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# FINITE TEMPERATURE FIELD THEORY AND QUANTUM NOISE IN INDUCTIVELY COUPLED LRC CIRCUITS

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Finite temperature ($0 \leq T < \infty$) field (FTF) theory with an effective spectral Lagrangian density formulation is used to study quantum noise in an inductively coupled LRC circuits. Analytical solutions and numerical results for the finite second moments at temperature $T$ which satisfy the uncertainty principle bound are given. From the numerical results, one can see the presence of a squeezed quantum state which depends upon the strength of the mutual inductance between the coupled circuits.

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## I. INTRODUCTION

The study of noise in electrical and optical systems plays an increasingly important role in modern technology. This is the result of the desire to produce systems which operate at low power with weak signals so as to produce small and sensitive systems. The laws of quantum physics place constraints on the design and performance of such systems. A large body of literature is developing which is related to this programme and describes quantum aspects of various types of physical devices.

In this letter the results of an investigation are presented which are concerned with the quantum noise and thermal noise associated with interacting dissipative oscillators where the interaction results from the mutual inductance coupling of the two separate circuits. General expressions are given for the variances in the charge and current within the separate circuits. In order to better understand the characteristics of the noise associated with the interacting circuits, numerical results are given in Table I. for a special example when both of the dissipative linear oscillators are identical and contain the same inductance $L$, capacitance $C$, and resistance $R$. These results are given for various values of the dimensionless temperature $z = k_B T/\hbar\omega_o$ and quality factor $q_o$. Here $k_B$ and $\hbar$ are respectively Boltzmann's constant, $2\pi$ times Plank's constant, and $\omega_o = (LC)^{-/12}$ is the natural frequency of the non-dissipative oscillator. As can be seen from the table, there are values for the variances in charge $q$ which are below the standard quantum limit. This indicates the presence of a squeezed quantum state associated with the interacting systems which depends upon the strength of the mutual inductance coupling.

## II. METHODS AND RESULTS

In this analysis, finite temperature field theory (FTF) methods [1] are employed so as to include temperature dependance in the expressions for quantum noise. These methods are based upon the zero temperature quantum field theory methods for a scalar boson field [2]; however, the creation and annihilation operators associated with the zero temperature quantum fields are transformed so as to include a finte temperature ($0 \leq T \leq \infty$.) dependance. This permits the determination of finite temperature ensemble averages of the operators of physical interest using algebraic methods based on quantum field theory. In order to obtain results for the variances which are consistent with the Dirac quantization condition, a regularization method is employed which is based upon the replacement of the dissipative elements of the circuit with semi-infinite low-pass filters. This method has been used to study the nature of quantum and thermal noise for a dissipative $LRC$ oscillator Ref.[3 and 4]. The FTF method and the regularization procedure are described in these references, and the notation and quantization procedure of these references are followed in the present study.

The regularization method has been introduced so as to produce a natural cut- off frequency so that the variance in the current $L\dot{q}$ remains finite. It also has the effect of introducing a frequency dependant damping coefficient. The semi-infinite low-pass filter consists of basic elements of inductance ($L_o = L_T \Delta x$)

and capacitance ($C_o = C_T \Delta x$). The characteristic impedance of the filter, obtained from Ref.[4],

$$Z_o(a,b) = i\omega L_o/2 + (L_o/Co - \omega^2 L_o^2/4)^{1/2}, \qquad (2.1)$$

implies a maximum frequency above which the voltage wave becomes damped. Below this frequency the voltage in the $n^{th}$ element of the filter is

$$V(n) = exp(ikn\Delta x)V(0) \qquad (2.2)$$

where

$$k/2 = \omega/2v(\omega)$$

relates the phase velocity $v(\omega) = R(\omega)/L_T$, the wave number $k$, and the real part $R(\omega)$ of the characteristic impedance (2.1).

