The Optical Soliton Contents of some Special Input Pulses*

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Abstract

The optical soliton contents of some special input pulses and their Galilei transforms will be determined by solving the linear eigenvalue problem associated with the non-linear Schrödinger equation. The special cases discussed are the initial envelope function of width α and height β, the initial envelope function $-i\beta\exp(-\alpha|x|)$ and the super-Gaussian initial pulse. Throughout, we compare our problem to the Korteweg-de Vries problem where a good understanding can be gained through Sturm-Liouville theory.

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The question we are going to address ourselves to is , given an input pulse $u(x,0)$, does it contain solitons, and if so, what type of solitons. (Here, $u(x,0)$ is the change in time of the initial envelope function at the point where the pulse is injected into the fiber.) A satisfactory detailed answer to our question should help to choose or build the laser best suited to injecting solitons into optical fibers at lowest cost.

In principle, the optical soliton contents of an input pulse is determined by the L^2-integrable solutions of the linear eigenvalue problem [1]

$$A\overrightarrow{v} = \lambda \overrightarrow{v} \quad with \quad A = \begin{pmatrix} id/dx & u(x,0) \\ -u^*(x,0) & -id/dx \end{pmatrix} . \tag{1}$$

Satsuma and Yajima [2] have started a detailed study of this eigenvalue problem and have solved it for the special initial envelope function of sech(x) form. To gain a better understanding we want to add more solvable cases to the one discussed by Satsuma and Yajima.

To explain what we are trying to achieve finally let us compare the theory of optical solitons to that of Korteweg-deVries solitons. For Korteweg-deVries solitons the question analogous to the one addressed here is, given an initial water wave with amplitude $u(x,0)$, does it contain KdV solitons. The answer is yes, if the eigenvalue problem

$$\left[-\frac{d^2}{dx^2} + u(x,0)\right]\psi = \lambda\psi \tag{2}$$

has L^2-integrable solutions, i.e. if the Schrödinger equation (2) has bound state solutions.

It is well-known that in this case theory does not end with this general answer. Every student of quantum mechanics has a good understanding of the eigenvalue problem (2). He or she can certainly solve eq. (2) for a square well potential, and possibly for a $\text{sech}^2 x$ potential. An advanced student may even be able to use Sturm-Liouville theory to prove that if a potential is bounded above by a square well potential and bounded below by a $\text{sech}^2 x$ potential the same is true for the number of the corresponding bound states. Sturm-Liouville theory also yields, among other results, that the ground state has no nodes and that the number of nodes increases as the eigenvalues increase.

In the case of optical solitons, we want to achieve an understanding as deep as in the case of KdV solitons and begin by solving the eigenvalue problem for special input pulses. For these special pulses we determine the transition matrix, i.e. find $a(\lambda)$ and $b(\lambda)$ where

$$e^{-i\lambda x}\begin{pmatrix}1\\0\end{pmatrix} \underset{x\to-\infty}{\longleftarrow} \begin{pmatrix}v_1\\v_2\end{pmatrix} \underset{x\to+\infty}{\longrightarrow} a(\lambda)e^{-i\lambda x}\begin{pmatrix}1\\0\end{pmatrix} + b(\lambda)e^{i\lambda x}\begin{pmatrix}0\\1\end{pmatrix} \qquad (3)$$

with $\lambda = \kappa + i\eta$, $\eta > 0$. Note that if $u(x,0)$ has eigenvalue $\lambda = \kappa + i\eta$, then the Galilei transform $u(x,0)\exp(-iVx)$ has eigenvalue $\kappa + V/2 + i\eta$. Therefore, solving the eigenvalue problem for $u(x,0)$ yields a solution for the whole family of Galilei transforms as well.

In our first case, $u(x,0)$ is given by

$$iu(x,0) = \begin{cases} 0 & for\ \ |x| > \alpha/2 \\ \beta & for\ \ |x| \leq \alpha/2 \end{cases} \qquad \alpha, \beta > 0\,. \qquad (4)$$

(In ref. 3, the L^2-integrable solutions with pure imaginary eigenvalue have been found for this case and the next one discussed below.) For this initial envelope function, $a(\lambda)$ and $b(\lambda)$ read

$$a(\lambda) = \frac{e^{-i\lambda\alpha}}{\sqrt{\beta^2+\lambda^2}}\left(i\lambda\sin\sqrt{\beta^2+\lambda^2}\alpha - \sqrt{\beta^2+\lambda^2}\cos\sqrt{\beta^2+\lambda^2}\alpha\right)\,,$$

$$b(\lambda) = \frac{\beta}{\sqrt{\beta^2+\lambda^2}}\sin\sqrt{\beta^2+\lambda^2}\alpha\,. \qquad (5)$$

