\alpha|^2}{\bar{n}(\beta)(\bar{n}(\beta)+1)}\right). \tag{3.3}$$

Another special case occurs when $\bar{n}(\beta) = 0$ and when α and ξ are real so that

$$\begin{aligned} P_n(\beta \rightarrow \infty)_{\xi\alpha} &= \frac{e^{-\alpha^2 e^{-\xi}/cosh\xi}}{cosh\xi}(tanh\xi)^n \\ &\otimes \sum_{l=0}^{n}(-)^l L_l^{-1/2}\left(\frac{2\alpha^2}{sinh2\xi}\right) L_{n-l}^{-1/2}(0). \end{aligned} \tag{3.4}$$

Using the identity

$$H_n^2(y) = n!2^n \sum_{l=0}^{n}(-)^l L_l^{-1/2}(2y^2) L_{n-1}^{-1/2}(0), \tag{3.5}$$

one finds[14]

$$P_n(\beta \to \infty)_{\xi\alpha} = \frac{e^{-\alpha^2 e^{-\xi}/cosh\xi}}{cosh\xi}(\frac{tanh\xi}{2})^n \frac{1}{n!} H_n^2(\frac{\alpha}{\sqrt{sinh 2\xi}}). \quad (3.6)$$

Using $\alpha \to \alpha e^{\xi}$, the corresponding case for the state (1.1b) is found to be[15]

$$\begin{aligned} P_n(\beta \to \infty)_{\alpha\xi} = & \frac{\sqrt{1 - tanh^2\xi}}{n!}(\frac{tanh\xi}{2})^n \\ & \times H_n^2(\sqrt{\frac{\alpha^2(1+tanh\xi)^2}{2tanh\xi}})e^{-\alpha^2(1+tanh\xi)}. \end{aligned} \quad (3.7)$$

IV. JAYNES-CUMMINGS MODEL

The probability distribution (3.1) found from the characteristic function (2.15) has a particularly timely application concerned with the revival and collapse properties of a two level system as described by the Jaynes-Cummings model[16]. Using the rotating wave approximation on resonance, the Hamiltonian for the interaction of a single mode radiation field of photon state $|n\rangle$ with a two level system of excited state $|+\rangle$ and ground state $|-\rangle$ is

$$H = \hbar\Omega(a^\dagger a + 1/2) + \hbar \begin{pmatrix} \omega_+ & 0 \\ 0 & \omega_- \end{pmatrix} + V_I(0) \quad (4.1)$$

where the Pauli spin matrices define the projection operators

$$2\sigma_\pm = (\sigma_1 \pm i\sigma_2) \quad (4.2)$$

which have the properties $\sigma_\pm|\pm) = 0$, and $\sigma_\pm|\mp) = |\pm)$ and where $\Omega = \omega_+ - \omega_-$. The interaction Hamiltonian is

$$V_I(0) = g(\sigma_+ a + \sigma_- a^\dagger) \tag{4.3}$$

where g denotes the coupling strength of the radiation field with the two level system. If in the initial state at $t = 0$ the two level system is in the excited pure state $|+)$ and if the photon field is characterized by the mixed state (1.1a) with density matrix $\rho(\beta)_{\xi\alpha}$, then the probability to observe the two level ststem in the state $|+)$ at time t is found from

$$P_{++}(t) = Tr|+)(+|U_I(t)\rho(\beta)_{\xi\alpha} \otimes |+)(+|U_I^\dagger(t) \tag{4.4}$$

where the time development operator in the interaction picture is

$$U_I(t) = e^{-itV_I(0)} \tag{4.5}$$

An elementary calculation yields the results

$$(+|U_I(t)|+) = \sum_{p=0}^{\infty} \frac{(-itg)^{2p}}{(2p)!}(aa^\dagger)^{2p} \tag{4.6a}$$

and

$$(n+|U_I(t)|+n) = \sum_{p=0}^{\infty} \frac{(-itg\sqrt{n+1})^{2p}}{(2p)!} \tag{4.6b}$$

so that when the density matrix in (4.4) is expressed as

$$\rho(\beta)_{\xi\alpha} = \sum_{n=0}^{\infty} P_n(\beta)_{\xi\alpha}|n)(n|, \tag{4.7}$$

one finds

$$P_{++}(t) = \sum_{n=0}^{\infty} P_n(\beta)_{\xi\alpha} \, cos^2(gt\sqrt{n+1}). \tag{4.8}$$

Numerical results for various special cases of this result have appeared in the literature [1,17, and 18]. In addition, experimental confirmation of the special case when $\xi = \alpha = 0$ has been achieved [19].

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