The quantum properties of the inductively coupled system of two $LRC$ circuits are obtained from an effective spectral Lagrangian density. At frequency $\omega$ the effective Lagrangian density for the inductively coupled LCR network is

$$\mathcal{L}_\omega(Q_{1\omega}, \partial_x Q_{1\omega}, \partial_t Q_{1\omega},, Q_{2\omega}, \partial_x Q_{2\omega}, \partial_t Q_{2\omega}, \beta) =$$
$$\frac{\delta(x)}{2}[L_1(\frac{\partial}{\partial t}Q_{1\omega}(x,t,\beta))^2 - Q_{1\omega}^2(x,t,\beta)/C_1]$$
$$+ L_{1T}\frac{H(x)}{2}[(\frac{\partial}{\partial t}Q_{1\omega}(x,t,\beta))^2 - v_1^2(\omega)(\frac{\partial}{\partial x}Q_{1\omega}(x,t,\beta))^2]+$$
$$\frac{\delta(x)}{2}[L_2(\frac{\partial}{\partial t}Q_{2\omega}(x,t,\beta))^2 - Q_{2\omega}^2(x,t,\beta)/C_2]$$
$$+ L_{2T}\frac{H(x)}{2}[(\frac{\partial}{\partial t}Q_{2\omega}(x,t,\beta))^2 - v_2^2(\omega)(\frac{\partial}{\partial x}Q_{2\omega}(x,t,\beta))^2]$$
$$+ L_{12}Q_{1\omega}Q_{2\omega} \tag{2.3}$$

where $\delta(x)$ and $H(x)$ are respectively the Dirac and Heaviside distributions, and $Q_{1\omega}(x,t,\beta)$, and $Q_{2\omega}(x,t,\beta)$ with $\beta = 1/k_B$, are the charge densities at finite temperature of the two separate oscillator circuits of inductances $L_1, L_2$ and capacitances $C_1, C_2$ respectively. The resistances $R_1$ and $R_2$ have been replaced with semi-infinite low-pass filters of inductance elements $L_{iT}$,(i=1 or 2) which have frequency dependent propagation speeds $v_i(\omega)$ given by

$$v_i(\omega) = R_i(\omega)/L_{iT}(i = 1or2) \tag{2.4}$$

with

$$R_i(\omega)/L_{iT} = (R_i\nu_i/L_{iT}\Lambda_i)/tan^{-1}(\frac{(\nu_i/\Lambda_i)}{(1-(\nu_i/\Lambda_i)^2)^{1/2}}) \tag{2.5}$$

where $R_i = (L_{io}/C_{io})^{1/2}, \nu_i = \omega_i/\omega_{io}, \Lambda_i = 2Q_{io}C_i/C_{io}, \omega_{io} = (L_iC_i)^{-1/2}$ and $Q_{io} = L_i\omega_{io}/R_i$. Here $\omega_{io}$ is the natural frequency of the $i^{th}LC$ oscillator, $Q_{io}$, the quality factor ,and $\Lambda_i\omega_{io}$ the highest frequency passed by the filter. In the effective Lagrangian density, the strength of the coupling is represented by the mutual inductance $L_{12}$.

The postulates of quantization [5] suggest for the spectral integrals of the charge density and its conjugate momentum $\Phi_\omega(x,t,\beta)$ the commutation realtion

$$[Q_i(x,t,\beta),\Phi_j(x',t,\beta)] = i\hbar\delta(x-x')\delta_{ij} \tag{2.6}$$

where the spectral integral is defined as

$$Q_i(x,t,\beta) = \int_o^{\omega_{imax}} Q_{i\omega}(x,t,\beta)d\omega. \tag{2.7}$$

The equations of motion associated with the effective Lagrangian density (2.3), are found from the Euler-Lagrange equation. The action for the effective Lagrangian density is

$$S = \int dt \int_0^\infty dx \int_0^{\omega_o \Lambda} d\omega \mathcal{L}_\omega(\partial_x Q_\omega, \partial_t Q_\omega, Q_\omega). \tag{2.8}$$