To find the eigenvalues, we set $a(\lambda)$ equal to zero. The conditions which follow are

$$\sqrt{\beta^2+\lambda^2}\alpha = \pm\alpha\beta\sin\sqrt{\beta^2+\lambda^2}\alpha\,,\ \ i\lambda = \pm\beta\cos\sqrt{\beta^2+\lambda^2}\alpha\,. \qquad (6)$$

For $\rho_1 + i\rho_2 := i\sqrt{\beta^2+\lambda^2}\alpha$, $\kappa \neq 0$, $\rho_1 \neq 0$, they lead to the equations

$$\cos\rho_2 = \pm\frac{\rho_1}{\alpha\beta\sinh\rho_1}\,,\ \ \cos\rho_2 = \mp\frac{\eta}{\beta\cosh\rho_1} \qquad (7)$$

which are controdictory for $\alpha > 0$, $\eta > 0$. The eigenvalues are therefore purely imaginary. (Arguments in ref. 2 go some way towards proving that this is always the case for even purely imaginary initial envelope functions. We do not think that the arguments are conclusive and prove in each special case that the eigenvalues are purely imaginary.)

If we define $\rho = \sqrt{\beta^2 - \eta^2}$, the conditions for the eigenvalues $\lambda = i\eta$ are

$$\rho^2 + \alpha^2\eta^2 = \alpha^2\beta^2 , \quad \eta = -\frac{\rho}{\alpha}\cot\rho . \tag{8}$$

These conditions imply that

$$N = \langle 1/2 + F/\pi \rangle \tag{9}$$

holds for the soliton number N, where

$$F = \int_{-\infty}^{+\infty} |u(x,0)|\, dx , \tag{10}$$

and $\langle ... \rangle$ denotes the integer smaller than the argument. In terms of F, N is the same in this case as in the case of an envelope function of $\mathrm{sech}(x)$ form [2].

As a second special case we solve the eigenvalue problem (1) for

$$iu(x,0) = \beta \exp(-\alpha|x|) , \quad \alpha, \beta > 0 , \tag{11}$$

by solving the second order equations corresponding to (1). For v_1, the first component of $\overrightarrow{v}$, the second order equation reads

$$v_1'' - \frac{u'}{u}v_1' + (\lambda^2 - i\lambda\frac{u'}{u} + |u|^2)v_1 = 0 . \tag{12}$$

The transformation $s = \beta e^{\alpha x}/\alpha$, $\psi = v_1/\sqrt{s}$ transforms this equation to Bessel's equation

$$\frac{d^2\psi}{ds^2} + \frac{1}{s}\frac{d\psi}{ds} + \left(1 - \frac{\nu^2}{s^2}\right)\psi = 0 , \tag{13}$$

where $\nu = -1/2 - i\lambda/\alpha$.

For $x < 0$, we thus find v_1 in terms of Bessel functions. By using eq. (1) we find v_2. An analogous derivation yields v_2 for $x > 0$ and through eq. (1) v_1. The matching conditions at $x = 0$ lead to

$$a(\lambda) = \frac{J_{\nu+1}^2(\frac{\beta}{\alpha}) - J_\nu^2(\frac{\beta}{\alpha})}{J_\nu(\frac{\beta}{\alpha})Y_{\nu+1}(\frac{\beta}{\alpha}) - Y_\nu(\frac{\beta}{\alpha})J_{\nu+1}(\frac{\beta}{\alpha})} ,$$

$$b(\lambda) = \frac{J_\nu(\frac{\beta}{\alpha})Y_\nu(\frac{\beta}{\alpha}) - J_{\nu+1}(\frac{\beta}{\alpha})Y_{\nu+1}(\frac{\beta}{\alpha})}{J_\nu(\frac{\beta}{\alpha})Y_{\nu+1}(\frac{\beta}{\alpha}) - Y_\nu(\frac{\beta}{\alpha})J_{\nu+1}(\frac{\beta}{\alpha})} \,. \tag{14}$$

The condition the eigenvalues have to satisfy is therefore

$$J_\nu(\frac{\beta}{\alpha}) = \pm J_{\nu+1}(\frac{\beta}{\alpha}) \,. \tag{15}$$