The Euler-Lagrange equation obtained from this action at frequency $\omega$ is

$$\partial_t\left(\frac{\partial \mathcal{L}_\omega}{\partial(\partial_t Q_{i\omega})}\right) + \partial_x\left(\frac{\partial \mathcal{L}_\omega}{\partial(\partial_x Q_{i\omega})}\right) - \frac{\partial \mathcal{L}_\omega}{\partial Q_{i\omega}} = 0. \tag{2.9}$$

The field equations for this system are found when (2.3) is substituted into (2.9) and when the resulting equations are integrated with respect to $x$ from $-\epsilon$ to $+\epsilon$. The discontinuous distribution $H(x)$ produces a term which depends upon the partial derivative $\partial_x Q_{i\omega}(x,t,\beta)$. If the interacting field associated with frequency $\omega$ is written in terms of the in-field and out-field as

$$Q_{i\omega}(x,t,\beta) = Q^{in}_{i\omega}(U,\beta) + Q^{out}_{i\omega}(V,\beta) \tag{2.10}$$

where $U = t + x/v(\omega)$ and $V = t - x/v(\omega)$, then this partial derivative becomes

$$\begin{aligned} \partial_x Q_{i\omega}(x,t,\beta) &= \frac{1}{v_i(\omega)}(\partial_U Q^{in}_{i\omega}(U,\beta) - \partial_V Q^{out}_{i\omega}(V,\beta)) \\ &= \frac{1}{v_i(\omega)}(\dot{Q}^{in}_{i\omega}(U,\beta) - \dot{Q}^{out}_{i\omega}(V,\beta)). \end{aligned} \tag{2.11}$$

Using the notation $Q_{i\omega}(t,\beta) = Q_{i\omega}(0,t,\beta)$, the operator field equations resulting form the above method become

$$
\begin{aligned}
&L_1\ddot{Q}_{1\omega}(t,\beta) + L_{12}\ddot{Q}_{2\omega}(t,\beta) + \frac{Q_{1\omega}(t,\beta)}{C_1} + \\
&R_1(\omega)\dot{Q}_{1\omega}(t,\beta) = 2R_1(\omega)\dot{Q}^{in}_{1\omega}(t,\beta) \\
&L_2\ddot{Q}_{2\omega}(t,\beta) + L_{21}\ddot{Q}_{1\omega}(t,\beta) + \frac{Q_{2\omega}(t,\beta)}{C_2} + \\
&R_2(\omega)\dot{Q}_{2\omega}(t,\beta) = 2R_2(\omega)\dot{Q}^{in}_{2\omega}(t,\beta).
\end{aligned}
\tag{2.12}
$$

When $Q_{i\omega}(t,\beta)$ and $Q^{in}_{i\omega}$ are represented in the form

$$
\begin{aligned}
Q_{i\omega}(t,\beta) &= \hat{Q}_{i\omega}e^{-i\omega t} + \hat{Q}^{\dagger}_{i\omega}e^{i\omega t} \\
Q^{in}_{i\omega}(t,\beta) &= \hat{Q}^{in}_{i\omega}e^{-i\omega t} + \hat{Q}^{in\dagger}_{i\omega}e^{i\omega t} \\
\hat{Q}^{in}_{i\omega} &= \left(\frac{\hbar}{4\pi\omega R_1(\omega)}\right)^{\frac{1}{2}} A^{in}_i(\omega,\theta)
\end{aligned}
\tag{2.13}
$$

and when $Q^{out}_{i\omega}(t,\beta)$ is represented in the same form, then (2.12) and (2.13) can be used to obtain $\hat{Q}^{out}_{i\omega}$. This results in the expression for the operator $\hat{Q}^{out}_{i\omega}$

$$
\hat{Q}^{out}_{i\omega} = \left(\mathcal{Z}^{-1}(\omega)\mathcal{R}(\omega) - \mathcal{I}\right)\hat{Q}^{in}_{i\omega} \tag{2.14}
$$

with

$$
\mathcal{Z}(\omega) = \begin{pmatrix} z_1(\omega) & z_{12}(\omega) \\ z_{21}(\omega) & z_2(\omega) \end{pmatrix}
$$

$$z_1(\omega) = \frac{1}{C_1} - \omega^2 L_1 - i\omega R_1(\omega)$$

$$z_2(\omega) = \frac{1}{C_2} - \omega^2 L_2 - i\omega R_2(\omega) \tag{2.15}$$

$$z_{12}(\omega) = z_{21}(\omega) = -\omega^2 L_{12}$$

The corresponding operators for the $Q_{i\omega}(t, \beta)$ fields given in (2.12) are found from (2.10) and (2.14) so that the fields $Q_{i\omega}(x, t)$ can be expressed in terms of the finite temperature annihilation operators $A_i^{in}(\omega, \beta)$ and their adjoints. These operators satisfy the spectral commutation relation

$$[A_i^{in}(\omega, \theta), A_j^{in\dagger}(\omega', \theta)] = \delta_{ij}\delta(\omega - \omega'] \tag{2.16}$$

In these expression, $A_i^{in}(\omega, \theta)$ is obtained from the zero temperature boson annihilation operator $A^{in}(\omega)$ upon using the Bogoliubov transformation $G(\theta)$ defined in Ref.[1 and 4], and the parameter $\theta$ is related to the inverse temperature $\beta$ through the expression $sinh\theta = f^{1/2}(\beta)$.

The time dependant operators $Q_i(t, \beta)$ and their conjugate momenta $L_i Q(t, \beta)$ which are found from the spectral sum of $Q_{i\omega}(t, \beta)$ must satisfy the Dirac bracket relationship

$$[Q_i(t, \theta), L_i \dot{Q}_i(t)] = i\hbar \tag{2.17a}$$

which implies for the variances the uncertainty inequality

$$\sigma(Q_i,\beta)\sigma(L\dot{Q}_i,\beta) \geq \hbar/2 \tag{2.17b}$$

This requires the fields to be renormalized so that the operators of physical interest are

$$\begin{aligned} q_1(t,\theta) &= Q_1(t,\theta)/K_2^{1/2}(1,2) \\ q_2(t,\theta) &= Q_2(t,\theta)/K_2^{1/2}(2,1) \end{aligned} \tag{2.18}$$

where

$$K_2(1,2) = (L_1/\pi)\int_o^{\Omega} d\omega\omega^2 \frac{[|z_2(\omega)|^2 R_1(\omega) + (L_{12}\omega^2)^2 R_2(\omega)]}{|Det Z(\omega)|^2} \tag{2.19}$$

The variances of these operators $q_i$ are defined according to the formula

$$\sigma^2(q_i)_{0\beta} = (\beta 0|q_i^2|0\beta) - (\beta 0|q_i|0\beta)^2. \tag{2.20}$$

where $|0\beta)$ represents the finite temperature vacuum state described in Ref.[1]. A similar definition is used for the operators $L_i\dot{q}_i$. From (2.10), (2.13), (2.14), and (2.15) one finds the results for the variances

$$\sigma^2(q_1, z, l_{12}) =$$
$$\frac{\hbar}{\pi K_2(1,2)} \int_o^{\omega} d\omega\omega coth(\frac{\hbar\omega}{2k_BT}) \frac{[|z_2(\omega)|^2 R_1(\omega) + |z_{12}(\omega)|^2 R_2(\omega)]}{|Det Z(\omega)|^2}$$
$$\sigma^2(L_1\dot{q}_1, z, l_{12}) =$$
$$\frac{\hbar}{\pi K_2(1,2)} \int_o^{\omega} d\omega\omega^3 coth(\frac{\hbar\omega}{2k_BT}) \frac{[|z_2(\omega)|^2 R_1(\omega) + |z_{12}(\omega)|^2 R_2(\omega)]}{|Det Z(\omega)|^2} \quad (2.21)$$

The analogous results for the second circuit are found from the above with the replacement $(1 \leftrightarrow 2)$. Although the results are not given, the same method can be easily used to obtain expressions for the corrolation functions of the type $(\beta 0|q_1(t)q_2(t)|0\beta)$.