That condition (15) cannot be satisfied for non-real ν with $Re\,\nu > -1/2$ can be proved as follows [4]: Assume that $J_\nu \pm J_{\nu+1}$ has a real zero s for non-real ν. Then using the Mittag-Leffler expansion [5, p.497]

$$1 \pm \sum_{n=1}^{\infty} \frac{2s}{j_{\nu n}^2 - s^2} = 0 \,, \tag{16}$$

and therefore,

$$\sum_{n=1}^{\infty} \frac{Re\, j_{\nu n}\ Im\, j_{\nu n}}{|j_{\nu n}^2 - s^2|^2} = 0 \tag{17}$$

follows, where $j_{\nu n}$ are the zeros of $s^{-\nu}J_\nu(s)$. This equation cannot hold because $Re\, j_{\nu n}/Im\, j_{\nu n} \gtrless 0$ for $\kappa \lessgtr 0$ for all n with $Im\, j_{\nu n} \neq 0$ [6], and because there are $j_{\nu n}$ with $Re\, j_{\nu n} \neq 0$ and $Im\, j_{\nu n} \neq 0$.

We are left with studying the points of intersection of J_ν and $\pm J_{\nu+1}$, which we denote as $s_n(\nu)$, for real order $\nu = \eta/\alpha - 1/2 > -1/2$. It is easy to prove that labelling the points of intersection by $s_n(\nu)$ makes sense because, if ν changes, the number of points of intersection stays the same, and s_n changes continuously with ν. Furthermore, $s_n \to \infty$ for $n \to \infty$ and for $\nu \to \infty$, and s_n increases monotonically with ν[4; consequence of Lemmas 2.3 and 2.5 in ref. 7]. This implies that $s_n(-1/2) = (2n-1)\pi/2$ determines the soliton number, which, in terms of F, turns out to be again given by eq. (9).

In all three solvable examples the number of L^2-integrable solutions is given by the same formula in terms of the pulse area. This is a much simpler result than one would expect from studying the Schrödinger equation in the Korteweg-deVries case. In both cases one expects that the "stronger" the potential the easier it is to "pull down" exponentially increasing solutions at minus infinity and turn them into exponentially decreasing solutions at plus infinity, i.e. the number of solitons should increase with the "strength" of the potential. However, for the Schrödinger equation no formula as simple as eq. (9) exists.

The input pulses studied so far are not very good models of realistic input pulses. That is why, as another example, we discuss the super-Gaussian pulse

$$u(x,0) = A_o \exp\left[-\frac{1}{2}(1-i\alpha)(\frac{x}{\sigma})^{2m}\right] . \tag{18}$$

To solve the eigenvalue problem (1), we set

$$v_1 = e^{-i\lambda x}\sum_{n=0}^{\infty} u_n(x) , \quad u_0 = 1 , \tag{19}$$

and determine the functions u_n recursively as

$$u_n(x) = \frac{A_o^2}{2i\lambda}\int_{-\infty}^{x} e^{-\frac{1}{2}(1-i\alpha)(s/\sigma)^{2m}} \int_{-\infty}^{s} e^{-\frac{1}{2}(1+i\alpha)(t/\sigma)^{2m}} u_{n-1}(t)\, dt\, ds . \tag{20}$$

By induction, we can prove that

$$|u_n(x)| \le \frac{\sqrt{\pi} A_o^{2n} \sigma^{2(2n-1)m} e^{-n(\epsilon/\sigma)^{2m}\epsilon}}{2^n(n-1)!\,\Gamma(\frac{2n+1}{2})|\lambda|^n \epsilon^{4nm}} \tag{21}$$

holds for $x \in (-\infty, -\epsilon)$, $\epsilon > 0$. Using the Weierstrass M-test we conclude that the series (19) is convergent on $(-\infty, -\epsilon)$, $\epsilon > 0$. Then, eq. (1) allows us to calculate v_2 on $(-\infty, -\epsilon)$. Analogously, we find v_2 and v_1 in form of convergent series on (ϵ, ∞).

From a physical point of view, all that remains is to match the functions at $x = 0$ for realistic parameters α, σ and m, which model pulses from a semiconductor laser. To solve the mathematical eigenvalue problem completely, an analysis like the one we have done for the intersections of J_ν and $J_{\nu+1}$ has to be added. Only then can the interesting question, whether eq. (9) holds in this case as well, be answered rigorously.

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