To obtain a better understanding of the nature of the noise associated with interacting systems, the above model is simplified with the assumption that $R_1 = R_2 = R, L_1 = L_2 = L$, $C_1 = C_2 = C$, and $l = L_{12}/L$. For this case, numerical resuslts for the dimensionless variances $\sigma(q, z, l)(L\omega_o/\hbar)^{1/2}$ and $\sigma(L\dot{q}, z, l)/(\hbar L\omega_o)^{1/2}$ are given in Table I. As a result of the simplification, the variances which appear in (2.21) take on the special form

$$\sigma^2(q,z,l)(L\omega_o/\hbar) = \frac{K_1(q_o,z,l)}{2K_2(q_o,0,l)}$$
$$\sigma^2(L\dot{q},z,l)/\hbar L\omega_o = \frac{K_3(q_o,z,l)}{2K_2(q_o,0,l)} \tag{2.22}$$

where

$$K_m(q_o,z,l) = \int_o^\Lambda \frac{d\nu\nu^m[(1-\nu^2)^2+(\nu/q_o(\nu))^2+(l\nu^2)^2]coth(\nu/z)}{\pi q_o(\nu)D(\nu,q_o,l)} \tag{2.23a}$$

with

$$D(\nu,q_o,l) = [((1-\nu^2)^2-(\nu/q_o(\nu))^2-(l\nu^2)^2)^2+4(1-\nu^2)^2(\nu/q_o(\nu))^2] \tag{2.23b}$$

and

$$q_o(\nu) = (q_o\Lambda/\nu)tan^{-1}(\frac{\nu/\Lambda}{(1-(\nu/\Lambda)^2)^{1/2}}). \tag{2.23c}$$

In these expressions, the definitions $\nu = \omega/\omega_o, \Lambda = 2(C/C_o)q_o$, and $q_o = L\omega_o/R$ have been introduced. The results for the zero temperature ($z = 2k_BT/\hbar\omega_o = 0$) vacuum are found with the replacement $coth(\nu/z) \rightarrow 1$. Furthermore, in the large $q_o (R \rightarrow 0)$ limit, one finds the $LC$ oscillator result from (2.22) and (2.23). Finally, in the limit $l \rightarrow 0$, the expressions (2.22) and (2.23) reduce to the previously presented results of Ref.[4].

The numerical results given in TABLE 1. for the variances (2.22) are particularly interesting for the operators $q(t, \beta)$ where it can be seen for small values of the quality factor that there are values below the standard quantum limit value $2^{-1/2}$. This indicates the presence of a squeezed quantum state [6] that depends upon the strength of the mutual inductance. It has already been demonstrated [4 and 7] that a squeezed quantum state appears in the dissipative linear oscillator; however, here it can be seen from the present results that an increase in the strength of the mutual inductance coupling produces a tendency towards stronger squeezing.

## III. CONCLUSIONS

In this letter, I have used the methods of finite temperature field theory to obtain the variances for the charge and current in a dissipative $LRC$ circuit which interacts with a similar circuit through a mutual inductance coupling. An effective Lagrangian density has been used and a regularization method based on a semi-infinite transmission line has been used to produce frequency dependant damping. The dissipative nature of the problem appears as the result of the discontinuous coupling associated with the Heaviside distribution which appears in the Lagrangian density (2.3).

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TABLE I.

| $l$ | $(\frac{L\omega_o}{\hbar})^{1/2}\sigma(q,z,l)$ | $\frac{\sigma(L\dot{q},z,l)}{(\hbar L\omega_o)^{1/2}}$ |
|---|---|---|
| 0.1 | 0.685 | 0.757 |
| 0.2 | 0.683 | 0.773 |
| 0.3 | 0.680 | 0.800 |
| 0.4 | 0.676 | 0.842 |
| 0.5 | 0.666 | 0.924 |
| 0.6 | 0.662 | 1.004 |
| 0.7 | 0.648 | 1.063 |
| 0.8 | 0.613 | 1.393 |
| 0.9 | 0.573 | 1.797 |

$q_o = 5.0$; $z = 0.1$; $C/C_o = 100$

Table I. Numerical results for the dimensionless seconds moments.